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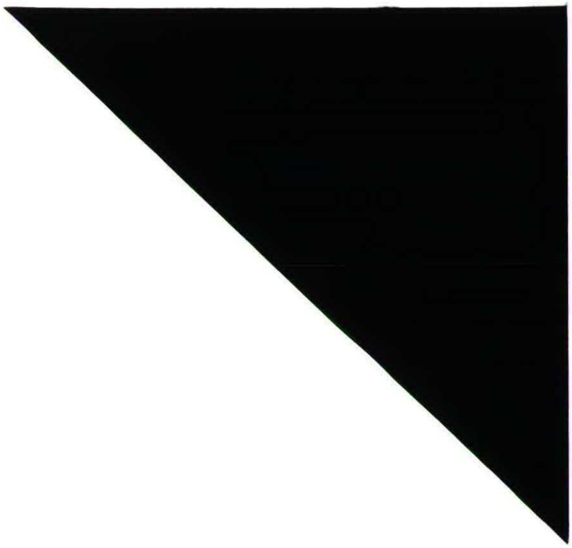

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Antoon van den
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Antoon van den Elzen †

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†A.H. van den Elzen, Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.

Abstract

We consider a general equilibrium model of an exchange economy with a finite number of commodities, and a finite number of price-taking utility maximizing consumers and price-taking profit maximizing producers. Furthermore, the economy satisfies the conditions given by Debreu (1959) to guarantee the existence of an equilibrium, i.e., a price vector at which demand equals supply. Crucial is the convexity of the total production set. In this paper we present a tâtonnement process that describes price adaptations towards an equilibrium. The essential idea is the compactification of the total production set as done by Debreu. This enables us to derive a well-defined profit function and supply correspondence. The process adapts prices according to the starting price vector and the state of the market, i.e., excess demand versus excess supply. The first feature distinguishes this type of processes from iterative processes like Walras' process and Smale's process. Along the path of prices generated, producers and consumers are assumed to behave competitively.

In the paper it is shown that this process generically exists and converges to an equilibrium from any starting vector and for any economy out of some well-known classes of convex economies with producers and consumers characterized by supply correspondences and demand functions, respectively. Furthermore, each convex economy of the Arrow/Debreu type with production and consumers demand represented by a demand function can be approximated arbitrary close by an economy with a convex production structure for which our process is well-defined. In that way we generalize all existing results on converging processes. Also the inclusion of a demand correspondence derived from a specific class of utility functions can be handled.

Finally, it will be shown that for any economy out of a broad class of so-called semi-algebraic convex economies and for any starting vector, there exists at least one path connecting the starting price vector and an equilibrium and satisfying all properties sketched.

Keywords: General equilibrium model, convex production, tâtonnement adjustment process, generic convergence, semi-algebraic economy.

1 Introduction

In this paper we consider a general equilibrium model with production of the Arrow/Debreu type. The most important feature of the model is the presence of a finite number of consumers and producers, all being price takers. The consumers maximize utility under a budget constraint whereas producers maximize profits over their convex production set. The possible profits accrue to the owners-consumers according to fixed shares. An equilibrium in such an economy is then a state in which, given certain prices and given these behavioral assumptions, all markets clear. In this convex environment each equilibrium is Pareto optimal. We are interested in the way how such an equilibrium is reached. More specifically, think of a situation in which an ongoing equilibrium is disturbed by the occurrence of an exogenous shock, leading to for example a change in the production structure. How does the economy adapt to the new situation and finds a new equilibrium? In another context we may want to evaluate the effect of a certain economic policy in the presence of multiple equilibria. The question concerning which equilibrium will be reached becomes then relevant. In this paper we present an adjustment process that, from almost every starting price vector, to be interpreted for example as the old equilibrium prices, reaches a new equilibrium by adaptations of prices and so inducing changes in demand and supply.

The process is of the tâtonnement type, i.e., no trade takes place before the equilibrium is reached. In other words, along the path followed by the process we have disequilibrium prices with no trade. Furthermore, the process resembles the Walrasian price adjustment processes (see Samuelson (1947)) in the sense that prices are adapted according to the sign of the related excess demand. However, contrary to the process of Samuelson, prices of goods in equilibrium are adjusted in order to keep them in equilibrium.

The main and distinguishing property of our process is that the price changes are taken relative to the starting prices. Thus, relative prices are considered. It is because of this that the process is generically converging, i.e., our process converges generically from any starting vector and for any economy out of some broad classes of convex economies. This in contrast to the procedures of Samuelson (1947) and Smale (1976) for pure exchange economies. See also Kehoe (1991), Section 4.3, for a Walrasian type of process for an economy with production and Mas-Colell (1986) who presented a mixed price and quantity adjustment process based on Walras. In other words, the process presented here is not purely iterative but also considers some global information, i.e., the location of the starting price vector. Saari and Simon (1978) gave necessary informational requirements for always converging processes. Here we obtain the positive result of an almost always converging process requiring the same amount of information plus the location of the starting vector.

This type of processes were originally developed by van der Laan and Talman (1987) who considered a model of a pure exchange economy. A rigorous analysis was performed by Herings (1994) who established the existence and convergence for generic starting vectors in generic pure exchange economies. Here we extend this analysis to convex production economies with the consumers characterized by a demand function. We obtain two main results. Firstly, we prove the generic existence and convergence of the adjustment process for the most important classes of models as for example the constant returns to scale case. Secondly, we prove that for a broad class of so-called semi-algebraic economies and for any starting vector there exists at least one adjustment path satisfying appealing properties and connecting the starting vector and an equilibrium. This property is not merely generic but holds for any economy in that class and for any starting vector. The semi-algebraicness roughly means that the graph of the excess demand correspondence can be described by polynomial (in)equalities. The consumption sector is described by a demand correspondence.

