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A GLOBALLY CONVERGENT SIMPLICIAL ALGORITHM FOR STATIONARY POINT PROBLEMS ON POLYTOPES

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by

A.J.J. Talman[1] and Y. Yamamoto[2]

Abstract: We propose a simplicial variable dimension restart algorithm for the stationary point problem or variational inequality problem: given a convex polytope C of R^n and a continuous function $f: C \to R^n$, find a point $\hat{x}$ of C such that $f(\hat{x}).\hat{x} \geqq f(\hat{x}).x$ for any point x in C. The algorithm is globally and finitely convergent. Namely, starting from an arbitrary point in C, it always gives an approximate stationary point after a finite number of function evaluations and linear programming pivot operations. The algorithm leaves the starting point v along one of the directions pointing to the vertices of C according to the optimum solution of the linear programming problem maximize $f(v).x$ subject to $x \in C$. In general the algorithm follows a piecewice linear path of points x satisfying for some t between 0 and 1, $x \in (1-t)\{v\} + tC$ and $\bar{f}(x).x \geqq \bar{f}(x).z$ for all $z \in (1-t)\{v\} + tC$, where $\bar{f}$ is the piecewise linear approximation to f with respect to some specific triangulation of C. The algorithm terminates at the moment it hits the boundary of C or it finds an approximate zero of f.

Key words: stationary point, variational inequality, simplicial algorithm, piecewise linear approximation, global convergence, triangulation

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1. INTRODUCTION

In order to compute zero points of continuous functions on the Euclidean space R^n a lot of so-called simplicial variable dimension algorithms have been introduced. Such an algorithm subdivides R^n into n-dimensional simplices and searches for a simplex which yields an approximate zero point or solution. More precisely, starting in an arbitrarily chosen grid point of the triangulation the algorithm generates, by alternating linear programming pivot steps in a system of typically n+1 linear equations and replacement steps in the triangulation, a sequence of adjacent simplices of varying dimension. Given some coercivity condition the algorithm then generates within a finite number of steps an approximate solution. When the accuracy at the approximate solution is not satisfactory, the algorithm can be restarted at the approximate solution with a finer triangulation in the hope that within a few number of iterations a better approximate solution is found, etc.

The several number of simplicial variable dimension restart algorithms differ from each other in the number of rays along which the algorithm may leave the starting point, say v, with one-dimensional simplices. Such an algorithm with n+1 rays, the so-called (n+1)-ray algorithm, was introduced in van der Laan and Talman [8]. The 2n-ray algorithm was also introduced in [8], the 2^n-ray algorithm in [2], and the (3^n-1)-ray algorithm in [6]. A unifying approach for these algorithms was given in van der Laan and Talman [9], see also Yamamoto [15]. In [9], the piecewise linear (abbreviated by pl) path traced by the algorithms when generating the sequence of adjacent simplices of varying dimension is interpreted as a curve of stationary points to the underlying problem with respect to an expanding set containing the starting point v in its interior. In fact the expanding set differs for each algorithm and is a polytope for which the number of vertices is equal to the number of rays along which the algorithm may leave the starting point. More precisely, let f be the function from R^n into R^n whose zero point is to be computed and let $\bar{f}$ be the pl approximation to f with respect to the underlying triangulation, i.e., $\bar{f}(x) = \Sigma_{i=1}^{n+1} \lambda_i f(w^i)$ if $x = \Sigma_{i=1}^{n+1} \lambda_i w^i$ lies in the n-dimensional simplex with vertices $w^1,\dots,$

w^{n+1}. Then a simplicial variable dimension algorithm traces the pl path of points x satisfying for some t, t ≥ 0,

$$\bar{f}(x).x \geq \bar{f}(x).z \text{ for any } z \in \{v\} + tB \tag{1.1}$$

which originates for t = 0 at x = v, where B is the appropriately chosen polytope containing the point $0 \in R^n$ in its interior. Of course t needs not to be monotonic on the path and the path terminates with an approximate solution $\bar{x}$ as soon as $\bar{f}(\bar{x}) = 0$. In case of the (n+1)-ray algorithm the set B is equal to an n-simplex. For the 2n-ray algorithm the set is the n-dimensional octahedron, for the 2^n-ray algorithm it is the n-dimensional unit cube, and for the (3^n-1)-ray algorithm the set B is a polytope with 3^n-1 vertices. For all these algorithms the so-called K'-triangulation of R^n, due to Todd [13], might be the underlying triangulation.

In order to solve the nonlinear complementarity problem or the stationary point problem on the unit simplex and on the product space S of unit simplices, called the simplotope, also several variable dimension algorithms have been developed in the last couple of years. These algorithms can be seen as adaptions or generalizations of the algorithms mentioned above for problems on R^n. The main difference between these methods on R^n and S is that the set R^n is unbounded whereas S is bounded. In the latter case the solution of the underlying problem may be on the boundary of S so that the triangulation must be chosen in such a way that a triangulation of S itself is obtained. The product-ray algorithm on S presented in Doup and Talman [1] traces a pl path of points satisfying the same condition as in (1.1) with the set B equal to S-{v}, i.e., this algorithm traces for varying t, 0 ≤ t ≤ 1, the pl path of points x in (1-t){v} + tS satisfying

$$\bar{f}(x).x \geq \bar{f}(x).z \text{ for any } z \in (1-t)\{v\} + tS \tag{1.2}$$

which originates for t = 0 at v and terminates for t = 1 with an approximate solution to the stationary point problem on S of finding an $\hat{x}$ in S such that

$$f(\hat{x}).\hat{x} \geq f(\hat{x}).z \text{ for any } z \in S. \tag{1.3}$$

Here $\bar{f}$ is again the pl approximation to the continuous function f on the simplotope S with respect to some specific triangulation of S and v is the arbitrarily chosen starting point in S of the algorithm. This so-called V-triangulation of S seems to be the most natural triangulation of S to underly the algorithm and is in fact a generalization of the K'-triangulation of R^n to S. The V-triangulation of S may also underly the other simplicial variable dimension algorithms on S (see e.g. [2]). The other algorithms on S, however, trace their p.l. path with respect to a different expanding set. Whereas for the zero point problem on R^n the set B can be chosen freely, a natural set for problem (1.3) seems to be $S-\{v\}$. The number of rays of the product-ray algorithm on S is equal to the number of vertices of S. The algorithm leaves the starting point v in the direction of one of the vertices of S depending on the values of the components of f(v).

In this paper we will combine the ideas of a simplicial variable dimension algorithm on R^n and the product-ray algorithm on the simplotope S to obtain a simplicial variable dimension restart algorithm for solving the stationary point problem on an n-dimensional convex polytope $C = \{x \in R^n | a^i.x \leq b_i \text{ for } i=1,\ldots,m\}$ in R^n of finding a point $\hat{x}$ in C for which

$$f(\hat{x}).\hat{x} \geq f(\hat{x}).z \text{ for any } z \in C, \tag{1.4}$$

where f is a continuous function from C to R^n. This problem arises e.g. from economic equilibrium problems, noncooperative games, traffic assignment problems and nonlinear optimization problems (see e.g. [4] and [12]). In each application the polytope C must be chosen in an appropriate way.

The algorithm to be presented in this paper will generate the pl path of points x satisfying for some t, $0 \leq t \leq 1$,

$$\begin{aligned} & x \in (1-t)\{v\} + tC \\ \text{and} & \\ & \bar{f}(x).x \geq \bar{f}(x).z \text{ for any } z \in (1-t)\{v\} + tC \end{aligned} \tag{1.5}$$

which originates for $t = 0$ at v and terminates for $t = 1$ with an approximate solution to (1.4). Again v is the arbitrarily chosen starting point in C and $\bar{f}$ is the pl approximation of f with respect to some specific triangulation of C. The underlying triangulation is a generalization of the V-triangulation of S and its mesh size can be made arbitrarily small by taking the grid size small enough. The algorithm can be restarted from the found approximate solution with a finer triangulation in order to find a better approximate solution hopefully within a few number of iterations.

The algorithm leaves the point v in the direction of one of the vertices of C. In order to determine which vertex first the linear programming problem

$$\text{maximize } f(v).z \text{ subject to } z \in C$$

has to be solved. Barring degeneracy there is a unique vertex u of C which optimizes this problem. This means that $f(v)$ is in the relative interior of the cone $\{y \mid y = \Sigma_{i\in I} \mu_i a^i,\ \mu_i > 0 \text{ for } i \in I\}$, where $I= \{i \mid 1 < i < m,\ a^i.u = b_i\}$. When the parameter t is close to zero, $\bar{f}((1-t)v+tu)$ is also close to $f(v)$ and remains in this cone. Therefore for small t the point $(1-t)v+tu$ is an optimum solution of the problem

$$\text{maximize } \bar{f}((1-t)v+tu).z \text{ subject to } z \in (1-t)\{v\} + tC.$$

That is, $(1-t)v+tu$ is a solution to (1.5) as long as t is small enough. If t attains 1 while the function value $\bar{f}((1-t)v+tu)$ remains in the cone, we find that the vertex u is an (approximate) stationary point. Otherwise $\bar{f}((1-t)v+tu)$ hits the boundary of the cone at some $\bar{t} < 1$. This means that the optimum solutions of the problem

$$\text{maximize } \bar{f}((1-\bar{t})v+\bar{t}u).z \text{ subject to } z \in (1-\bar{t})\{v\} + \bar{t}C$$

form a one-dimensional face $(1-\bar{t})\{v\} + \bar{t}F$ of $(1-\bar{t})\{v\} + \bar{t}C$ having $y = (1-\bar{t})v + \bar{t}u$ as a vertex, where F is some 1-face of C. When v lies in F the point y is an approximate solution. Otherwise the algorithm turns to the direction of the face F of C and moves in the two-dimensional set $\{(1-t)\{v\} + tF \mid t \in [0,1]\}$. In this way the pl path leaves the starting point v along one of the directions pointing to the vertices of C and then steps on to some two-dimensional set. In general, the algorithm follows the pl path given in (1.5) by generating for varying boundary faces F of C points x in the set $vF = \{(1-t)\{v\} + tF \mid t \in [0,1]\}$ such that $\bar{f}(x)$ lies in the cone $\{y \mid y = \Sigma_{i \in I} \mu_i a^i, \mu_i \geq 0 \text{ for } i \in I\}$ with $I = \{i \mid 1 \leq i \leq m, a^i.x = b_i \text{ for any point } x \text{ in } F\}$. In fact this pl path is followed by making linear programming pivot steps and replacement steps with respect to a sequence of adjacent simplices of varying dimension. The dimension of the sets vF where the path moves is not necessarily monotonic. The algorithm terminates with an approximate solution x when t becomes 1 or $\bar{f}(x)$ becomes 0 or F becomes such that v lies in F.

