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AN INPUT-OUTPUT LIKE CORPORATE MODEL INCLUDING MULTIPLE TECHNOLOGIES AND MAKE-OR-BUY DECISIONS

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Contents

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A substantial part of our research project "The analysis of multilevel decisions" will be devoted to delegation within the firm, with transfer prices and budgets as coordinating instruments. To provide for the basic framework for this research, an integral model of the firm is to be developed, covering three issues, namely multiple technologies for intermediate and end products, "make-or-buy" decisions with respect to technical services, and common cost allocation due to the presence of general services.

In this paper, we concentrate upon the first and the second of these issues.

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1. Introduction

Input-output analysis as introduced by Leontief (1936), was originally applied in macroeconomics, but has become a wide-spread and fruitful approach in corporate modelling as well (e.g. Van Halem (1981) for a survey). In this paper, the input-output view on the firm forms the starting point for the development of a decision model. It is correct to say that here the original input-output setting is generalized by the allowance for multiple technologies and make-or-buy decisions. In a later stage of research, to be reported on in a forthcoming paper, the issue of common cost allocation will be incorporated, thus leading to what will be referred to as an "integral model of the firm".

The paper is organized as follows. Chapter 2 embodies a first rough sketch of the corporate model to be developed, stated in common input-output terminology. After some formal definitions (Chapter 3), the decision model will be developed. By the allowance for multiple technologies, an LP model is obtained (Chapter 4). The incorporation of the make-or-buy aspect leads to a mixed-integer programming formulation (Chapter 5). Some concluding remarks can be found in Chapter 6.

2. Problem formulation

2.1. The firm in input-output terms

We consider a firm that consists of a number of producing sub-units. These sub-units can be grouped into four sectors:

1. end products (EP),
2. intermediate products (IP),
3. technical services (TS), and
4. general services (GS).

In the EP sector, there is a number of sub-units each of them producing one end product. An end product is a product which can be sold externally.

In the IP sector, each sub-unit produces one intermediate product; for this type of products there is no outside market.

We assume that the sub-units in the EP and IP sector only incur variable costs, i.e. production-volume dependent costs.

Both EP and IP sector require certain technical services. Production of TS leads to variable and fixed costs.

Finally, there is a sector producing certain general services (e.g. R&D, PR). In this sector, only fixed costs, the so-called common costs, are incurred.

A different term for 'product' or 'service' will be 'commodity'. Sub-units that produce products are called divisions; subunits producing services are called departments.

As output from one sub-unit may be input elsewhere in the firm, there exist complex interrelationships between the sub-units of various sectors. The transactions within the firm resulting from the internal deliveries of commodities can be taken together in the well-known input-output table (see figure 1). Here two auxiliary sectors, viz. 'primary input' and 'final output', are present. Primary input is input which is not the output of any sub-unit of the firm. Final output, reversely, is output which is not input to any sub-unit of the firm.

In common input-output analysis (e.g. Livingstone (1969), Smits and Verheyen (1976)), it is assumed that:

- there is a constant final demand for end products;
- the market prices for primary input are constants;
- a transfer-price scheme for internal deliveries and an allocation scheme for common costs have been established;
- the production of every commodity obeys a linear production function (constant returns to scale).

Based on the knowledge of final demand and production function, an input-output table on a real basis (quantities of products and services) can be formed. Using the prices for primary input, the transfer prices for internal deliveries and the allocation scheme for common costs, the table can be transformed into an input-output table on a nominal basis, i.e. all transactions expressed in monetary terms.

2.2. Central Problem

So far we have outlined the basic elements of our corporate model, mainly in well-known input-output language. Now we are ready to introduce the essential features of the model to be discussed in the course of the paper.

Firstly, we do not intend to build in the GS sector, yet. (The GS sector and the issue of common cost allocation will be treated in a forthcoming paper.) Furthermore, the development of our model takes place in two phases:

- In phase 1, we only work with an EP and IP sector. Each division in these sectors can choose among a finite number of (linear) technologies in order to produce its products. Products made under different technologies but within the same division are identical (and hence physically equivalent).

Definition 1:

Some collection of divisional technologies, one for each product, will be referred to as a subset of assigned technologies.

- In phase 2, the model of phase 1 is extended with a TS sector. Every TS can be produced internally, by the firm itself, as well as bought externally. Internal production of some TS leads to fixed and variable costs; external acquisition only to variable costs due to the per-unit market price of this service.

Definition 2:

A particular combination of 'make-or-buy' decisions, stating which TS will be produced internally and which TS will be bought externally, is called an internal-external alternative.

Loosely speaking, we are generalizing the input-output concept by the introduction of multiple technologies and make-or-buy decisions.

The Central Problem to be analysed is:

"Given a constant final demand for end products and given constant prices for primary input and external technical services, which subset of assigned technologies together with which internal-external alternative will lead to minimal total costs for the firm as a whole?"

After an optimal (i.e. cost-minimizing) technology and internal-external alternative are determined, the optimal input-output table can be formed, on a real basis or on a nominal basis. Computing the nominal table, which is based on the real table and the transfer prices for internal deliveries, is in fact only a matter of bookkeeping. The transfer prices influence the allocation of the firm's costs to the sub-units, but have no effect on the firm's total costs. For this reason, we did not include the transfer prices as parameters that should remain constant, in the statement of the Central Problem.

2.3. A decision model

One possibility to solve the Central Problem would be to re-compute the input-output table for every possible subset of assigned technologies and internal-external alternative, and then select the one with lowest total costs. But, in general, the number of technologies and internal-external alternatives can be very large, giving rise to an equally large number of input-output tables to be computed.

Therefore we will proceed in another way. It is possible to integrate all possible technology and internal-external alternatives into one "overall" model. This decision model can be viewed as an extension to the input-output approach. It will appear to belong to the class of mixed-integer programming problems.

3. Formalization

The introductory verbal description of the firm, as presented in 2.1, will now be formalized. We introduce notational conventions to be used troughout the remaining part of the paper.