Up till now only generic converging processes for some specific production models exist. In addition, they suffer from the drawback that the set of allowed starting vectors is restricted. Van den Elzen, van der Laan and Talman (1994) considered a model with a finite number of producers, each characterized by a linear production technique. Van der Laan and Kremers (1993) described a process for a model with general, i.e., (non)-linear, constant returns to scale technologies excluding joint production. The main drawback of these processes is that they have to start from a price vector at which all firms make zero or negative profit. This because at other price vectors the supply would be undefined. However, when we consider the starting vector as the previous equilibrium after a shock, it may well be the case that some activities make profit at that price vector. This will for example happen in case of a technical innovation.

In this paper we describe an adjustment process for a model that encompasses the models described above. In addition, we allow for an arbitrary starting price vector. The basic idea is to compactify the total production set by taking into consideration the available endowments. This compactified production set then contains all productions attainable in equilibrium (see Debreu (1959)). Related to this compact set, supply and profit are well-defined for all price vectors which allows us to define a price adjustment process starting from any starting price vector.

For expository reasons we first present the process for a model with (piecewise)-linear production structure. This gives a direct generalization of the process of van den Elzen, van der Laan and Talman (1994) by allowing for an arbitrary starting point. In the case that at the starting price vector all activities make nonpositive profit, both processes coincide. Next, we consider models with other production structures, and show that the process is also then well-defined and converges generically for broad classes of convex economies, such as economies in which the total production set is a polyhedral set. From this we derive that any convex economy can be approximated arbitrary close by an economy for which the process is well-defined.

The paper consists of five sections. In Section 2 we introduce some notation, describe the general convex model, and define the process. Next, we prove in Section 3 the generic existence of the price adjustment process for the model with linear production technologies. Section 4 considers the existence of the process for models with other standard convex production structures. In an example we also consider a model with a consumers demand correspondence. Finally, in Section 5 we deal with the class of semi-algebraic models and show that in that case the path of the process may be not unique.

2 The basic model

All models treated in this paper fit in the framework of the standard Arrow/Debreu-model of an exchange economy with production. Therefore we start by shortly reviewing that model. In the next sections we consider several specifications for the production structure. The Arrow/Debreu model represents a competitive exchange economy with a finite number of consumers and producers trading and producing a finite number of commodities, while striving for utility-maximization and profit-maximization, respectively.

We first introduce some notation. For any positive integer k we denote the set $\{1,\ldots,k\}$ by I_k . By $\mathbf{R}^k_+$ we denote the nonnegative orthant of the k-dimensional Euclidean space, i.e., $\mathbf{R}^k_+ = \{x \in \mathbf{R}^k | x \geq \underline{0}\}$. Here, $\underline{0}$ denotes the vector of zeros of appropriate length, whereas $x \geq \underline{0}$ indicates that all elements of the vector x are greater than or equal to zero. Furthermore, for $x, y \in \mathbf{R}^k$, $x > y$ equals $x \geq y$ and $x \neq y$, while $x \gg y$ indicates $x_h > y_h$, $\forall h \in I_k$. Accordingly, $\mathbf{R}^k_{++} = \{x \in \mathbf{R}^k | x \gg \underline{0}\}$. For $x, y \in \mathbf{R}^k$ we denote the set $\{z \in \mathbf{R}^k | z = \lambda x + (1-\lambda)y,\ 0 \leq \lambda \leq 1\}$ by $[x,y]$, which is called the line segment between x and y. With $[x,y)$ we indicate $[x,y]\backslash\{y\}$. Similarly, we denote $(x,y]$ and (x,y) with obvious meaning.

A sign vector $s \in \mathbf{R}^k$ denotes a vector with components $s_h \in \{-1,0,+1\}$, $\forall h \in I_k$.

Related to $x \in \mathbf{R}^k$ we define $\mathrm{sgn}(x)$, the sign vector of x, as the sign vector $s \in \mathbf{R}^k$ such that $s_h = -1$ $(+1)$ if and only if $x_h < 0$ (> 0), $\forall h \in I_k$. Given a sign vector $s \in \mathbf{R}^k$ we denote by $I^-(s)$ the set $\{h \in I_k | s_h = -1\}$. Similarly, $I^+(s) = \{h \in I_k | s_h = +1\}$ and $I^0(s) = \{h \in I_k | s_h = 0\}$. Furthermore, the cardinality of $I^-(s)$, $I^+(s)$ and $I^0(s)$ is denoted by $k^-(s)$, $k^+(s)$ and $k^0(s)$, respectively.

Furthermore, given a set $A \subset \mathbf{R}^k$ we denote by $\mathrm{int}(A)$ and $\mathrm{bd}(A)$ the interior and the boundary of A relative to its affine hull. Similarly, $\mathrm{cl}(A)$ and $\mathrm{co}(A)$ denote the closure and the convex hull of the set A, respectively. By $\dim(A)$ we denote the dimension of A. The cardinality of a finite set A is denoted by $|A|$. By $e(k)$ and e we denote the k-th standard unit vector and the vector of ones of appropriate length, respectively. Given a (row)vector x, the transpose of x is denoted by $x^\top$. Finally, we say that a function f is of class C^r whenever f is r times continuously differentiable.

Let us now start with the description of the model. In the model there are $n+1$ commodities indexed by $\ell \in I_{n+1}$. Furthermore, we have c consumers indexed by $i \in I_c$ and m firms indexed by $j \in I_m$. The economy is then denoted by E and specified as $E = \left\{\left(X^i, u^i, w^i, (\theta_{ij})_{j=1}^m\right)_{i=1}^c, (Y^j)_{j=1}^m\right\}$. For each producer $j \in I_m$ we have a production set Y^j . Each consumer $i \in I_c$ is specified by a consumption set X^i, a utility function u^i, initial endowments w^i, and profit shares $(\theta_{ij})_{j=1}^m$, where θ_{ij} denotes the share of consumer i in the profit of firm j. Of course $\sum_{i=1}^c \theta_{ij} = 1$, $\forall j \in I_m$.