The organization of this paper is as follows. In Section 2 we review the unifying framework for restart fixed point algorithms based on the primal-dual pair of subdivided manifolds proposed in Kojima and Yamamoto [5]. In Section 3 we specify the primal-dual pair of subdivided manifolds for our algorithm and present the basic system. We also prove the convergence of the algorithm and show the accuracy of the approximate solution obtained. In Section 4 we give a formal description of the algorithm under the assumption that the polytope C is simple and the linear inequalities defining the polytope are nonredundant. Section 5 is devoted to the explanation of the new triangulation of the polytope C underlying the algorithm. In Section 6 we discuss the relation between the basic idea in this section and the algorithm and we consider some special cases.

2. PRELIMINARIES

In this section we give a brief description of the subdivided manifold, the basic theorem for simplicial algorithms and the primal-dual pair of subdivided manifolds.

We call a convex polyhedral set a cell. A cell of dimension m is abbreviated by m-cell. If a cell B is a face of a cell C, we write $B < C$.

Let M be a finite or countable collection of m-cells. We denote $\{B \mid B$ is a face of some m-cell of $M\}$ by $\bar{M}$ and $\cup\{C \mid C \in M\}$ by $|M|$. We call M a subdivided m-manifold if and only if

(2.1) for any $B, C \in M$, $B \cap C = \emptyset$ or $B \cap C < B$ and C,

(2.2) for each (m-1)-cell B of $\bar{M}$ at most two m-cells of M have B as a facet,

(2.3) M is locally finite: each point x of $|M|$ has a neighborhood which intersects only a finite number of m-cells of M.

We call the collection of (m-1)-cells of $\bar{M}$ that lie in exactly one m-cell of M the boundary of M and denote it by ∂M.

A continuous function H from $|M|$ into some Euclidean space is said to be a pl function on M if the restriction of H to each cell of M is an affine function. For a subdivided (n+1)-manifold M and a pl function H on M into R^n we say that $c \in R^n$ is a regular value of $H : |M| \to R^n$ if $B \in \bar{M}$ and $H^{-1}(c) \cap B \neq \emptyset$ always imply $\dim H(B) = n$.

The following theorem is a basic theorem for fixed point algorithms (see Eaves [3]).

<u>Theorem 2.1.</u> Let M be a subdivided (n+1)-manifold, H be a pl function on M into R^n. Suppose that $c \in R^n$ is a regular value of H. Then $H^{-1}(c)$ is a disjoint union of paths and loops, where a path is a subdivided 1-manifold homeomorphic to one of the intervals (0,1), (0,1] and [0,1] and a loop is a subdivided 1-manifold homeomorphic to the 1-dimensional sphere. Furthermore $H^{-1}(c)$ satisfies the following conditions.

(2.4) $H^{-1}(c) \cap B$ is either empty or a 1-cell for each $B \in \mathcal{M}$.

(2.5) A loop of $H^{-1}(c)$ does not intersect $|\partial\mathcal{M}|$.

(2.6) If a path S of $H^{-1}(c)$ is compact, the boundary ∂S of S consists of two distinct points in $|\partial\mathcal{M}|$.

Let $\mathcal{P}$ and $\mathcal{D}$ be subdivided manifolds. If $\mathcal{P}$ and $\mathcal{D}$ satisfy the following conditions with some positive integer m and an operator d: $\bar{\mathcal{P}} \cup \bar{\mathcal{D}} \to \bar{\mathcal{P}} \cup \bar{\mathcal{D}} \cup \{\emptyset\}$, we say that $(\mathcal{P},\mathcal{D};d)$ is a primal-dual pair of subdivided manifolds (abbreviated by PDM) with degree m.

(2.7) for every $X \in \bar{\mathcal{P}}$, $X^d = \emptyset$ or $X^d \in \bar{\mathcal{D}}$.

(2.7)' for every $Y \in \bar{\mathcal{D}}$, $Y^d = \emptyset$ or $Y^d \in \bar{\mathcal{P}}$.

(2.8) if $Z \in \bar{\mathcal{P}} \cup \bar{\mathcal{D}}$ and $Z^d \neq \emptyset$, then $(Z^d)^d = Z$ and $\dim Z + \dim Z^d = m$.

(2.9) if $X_1, X_2 \in \bar{\mathcal{P}}$, $X_1 < X_2$, $X_1^d \neq \emptyset$ and $X_2^d \neq \emptyset$, then $X_2^d < X_1^d$.

(2.9)' if $Y_1, Y_2 \in \bar{\mathcal{D}}$, $Y_1 < Y_2$, $Y_1^d \neq \emptyset$ and $Y_2^d \neq \emptyset$, then $Y_2^d < Y_1^d$.

We call the operator d the dual operator.

For a PDM $(\mathcal{P},\mathcal{D};d)$ with degree m let

$$\langle\mathcal{P},\mathcal{D};d\rangle = \{X \times X^d \mid X \in \bar{\mathcal{P}},\ X^d \neq \emptyset\},$$

or equivalently

$$\langle\mathcal{P},\mathcal{D};d\rangle = \{Y^d \times Y \mid Y \in \bar{\mathcal{D}},\ Y^d \neq \emptyset\}.$$

Then we have the following theorems. See Kojima and Yamamoto [5] for the proofs and more details.

Theorem 2.2. (Theorem 3.2 and 3.3 in [5]) Let $(\mathcal{P},\mathcal{D};d)$ be a PDM with degree m. Then $\mathcal{L} = \langle\mathcal{P},\mathcal{D};d\rangle$ is a subdivided m-manifold and

$$\partial\mathcal{L} = \{X \times Y \mid X \times Y \text{ is an (m-1)-cell of } \bar{\mathcal{L}},\ X \in \bar{\mathcal{P}},\ Y \in \bar{\mathcal{D}} \text{ and either } X^d \text{ or } Y^d \text{ is empty}\}.$$

Let $\mathcal{Q}$ be a refinement of $\mathcal{P}$, namely $\mathcal{Q}$ is a subdivided manifold of the same dimension as $\mathcal{P}$, each cell of $\mathcal{Q}$ is contained in some cell of $\mathcal{P}$ and $|\mathcal{Q}| = |\mathcal{P}|$. For each cell X of $\bar{\mathcal{P}}$ let

$$X = \{\sigma \; \sigma \in \bar{Q}, \; \sigma \subset X, \; \dim \sigma = \dim X\}.$$

Theorem 2.3. (Theorem 4.1 in [5]) Let $(P,D;d)$ be a PDM with degree m and Q be a refinement of P. Then

$$M = \{\sigma \times Y | Y \in \bar{D}, \; Y^d \neq \emptyset, \; \sigma \in Q|Y^d\}$$

is a subdivided m-manifold and a refinement of $L = \langle P,D;d\rangle$.

Note that ∂M is also a refinement of ∂L and $|\partial M| = |\partial L|$.

Now consider a PDM $(P,D;d)$ with degree n+1, a refinement Q of P and a pl function $F : |Q| \to R^n$. Let

$$H(x,y) = y - F(x) \text{ for each } (x,y) \in |M| , \tag{2.10}$$

where M is the refinement of $L = \langle P,D;d\rangle$ in Theorem 2.3. Then H is a pl function on M. If we assume that $0 \in R^n$ is a regular value of H, then we can apply Theorem 2.1 to the system of pl equations

$$H(x,y) = 0, \; (x,y) \in |M| . \tag{2.11}$$

This system is a basic model of the class of variable dimension algorithms and also gives the foundation of the algorithm to be presented.

3. THE ALGORITHM FOR STATIONARY POINT PROBLEMS

Before giving the PDM for our algorithm we rewrite the stationary point problem (1.4). Let $\mathcal{F}$ be the collection of all faces of the polytope C. For each face $F \in \mathcal{F}$ let F^* be the set of all n-dimensional coefficient vectors y such that any point of F is an optimum solution of the linear programming problem

$$\text{maximize } y \cdot x \text{ subject to } x \in C.$$

Then the stationary point problem on C is the problem of finding a point $\hat{x}$ in C such that

$$f(\hat{x}) \in \cup\{F^* \mid \hat{x} \in F \in \mathcal{F}\}.$$

By the inclusion reversing property of F and F^* it is equivalent to $f(\hat{x}) \in F^*$ for a minimum face F of C having the point $\hat{x}$. Note that if we define $I = \{i \mid 1 \leq i \leq m,\ a^i \cdot x = b_i \text{ for any point } x \text{ in } F\}$, then $F^* = \{y \mid y = \Sigma_{i \in I}\, \mu_i a^i,\ \mu_i \geq 0 \text{ for } i \in I\}$, and also $C^* = \{0\}$.