3.1. Indices

We define:

L := number of technical services,

M := number of intermediate products,

N := number of end products

In general, the indices, ℓ, m, n, are elements of the following index sets:

$$\ell \in \{1,\ldots,L\},$$

$$m \in \{L+1,\ldots,L+M\},$$

$$n \in \{L+M+1,\ldots,L+M+N\}.$$

3.2. Names

The commodities will be denoted as follows:

$\{T_\ell \mid \ell = 1,\dots,L\}$ is the set of technical services,

$\{X_m \mid m = L+1,\dots,L+M\}$ is the set of intermediate products,

$\{Y_n \mid n = L+M+1,\dots,L+M+N\}$ is the set of end products.

Sub-units are named as follows:

ET_ℓ denotes the external supplier of T_ℓ,

DT_ℓ denotes the (internal) department that produces T_ℓ,

DX_m denotes the division that produces X_m,

DY_n denotes the division that produces Y_n.

3.3. Internal structure

In order to concentrate upon the internal deliveries within the firm, we present a "truncated" input-output table, in the sense, that the sectors 'primary input' and 'final output' are omitted.

We presume the absence of

- deliveries from IP to TS sector,
- deliveries from EP to TS sector, and
- deliveries from EP to IP sector.

In the usual input-output setting, with one single technology for every product and all TS internally produced, the truncated input-output table would be as depicted in figure 2. Every A-matrix represents the deliveries between the corresponding sectors, expressed in physical quantities.

More specifically:

A_{LL}, A_{MM}, A_{NN} contain the deliveries within the TS, IP and EP sector, respectively,

A_{LM} contains the deliveries from TS to IP sector,

A_{LN} contains the deliveries from TS to EP sector,

A_{MN} contains the deliveries from IP to EP sector.

As an example, matrix A_{MN} is further analyzed. It is an M×N matrix, whose (m,n)-th element A_{mn} represents the flow of commodities from DX_m to DY_n.

It is assumed that there is a <u>linear production function</u> so A_{mn} is a fixed proportion of the amount of produced Y_n, i.e.

$$A_{mn} = a_{mn}\, y_n \;, \quad m \in \{L+1,\ldots,L+M\},\; n \in \{L+M+1,\; L+M+N\}$$

where y_n denotes the amount of produced Y_n.

The coefficient a_{mn} is called an <u>intermediate input coefficient</u>. We require $a_{mn} \geq 0$.

It is trivial to state similar formulas for the other A-matrices. Briefly, this would yield

$$A_{\ell_1 \ell_2} = a_{\ell_1 \ell_2}\, t_{\ell_2}, \quad \ell_1, \ell_2 \in \{1,\ldots,L\} \;,$$

$$A_{\ell m} = a_{\ell m}\, x_m \;, \quad \ell \in \{1,\ldots,L\},\; m \in \{L+1,\ldots,L+M\} \;,$$

$$A_{\ell n} = a_{\ell n} \; y_n \;, \quad \ell \in \{1,\ldots,L\},\; n \in \{L+M+1,\ldots,L+M+N\} \;,$$

$$A_{m_1 m_2} = a_{m_1 m_2} \; x_m \;, \quad m_1,\; m_2 \in \{L+1,\ldots,L+M\} \;,$$

$$A_{n_1 n_2} = a_{n_1 n_2} \; y_n \;, \quad n_1,\; n_2 \in \{L+M+1,\ldots,L+M+N\} \;,$$

where t_{ℓ_2} and x_m denote the amount of produced T_{ℓ_2} and X_m, respectively. For completeness, we define

$$a_{m\ell} = 0,\; a_{n\ell} = 0,\; a_{nm} = 0 \quad \text{for the usual } \ell,\ m \text{ and } n \text{ values.}$$

As there is one single technology for each commodity, the intermediate input coefficients are uniquely determined.

3.4. Conventions for vector inequalities

Let $\underline{x} = (x_1,\ldots,x_q)'$ and $\underline{y} = (y_1,\ldots,y_q)'$ be two vectors of length q. Now we adopt the following conventions:

$$\underline{x} \geqq \underline{y} \;:\Leftrightarrow\; x_i \geq y_i,\; i = 1,\ldots,q \;.$$

$$\underline{x} \geq \underline{y} \;:\Leftrightarrow\; \underline{x} \geqq \underline{y} \text{ and } \underline{x} \neq \underline{y} \;.$$

$$\underline{x} > \underline{y} \;:\Leftrightarrow\; x_i > y_i,\; i = 1,\ldots,q \;.$$

4. Development of the decision model: the sectors 'End products' and 'Intermediate products', with multiple technologies

4.1. Introduction

In this chapter a L(inear) P(rogramming) framework is proposed that generalizes the input-output model of a firm by allowing multiple technologies for products. The sector TS is not included here; the firm consists of intermediate and end product divisions.

Before the actual development of the model is presented, some remarks on the sectors 'primary input' and 'final output' are in order, and the notion of technology is more thoroughly investigated.
The chapter is concluded with a section devoted to certain properties of the model.

4.2. Primary input; final output

Suppose there are K different primary input categories (like raw materials, labour, etc.) to be denoted by $B_1,\dots,B_K$. Because of the linearity assumption, the amount of primary input involved in the production of some product (e.g. X_m), is a fixed proportion of x_m, the produced amount of X_m. Hence, if B_{km} denotes the flow of B_k towards DX_m, we have the formula

$$B_{km} = b_{km}\, x_m$$

and, similarly

$$B_{kn} = b_{kn}\, y_n$$

for the flow of B_k towards DY_n.
The coefficients b_{kj} $(k = 1,\dots,K;\ j = L+1,\dots,L+M+N)$, which are assumed to be non-negative, are called the primary input coefficients. For notational convenience, we define

$$\underline{b}_j := (b_{1j},\dots,b_{Kj})' \ , \qquad j \in \{L+1,\dots,L+M+N\} \ .$$

Vector $\underline{b}_j$ represents the bundle of primary input required for the production of one unit X_j, if $j \in \{L+1,\dots,L+M\}$, or one unit Y_j, if $j \in \{L+M+1,\dots,L+M+N\}$.
If β_k (> 0) is the unit price for B_k, this bundle costs:

$$c_j := \sum_{k=1}^{K} \beta_k\, b_{kj} \ , \quad j \in \{L+1,\dots,L+M+N\} \tag{4.1}$$

Definition 3:

The cost coefficients c_j as defined in formula (4.1) are called the per-unit direct costs of $X_{L+1},\dots,X_{L+M}$, $Y_{L+M+1},\dots,Y_{L+M+N}$, respectively.