First we consider the firms in more detail. For $j \in I_m$, the set Y^j is a subset of $\mathbf{R}^{n+1}$ with characteristic element y^j representing a production plan. Positive components of y^j denote outputs, whereas $y^j_\ell < 0$ indicates that commodity ℓ serves as input. By Y we denote the total production set $\sum_{j=1}^m Y^j$ with corresponding generic element y. We assume the following technological assumptions on the production sets

$$
\begin{array}{lll}
\text{a)} & \underline{0} \in Y^j,\ \forall j \in I_m & \text{(possibility of inaction)} \\
\text{b)} & Y \cap (-Y) \subset \{\underline{0}\} & \text{(irreversibility of production)} \\
\text{c)} & -\mathbf{R}^{n+1}_+ \subset Y & \text{(free disposal)} \\
\text{d)} & Y \text{ is closed} & \text{(continuity)} \\
\text{e)} & Y \text{ is convex} & \text{(non-increasing returns to scale).}
\end{array}
\tag{2.1}
$$

Note that most assumptions are related to the total production set.

The crucial assumption concerns the convexity of Y. Technologically, this corresponds to non-increasing returns to scale meaning that increasing the amounts of inputs with a certain factor leads to relatively less increase in the outputs. Economically, it is important because it fits well in the context of price-taking profit maximizing behaviour. This does not hold in case of increasing returns to scale, for which profit maximizing production is in general not defined. The closedness of Y is a technical assumption needed for the derivation of the supply correspondence. Furthermore, note that (2.1.a) and (2.1.b) imply $Y \cap (-Y) = \{\underline{0}\}$.

Concerning the consumer side of the model we will be very brief because our main interest lies in the production side. Also here we take the standard assumptions made by Debreu (1959). Thus, the consumption sets are closed, convex and bounded from below, whereas the preference relations are complete, transitive, continuous, strictly monotonic and convex. Equivalently, the preferences can be represented by a continuous, strictly increasing and quasi-concave utility function. In the sequel we assume that $X^i = \mathbf{R}_+^{n+1}$, $\forall i \in I_c$. Furthermore, $w^i \in \text{int}(X^i)$, $\forall i \in I_c$. The total endowments $\sum_{i=1}^c w^i$ are denoted by W.

Following Debreu (1959) we observe that we may compactify the total production set. This because the production vectors and the consumptions that can be reached in equilibrium form a compact set (see Debreu (1959), Theorem 1, page 77). Thus, by restricting ourselves to those compactified sets, the set of equilibria remains unchanged.

Concerning the economic behaviour of the firms we assume that they are profit-maximizing price takers. Thus, given a price vector $p = (p_1, \ldots, p_{n+1})^\top \in \mathbf{R}_+^{n+1} \backslash \{\underline{0}\}$, producer $j \in I_m$ utilizes a production vector $y^j \in Y^j$ such that the related profit $p^\top y^j$ is maximal. To model this behaviour the total production set Y entails sufficient information. This because $y \in Y$ is maximizing total profit given a certain price vector p if and only if for all $j \in I_m$, y^j is profit-maximizing over Y^j, where $y = \sum_{j=1}^m y^j$. Note that because of the free disposal assumption we may restrict ourselves to nonnegative prices.

Related to the compactified total production set, notation $\hat{Y}$, the supply correspondence is well-defined for all prices. Here, we implicitly assume that the producers observe the restrictive amount of endowments available, and consider as their production possibilities $\hat{Y}$ instead of Y. Furthermore, we assume that $\hat{Y}$ is convex, compact and $\hat{Y} \cap (-\hat{Y}) = \{\underline{0}\}$. A suitable compactification which will be used later on in some examples, is given by

$$\hat{Y} = Y \cap \{y \in \mathbf{R}^{n+1} | y \geq -\bar{e}\} \text{ with } \bar{e} \gg W \gg \underline{0}. \tag{2.2}$$

Profit-maximizing over $\hat{Y}$ gives rise to the following definition.

Definition 2.1. The correspondence $S : \mathbf{R}^{n+1}_{++} \mapsto \hat{Y}$ denotes the total supply correspondence, with $S(p) = \{\hat{y} \in \hat{Y} | p^\top \hat{y} \geq p^\top y,\ \forall y \in \hat{Y}\}$ the total supply at prices p. The function $\pi : \mathbf{R}^{n+1}_{++} \mapsto \mathbf{R}_+$ is the total profit function with $\pi(p)$ the profit at prices p.

Observe that the supply and profit are also defined for prices with zero components. However, later on we show that zero prices are not encountered by our process whenever started from a strictly positive price vector.

We can be somewhat more specific about the set of possible supply vectors. This is the set of weakly efficient production vectors, Eff$(\hat{Y})$, being the set $\{y \in \hat{Y} | (y + \mathbf{R}^{n+1}_{+}) \cap$ int $(\hat{Y}) = \emptyset\}$. From this we see that Eff$(\hat{Y}) \subset$ bd$(\hat{Y})$ and $\underline{0} \in$ Eff$(\hat{Y})$. This follows from 2.1.a) - c).

Both S and π are well-defined because we are maximizing a continuous function over a compact set. From the maximum-theorem it follows that S is upper hemicontinuous (u.h.c.), whereas π is continuous. It is easy to verify that S is homogeneous of degree 0, i.e., $S(\lambda p) = S(p)$, $\forall \lambda > 0$, whereas π is homogeneous of degree 1 $(\pi(\lambda p) = \lambda \pi(p),\ \forall \lambda > 0)$. Furthermore, it is straightforward that S has nonempty, convex and compact values, whereas $\pi(p) \geq 0$, $\forall p$.