Now let v be a starting point in C of the algorithm. Take an initial guess of a stationary point as v. Since an initial guess normally lies on the boundary of C, we allow the starting point v to lie on the boundary of C. For a face F of C which does not have the starting point v let vF be the join of v and F, i.e.,

$$vF = \{x \mid x = \alpha v + (1-\alpha)z \text{ for some } z \in F \text{ and some } \alpha \in [0,1]\}.$$

Note that dim vF = dim F + 1. To make a PDM we define

$$\mathcal{P} = \{vF \mid v \notin F \in \mathcal{F},\ \dim F = n - 1\}. \tag{3.1}$$

Then $\mathcal{P}$ is a subdivided n-manifold and

$$\bar{\mathcal{P}} = \{vF \mid v \notin F \in \mathcal{F}\} \cup \{F \mid v \notin F \in \mathcal{F}\} \cup \{\{v\}\}. \tag{3.2}$$

It should be noted that

$$|P| = C. \tag{3.3}$$

Figure 3.1 shows two examples of P for two distinct starting points, where the convex polytope C is a pentagon defined by five linear inequalities $a^i.x \leq b_i$, $i=1,\dots,5$ and $F(I) = \{x \mid x \in C,\ a^i.x = b_i$ for $i \in I\}$.

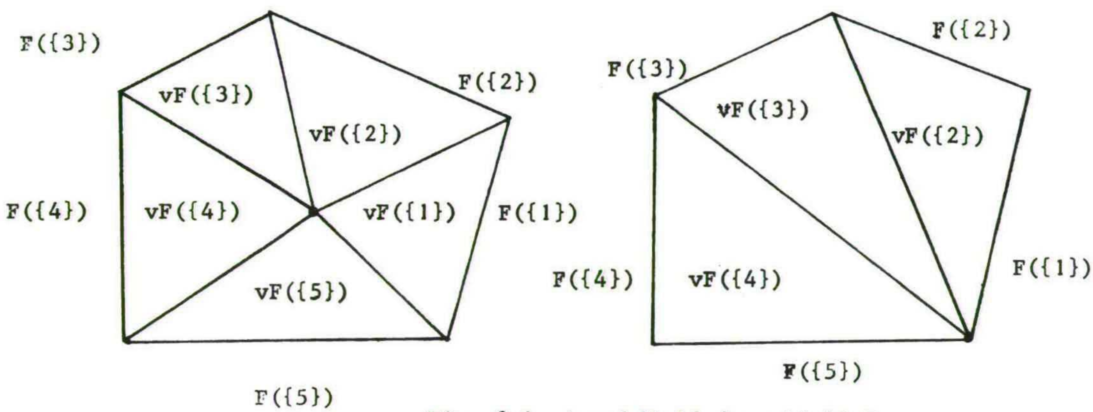


Fig. 3.1. A subdivided manifold P

Let

$$D = \{F^* | F \in F,\ \dim F = 0\}$$

$$= \{\ \{u\}^* | u \text{ is a vertex of } C\}. \tag{3.4}$$

Then D is also a subdivided n-manifold and

$$\bar{D} = \{F^* | F \in F\}, \tag{3.5}$$

$$|D| = R^n. \tag{3.6}$$

Now let the dual operator d be defined by

$$
\begin{aligned}
(vF)^d &= F^* \text{ if } v \notin F \in \mathcal{F} \\
F^d &= \emptyset \text{ if } v \notin F \in \mathcal{F} \\
\{v\}^d &= \emptyset \\
(F^*)^d &= vF \text{ if } v \notin F \in \mathcal{F} \\
&= \emptyset \text{ if } v \in F \in \mathcal{F}.
\end{aligned}
$$

It is readily seen that $(\mathcal{P},\mathcal{D};d)$ is a PDM with degree $n+1$. By Theorem 2.2 we have the following lemma.

Lemma 3.1. Let $\mathcal{L}=\langle\mathcal{P},\mathcal{D};d\rangle$. Then $\mathcal{L}$ is a subdivided $(n+1)$-manifold and

$$
\begin{aligned}
\partial\mathcal{L} = &\ \{\ \{v\} \times \{u\}^* \mid u \text{ is a vertex of C, } u \neq v\} \\
&\cup\{F \times F^* \mid v \notin F \in \mathcal{F}\} \qquad (3.7) \\
&\cup\{vE \times F^* \mid v \in F \in \mathcal{F},\ \dim F > 0,\ E \text{ is a facet of } F \\
&\qquad \text{and } v \notin E\}.
\end{aligned}
$$

Proof. We only prove (3.7). Suppose that an n-cell $X \times Y$ lies in the boundary $\partial\mathcal{L}$ of $\mathcal{L}$. Then by Theorem 2.2 $\dim X + \dim Y = n$ and either $X^d = \emptyset$ or $Y^d = \emptyset$. Suppose first $X^d = \emptyset$. Then the unique $(n+1)$-cell of $\mathcal{L}$ having $X \times Y$ is $Y^d \times Y$. When $X = \{v\}$, Y^d is a 1-cell of $\mathcal{P}$ having $\{v\}$ as a facet, i.e., $Y^d = v\{u\}$ for some vertex u of C with $u \neq v$. Therefore $Y = (v\{u\})^d = \{u\}^*$. When $X = F$ for some face $F \in \mathcal{F}$ with $v \notin F$, $Y^d = vF$. Therefore $Y = F^*$. Next suppose $Y^d = \emptyset$, i.e., $Y = F^*$ for some face F of C having v. Then $X \times Y$ lies in $X \times X^d$ and $X^d = E^*$ for some face E of C such that $v \notin E$ and E^* has F^* as a facet. Therefore $X = (E^*)^d = vE$. By the inclusion reversing property of F and F^* we see that E is a facet of F and $\dim F > 0$.

Since it is readily seen that these cells above are members of $\partial\mathcal{L}$, we have proved (3.7). □

Corollary 3.2.

$$|\partial L| = (\{v\} \times \cup(\{u\}^* | u \text{ is a vertex of C, } u \neq v))$$

$$\cup(\cup(F \times F^* | v \notin F \in \mathcal{F}))$$

$$\cup(\cup(F \times F^* | v \in F \in \mathcal{F}, \dim F > 0)). \tag{3.8}$$

If the starting point v is not a vertex of C, then

$$|\partial L| = (\{v\} \times R^n) \cup(\cup(F \times F^* | F \in \mathcal{F})). \tag{3.9}$$

Proof. Note that the set $\cup(vE | E$ is a facet of F and $v \notin E)$ is equal to F when F has the starting point v. Then we see that the third class of cells in (3.7) gives the third subset of $|\partial L|$ in (3.8). (3.9) is readily obtained from (3.6) and (3.8). □

Now let T be a triangulation of C such that the restriction $T|X$ also triangulates X for each cell X of $\bar{P}$. A specific triangulation is described in detail in Section 5. Then T is a refinement of P. We denote by M the subdivided (n+1)-manifold defined by the refinement T of P, D and the dual operator d (see Theorem 2.3). Let $\bar{f}$ be a pl approximation of the function f with respect to the triangulation T and let

$$H(x,y) = y - \bar{f}(x) \text{ for each } (x,y) \in |M|. \tag{3.10}$$

Then the function $H:|M| \to R^n$ is a pl function on M. We consider the system of pl equations

$$H(x,y) = 0, \ (x,y) \in |M| \tag{3.11}$$

as the basic model of our algorithm. By applying Theorem 2.1 to (3.11) we have the following main theorem.

Theorem 3.3. Suppose that the starting point v in G is not a stationary point. Then $(v,f(v))$ lies in $H^{-1}(0) \cap |\partial M|$. Suppose further that $0 \in R^n$ is a regular value of the function $H:|M| \to R^n$. Then the connected component S of $H^{-1}(0)$ having $(v,f(v))$ is a path and it leads to a point $(x,\bar{f}(x))$ in $|\partial M|$ such that x is in C and $\bar{f}(x)\cdot x \geq \bar{f}(x)\cdot z$ for any point z in C, i.e., x is a stationary point of the pl approximation $\bar{f}$ of f.

Proof. Since the starting point v is not a stationary point, f(v) does not lie in F^* for any face F of C having the point v. Therefore whether v may be a vertex of C or not, we have by (3.8) that $(v,f(v))$ is in $\{v\} \times \cup\{\{u\}^* \mid u \text{ is a vertex of } C,\ u \neq v\} \subset |\partial M|$. Since the point v is a vertex of the triangulation T, $\bar{f}(v) = f(v)$ and consequently $(v,f(v)) \in H^{-1}(0)$.

By (2.5) in Theorem 2.1 the connected component S of $H^{-1}(0)$ having $(v,f(v))$ is a path. Since $\bar{f}$ is continuous and C is compact, $\bar{f}(C)$ is also compact. Hence $H^{-1}(0)$ is bounded and so is S. It is easily seen that the intersection of S and each cell of M is a bounded 1-cell if it is not empty. By the local finiteness property (2.3) of M S intersects finitely many cells of M. Therefore S is compact and by (2.6) of Theorem 2.1 ∂S consists of two points in $|\partial M|$,one of which is $(v,f(v))$. Let (x,y) be the other point of ∂S and suppose $(x,y) \in \{v\} \times \{\cup\{u\}^* \mid u \text{ is a vertex of } C,\ u \neq v\}$. Since $(x,y) \in S \subset H^{-1}(0)$, $y = \bar{f}(x) = f(v)$. This contradicts that $(x,y) \neq (v,f(v))$. Then by Corollary 3.2 we have $(x,y) = (x,\bar{f}(x))$ lies in $\cup\{F \times F^* \mid F \in \mathcal{F}\}$. This implies that x is a stationary point for $\bar{f}$. □

Thus we have seen that we obtain an approximate stationary point x by tracing the pl path S from $(v,f(v))$. If $f(x)$ happens to lie in F^* for some face F of C having the point x, then x is a stationary point for f. Otherwise it is only an approximate stationary point. If the distance between $f(x)$ and F^* is not satisfactorily small, we take x as a new starting point, take a finer triangulation of C and restart the algorithm. In the following lemma we give the accuracy of the approximate solution x.

Lemma 3.4. Let $\gamma = \sup\{\operatorname{diam} f(\sigma) \mid \sigma \in T\}$, where $\operatorname{diam} B = \sup\{\|z^1 - z^2\| \mid z^1, z^2 \in B\}$. Let x be an approximate stationary point obtained by the algorithm, i.e., $x \in F$ and $\bar{f}(x) \in F^*$ for some face F of C. Then $f(x)$ lies in the γ-neighborhood of F^*.