The conversion of primary input coefficients into direct cost coefficients can be seen as the replacement of the K primary input categories by one fictitious primary input, say "labour", with input coefficient c_j and price 1. It is presumed that all $c_j > 0$.

With respect to the sector 'final output' we presume a non-negative outside demand d_n for every end product Y_n. We denote the final demand vector $\underline{d}_{final}$ as follows:

$$\underline{d}_{final} := (d_{L+M+1},\dots,d_{L+M+N})' .$$

4.3. Technology

Replace every element of the compound matrix A, defined as

$$A := \left(\begin{array}{c|c} A_{MM} & A_{MN} \\ \hline 0 & A_{NN} \end{array}\right) \quad , \text{ dimensions } (M+N) \times (M+N) ,$$

by its corresponding intermediate input coefficient. We obtain a matrix $\hat{A}$ of the same structure, but independent of the actual production volume. Partition $\hat{A}$ in columns, i.e.

$$\hat{A} = (\underline{a}_{L+1} \mid \dots \mid \underline{a}_{L+M} \mid \underline{a}_{L+M+1} \mid \dots \mid \underline{a}_{L+M+N})$$

where $\underline{a}_j := (a_{L+1,j},\dots,a_{L+M+N,j})'$, $j \in \{L+1,\dots,L+M+N\}$.

Definition 4:

The columnvector $(\underline{a}_m',\underline{b}_m')'$ is called the technology column for product X_m, $m \in \{L+1,\dots,L+M\}$.

The columnvector $(\underline{a}_n',\underline{b}_n')'$ is called the technology column for product Y_n, $n \in \{L+M+1,\dots,L+M+N\}$.

Using the market prices for primary input, technology columns are easily transformed into cost-oriented technology columns.

Definition 5:

The columnvector $(\underline{a}_m', c_m)'$ is called the cost-oriented technology column for product X_m, $m \in \{L+1,\ldots,L+M\}$.
The columnvector $(\underline{a}_n', c_n)'$ is called the cost-oriented technology column for product Y_n, $n \in \{L+M+1,\ldots,L+M+N\}$.

The allowance for multiple technologies means that every product is allowed to have more than one technology column. In other words, input factors can be combined in more than one way.
Let $j \in \{L+1,\ldots,L+M+N\}$ and $\Lambda_j \in \{1,2,3,\ldots\}$. We assume that division j has a choice among Λ_j technology columns:

$$(\underline{a}_j'(\lambda),\ \underline{b}_j'(\lambda))',\quad \lambda \in \{1,2,\ldots,\Lambda_j\}\ .$$

The amount of X_m that is produced by applying technology $((\underline{a}_m'(\lambda), \underline{b}_m'(\lambda))'$ is denoted by $x_m(\lambda)$. A slightly different phrase we might use is: "Technology $(\underline{a}_m'(\lambda),\ \underline{b}_m'(\lambda))'$ is used at intensity $x_m(\lambda)$." For the total amount of produced X_m, we have

$$x_m = \sum_{\lambda=1}^{\Lambda_m} x_m(\lambda).$$

With respect to end products, we have similar statements for Y_n, $y_n(\lambda)$, y_n. If a certain technology, say λ, is applied for product j, then the coefficients

$$a_{ij}(\lambda),\ i,j \in \{L+1,\dots,L+M+N\}\ ;$$

$$b_{kj}(\lambda),\ k \in \{1,\dots,K\},\ j \in \{L+1,\dots,L+M+N\}\ ;$$

$$c_j(\lambda),\ j \in \{L+1,\dots,L+M+N\}$$

are the corresponding intermediate input, primary input and per-unit direct cost coefficients, respectively.

In section 2.1 we have introduced the notion of a subset of assigned technologies (cf. definition 1). From a more formal point of view, each subset of assigned technologies can be characterized by a vector

$$\underline{\lambda}^* := (\lambda^*_{L+1},\dots,\lambda^*_{L+M+N})'\ ,$$

where $\lambda^*_{L+j} \in \{1,\dots,\Lambda_{L+j}\}$ for $j = 1,\dots,M+N$. Vector $\underline{\lambda}^*$ contains one technology index per product and hence determines one technology column

$$(\underline{a}'_j(\lambda^*_j),\ \underline{b}'_j(\lambda^*_j))'$$

for each product.

We define

$$A(\underline{\lambda}^*) := (\underline{a}_{L+1}(\lambda^*_{L+1})\ |\dots|\ \underline{a}_{L+M+N}\ (\lambda^*_{L+M+N}))$$

so $A(\underline{\lambda}^*)$ is the matrix of intermediate input coefficients of assigned technologies as characterized by $\underline{\lambda}^*$.

Similarly,

$$B(\underline{\lambda}^*) := (\underline{b}_{L+1}(\lambda^*_{L+1}) \mid \ldots \mid \underline{b}_{L+M+N}(\lambda^*_{L+M+N}))$$

is the matrix of primary input coefficients of assigned technologies as characterized by $\underline{\lambda}^*$.