Given the assumptions concerning the consumers stated earlier and adopting the hypothesis of utility maximization, we can summarize their behaviour by the total demand correspondence $D : \mathbf{R}^{n+1}_{++} \mapsto \mathbf{R}^{n+1}$, with $D(p)$ the set of demand vectors at prices p. Furthermore, the correspondence D is homogeneous of degree 0 and u.h.c. with nonempty, convex and compact values. Finally, the consumers spend their total income, i.e., $p^\top x = p^\top W + \pi(p)$, $\forall x \in D(p)$. The correspondence $\tilde{Z} : \mathbf{R}^{n+1}_{++} \mapsto \mathbf{R}^{n+1}$ defined by $\tilde{Z}(p) = D(p) - \{W\}$ is called the consumers excess demand. Note that $\tilde{Z}$ is well-defined because π is. Together with $p \in \mathbf{R}^{n+1}_{++}$ and $w^i \in$ int(X^i), $\forall i \in I_c$, this makes that consumers maximize utility over a compact set. We remark that our restriction to a finite number of consumers is not essential. A standard model with a continuum would give us a demand correspondence satisfying the same properties (see Mas-Colell (1985), Proposition 5.25).

Crucial behind this derivation is the assumption that the consumers are the owners of the firms and receive the profits. Usually we assume fixed shares but slightly more general profit distribution schemes are also allowed (see Bonnisseau and Cornet (1988)). In case the demand is given by a function it is denoted by d whereas the consumers excess demand is then indicated as $\tilde{z}$.

At this point we summarize the main elements of the model by defining the total excess demand correspondence $Z : \mathbf{R}^{n+1}_{++} \mapsto \mathbf{R}^{n+1}$, with $Z(p) = D(p) - S(p) - \{W\}$ and satisfying

a) upper hemi-continuity
b) nonempty, convex and compact valued
c) homogeneity of degree 0
d) $p^\top z = 0$, $\forall z \in Z(p)$.

Properties a), b), and c) follow directly from corresponding properties of S and D. Property d) results from $p^\top(x - W) = \pi(p)$, $\forall x \in D(p)$, and $p^\top y = \pi(p)$, $\forall y \in S(p)$.

In the sequel we occasionally identify the economy E with Z or z. With all this we are ready to define an equilibrium.

Definition 2.2. The tuple (p^*, x^*, y^*) is a Walrasian equilibrium of the economy E if

a) $x^* \in D(p^*)$
b) $y^* \in S(p^*)$
c) $x^* - y^* = W$.

From Debreu (1959) we know that such an equilibrium exists under the assumptions given. Because of the strict monotonicity of the preferences all equilibrium prices are strictly positive. Note that each equilibrium corresponds to a zero point of Z. Because Z is homogeneous of degree 0, to each equilibrium allocation corresponds a ray of equilibrium price vectors. To get rid of this indeterminacy we normalize price vectors by dividing each price through the sum of the prices. This makes the prices to lie in the n-dimensional unit simplex S^n being the set $\{p \in \mathbf{R}^{n+1}_{+} | \sum_{\ell=1}^{n+1} p_\ell = 1\}$. In the sequel we define a tâtonnement price adjustment process for a broad class of models falling in the framework sketched, that reaches an equilibrium price vector $p^* \in S^n$ from almost any price vector $p^0 \in \text{int}(S^n)$. Besides, the process has an appealing economic interpretation. For simplicity we assume throughout the paper the consumer demand to be a function instead of a correspondence. However, in Sections 4 we briefly consider some cases in which our process also works for a demand correspondence. Finally, in Section 5 the consumers are characterized by a semi-algebraic demand correspondence.

Let us now define the adjustment process for an economy as described before. First, choose a price vector $p^0 \in \text{int}(S^n)$ for which $S(p^0)$ consists of a unique element y^0. Then from (p^0, y^0) a path of price vectors $p \in S^n$ and production vectors $y \in S(p)$ is generated for which $\forall \ell \in I_{n+1}$

$$\begin{aligned} p_\ell/p_\ell^0 &= \max_r p_r/p_r^0 && \text{if} \quad z_\ell(p|y) > 0 \\ \min_r p_r/p_r^0 \leq p_\ell/p_\ell^0 &\leq \max_r p_r/p_r^0 && \text{if} \quad z_\ell(p|y) = 0 \\ \min_r p_r/p_r^0 &= p_\ell/p_\ell^0 && \text{if} \quad z_\ell(p|y) < 0, \end{aligned} \tag{2.3}$$

where $z(p|y) = d(p) - y - W$.

We prove in this paper that the set of vectors (p, y) satisfying (2.3) generically contains a path connecting (p^0, y^0) and an equilibrium (p^*, y^*), for any (p^0, y^0) and for any economy z, within certain important classes of convex economies. For this we need to impose some differentiability requirements on the economy. Therefore we cannot consider all convex economies. However, we prove a weaker property of the set (2.3) holding for all semi-algebraic economies.

The adjustment process that generates vectors (p, y) satisfying (2.3) has an appealing economic interpretation. Recall that we assume that the consumers express a unique demand at any price vector. Along the path, price adaptations occur according to the relation of the ongoing price and the starting price on the one hand and market situations on the other. From the start, prices related to commodities in excess demand (supply) are increased (decreased). Generally, if a market is in excess demand (supply) the related price is relatively to the starting price maximal (minimal). As soon as the market for commodity ℓ becomes in equilibrium ($z_\ell(p|y) = 0$) it is in principle kept in equilibrium. However, when the price at a market being kept in equilibrium becomes relatively maximal (minimal), then the equilibrium on this market is distorted and the market becomes in excess demand (supply), while keeping the prices relatively maximal (minimal). All along the path consumers and producers behave optimally. However, the behaviour of the producer is somewhat arbitrarily at prices p at which he is indifferent among a subset of production plans, i.e., when $S(p)$ is a set. Then a specific vector out of that set is prescribed for him.