Proof. Let $w^1,\ldots,w^{t+1}$ be the vertices of the simplex of T having x. Then $\bar{f}(x) = \Sigma_{j=1}^{t+1} \lambda_j f(w^j)$, where $\lambda_1,\ldots,\lambda_{t+1}$ are the convex combination coefficients such that $x = \Sigma_{j=1}^{t+1} \lambda_j w^j$ and $\Sigma_{j=1}^{t+1} \lambda_j = 1$. Therefore

$$\|\bar{f}(x) - f(x)\| = \|\Sigma_{j=1}^{t+1} \lambda_j f(w^j) - f(x)\| = \|\Sigma_{j=1}^{t+1} \lambda_j (f(w^j)$$

$$- f(x))\| \leq \Sigma_{j=1}^{t+1} \lambda_j \|f(w^j) - f(x)\| \leq \gamma.$$

□

Since the polytope C is compact and f is continuous on C, the error γ goes to zero as the mesh size $\delta = \sup \{\text{diam } \sigma \mid \sigma \in T\}$ of the triangulation T goes to zero. Let x^h be an approximate stationary point and γ^h be the error in Lemma 3.4 for a triangulation with mesh size δ^h. Suppose δ^h converges to zero as h goes to infinity. Then the sequence $\{x^h \mid h=1,2,...\}$ has a cluster point $\bar{x}$ in C. For the simplicity of notations we assume that this sequence itself converges to $\bar{x}$. Since the number of faces of C is finite, there is a face F of C and a subsequence $\{z^h \mid h=1,2,...\}$ such that $z^h \in F$ and $f(z^h)$ is in the γ^h-neighborhood of F^* for all h. Therefore by the closedness of F and F^* we obtain that $\bar{x} \in F$ and $f(\bar{x}) \in F^*$.

<u>Theorem 3.5.</u> Let x^h be an approximate stationary point obtained by the algorithm on a triangulation with mesh size δ^h for h=1,2,... . Suppose δ^h converges to zero as h goes to infinity. Then the sequence $\{x^h \mid h=1,2,...\}$ has a cluster point and any cluster point is a stationary point.

In section 6 we will show that the path S when projected on C yields the pl path of points satisfying (1.5) which originates for t = 0 at the point v and terminates with an approximate solution x.

4. DESCRIPTION OF THE ALGORITHM

In this section we will give a formal description of the algorithm for following the path S under the assumption that the polytope C is simple and the linear inequalities defining the polytope are not redundant.

The system (3.11) is equivalent to the following family of systems

$$y - \bar{f}(x) = 0, \ (x,y) \in \sigma \times F^*, \tag{4.1}$$

where σ is a simplex of $T|vF$ and F is a face of C not having the starting point v. Let $I = \{i | 1 \leq i \leq m, \ a^i.x = b_i$ for any point x of $F\}$ and let $w^1,\ldots,w^{t+1}$ be the vertices of the simplex σ. Then (4.1) has a solution (x,y) if and only if the following system (4.2) has a solution $(\mu,\lambda) \in R^{m+t+1}$

$$\begin{aligned} &\Sigma_{i=1}^m \mu_i a^i - \Sigma_{j=1}^{t+1} \lambda_j f(w^j) = 0 \\ &\Sigma_{j=1}^{t+1} \lambda_j = 1 \\ &\mu_i \geq 0 \text{ for } i=1,\ldots,m, \ \mu_i = 0 \text{ for } i \notin I \\ &\lambda_j \geq 0 \text{ for } j=1,\ldots, t+1. \end{aligned} \tag{4.2}$$

A line segment of solutions (μ,λ) to (4.2) can be followed by making a linear programming (lp) pivot step in (4.2). At the start of the algorithm we have to find the simplex σ and the cone F^* such that $(v,f(v)) \in \sigma \times F^*$. To find the cone F^* we solve the linear programming problem

$$\begin{aligned} \text{minimize } b.\mu \text{ subject to } &\Sigma_{i=1}^m \mu_i a^i - \lambda f(v) = 0 \\ &\mu_i \geq 0 \text{ for } i=1,\ldots,m \\ &\lambda = 1, \end{aligned} \tag{4.3}$$

which is the dual problem of

maximize $f(v).z$ subject to $z \in C$.

The optimum solution of (4.3) gives us the cell of L in which the end point $(v,f(v))$ of the path S lies. Namely, let I be the set of indices such that $\mu_i > 0$ at the optimum solution and let F be the face of C defined by the system of equations $a^i.x = b_i$ for $i \in I$. Then $(v,f(v)) = (v,\Sigma_{i \in I}\ \mu_i a^i)$ lies in $\{v\} \times F^* \subset vF \times F^* \in L$. Barring degeneracy of the linear programming problem (4.3), the set I has exactly n elements, so that F is a vertex of C and dim vF = 1. Then the simplex σ is a 1-simplex of $T|vF$ having the starting point v as a facet, i.e., $\sigma = \{v,w\}$ for some vertex w of $T|vF$. Thus we leave the starting point v along the line segment vF.

We now show that in general the set of solutions of (4.2) is bounded. Suppose on the contrary that the set of solutions has an unbounded ray $\{ (\mu^o,\lambda^o) + \alpha(\Delta\mu,\Delta\lambda) | \alpha > 0\}$. Since $\Sigma_{j=1}^{t+1} (\lambda_j^o+\alpha\Delta\lambda_j) = 1$ and $\lambda_j^o+\alpha\Delta\lambda_j > 0$ for any $\alpha > 0$, we have $\Delta\lambda = 0$. Therefore $\Sigma_{i \in I}\ \Delta\mu_i a^i = 0$ and $\Delta\mu_i > 0$ for $i \in I$. Since a point of F satisfies $a^i.x = b_i$ for any $i \in I$, this implies dim C < n, a contradiction. Hence the set of solutions to (4.2) is bounded and consequently has two distinct basic solutions. When some λ_j vanishes at the basic solutions, the point $(x,y) = (\Sigma_{j=1}^{t+1}\ \lambda_j w^j,\ \Sigma_{i \in I}\ \mu_i a^i)$ lies in a facet $\tau \times F^*$ of $\sigma \times F^*$, where τ is a facet of σ. Then either τ is a facet of just one other simplex $\bar{\sigma}$ of the triangulation of vF or τ lies in the boundary of vF. On the other hand, when some μ_i vanishes, we can in general not conclude that (x,y) lies on a facet of $\sigma \times F^*$. This is due to the fact that the cone F^* could have more vectors a^i's than its dimension. When the polytope C is a simple polytope and the system of linear inequalities defining C is nonredundant, the number of inequalities such that $a^i.x = b_i$ for any point x of F is equal to n − dim F = dim F^* so that F^* has exactly dim F^* coefficient vectors a^i's. In this case we can conclude that (x,y) is on a facet $\sigma \times E^*$ of $\sigma \times F^*$ when some μ_i vanishes at the basic solution of (4.2). By this reason we assume that the polytope $C = \{x \in R^n | a^i.x \leq b_i \text{ for } i=1,\dots,m\}$ is a simple polytope and that the linear inequalities are not redundant.

For a subset I of the index set $\{1,\dots,m\}$ let

$$F(I) = \{x \in C \mid a^i.x = b_i \text{ for any } i \in I\}.$$

Then F(I) is a face of C unless it is empty. Let $\mathcal{I}$ be the class of index sets $I \subset \{1,\dots,m\}$ such that F(I) is a nonempty face of C. Under the above assumption that C is a simple polytope and the linear inequalities defining C are nonredundant we have the following properties.

(i) For each face F of C the set $I \in \mathcal{I}$ such that $F(I) = F$ is unique and identical with the set $\{i \mid 1 \leq i \leq m,\ a^i.x = b_i$ for any point x in $F\}$.

(ii) $\dim F(I) = n - |I|$.

(iii) G is a facet of F(I) if and only if $G = F(I \cup \{j\})$ for some $j \notin I$ with $I \cup \{j\} \in \mathcal{I}$.

(iv) G has F(I) as a facet if and only if $G = F(I \setminus \{k\})$ for some $k \in I$.