4.3. Linear programming formulation of the decision model

In this section, the LP model for the firm will be derived. The amount of produced X_{L+1} is:

$$x_{L+1} = \sum_{\lambda=1}^{\Lambda_{L+1}} x_{L+1}(\lambda) \tag{4.2}$$

This amount should meet the internal demand requirements, i.e.

$$x_{L+1} = \sum_{m=1}^{M} \sum_{\lambda=1}^{\Lambda_{L+m}} a_{L+1,\ L+m}(\lambda)\ x_{L+m}(\lambda) + $$

$$+ \sum_{n=1}^{N} \sum_{\lambda=1}^{\Lambda_{L+M+n}} a_{L+1,\ L+M+n}(\lambda)\ x_{L+M+n}(\lambda) \tag{4.3}$$

Combining (4.2) and (4.3) yields:

$$\sum_{\lambda=1}^{\Lambda_{L+1}} (1-a_{L+1,L+1}(\lambda))x_{L+1}(\lambda) - \sum_{m=2}^{M} \sum_{\lambda=1}^{\Lambda_{L+m}} a_{L+1,L+m}(\lambda)x_{L+m}(\lambda)$$

$$- \sum_{n=1}^{N} \sum_{\lambda=1}^{\Lambda_{L+M+n}} a_{L+1,L+M+n}(\lambda)x_{L+M+n}(\lambda) = 0 \tag{4.4}$$

For X_{L+j}, $j \in \{2,\ldots,M\}$, we similarly have

$$\sum_{\lambda=1}^{\Lambda_{L+j}} (1-a_{L+j,L+j}(\lambda))x_{L+j}(\lambda) - \sum_{\substack{m=1\\ m\neq j}}^{M} \sum_{\lambda=1}^{\Lambda_{L+m}} a_{L+j,L+m}(\lambda)x_{L+m}(\lambda)$$

$$- \sum_{n=1}^{N} \sum_{\lambda=1}^{\Lambda_{L+M+n}} a_{L+1,L+M+n}(\lambda)x_{L+M+n}(\lambda) = 0 \tag{4.5}$$

For end products, there are no "back deliveries" to the IP sector. Hence

$$a_{L+M+n,L+m}(\lambda) = 0, \quad n = 1,\dots,N, \quad m = 1,\dots,M, \quad \lambda = 1,\dots,\Lambda_{L+m}.$$

In order to meet internal and external demand, it should hold that, $j \in \{1,\dots,N\}$:

$$\sum_{\lambda=1}^{\Lambda_{L+M+j}}(1-a_{L+M+j}(\lambda))y_{L+M+j}(\lambda) - \sum_{\substack{n=1\\ n\neq j}}^{N} \sum_{\lambda=1}^{\Lambda_{L+M+n}} a_{L+M+j,L+M+n}(\lambda)y_{L+M+n}(\lambda) =$$

$$= d_{L+M+j} \tag{4.6}$$

The costs of production are equal to

$$\sum_{m=1}^{M} \sum_{\lambda=1}^{\Lambda_{L+m}} c_{L+m}(\lambda)x_{L+m}(\lambda) + \sum_{n=1}^{N} \sum_{\lambda=1}^{\Lambda_{L+M+n}} c_{L+M+n}(\lambda)y_{L+M+n}(\lambda) \tag{4.7}$$

Now recall the Central Problem (cf. section 2.2). Within the context of a firm without a TS sector, the Central Problem reduces to the question:

"Given a constant final demand for end products and given constant prices for primary input, which subset of assigned techologies will lead to minimal total costs for the firm as a whole?"

It is not surprising that, in order to answer this question, we concentrate upon solving the following problem:

$$\text{Minimize (4.7) subject to } \begin{cases} (4.5) & j = 1,\dots,M\ , \\ (4.6) & j = 1,\dots,N\ , \\ \text{all } x_j(\lambda),\ y_j(\lambda) \geq 0\ . \end{cases} \tag{4.8}$$

which is actually an LP problem.

Altogether, the choice of the optimal, i.e. cost-minimizing, technology alternative, will be analyzed within an LP framework where:

1. the constraints describe physical relationships with respect to products and technologies;
2. the objective function, which is to be minimized, equals the direct variable costs for primary input involved in the production process.

The formulation does not a-priori exclude combinations of two or more technologies for one single product, i.e. the simultaneous use of more than one process per product.

In order to enable a compact representation of problem (4.8), we redefine matrix $\hat{A}$ (cf. section 4.2):

$$\hat{A} := (\underline{a}_{L+1}(1)|\dots|\underline{a}_{L+1}(\Lambda_{L+1})|\dots\dots|\underline{a}_{L+M+N}(1)|\dots|\underline{a}_{L+M+N}(\Lambda_{L+M+N})) \tag{4.9}$$

so $\hat{A}$ consists of all columns of intermediate coefficients (dimensions: $(M+N) \times (\sum_{j=L+1}^{L+M+N} \Lambda_j)$).

It can be subdivided as follows:

$$\hat{A} = \left[\begin{array}{c|c} \hat{A}_{MM} & \hat{A}_{MN} \\ \hline 0 & \hat{A}_{NN} \end{array}\right], \quad \text{where} \quad \begin{array}{l} \hat{A}_{MM} \text{ is an } M \times (\sum_{j=L+1}^{L+M} \Lambda_j) \text{ matrix} \\ \hat{A}_{MN} \text{ is an } M \times (\sum_{j=L+M+1}^{L+M+N} \Lambda_j) \text{ matrix} \\ \hat{A}_{NN} \text{ is an } N \times (\sum_{j=L+M+1}^{L+M+N} \Lambda_j) \text{ matrix} \end{array}$$

Furthermore, a "generalized identity matrix" $\hat{I}$ is required:

$$\hat{I} := \mathrm{diag}(1_j'), \quad j = 1,\dots,M+N$$

where $1_j' := (1,\dots,1)$ is a row vector of length Λ_{L+j}, with all elements equal to 1.

So $\hat{I}$ is of the same dimensions as $\hat{A}$. Matrix $\hat{I}$ looks like:

$$\begin{matrix} 1 \dots 1 & & & \\ & 1 \dots 1 & & \\ & & \ddots & \\ & & & 1 \dots 1 \end{matrix}$$

Now define

$$\underline{x} := (x_{L+1}(1),\dots,x_{L+1}(\Lambda_{L+1}),\dots,x_{L+M}(1),\dots,x_{L+M}(\Lambda_{L+M}))' ,$$

$$\underline{y} := (y_{L+M+1}(1),\dots,y_{L+M+1}(\Lambda_{L+M+1}),\dots,y_{L+M+N}(1),\dots,y_{L+M+N}(\Lambda_{L+M+N}))' ,$$

$$\underline{z} := (\underline{x}',\underline{y}')' .$$

$\underline{x}$, $\underline{y}$ and $\underline{z}$ may be referred to as the intensity vectors of the IP sector, the EP sector and the whole firm, respectively.