We note that there is a strong relationship between the process defined here and the simplicial algorithm as defined in Doup, van der Laan and Talman (1987) and applied by Talman (1990) to an economy with production. In fact our process can be followed arbitrary close by this simplicial algorithm. The simplicial algorithm subdivides the price space into simplices. Corresponding to each price vector p it chooses an element out of the set of possible total excess demand vectors $Z(p)$. Then the algorithm generates a piecewise linear path in a sequence of simplices. Our process can be seen as the limit path corresponding to the algorithm, i.e., the path that is generated when the diameter of the simplices approaches zero. We make use of this in the proof of Theorem 5.1.

3 Exchange economies with linear production technologies

In this section we deal with the case in which Y is a cone, i.e., if $y \in Y$ then also $\alpha y \in Y$ for all $\alpha \geq 0$. Thus, the technological possibilities reveal constant returns to scale. Note that the total supply vector is not defined for prices at which profit is positive. With $\underline{0} \in Y$ we derive that equilibrium profit has to be zero. In this model the total production set can be thought of as being generated by a finite number of elementary activities $\{a^1, \ldots, a^m\} \subset \mathbf{R}^{n+1}$. More precisely, the individual production set Y^j, $j \in I_m$, can be seen as the set $Y^j = \{y^j \in \mathbf{R}^{n+1} | y^j \leq \alpha_j a^j,\ \alpha_j \geq 0\}$. This leads to the total production set $Y = \{y \in \mathbf{R}^{n+1} | y \leq \sum_{j=1}^{m} \alpha_j a^j,\ \alpha_j \geq 0\}$.

We circumvent the problems related to non-defined supply by using a compact set $\hat{Y}$, obtained by intersecting Y and a polyhedral set containing the set $\{x \in \mathbf{R}^{n+1} | x \geq -W\}$. An example is given in (2.2). The compactified production set $\hat{Y}$ is then a polytope, being the convex hull of a finite number of extreme points. The precise shape of $\hat{Y}$ is determined by the method of compactification and the assumptions concerning the production structure. In the standard activity analysis model without intermediate production and mergers we have that related to each activity a^j there is an extreme point on the intersection of bd$(\hat{Y})$ and the ray containing the origin and a^j. Also the origin is an extreme point and there are extreme points strictly related to the lower bound used for the compactification. However, in a more general model there are additional extreme points. They occur when one technique produces an output that serves as an input for another technique (intermediate production is allowed for).

For example, consider an economy with 3 commodities, 2 activities, namely $a^1 = (0,-1,1)^\top$ and $a^2 = (-1,1,0)^\top$, whereas the endowments W equal $(1,1,1)^\top$. Furthermore, we compactify Y by only considering productions needing endowments equal to or less than $\bar{e} = 2W$. Then in the standard model the set $\hat{Y}$ equals the convex hull of the vectors $(0,-2,2)^\top$, $(-2,2,0)^\top$, $(0,0,0)^\top$, $(0,0,-2)^\top$, $(0,-2,0)^\top$, $(-2,0,0)^\top$, and $(-2,-2,-2)^\top$, whereas Eff$(\hat{Y})$ is equal to co$\{(0,-2,2)^\top, (-2,2,0)^\top, (0,0,0)^\top\}$. In case intermediate production is allowed we obtain one additional extreme point, i.e.,

$(-2,-2,4)^\top$, and $\mathrm{Eff}(\hat{Y})$ becomes $\mathrm{co}\{(0,-2,2)^\top,(-2,2,0)^\top,(0,0,0)^\top,(-2,-2,4)^\top\}$. It is important to note that $\mathrm{Eff}(\hat{Y})$ is homeomorphic to a unit simplex whose dimension equals the minimum of n and m, under the assumption that the activity vectors are linearly independent, see Bonnisseau and Cornet (1988). In the example $\dim(\mathrm{Eff}(\hat{Y}))$ equals 2.

Here we generalize the process of van den Elzen, van der Laan and Talman (1994) in two respects. First of all, the linear model considered here is more general in that it allows for intermediate production. Furthermore, the starting price vector of the process is not restricted to be a vector at which each activity makes nonpositive profits.

In van den Elzen, van der Laan and Talman (1994) it is assumed that there can be no production without input. Theorem 3.1 states that this is equivalent with the irreversibility condition.

Theorem 3.1. Under free disposal (2.1.c), the no production without input condition, i.e., $Ax \geq \underline{0}$ and $x \geq \underline{0}$ implies $x = \underline{0}$, where $A = [a^1, \ldots, a^m]$, with $a^j \neq \underline{0}$, $\forall j \in I_m$, is equivalent to the strong irreversibility condition $Y \cap (-Y) = \{\underline{0}\}$, with Y as above.

Proof. Assume that $Y \cap (-Y) = \{\underline{0}\}$. From $Ax \geq \underline{0}$, $x \geq \underline{0}$ we want to conclude $x = \underline{0}$. First, we consider the case in which $Ax > \underline{0}$ and $x > \underline{0}$. This is in contradiction with strong irreversibility which gives $Y \cap \mathbf{R}^{n+1}_+ \backslash \{\underline{0}\} = \emptyset$. Next, we consider the case $Ax = \underline{0}$ and $x > \underline{0}$. Assuming $x_1 > 0$ we get that $a^1 x_1 = -\sum_{j\neq 1} a^j x_j \neq \underline{0}$. Besides, $y = a^1 x_1 \in Y$ because $y = A\bar{x}$ with $\bar{x} = e(1)x_1$. Similar we obtain $-\sum_{j\neq 1} a^j x_j \in -Y$. Contradiction with $Y \cap (-Y) = \{\underline{0}\}$.