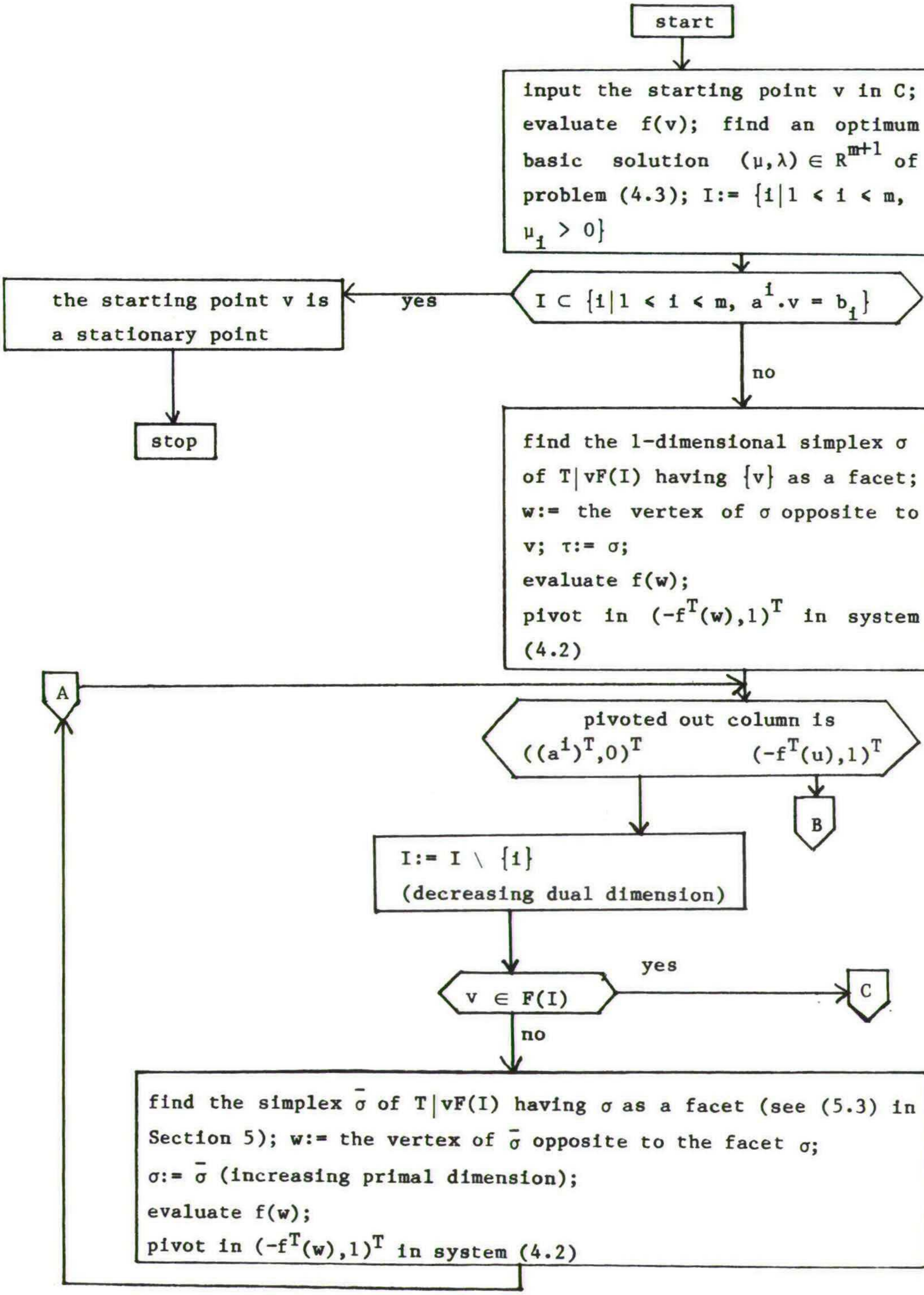
Note that $I \setminus \{k\} \in \mathcal{I}$ for any $I \in \mathcal{I}$ and any $k \in I$. Now starting at v the algorithm generates the path S by making alternating lp pivot steps in (4.2) and replacement steps in the triangulation of vF(I), for varying $I \in \mathcal{I}$, as described in the flow chart given in Figure 4.1.

The algorithm terminates as soon as one of the following cases occurs.

1) τ lies in F(I).

2) μ_i becomes 0 for some $i \in I$ and $F(I \setminus \{i\})$ has the starting point v (including the case where $I \setminus \{i\}$ becomes empty).

In both cases let x be equal to $\Sigma_{i=1}^{t+1} \lambda_i w^i$. In case 1 $(x, f(x)) = (x, y)$ lies in $F(I) \times F(I)^*$. In case 2 we have $vF(I) \subset F(I \setminus \{i\})$ because F(I) is a facet of $F(I \setminus \{i\})$ and $F(I \setminus \{i\})$ has the starting point v. Since x lies in a simplex σ in vF(I), we have $(x, \bar{f}(x)) = (x, y)$ lies in $F(I \setminus \{i\}) \times F(I \setminus \{i\})^*$. Thus in either case we obtain a stationary point for the pl approximation $\bar{f}$ of f.


start
input the starting point v in C; evaluate f(v); find an optimum basic solution $(\mu,\lambda) \in R^{m+1}$ of problem (4.3); I:= $\{i \mid 1 < i < m, \mu_i > 0\}$
$I \subset \{i \mid 1 < i < m, a^i.v = b_i\}$
yes
the starting point v is a stationary point
stop
no
find the 1-dimensional simplex σ of $T|vF(I)$ having $\{v\}$ as a facet; w:= the vertex of σ opposite to v; $\tau := \sigma$;
evaluate f(w);
pivot in $(-f^T(w),1)^T$ in system (4.2)
A
pivoted out column is
$((a^i)^T,0)^T$
$(-f^T(u),1)^T$
B
I:= I \ {i}
(decreasing dual dimension)
$v \in F(I)$
yes
C
no
find the simplex $\bar{\sigma}$ of $T|vF(I)$ having σ as a facet (see (5.3) in Section 5); w:= the vertex of $\bar{\sigma}$ opposite to the facet σ;
$\sigma := \bar{\sigma}$ (increasing primal dimension);
evaluate f(w);
pivot in $(-f^T(w),1)^T$ in system (4.2)

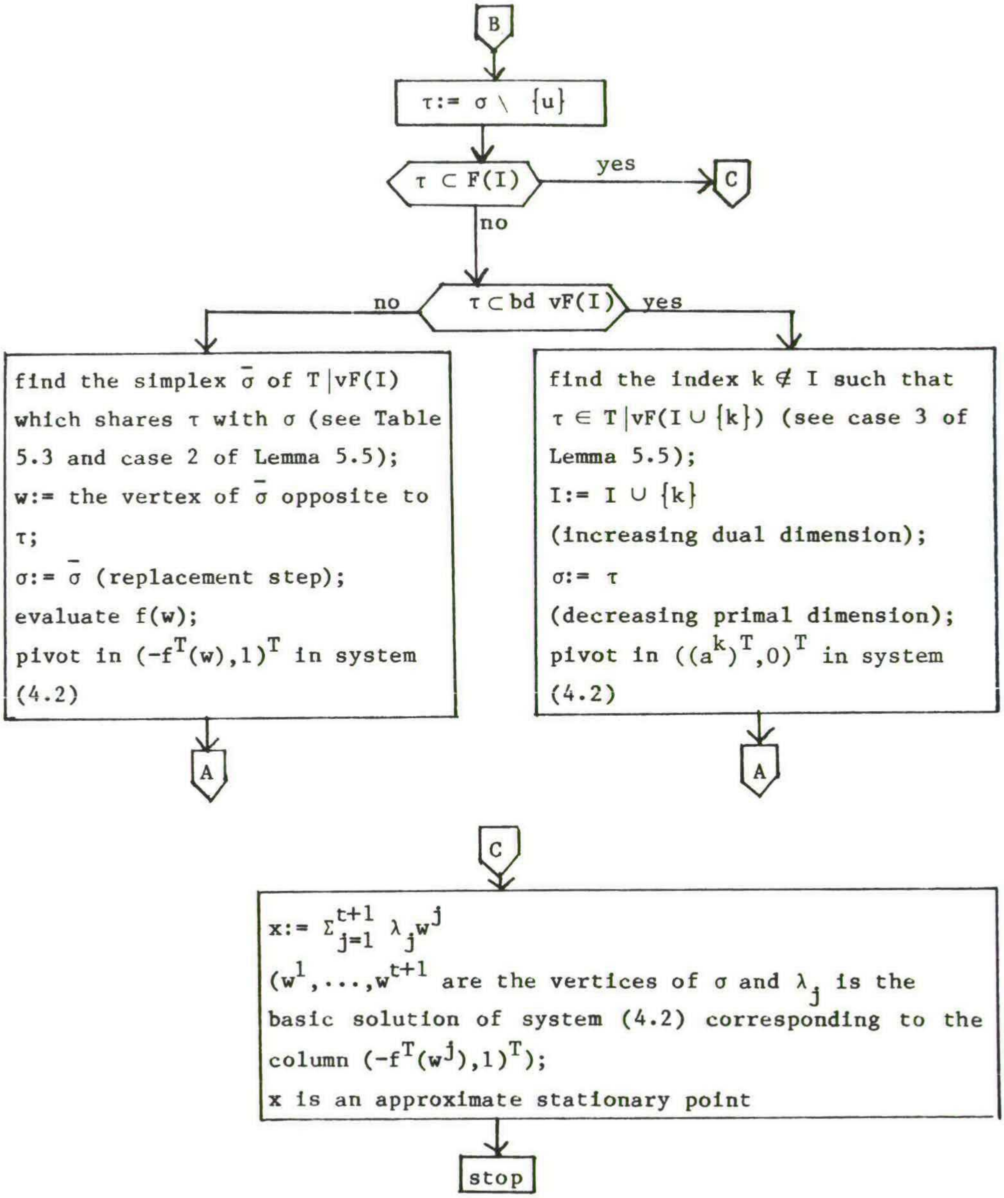


Fig. 4.1. Flow chart of the steps of the algorithm

We show two examples of the trajectory of the algorithm in Figure 4.2. In the first example $f(v)$ lies in $F(\{1,2\})^*$. Then the algorithm leaves the starting point v along the line segment $vF(\{1,2\})$. When the column $(-f^T(w^2),1)^T$ has been pivoted in, the column $((a^1)^T,0)^T$ is pivoted out, i.e., I becomes $\{2\}$ and the dual dimension decreases. To increase the primal dimension the vertex w^3 is found and $(-f^T(w^3),1)^T$ is pivoted in. After several pivot operations and replacements in $vF(\{2\})$ we have the 1-simplex $\tau = \{w^7,w^8\}$ in $vF(\{2,3\})$, and the primal dimension decreases. To increase the dual dimension we pivot in the column $((a^3)^T,0)^T$. Then the column $((a^2)^T,0)^T$ is pivoted out and the dual dimension does not change. To move in $vF(\{3\})$ we find the vertex w^9. After several iterations we have the simplex $\tau = \{w^{12},w^{13}\}$ which lies in $F(\{3\})$. Case 1 occurs and the algorithm terminates with an approximate stationary point in $F(\{3\})$.

In the second example $f(v)$ lies in $F(\{3,4\})$, and we go along $vF(\{3,4\})$. After several iterations we have $\tau = \{u^7,u^8\}$ in $vF(\{4,5\})$ and the primal dimension decreases. To increase the dual dimension $((a^5)^T,0)^T$ is pivoted in, $(-f^T(u^7),1)^T$ is out and we go to the left. When $(-f^T(u^9),1)^T$ is pivoted in, $((a^4)^T,0)^T$ is out and I becomes $\{5\}$. The face $F(\{5\})$ of C has the starting point v. Case 2 occurs so that the algorithm terminates with an approximate stationary point in the simplex $\{u^8,u^9\}$ in $F(\{5\})$.