Finally,

$$\underline{c}_x' := (c_{L+1}(1),\ldots,c_{L+1}(\Lambda_{L+1}),\ldots,c_{L+M}(1),\ldots,c_{L+M}(\Lambda_{L+M}))\ ,$$

$$\underline{c}_y' := (c_{L+M+1}(1),\ldots,c_{L+M+1}(\Lambda_{L+M+1}),\ldots,c_{L+M+N}(1),\ldots,c_{L+M+N}(\Lambda_{L+M+N}))\ ,$$

$$\underline{c}' := (\underline{c}_x',\underline{c}_y')\ .$$

The LP problem (4.8) can be written as

$$\begin{aligned} &\text{Minimize } \underline{c}'\ \underline{z} \\ &\text{s.t. } (\hat{I}-\hat{A})\ \underline{z} = (\underline{0}',\ \underline{d}_{final}')' \\ &\qquad\qquad \underline{z} \geqq \underline{0} \end{aligned} \tag{4.10}$$

Using matrices of the same form as $\hat{I}$, namely

$$\hat{I}_M := \text{diag}(1_m'),\ m = L+1,\ldots,L+M\ ,$$

$$\hat{I}_N := \text{diag}(1_n'),\ n = L+M+1,\ldots,L+M+N\ ,$$

where $1_j' := (1,\ldots,1)$ is a row vector of length Λ_{L+j}, with all elements equal to 1 ($j \in \{1,\ldots,M+N\}$), we can give a more "partitioned" representation of (4.10), i.e.

$$\begin{aligned} &\text{Minimize} \qquad \underline{c}_x'\ \underline{x} + \underline{c}_y'\ \underline{y} \\ &\text{s.t.} \qquad (\hat{I}_M-\hat{A}_{MM})\underline{x} - \hat{A}_{MN}\ \underline{y} = \underline{0} \\ &\qquad\qquad\qquad (\hat{I}_N-\hat{A}_{NN})\underline{y} = \underline{d}_{final} \\ &\qquad \text{all elements of } \underline{x},\ \underline{y} \geq 0. \end{aligned} \tag{4.11}$$

Note that in the LP formulation, matrix $\hat{B}$ which is defined as

$$\hat{B} := (\underline{b}_{L+1}(1)|\ldots|\underline{b}_{L+1}(\Lambda_{L+1})|\ldots\ldots|\underline{b}_{L+M+N}(1)|\ldots|\underline{b}_{L+M+N}(\Lambda_{L+M+N}))$$

and hence consists of all columns of primary input coefficients, is implicitly present, because

$$\underline{c}' = \underline{\beta}' \hat{B}$$

(Recall that $\underline{\beta}' = (\beta_1,\ldots,\beta_K)$ contains the unit prices for primary input.)

4.5. Properties of the LP model

Definition 6:

Let A be a square matrix with entries $a_{ij} \geq 0$. Matrix A will be called feasible iff

$$\underset{\underline{d} \geq \underline{0}}{\forall} \quad \underset{\underline{z} \geq \underline{0}}{\exists} \quad [(I-A)\underline{z} = \underline{d}] .$$

Definition 7:

Let B be a square matrix of the form

$$B = sI-A, \quad s > 0, \quad \text{all entries of } A \geq 0 .$$

Let $\rho(A)$ be the spectral radius of A.

Matrix B is called an M-matrix iff $\rho(A) \leq s$.

Matrix B is called a non-singular M-matrix iff $\rho(A) < s$.

Now we have:

Theorem 1 (cf. Berman and Plemmons (1979); theorem 9.3.9):

Matrix A is feasible $\Leftrightarrow$ Matrix I-A is a non-singular M-matrix.

Berman and Plemmons (1979; theorem 6.2.3) give 50 characterizations of non-singular M-matrices. Using the inequality conventions as introduced in 3.4, one of these characterizations is

$$\exists_{\underline{z} \geq \underline{0}} \; [(I-A)\,\underline{z} > \underline{0}] \; .$$

For the multiple-technology case, where $\hat{A}$ is not square in general, we will define feasibility for a broader class of right-hand sides than $(\underline{0}', \underline{d}'_{final})'$ (as occurring in problem 4.10). Let

$$\underline{d}_{tot} := (d_{L+1},\ldots,d_{L+M}, \underline{d}'_{final})', \quad \text{all } d_m \geq 0.$$

Definition 8:

Matrix $\hat{A}$, as defined by formula (4.9), will be called feasible iff

$$\forall_{\underline{d}_{tot} \geqq 0} \; \exists_{\underline{z} \geqq \underline{0}} \; [(\hat{I}-\hat{A})\underline{z} = \underline{d}_{tot}] \; .$$

It is immediately clear, that the feasibility of $\hat{A}$ implies the existence of feasible solutions of problem (4.10) (or, equivalently, (4.11)).

Let $\underline{\lambda}^*$ represent some subset of assigned technologies and form the corresponding $A(\underline{\lambda}^*)$ (cf. section 4.3).

Theorem 2:

If $A(\underline{\lambda}^*)$ is feasible, in the sense of definition 6, then $\hat{A}$ is feasible, in the sense of definition 8.

Proof:

Let $\underline{d}_{tot} \geqq \underline{0}$. Then $(I-A(\underline{\lambda}^*))\underline{z} = \underline{d}_{tot}$ has a non-negative solution. This solution is easily extended to a non-negative solution of $(\hat{I}-\hat{A})\underline{z} = \underline{d}_{tot}$.

□

Now we will cite a theorem due to Cassels (1981) which was originally proved for the case with just one primary input. Nevertheless, it also applies here if we suppose that the initial K primary input categories of our model are replaced by one fictitious primary input (cf. section 4.1).

Furthermore, internal prices $p_{L+1},\dots,p_{L+M+N}$ for $X_{L+1},\dots,X_{L+M}$, $Y_{L+M+1},\dots,Y_{L+M+N}$, respectively, play a crucial role in the following definition and theorem:

Definition 9:

Under a certain price $\underline{p}' = (p_{L+1},\dots,p_{L+M+N})$, a technology column, $(\underline{a}_j'(\lambda),\ \underline{b}_j'(\lambda))'$ or $(\underline{a}_j'(\lambda),\ c_j(\lambda))'$, is said to make a loss if $\underline{p}'\ \underline{a}_j(\lambda) < c_j(\lambda)$,

break even if $\underline{p}'\underline{a}_j(\lambda) = c_j(\lambda)$,

make a profit if $\underline{p}'\underline{a}_j(\lambda) > c_j(\lambda)$.