To prove the only if part we assume that $Y \cap (-Y) \neq \{\underline{0}\}$. Thus, there is a nonzero production vector $y \in Y \cap (-Y)$, i.e., $y = Ax$, $x \geq \underline{0}$ and $y = -A\tilde{x}$, $\tilde{x} \geq \underline{0}$. We get that $A(x + \tilde{x}) = \underline{0}$ with $x + \tilde{x} \geq \underline{0}$. From the no production without input condition we obtain $x + \tilde{x} = \underline{0}$ and so $x = \tilde{x} = \underline{0}$. Contradiction with y being nonzero. □

We now want to prove the generic existence and convergence of the adjustment process defined in Section 2, for the class of economies $E = \left\{\left(X^i, u^i, w^i, (\theta_{ij})_{j=1}^m\right)_{i=1}^c, Y\right\}$ considered above. Furthermore, we assume that there are no redundant activity vectors, i.e., they are independent. To make the analysis more tractable we rewrite (2.3) by making use of the conical production structure. We denote the set of vectors (p, y) satisfying (2.3) by $\mathcal{B}(p^0; Z, \hat{Y})$. Furthermore, the extreme vectors of $\mathrm{Eff}(\hat{Y})$ are denoted by $\{\hat{y}^1, \ldots, \hat{y}^q\}$, with $q \geq \min\{(n+1), (m+1)\}$. Related to a nonempty set $U \subset I_q$, we denote by $\mathrm{Eff}(\hat{Y}(U))$ the set $\mathrm{co}\{\hat{y}^k,\ k \in U\}$. Now we can regard $\mathcal{B}(p^0; Z, \hat{Y})$ as the union of sets $B(s, U)$ over pairs (s, U), with s a sign vector in $\mathbf{R}^{n+1}$ and $\emptyset \neq U \subset I_q$, given by

$$B(s,U) = \{(p,y) \in A(s,U) | \mathrm{sgn}(d(p) - y - W) = s\}, \text{ where}$$

$$A(s,U) = \{(p,y) \in S^n \times \mathrm{Eff}(\hat{Y}) | \begin{array}{l} p^\top \hat{y}^k = \max_{\bar{y} \in \hat{Y}} p^\top \bar{y},\ k \in U \\ y = \sum_{k \in U} \alpha_k \hat{y}^k \text{ with } \alpha_k \geq 0 \text{ and } \sum_{k \in U} \alpha_k = 1 \\ p_\ell / p_\ell^0 = \max_r p_r / p_r^0 \text{ if } s_\ell = +1 \\ p_\ell / p_\ell^0 = \min_r p_r / p_r^0 \text{ if } s_\ell = -1\}. \end{array} \tag{3.1}$$

However, not all possible sign vectors s and subsets U are relevant. Due to the complementarity condition holding for Z, a relevant sign vector needs to contain at least one pair of components $(+1, -1)$. We denote the set of allowed sign vectors by $\mathcal{S}$. Furthermore, we only consider those subsets U for which $\dim(\mathrm{Eff}(\hat{Y}(U))) = |U| - 1$. Because $\mathrm{Eff}(\hat{Y})$ is homeomorphic to a simplex we can subdivide $\mathrm{Eff}(\hat{Y})$ into simplices. In the sequel we fix one subdivision and denote the set of subsets U involved by $\mathcal{U}$. Concerning the pair (s, U) we have to assume that $|U| - 1 \leq k^0(s) + 1$. In case $|U| > k^0(s) + 2$ there would be more than $n + 1$ independent restrictions on the price vector p in the definition of $A(s, U)$, making $B(s, U)$ equal to the empty set. We denote the set of pairs $(s, U) \in \mathcal{S} \times \mathcal{U}$ with $|U| \leq k^0(s) + 2$ by $\overline{\mathcal{S} \times \mathcal{U}}$. The subsets $A(s, U)$, $(s, U) \in \overline{\mathcal{S} \times \mathcal{U}}$, form a subdivision of $S^n \times \mathrm{Eff}(\hat{Y})$. That indeed also S^n is subdivided is illustrated for example in van der Laan and Talman (1987). We provide some intuition in Example 3.1.

Note that the right-hand side of (2.3) is captured by the sign vector in (3.1), whereas the left-hand side is included in the description of $A(s, U)$. Expression (3.1) is more explicit concerning the location of a supply vector at price vector p. Given p the profit maximizing supplies lie on a face of $\mathrm{Eff}(\hat{Y})$ having the vectors indexed by $k \in U$ as extreme points. Furthermore, the set U is nonempty because the continuous profit function always attains a maximum on the compact set $\mathrm{Eff}(\hat{Y})$.

We argue that $\mathcal{B}(p^0; Z, \hat{Y}) = \cup_{s,U} B(s, U)$, $(s, U) \in \overline{\mathcal{S} \times \mathcal{U}}$, generically contains a path connecting (p^0, y^0) and an equilibrium (p^*, y^*). To make this genericity more precise, we define Ω being the set of all possible distributions of endowments, i.e., $\Omega = \{w = (w^1, \ldots, w^c) | w^i \gg \underline{0},\ \forall i \in I_c\}$. Now we can state the following theorem.