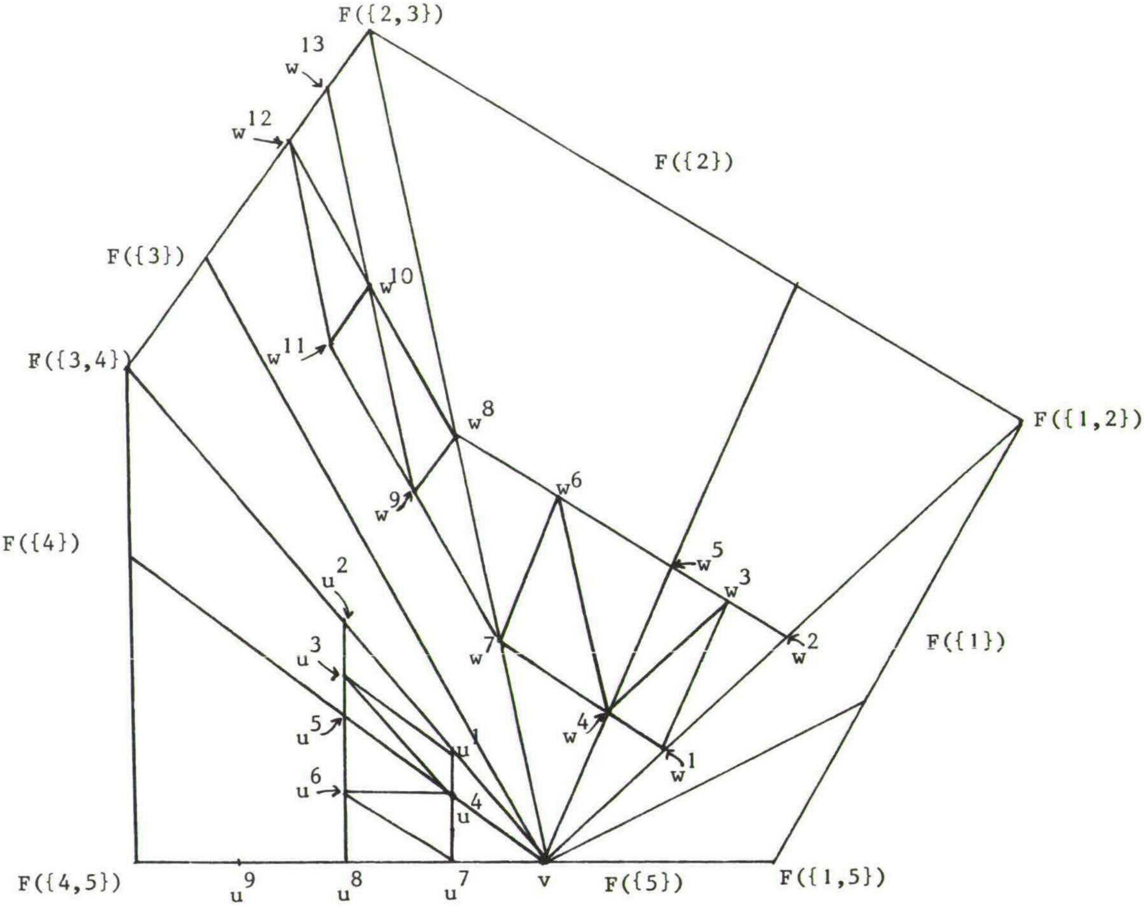


Fig. 4.2. Two possible trajectories of the algorithm

5. TRIANGULATION OF C

To describe the triangulation of C which underlies the simplicial algorithm we have to know some face structure of C in advance. In case C is just an n-simplex or the product of several simplices this face structure need not be known a priori since in that case a face of C can easily be obtained at any time from just the vertices of the set C.

Again let $C = \{x \in R^n | a^i.x \leq b_i \text{ for } i=1,\dots,m\}$ be an n-dimensional simple convex polytope in R^n and assume that the linear inequalities defining C are not redundant. Let $\mathcal{I}$ be the set of index sets I such that F(I) is a nonempty face of C. For any I in $\mathcal{I}$, let v(I) be a relative interior point of F(I) if $v \notin F(I)$ and $v(I) = v$ if $v \in F(I)$. We call v(I) the projection of the starting point v on the face F(I) of C. These projections of v on the faces of C will determine a triangulation of C such that each $vF(I) = \{x | x = (1-\alpha)v + \alpha z \text{ for some } z \in F(I) \text{ and some } \alpha \in [0,1]\}$ is triangulated for any I in $\mathcal{I}$. Now let I(n) be a subset of n indices of $\{1,\dots,m\}$ such that $I(n) \in \mathcal{I}$, i.e., F(I(n)) is a vertex of C. Suppose this vertex is not equal to v. Clearly, vF(I(n)) is a 1-dimensional face of vF(I) for all subsets I of I(n). Let I be a subset of I(n) and let $\gamma(I(n) \setminus I)$ be a permutation vector of the $t - 1 = n - |I|$ elements of the set $I(n) \setminus I$. The subset $F(I,\gamma(I(n) \setminus I))$ of $F(I) \in \mathcal{F}$ is the convex hull of the points v(I), $v(I \cup \{\gamma_{t-1}\})$, $v(I \cup \{\gamma_{t-2},\gamma_{t-1}\})$, $\dots,v(I \cup \{\gamma_2,\dots,\gamma_{t-1}\})$ and v(I(n)), which all lie in F(I). Observe that $v(I(n)) = v(I \cup \{\gamma_1,\dots,\gamma_{t-1}\})$. Clearly the t-dimensional set $vF(I,\gamma(I(n) \setminus I))$ is a t-simplex in vF(I). Now let $F(I^1(n)),\dots,F(I^K(n))$ denote the say K vertices of F(I).

<u>Lemma 5.1.</u> The union of the t-simplices $vF(I,\gamma(I^k(n) \setminus I))$ over all permutations $\gamma(I^k(n) \setminus I)$ and over all k, k=1,...,K, is a triangulation of the set vF(I). Furthermore, the union of the n-simplices $vF(\{i\}, \gamma(I(n) \setminus \{i\}))$ over all permutations $\gamma(I(n) \setminus \{i\})$ such that $i \in I(n)$ and over all $I(n) \in \mathcal{I}$, is a triangulation of C.

For n = 2 Lemma 5.1 is illustrated in Figure 5.1. The only vertices of this triangulation are the projections v(I) for $I \in \mathcal{I}$. Notice that $F(I(n)) = \{v(I(n))\}$ for all $I(n) \in \mathcal{I}$. In Figure 5.1 $v(\{1,2\})$ and $v(\{2,3\})$ are the vertices of the facet $F(\{2\})$ of C so that the two-dimensional set $vF(\{2\})$ is triangulated in the two 2-simplices $vF(\{2\},(1))$ and $vF(\{2\},(3))$.

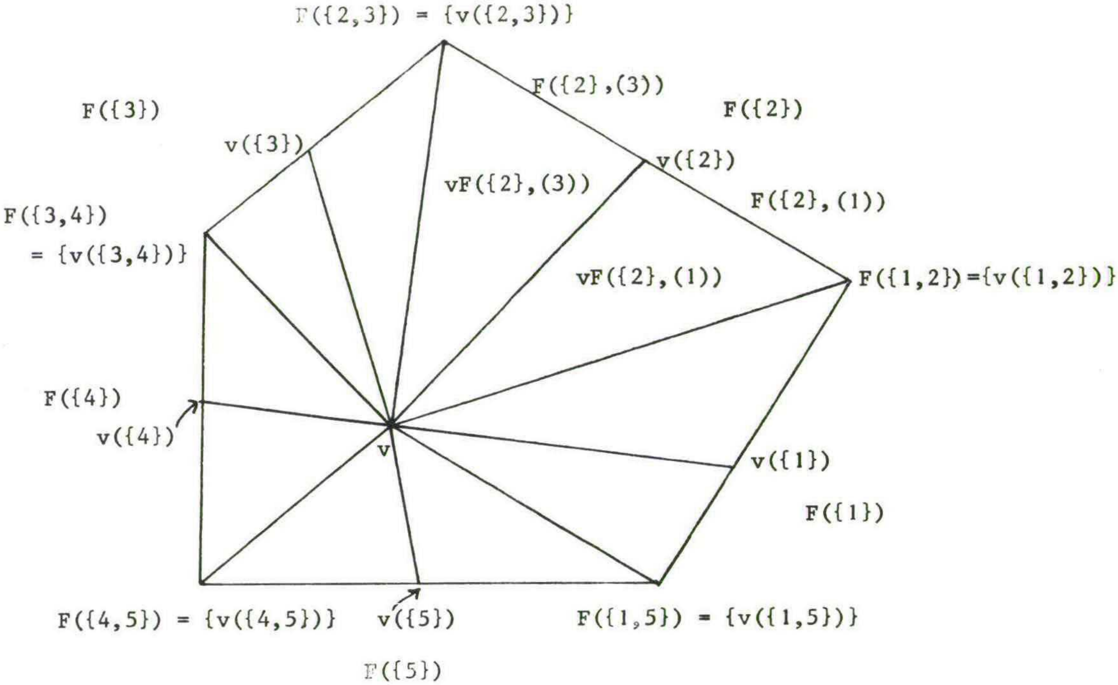


Fig. 5.1. Triangulation of C in $vF(I,\gamma(I(n) \setminus I))$'s

Before we describe a triangulation of C with arbitrary mesh size, we first discuss some properties of $vF(I,\gamma(I(n)\setminus I))$. For a given index set I and permutation $\gamma(I(n)\setminus I)$ such that $I \subset I(n)$, $I(n) \in \mathcal{I}$ and $v \notin F(I)$, define

$$q(0) = v(I(n)) - v$$

and for $j = 1,\ldots, t-1$

$$q(j) = v(I\cup\{\gamma_{j+1},\ldots,\gamma_{t-1}\}) - v(I\cup\{\gamma_j,\ldots,\gamma_{t-1}\}),$$

i.e., $q(j)$ is the vector between the projection of v on $F(I\cup\{\gamma_{j+1},\ldots,\gamma_{t-1}\})$ and the one on $F(I\cup\{\gamma_j,\ldots,\gamma_{t-1}\})$. Notice that

$q(1) = v(I(n) \setminus \{\gamma_1\}) - v(I(n))$ and that $q(t-1) = v(I) - v(I \cup \{\gamma_{t-1}\})$.