Theorem 3 (Cassels (1981); chapter 5, theorem 2.3):

Consider the following statements:

(i) there exists an intensity vector $\underline{z} \geq \underline{0}$, such that $(\hat{I}-\hat{A})\underline{z} > \underline{0}$;

(ii) there exists a price vector $\underline{p} \geq \underline{0}$, such that for every product there is a technology column that does not make a loss;

(a) $\hat{A}$ is feasible (in the sense of definition 8);

(b) there exists a price vector $\underline{v} > \underline{0}$ such that no technology column makes a profit; for every product j there is a technology column $\underline{a}_j(\lambda_j^*)$ which breaks even, under this price $\underline{v}$;

(c) for every demand vector $\underline{d}_{tot} \geqq \underline{0}$, a combination of the $\underline{a}_j(\lambda_j^*)$ only, can produce $\underline{d}_{tot}$ efficiently, i.e. against minimal total costs. Moreover, these minimal costs are equal to $\underline{v}'\underline{d}_{tot}$.

Now it holds that:

(i) $\Leftrightarrow$ (ii), and

(i) $\vee$ (ii) $\Rightarrow$ (a), (b), (c).

From theorem 3, we can deduce several results:

1) The implication (i) $\Rightarrow$ (c) says in particular:

(i) $\Rightarrow$ { there exists a subset of assigned technologies,
represented by $\underline{\lambda}^*$,
such that the corresponding $A(\underline{\lambda}^*)$ is feasible}.

Of course, (a) $\Rightarrow$ (i), so:

$$(a) \Rightarrow \{\text{there exists a } \underline{\lambda}^* \text{ such that } A(\underline{\lambda}^*) \text{ is feasible}\}.$$

Combining this with theorem 2 yields

Theorem 4:

$\hat{A}$ is feasible $\Leftrightarrow$ there exists a $\underline{\lambda}^*$ such that $A(\underline{\lambda}^*)$ is feasible.

2) Statement (b) in theorem 3 expresses that

$$\underline{v}'\underline{a}_j(\lambda) \leq c_j(\lambda), \quad j = L+1,\ldots,L+M+N, \quad \lambda = 1,\ldots,\Lambda_j,$$

equivalent with

$$\underline{v}'(\hat{I}-\hat{A}) \leqq \underline{c}' \ ,$$

so: $\underline{v}$ is dual feasible (see later part of this section: problem (4.12)). In particular:

$$\underset{j\in\{L+1,\ldots,L+M+N\}}{\forall} \ \underset{\lambda_j^*\in\{1,\ldots,\Lambda_j\}}{\exists} \ [\underline{v}'\underline{a}_j(\lambda_j^*) = c_j(\lambda_j^*)] \ ,$$

so $$\underline{v}'(I-A(\underline{\lambda}^*)) = (\underline{c}^*)' \ .$$

with $A(\underline{\lambda}^*)$ as usual and $\underline{c}^* = (c_{L+1}(\lambda^*_{L+1}),\ldots,c_{L+M+N}(\lambda^*_{L+M+N}))'$.
Apparently $A(\underline{\lambda}^*)$ is feasible, so that $I-A(\underline{\lambda}^*)$ is non-singular. Hence

$$\underline{v}' = (\underline{c}^*)' \ (I-A(\underline{\lambda}^*))^{-1} \ .$$

3) From statement (c), we conclude that $\underline{v}$ is even dual optimal. As for every demand vector $\underline{d}_{tot}$, the optimal costs are equal to $\underline{v}'\underline{d}_{tot}$, this price vector $\underline{v}$ is uniquely determined.
Nevertheless, it is not necessarily true that the dual optimum is uniquely determined. E.g., it is true that if there is a unique but degenerate primal optimum, then the dual optimum is not uniquely determined. On the other hand, if at least one primal optimum is non-degenerate, then there exists exactly one dual optimum (namely $\underline{v}$!). See Papatimitriou and Steiglitz (1982; chapter 3, exercises 6,7).

We conclude this section with some remarks on the dual of problem (4.10), i.e.

$$\begin{aligned} &\text{Minimize } \underline{d}'\underline{p} \\ &\text{s.t. } (\hat{I}-\hat{A})'\, \underline{p} \leqq \underline{c} \end{aligned} \tag{4.12}$$

where $\underline{d}' := (\underline{0}', \underline{d}'_{final})$

For convenience, assume that there is a unique dual optimum $\underline{v}$. If we interprete its components $v_{L+1},\ldots,v_{L+M+N}$ as prices for $X_{L+1},\ldots,X_{L+M}$, $Y_{L+M+1},\ldots,Y_{L+M+N}$ respectively, the following verbalization of the dual problem is in order:
"Choose prices $v_{L+1},\ldots,v_{L+M+N}$ that maximize the returns $\underline{d}'\underline{v}$, subject to the restriction that for every product the price may not exceed the unit cost of that product under the optimal technology."

From (4.12) it is seen that the rows of the dual are of the form:

$$\underline{a}_j'(\lambda)\underline{p} \leq c_j \text{ .}$$

If $z_j(\lambda_j^*)$ is in the optimal primal basis, and $z_j(\lambda_j^*) > 0$, we have:

$$\underline{a}_j'(\lambda_j^*)\underline{v} = c_j$$

due to the complementary slackness conditions (cf. Papadimitriou and Steiglitz (1982; theorem 3.4).