Theorem 3.2. For all $i \in I_c$, let X^i be equal to $\mathbf{R}^{n+1}_{++}$ and let u^i be C^3, strictly increasing, strictly quasi-concave, let the indifference surfaces of u^i have nonzero Gaussian curvature at every $x \in X^i$ whereas the closure of the indifference surfaces in $\mathbf{R}^{n+1}$ is a subset of $\mathbf{R}^{n+1}_{++}$. Furthermore, let the set Y be a cone satisfying (2.1), and let the compactified production set $\hat{Y}$ satisfy (2.1) except free disposal. Let $p^0 \in \text{int}(S^n)$ be the starting price system. If $S(p^0) = \{y^0\}$ and $z_\ell(p^0, y^0) \neq 0$, $\forall \ell \in I_{n+1}$, then the price adjustment process defined by (2.3) for the economy $E = \left\{\left(X^i, u^i, w^i, (\theta_{ij})_{j=1}^m\right)_{i=1}^c, \hat{Y}\right\}$ generates a path of vectors (p, y) converging to an equilibrium (p^*, y^*), except for a set of initial endowments in Ω having a closure in Ω with Lebesque measure zero.

Remark 3.1. The set of economies is parametrized by their initial endowments. A motivation for this specific parametrization is for example given in Balasko (1988), Chapter 1. It is also very convenient because it delivers a natural topology.

Remark 3.2. Note that the conditions on $(X^i, u^i)_{i=1}^c$ are more strict than in Section 2 where only sufficient conditions for the existence of an equilibrium were given. Concerning u^i, some differentiability requirements are made, whereas the indifference curves should have some curvature (see Mas-Colell (1985), Proposition 2.5.1). Finally, some boundary condition is stated. Because of this, no consumer will ever demand zero amounts of some commodities. Thus, in fact X^i can be taken equal to $\mathbf{R}^{n+1}_+$ which satisfies all conditions of Section 2.

Remark 3.3. The proof follows the idea of Herings (1994). He proved the theorem for the special case of a pure exchange economy. For applying his argument we need the total excess demand to be of class C^2. This does not hold here because the profit function is not differentiable everywhere. However, we will be able to subdivide the problem into pieces for which the differentiability holds.

Proof theorem. To indicate the dependence of E on $w \in \Omega$ we denote $B(s, U)$ by $B^w(s, U)$ and Z by Z^w etc. We have to prove that the set $\cup_{s,U} B^w(s, U)$, with $(s, U) \in \overline{\mathcal{S} \times \mathcal{U}}$, contains for generic $w \in \Omega$ a unique path connecting (p^0, y^0) and an equilibrium (p^*, y^*). Observe from (3.1) that on $A(s, U)$ we have $S(p) = \text{Eff}(\hat{Y}(U))$, i.e., S is a continuous correspondence on $A(s, U)$. Furthermore, π is smooth on $A(s, U)$ because $\pi(p)$ equals $p^\top \hat{y}^k$ for some $k \in U$. Therefore, we may define the C^2-function $\hat{z}^w : A(s, U) \mapsto \mathbf{R}^{n+1}$ with $\hat{z}^w(p, y) = \tilde{z}^w(p) - y$, where $\tilde{z}^w(p)$ denotes the consumers excess demand at p for given w. In this manner we transform the correspondence Z^w into C^2-functions $\hat{z}^w$ on sets $A(s, U)$. That indeed $\tilde{z}^w$ is of class C^2 on $A(s, U)$ follows from the conditions on $(X^i, u^i)_{i=1}^c$. In the sequel we consider $\hat{z}$ to be also a function of w, i.e., $\hat{z} : A(s,U) \times \Omega \mapsto \mathbf{R}^{n+1}$, with $\hat{z}(p,y,w) = \tilde{z}^w(p) - y$.

Now, consider a pair $(s,U) \in \overline{\mathcal{S} \times \mathcal{U}}$. Without loss of generality we assume that $I^0(s) = I_{k^0(s)}$, $I^-(s) = I_{k^0(s)+k^-(s)} \backslash I_{k^0(s)}$, $I^+(s) = I_{n+1} \backslash I_{k^0(s)+k^-(s)}$, and $|U| = I_{|U|}$. Let some $\ell^- \in I^-(s)$ and $\ell^+ \in I^+(s)$ be given. Finally, observe that an element $y = \sum_{k \in U} \alpha_k \hat{y}^k$ is characterized by the vector $\alpha \in \mathbf{R}_+^{|U|}$. Related to each U with $|U| \neq 1$ we extend the set of possible vectors α and consider vectors $\alpha \in \mathbf{R}_{-\epsilon}^{|U|}$, where $\mathbf{R}_{-\epsilon}^{|U|} = \{\bar{\alpha} \in \mathbf{R}^{|U|} | \bar{\alpha}_k > -\epsilon,\ \forall k \in I_{|U|}\}$, and $\epsilon > 0$ arbitrary small. Thus, we extend $\mathbf{R}_+^{|U|}$ to a smooth manifold without boundary. We obtain that $(p,y) \in B^w(s,U)$ if and only if $(p,\alpha,w) \in \mathbf{R}_{++}^{n+1} \times \mathbf{R}_{-\epsilon}^{|U|} \times \Omega$ satisfies

$$
\begin{aligned}
p^\top \hat{y}^k - p^\top \hat{y}^1 &= 0 \quad , \quad \forall k \in U \backslash \{1\} && (3.2)\\
\sum_{k \in U} \alpha_k - 1 &= 0 && (3.3)\\
\hat{z}_\ell(p,\alpha,w) &= 0 \quad , \quad \forall \ell \in I^0(s) && (3.4)\\
p_\ell p_{\ell+1}^0 - p_{\ell+1} p_\ell^0 &= 0 \quad , \quad \forall \ell \in I_{k^0(s)+k^-(s)-1} \backslash I_{k^0(s)} && (3.5)\\
p_\ell p_{\ell+1}^0 - p_{\ell+1} p_\ell^0 &= 0 \quad , \quad \forall \ell \in I_n \backslash I_{k^0(s)+k^-(s)} && (3.6)\\
\sum_{\ell=1}^{n+1} p_\ell - 1 &= 0 && (3.7)\\
-\hat{z}_\ell(p,\alpha,w) &\geq 0 \quad , \quad \forall \ell \in I^-(s) && (3.8)\\
\hat{z}_\ell(p,\alpha,w) &\geq 0 \quad , \quad \forall \ell \in I^+(s) \text{ if } k^0(s) \leq n-2 && (3.9)\\
p_\ell p_{\ell^-}^0 - p_{\ell^-} p_\ell^0 &\geq 0 \quad , \quad \forall \ell \in I^0(s) && (3.10)\\
p_{\ell^+} p_\ell^0 - p_\ell p_{\ell^+}^0 &\geq 0 \quad , \quad \forall \ell \in I^0(s) && (3.11)\\
p_{\ell^+} p_{\ell^-}^0 - p_{\ell^-} p_{\ell^+}^0 &\geq 0 && (3.12)\\
-p^\top \hat{y}^k + p^\top \hat{y}^1 &\geq 0 \quad , \quad \forall k \notin U && (3.13)\\
\alpha_k &\geq 0 \quad , \quad \forall k \in U. && (3.14)
\end{aligned}
$$