Lemma 5.2. The t-simplex $vF(I,\gamma(I(n) \setminus I))$ in $vF(I)$ is equal to the set of points

$$\{x \in R^n \mid x = v + \Sigma_{j=0}^{t-1} a(j)q(j) \quad \text{with } 1 \geq a(0) \geq \ldots \geq a(t-1) \geq 0\}. \tag{5.1}$$

Proof. The set $vF(I,\gamma(I(n) \setminus I))$ is the convex hull of the point v and the projections $v(I \cup \{\gamma_{h+1},\ldots,\gamma_{t-1}\})$ for $h=0,1,\ldots,t-1$. On the other hand

$$x = v + \Sigma_{j=0}^{t-1} a(j)q(j)$$

implies with $a(t) = 0$

$$x = v - a(0)v + \Sigma_{j=0}^{t-1} (a(j)-a(j+1))\ v(I \cup \{\gamma_{j+1},\ldots,\gamma_{t-1}\}).$$

Therefore when $a(0) = 0$ the point x is equal to v and when $a(0) > 0$

$$x = \lambda v + (1-\lambda)z \text{ with } 0 \leq \lambda = 1 - a(0) < 1. \tag{5.2}$$

The point $z = \Sigma_{j=0}^{t-1} \lambda_j v(I \cup \{\gamma_{j+1},\ldots,\gamma_{t-1}\})$ lies in $F(I,\gamma(I(n) \setminus I))$, since $\lambda_j = (a(j)-a(j+1))/a(0) \geq 0$ for $j=0,\ldots,t-1$, and $\Sigma_{j=0}^{t-1} \lambda_j = a(0)/a(0) = 1$. □

The boundary of the t-dimensional set $vF(I,\gamma(I(n) \setminus I))$ is obtained by letting one of the inequalities in (5.1) be an equality.

Lemma 5.3. The (t-1)-facets of $vF(I,\gamma(I(n) \setminus I))$ are obtained by setting in (5.1) either $a(0) = 1$ or $a(h) = a(h+1)$ for some h, $h = 0,1,\ldots,t-2$ or $a(t-1) = 0$. More precisely we have the following four cases.

1) If $a(0) = 1$ in (5.1), then according to (5.2) the facet $F(I,\gamma(I(n) \setminus I))$ opposite to the vertex v is obtained.

2) If $a(t-1) = 0$ in (5.1), we obtain the $(t-1)$-dimensional set $vF(I \cup \{\gamma_{t-1}\},(\gamma_1,\ldots,\gamma_{t-2}))$ as a facet of $vF(I,\gamma(I(n) \setminus I))$. This facet lies opposite to vertex $v(I)$ of $vF(I,\gamma(I(n) \setminus I))$.

3) If $a(h) = a(h+1)$ in (5.1) for some h, $h=1,\ldots,t-2$ then the obtained facet opposite to vertex $v(I \cup \{\gamma_{h+1},\ldots,\gamma_{t-1}\})$ is also a facet of the t-simplex $vF(I,\bar{\gamma}(I(n)) \setminus I))$ with

$$\bar{\gamma}(I(n) \setminus I) = (\gamma_1,\ldots,\gamma_{h-1},\gamma_{h+1},\gamma_h,\gamma_{h+2},\ldots,\gamma_{t-1}).$$

4) If $a(0) = a(1)$ in (5.1), then the obtained facet opposite to vertex $v(I(n))$ is also a facet of the t-simplex $vF(I,\bar{\gamma}(\bar{I}(n) \setminus I))$ with $\bar{I}(n)$ and $\bar{\gamma}(\bar{I}(n) \setminus I)$ determined as follows. Consider the edge $F(I(n) \setminus \{\gamma_1\})$ of C. This one-face connects the two vertices $F(I(n))$ and $F(I(n)) \setminus \{\gamma_1\} \cup \{k\})$ for some k not in $I(n)$. Then $\bar{I}(n) = I(n)\setminus\{\gamma_1\} \cup \{k\}$ and $\bar{\gamma}(\bar{I}(n)\setminus I) = (k,\gamma_2,\ldots,\gamma_{t-1})$.

Lemma 5.3 is illustrated in Figure 5.2 for $n = 3$, $I(n) = \{1,2,4\}$ and $I = \{2\}$. In this figure the sets $F(\{2\}, (5,4))$, $F(\{2\}, (1,4))$, $F(\{2\}, (4,1))$ and $F(\{2\},(3,1))$ are 2-simplices in the facet $F(\{2\})$ of C. Consider the 3-simplex $vF(\{2\},(4,1))$. The set $F(\{2\},(4,1))$ obtained by letting $a(0) = 1$ in (5.1) is the facet of $vF(\{2\},(4,1))$ opposite to vertex v. When $a(2) = 0$, we have case 2 and the facet $vF(\{1,2\},(4))$ is opposite to vertex $v(\{2\})$. Case 3 of Lemma 5.3 occurs when $a(2) = a(1)$, in which case the facet of $vF(\{2\},(4,1))$ opposite to vertex $v(\{1,2\})$ is also a facet of $vF(\{2\},(1,4))$. If $a(0) = a(1)$, i.e., case 4 of Lemma 5.3 occurs, we obtain the facet of $vF(\{2\}, (4,1))$ opposite to the vertex $v(I(n)) = v(\{1,2,4\})$. Since $v(\{1,2,3\})$ is the other end point of the edge $F(\{1,2\})$ of C than $v(\{1,2,4\})$, this facet also lies in the 3-simplex $vF(\{2\},(3,1))$.

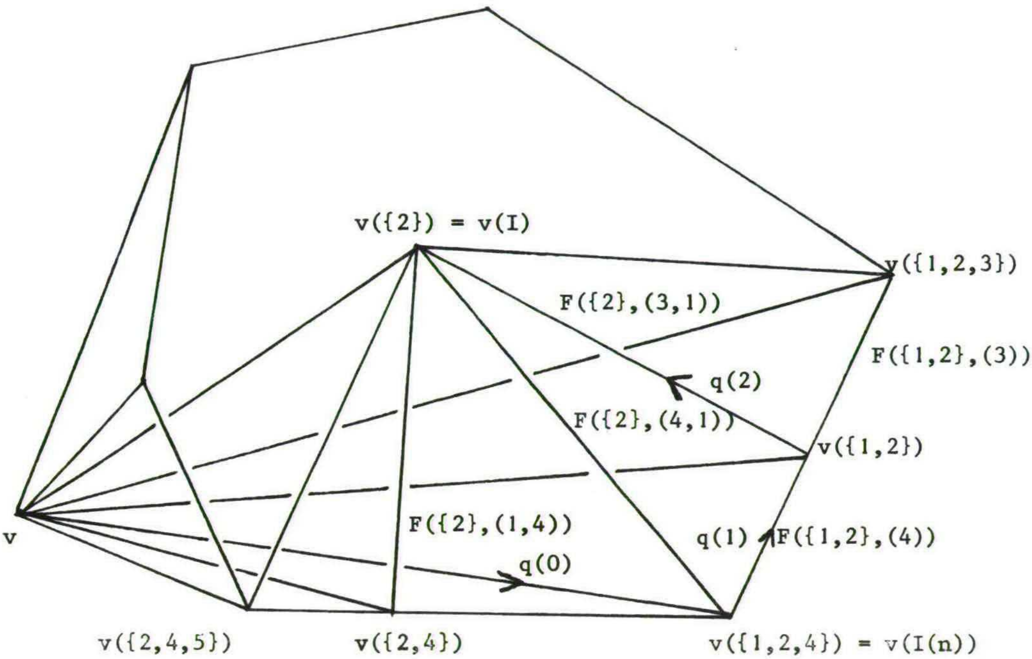


Fig. 5.2. Part of triangulation of $vF(\{2\})$ when $n = 3$

To obtain a simplicial subdivision of C with arbitrary mesh size each set $vF(I,\gamma(I(n) \setminus I))$ is triangulated as follows. Let d be an arbitrary positive number such that d^{-1} is an integer. We call d the grid size. A t-dimensional set $vF(I,\gamma(I(n) \setminus I))$ will be triangulated into $(d^{-1})^t$ simplices all having the same volume such that the union of these triangulations yields a triangulation of C with mesh size less than or equal to d.diam C.

<u>Definition 5.4.</u> The set $G^d(I,\gamma(I(n) \setminus I))$ is the collection of t-simplices $\sigma(w^1,\pi)$ in $vF(I,\gamma(I(n) \setminus I))$ with vertices $w^1,\dots,w^{t+1}$ such that

i) $w^1 = v + \Sigma_{j=0}^{t-1} R(j)dq(j)$ where $R(0),\dots,R(t-1)$ are integers such that
$0 \leq R(t-1) \leq \dots \leq R(0) \leq d^{-1}-1$.

ii) $\pi = (\pi_1,\dots,\pi_t)$ is a permutation of the t elements in the set $\{0,\dots,t-1\}$ such that $s < s'$ if $\pi_s = j$, $\pi_{s'} = j+1$ and $R(j+1) = R(j)$ for some $j \in \{0,1,\dots,t-2\}$.

iii) $w^{i+1} = w^i + dq(\pi_i)$ for $i=1,\ldots,t$.