Now let $\underline{z}^I$ and $\underline{z}^{II}$ be two primal optima, both $> \underline{0}$, and

$$\underline{z}^i = (z_{L+1}(\lambda_{L+1}^i),\ldots,z_{L+M+N}(\lambda_{L+M+N}^i))', \ i = I,II,$$

then it will hold that ($j = L+1,\ldots,L+M+N$)

$$\underline{a}_j'(\lambda_j^I)\underline{v} = c_j \quad \text{and} \quad \underline{a}_j'(\lambda_j^{II})\underline{v} = c_j \text{ .}$$

If $\underline{z}^I$ and $\underline{z}^{II}$ differ, then more than M+N constraints of (4.12) will be satisfied with equality. In other words, the occurrence of alternative primal optima leads to a dual optimum with more than M+N slack variables equal to zero. Reversely, we have:

Theorem 5:

Let $\underline{\lambda}^I$ and $\underline{\lambda}^{II}$ be two different vectors such that

$$\underline{a}_j'(\lambda_j^I)\underline{v} = c_j(\lambda_j^I) \text{ and } \underline{a}_j'(\lambda_j^{II})\underline{v} = c_j(\lambda_j^{II}) \tag{4.13}$$

Then the subset of assigned technologies as represented by $\underline{\lambda}^I$ as well as the one represented by $\underline{\lambda}^{II}$ can realize the minimal costs.

Proof:

From (4.13), we see that there exists a price vector $\underline{p} \geq 0$ (viz. $\underline{p} = \underline{v}$) such that

$$\underline{p}'(I-A(\lambda^I)) = \underline{c}'(\underline{\lambda}^I)$$

where $\underline{c}'(\underline{\lambda}^I) := (c_{L+1}(\lambda^I_{L+1}),\ldots,c_{L+M+N}(\lambda^I_{L+M+N}))' > \underline{0}'$.

Using theorem 6.2.3 from Berman and Plemmons (1979) yields the result that $I-A(\underline{\lambda}^I)$ is a non-singular M-matrix. Hence, $A(\underline{\lambda}^I)$ is feasible (cf. theorem 1).

Now consider the LP problem:

$$\begin{aligned} &\text{Minimize } \underline{c}'(\underline{\lambda}^I)\underline{y} \\ &\text{s.t. } (I-A(\underline{\lambda}^I))\underline{y} = \underline{d} \\ &\qquad\qquad \underline{y} \geq \underline{0} \end{aligned}$$

The optimal solution value will equal $\underline{v}'\underline{d}$, while the optimal solution $\underline{y}^I$ is easily extended to an (optimal!) solution of the original LP problem (4.10). Analoguously, $A(\underline{\lambda}^{II})$ leads to a different optimal solution of the original LP problem (4.10).

So $A(\underline{\lambda}^I)$ as well as $A(\underline{\lambda}^{II})$ can realize minimal costs. □

5. The incorporation of technical services

5.1. Introduction

It is our aim to extend the model of chapter 4 with a TS sector. As noted in chapter 3, we will account for L different technical services $T_1,\dots,T_\ell,\dots,T_L$.

- They are demanded for by the product divisions, and
- they can also be bought externally instead of producing them internally.

If (some of) these services are bought from external suppliers, the firm avoids the variable as well as the fixed costs of internal production. Instead, the firm incurs variable costs due to the per-unit price for externally supplied services.

From now on, the extension of our decision model with a TS sector in the above described sense will briefly be referred to as: "the incorporation of technical services". First of all, a few notational extensions are in order. Then we treat the case with a constant demand for technical services. Finally, the ultimate decision model can be presented, wherein the demand for technical services is a (linear) function of the divisions production intensities.

5.2. Make-or-buy decisions

For every TS, there are two mutually exclusive possibilities:

1) T_ℓ is produced internally.

The per-unit direct costs of T_ℓ are $c_\ell(\text{in})$.

The amount of internally produced T_ℓ is $t_\ell(in)$.

If $t_\ell(in) > 0$, then the firm incurs a fixed cost equal to $C_\ell(fix)$.

Hence, the total cost of internal production is:

$$c_\ell(in)t_\ell(in) + C_\ell(fix), \quad t_\ell(in) > 0 .$$

Internal production of T_ℓ requires an amount $a_{i\ell}t_\ell(in)$ of T_i, $i \in \{1,\dots,L\}$, where $a_{i\ell} \geq 0$. Such a rate $a_{i\ell}$ is again referred to as an <u>intermediate input coefficient</u>. We define the matrix $A_{LL}(in)$ as follows:

$$A_{LL}(in) := (\underline{a}_1 \ \cdots \ \underline{a}_L)$$

where $\underline{a}_\ell := (a_{1\ell},\dots,a_{L\ell})'$, $\ell \in \{1,\dots,L\}$.

It is presumed that $A_{LL}(in)$ is feasible (cf. definition 6).

2) T_ℓ is bought externally.

The amount of externally bought T_ℓ is $t_\ell(ex)$.

The firm incurs a variable cost equal to

$$c_\ell(ex) \ t_\ell(ex)$$

where $c_\ell(ex)$ is the per-unit external price of T_ℓ. Some TS that is bought externally, does not require certain amounts of other TS-ses.

5.3. The case with a constant demand for technical services

Definition 10:

The net demand for T_ℓ is defined as the amount of T_ℓ required by the IP and EP sector. Notation: t_ℓ. We require $t_\ell > 0$.

If t_ℓ is constant for every TS, the Central Problem splits into two independent subproblems, namely the problem of finding the optimal divisional technologies and the problem of finding the optimal internal-external alternative. This section treats the latter problem. To this end, we can simply adopt the approach as introduced by Manes, Park and Jensen (1982), which leads to a mixed-integer programming formulation. We refer to Manes et. al. for the actual development of the model. Furthermore, these authors provide for a historical overview of the research of the so-called reciprocal service-cost problem. Up to their contribution, especially the aspect of (eventually avoidable) fixed costs had received little attention in the literature.

Altogether, for the problem of finding the optimal internal-external alternative with all t_ℓ constant, we propose the following mixed-integer programming framework:

$$\text{Minimize } \sum_{\ell=1}^{L} c_\ell(ex)t_\ell(ex) + \sum_{\ell=1}^{L} c_\ell(in)t_\ell(in) + \sum_{\ell=1}^{L} C_\ell(fix)\delta_\ell$$

$$\text{s.t. } t_\ell(ex) + (1-a_{\ell\ell}(in))t_\ell(in) - \sum_{\substack{j=1 \\ j\neq\ell}}^{L} a_{\ell j}(in)t_j(in) = t_\ell, \quad \ell = 1,\ldots,L$$

$$t_\ell(in) - M\,\delta_\ell \leq 0\,, \quad \ell = 1,\ldots,L$$

$$\text{all } \delta_\ell \text{ 0-1 variables, all } t_\ell(in),\ t_\ell(ex) \geq 0$$

Here M is a large number that guarantees: $t_\ell(in) > 0 \Rightarrow \delta_\ell = 1$.

The following definitions enable a more compact formulation of this problem.

Let:

$$\underline{t}(ex) := (t_1(ex),\ldots,t_L(ex))'\ ,$$

$$\underline{t}(in) := (t_1(in),\ldots,t_L(in))'\ ,$$

$$\underline{t} := (t_1,\ldots,t_L)'\ ,$$

$$\underline{\delta} := (\delta_1,\ldots,\delta_L)', \text{ all } \delta_\ell \text{ 0-1 variables,}$$

$$\underline{c}'(ex) := (c_1(ex),\ldots,c_L(ex))\ ,$$

$$\underline{c}'(in) := (c_1(in),\ldots,c_L(in))\ ,$$

$$\underline{C}'(fix) := (C_1(fix),\ldots,C_L(fix))\ .$$

The problem can be written as follows:

$$\begin{array}{llr} \text{Minimize} & \underline{c}'(ex)\underline{t}(ex) + \underline{c}'(in)\,\underline{t}(in) + \underline{C}'(fix)\underline{\delta} & \\ \text{s.t.} & \underline{t}(ex) + (I-A_{LL}(in))\underline{t}(in) & = \underline{t} \\ & \underline{t}(in) - M\,\underline{\delta} & \leqq \underline{0} \\ & \text{all elements of } \underline{t}(ex),\ \underline{t}(in) & \geq 0 \end{array}$$

(Again: M a large number that guarantees: $t_\ell(in) > 0 \Rightarrow \delta_\ell = 1$.)

5.4. The case where the demand for technical services depends on the product divisions

This section is devoted to the actual incorporation of TS into the model as developed in section 4.4. It is not surprising that the coupling of the TS sector to the IP and EP sector again leads to an LP model of the mixed-integer type. The difference with the previous section is that from now on the net demand for TS is not constant anymore. Actually, it becomes a linear, homogeneous function of (some of) the divisions production intensities.

The final decision model, accounting for the sectors TS, IP and EP, can be given in a compact representation. Therefore, the following notational modification is in order.
Each of the columns of $\hat{A}$ (see formula (4.9)) is augmented with L components, being the intermediate input coefficients with respect to delive-

ries from TS to IP and EP sector. Such a column $\underline{a}_{L+j}^{new}(\lambda)$ is of the form $(j = 1,\dots,M+N)$:

$$\underline{a}_{L+j}^{new}(\lambda) := (a_{1,L+j}(\lambda),\dots,a_{L,L+j}(\lambda),\ \ \underline{a}'_{L+j}(\lambda))' .$$

We will not continue to write down the superscripts "new". The new $\hat{A}$ can be subdivided as follows:

$$\hat{A} = \left(\begin{array}{c|c} \hat{A}_{LM} & \hat{A}_{LN} \\ \hline \hat{A}_{MM} & \hat{A}_{MN} \\ \hline 0 & \hat{A}_{NN} \end{array}\right)$$

the dimensions of $\hat{A}_{MM}$, $\hat{A}_{MN}$, $\hat{A}_{NN}$ remain the same,

$\hat{A}_{LM}$ is an $L \times (\sum_{j=L+1}^{L+M} \Lambda_j)$ matrix,

$\hat{A}_{LN}$ is an $L \times (\sum_{j=L+M+1}^{L+M+N} \Lambda_j)$ matrix.

In figure 3, the ultimate M(ixed) I(integer) P(rogramming) model is presented.

6. Summary

In this paper, we have developed a model for the firm including multiple technologies, and "make-or-buy" decisions. Though we start in a typical input-output setting, we end up in a mixed integer programming formulation.

With respect to the "make-or-buy" aspect, there is a correspondence with recent literature, viz. Manes et. al. (1982). As noted in this article, there are practical limitations on the size of effective integer programs. A theoretical difficulty is that duality theory for integer programming is considerably weaker than it is for ordinary LP.

In future research, the issue of common cost allocation is to be incorporated in order to obtain a so-called "integral model of the firm".

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From \ To	GS	TS	IP	EP	final output
GS					
TS					
IP					
EP					
primary input					

Figure 1: General input-output table; only EP sector delivers to 'final output'.

To / From	TS 1,...,L	IP L+1,...,L+M	EP L+M+1,...,L+M+N
TS 1 ... L	A_{LL}	A_{LM}	A_{LN}
IP L+1 ... L+M		A_{MM}	A_{MN}
EP L+M+1 ... L+M+N			A_{NN}

Figure 2: Input-output table for corporate model, sectors 'primary input' and 'final output' deleted.
Cross X symbolizes the absence of certain types of deliveries.

Minimize

$$\underline{c}'(ex)\underline{t}(ex) + \underline{c}'(in)\ \underline{t}(in) + \underline{c}'_x\ \underline{x} + \underline{c}'_y\ \underline{y} + \underline{C}'(fix)\underline{\delta}$$

subject to

$$\underline{t}(ex) + (I - A_{LL}(in))\underline{t}(in) - \hat{A}_{LM}\ \underline{x} - \hat{A}_{LN}\ \underline{y} = \underline{0}$$

$$(\hat{I}_M - \hat{A}_{MM})\underline{x} - \hat{A}_{MN}\ \underline{y} = \underline{0}$$

$$(\hat{I} - \hat{A}_{NN})\underline{y} = \underline{d}_{final}$$

$$\underline{t}(in) - M\ \underline{\delta} \leqq \underline{0}$$

all elements of $\underline{x}$, $\underline{y}$, $\underline{t}(ex)$, $\underline{t}(in) \geq 0$

$\underline{\delta} = (\delta_1, \ldots, \delta_L)'$, all δ_ℓ 0-1 variables

M is a large number that guarantees: $t_\ell(in) > 0 \Rightarrow \delta_\ell = 1$

Figure 3: MIP-model, including EP, IP and TS sector

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