Observe the similarity with the V-triangulation of a simplotope defined in Doup and Talman [1]. Furthermore if d is equal to one $G^d(I,\gamma(I(n)\setminus I))$ consists of one simplex, $vF(I,\gamma(I(n)\setminus I))$. Notice that in that case $R(j) = 0$ for all j and $\pi_i = i - 1$ for $i=1,\ldots,t$. So, in fact, $G^d(I,\gamma(I(n)\setminus I))$ is a triangulation of $vF(I,\gamma(I(n)\setminus I))$ with refinement factor d^{-1}. Taking the union over all permutations γ and over all feasible I(n) and I, we obtain a triangulation of C with grid size d, i.e., with mesh size less than or equal to d.diam C.

As described in Section 4, the algorithm follows the pl path S defined in Theorem 3.3 leading from the starting point v to an approximate solution by alternating linear programming pivot steps in system (4.2) and replacement steps with respect to a sequence of simplices of varying dimension of the triangulation defined above. We discuss now in detail the replacement rules of the algorithm. So, let $\sigma = \sigma(w^1,\pi)$ be an arbitrary t-simplex in $G^d(I,\gamma(I(n)\setminus I))$ for some $I(n) \in \mathcal{I}$ generated by the algorithm. Suppose that by the linear programming pivot step in (4.2) with respect to this simplex μ_i becomes equal to zero for some $i \in I$. Let $\bar{I}$ be equal to $I\setminus\{i\}$. If v lies in the face $F(\bar{I})$ of C, the algorithm terminates and $x = \Sigma_{j=1}^{t+1} \lambda_j w^j$ is an approximate solution. Recall that this case includes the case where $\bar{I}$ is empty. If $v \notin F(\bar{I})$, then the algorithm continues with the unique (t+1)-simplex $\bar{\sigma}$ in $vF(\bar{I})$ having σ as a facet. Notice that the set $vF(\bar{I})$ is indeed (t+1)-dimensional if $v \notin F(\bar{I})$. According to Definition 5.4 $\bar{\sigma}$ is the (t+1)-simplex $\sigma(\bar{w}^1,\bar{\pi})$ in $G^d(\bar{I},\bar{\gamma}(I(n)\setminus\bar{I}))$, where $\bar{I} = I\setminus\{i\}$,

$$\bar{\gamma}(I(n)\setminus\bar{I}) = (\gamma_1,\ldots,\gamma_{t-1},i)$$

$$\bar{w}^1 = w^1 \tag{5.3}$$

$$\bar{R}(j) = R(j) \text{ for } j = 0,1,\ldots,t-1, \text{ and}$$

$$\bar{\pi} = (\pi_1,\ldots,\pi_t,i).$$

The algorithm continues by setting $\bar{R}(t) = 0$ and t equal to t+1 and by making a linear programming pivot step in (4.2) with $(-f^T(\bar{w}),1)^T$, where $\bar{w}$ is the new vertex of $\bar{\sigma}$ opposite to the facet σ. If not a μ_i becomes zero when a linear programming pivot step is made in (4.2) with respect to some t-simplex $\sigma(w^1,\pi)$ in $G^d(I,\gamma(I(n) \setminus I))$, then λ_s becomes zero for some s, $1 < s < t + 1$. In that case the vertex w^s of σ has to be replaced. Two cases can occur. Either the facet of σ opposite to vertex w^s lies in the boundary of $vF(I,\gamma(I(n) \setminus I))$ or it is a facet of another t-simplex $\bar{\sigma}(\bar{w}^1,\bar{\pi})$ in $G^d(I,\gamma(I(n) \setminus I))$. In the latter case the algorithm continues by making a linear programming pivot step in (4.2) with $(-f^T(\bar{w}),1)^T$, where $\bar{w}$ is the new vertex of $\bar{\sigma}$ opposite to the facet shared with σ. The parameters $\bar{w}^1$, $\bar{\pi}$ and $\bar{R}$ are obtained from the parameters w^1, π and R of σ as described in Table 1.

	$\bar{w}^1$	$\bar{\pi}$	$\bar{R}$
$s = 1$	$w^1 + dq(\pi_1)$	$(\pi_2,\dots,\pi_t,\pi_1)$	$R + e(\pi_1)$
$1 < s < t+1$	w^1	$(\pi_1,\dots,\pi_{s-2},\pi_s,\pi_{s-1},\dots,\pi_t)$	R
$s = t+1$	$w^1 - dq(\pi_t)$	$(\pi_t,\pi_1,\dots,\pi_{t-1})$	$R - e(\pi_t)$

Table 1. Parameters of $\bar{\sigma}$ if the vertex w^s of $\sigma(w^1,\pi)$ is replaced

Lemma 5.5. The facet τ opposite to the vertex w^s of $\sigma(w^1,\pi)$ in $vF(I,\gamma(I(n) \setminus I))$ lies in the boundary of the latter set if and only if one of the following three cases holds.

1) $s = 1$, $\pi_1 = 0$ and $R(0) = d^{-1} - 1$.
2) $1 < s < t+1$, $\pi_s = h + 1$, $\pi_{s-1} = h$ for some $h \in \{0,1,\dots,t-2\}$ and $R(h) = R(h+1)$.
3) $s = t + 1$, $\pi_t = t - 1$ and $R(t-1) = 0$.

In case 1 of Lemma 5.5 τ lies, according to Lemma 5.3, in $F(I,\gamma(I(n) \setminus I))$ and the algorithm terminates with an approximate solution in F(I). In case 2 of Lemma 5.5 the facet τ of σ is, according to Lemma 5.3 cases 3 and 4, a facet of a unique t-simplex $\bar{\sigma}$ lying in $vF(I,\bar{\gamma}(I(n) \setminus I))$ with $\bar{\gamma}$ defined as in case 3 of Lemma 5.3 when $h > 1$ and in $vF(I,\bar{\gamma}(\bar{I}(n) \setminus I))$ with $\bar{\gamma}$ and $\bar{I}(n)$ defined as in case 4 of that lemma when $h = 0$. In both cases w^1, π and R do not change. Notice, however, that some of the vectors $q(j)$, $j=0,\dots,t-1$, change when

$\gamma(I(n) \setminus I)$ changes. The algorithm continues in both cases with the new vertex $\bar{w}$ of $\bar{\sigma}$ opposite to the facet shared with σ by making a linear programming pivot step in (4.2) with $(-f^T(\bar{w}),1)^T$, setting $I(n)$ to the new $\bar{I}(n)$ when $h = 0$ and γ to the new $\bar{\gamma}$.

Finally, in case 3 of Lemma 5.5 the facet τ of σ opposite to the vertex w^{t+1} is, according to case 2 of Lemma 5.3, the (t-1)-simplex $\bar{\sigma}(\bar{w}^1,\bar{\pi})$ in $vF(I \cup \{\gamma_{t-1}\},(\gamma_1,\ldots,\gamma_{t-2}))$, where $\bar{w}^1 = w^1$, $\bar{\pi} = (\pi_1,\ldots,\pi_{t-1})$ and $\bar{R} = (R(0),\ldots,R(t-2))$. The algorithm continues by setting I equal to $I \cup \{\gamma_{t-1}\}$, $\gamma(I(n) \setminus I)$ equal to $(\gamma_1,\ldots,\gamma_{t-2})$, σ equal to $\bar{\sigma}$, t equal to t-1 and making a linear programming pivot step with $((a^k)^T,0)^T$ in system (4.2), where k is the new element in the set I.

This completes the description of the replacement step in the triangulation of C when the grid size is equal to d.

6. REMARKS

We now show that the algorithm traces the path of stationary points generated by the pl approximation $\bar{f}$ of f and the expanding set $C(t) = (1-t)\{v\} + tC$. Let (x,y) be an arbitrary point on the path S of $H^{-1}(0)$ that the algorithm traces. Then $x \in vF$ and $\bar{f}(x) = y \in F^*$ for some face F of C. Let $\alpha = \inf\{t | x \in C(t),\ t \in [0,1]\}$. If $x \neq v$, then $\alpha > 0$, $F(\alpha) = (1-\alpha)\{v\} + \alpha F$ is a face of $C(\alpha)$ and $x \in F(\alpha)$. Let

$$F(\alpha)^* = \{y \in R^n | \text{any point of } F(\alpha) \text{ is an optimum solution of the linear programming problem maximize } y.z \text{ subject to } z \in C(\alpha)\}.$$

Then it is readily seen that $F(\alpha)^* = F^*$. Therefore we have that x is a stationary point for the pl approximation $\bar{f}$ to f on the set $C(\alpha)$. If $x = v$, then $\alpha = 0$ and hence v is a trivial stationary point for $C(0) = \{v\}$. Thus we have seen that the algorithm follows the path of stationary points for the pl approximation $\bar{f}$ to f with respect to the underlying triangulation and the expanding set $C(t)$, $0 \leq t \leq 1$.

Special cases of the set C are cubes or simplices. In case the set C is the n-dimensional cube $C = \{x \in R^n | a \leq x \leq b\}$ for two vectors a and b in R^n with $a_i < b_i$, $i=1,\ldots,n$, the stationary point problem reduces to finding an x^* in C such that for all i

$$x_i^* = b_i \qquad \text{implies } f_i(x^*) \geq 0,$$

$$a_i < x_i^* < b_i \text{ implies } f_i(x^*) = 0$$

and

$$x_i^* = a_i \qquad \text{implies } f_i(x^*) \leq 0.$$

A simplicial algorithm for this problem was introduced in [10]. However that algorithm has only 2n rays to leave the arbitrarily chosen starting point, one ray to each facet of C. The algorithm deviced in this paper has 2^n rays, one to each vertex of C. The difference between both algorithms can be compared with the difference of Lemke's algorithm and the algorithm proposed in [11] for solving the linear complementarity problem with upper and lower bounds. In the latter paper it has been argued that the algorithm with 2^n rays is very natural and might be faster than the algorithm with 2n rays.

In case C is an n-dimensional simplex $\tau(w^1,\dots,w^{n+1})$ in R^n the algorithm proposed in this paper is similar to the algorithm proposed in [1]. However, the latter algorithm was developed for the n-dimensional unit simplex in R^{n+1} with w^i the i-th unit vector in R^{n+1}. The same remark holds when C is the product space of more than one simplex. Notice that the cube is the product of n one-dimensional simplices.

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