Apart from the inclusion of production via α in $\hat{z}$, the subsystem (3.4)-(3.12) is identical to the system given by Herings (1994). The extra conditions state that the profits at each extreme vector of $\mathrm{Eff}(\hat{Y}(U))$ are equal (equation (3.2)) and not less than at other vectors of $\hat{Y}$ (inequality (3.13)). Furthermore, the production vector ought to lie in $\mathrm{Eff}(\hat{Y}(U))$ (equations (3.3) and (3.14)). Observe that in case $|U| = 1$, the variable α is fixed to 1. What is left concerns a set of tuples (p,y) satisfying (3.4) - (3.13). That system is already discussed by Herings (1994).

Following Herings (1994), we first consider the set of points (p, α, w) satisfying the equations (3.2)-(3.7). For that we define for $(s, U) \in \overline{\mathcal{S} \times \mathcal{U}}$ the function $\Psi^{s,U}$: $\mathbf{R}^{n+1}_{++} \times \mathbf{R}^{|U|}_{-\epsilon} \times \Omega \mapsto \mathbf{R}^{n+|U|}$ such that $\Psi^{s,U}(p, \alpha, w)$ is the left hand side of (3.2)-(3.7). Similarly, we define $\forall (s, U, w) \in \overline{\mathcal{S} \times \mathcal{U}} \times \Omega$ the function $\Psi^{s,U,w} : \mathbf{R}^{n+1}_{++} \times \mathbf{R}^{|U|}_{-\epsilon} \mapsto \mathbf{R}^{n+|U|}$ by $\Psi^{s,U,w}(p, \alpha) = \Psi^{s,U}(p, \alpha, w)$. We argue that $(\Psi^{s,U,w}(\{\underline{0}\}))^{-1}$ is a C^2 1-dimensional manifold, except for a set of initial endowments $w \in \Omega$ with Lebesque measure zero. To show this we first prove that the Jacobian matrix at a point $(\tilde{p}, \tilde{\alpha}, \tilde{w})$ satisfying $\Psi^{s,U}(\tilde{p}, \tilde{\alpha}, \tilde{w}) = \underline{0}$ has full rank. The Jacobian, denoted by $\hat{M}$, is the $(n + |U|) \times (n + |U| + 1 + c(n + 1))$-matrix

$$
\begin{array}{cc}
 & \begin{array}{ccc} \quad n+1 \quad & \quad |U| \quad & \quad c(n+1) \quad \end{array} \\
\begin{array}{c} \\ |U|-1 \\ \\ 1 \\ \\ k^0(s) \\ \\ k^-(s)-1 \\ k^+(s)-1 \\ 1 \end{array} &
\begin{bmatrix}
(\hat{y}^2 - \hat{y}^1)^\top & & \\
\vdots & & \\
(\hat{y}^{|U|} - \hat{y}^1)^\top & & \\
\underline{0}^\top & e^\top & \underline{0}^\top \\
\partial_p \hat{z}_1(\tilde{p}, \tilde{\alpha}, \tilde{w}) & \partial_\alpha \hat{z}_1(\tilde{p}, \tilde{\alpha}, \tilde{w}) & \partial_w \hat{z}_1(\tilde{p}, \tilde{\alpha}, \tilde{w}) \\
\vdots & \vdots & \vdots \\
\partial_p \hat{z}_{k^0(s)}(\tilde{p}, \tilde{\alpha}, \tilde{w}) & \partial_\alpha \hat{z}_{k^0(s)}(\tilde{p}, \tilde{\alpha}, \tilde{w}) & \partial_w \hat{z}_{k^0(s)}(\tilde{p}, \tilde{\alpha}, \tilde{w}) \\
\bar{M}^1 & & \\
\bar{M}^2 & & \\
e^\top & &
\end{bmatrix}
\end{array},
$$

Empty places in the matrix above denote zeros. The first $n+1$ columns of $\hat{M}$ concern the derivatives to p_ℓ, $\ell \in I_{n+1}$. The next $|U|$ columns list the derivatives to α_k, $k \in I_{|U|}$, whereas the last $c(n + 1)$ columns are the derivatives to w^i_ℓ, $i \in I_c$, and $\ell \in I_{n+1}$. The submatrices $\bar{M}^1$ and $\bar{M}^2$ are explicitly given in Herings (1994). However, they are easily to derive by differentiating (3.5) and (3.6) towards p. We want to show that rank$(\hat{M})$ equals $n + |U|$. In other words, from $b \in \mathbf{R}^{n+|U|}$ such that $b^\top \hat{M} = \underline{0}^\top$ it has to follow that $b = \underline{0}$. First, following Herings (1994), we derive that $b_{|U|+1} = \ldots = b_{|U|+k^0(s)} = 0$. Then it trivially follows from considering the columns related to α, that also $b_{|U|} = 0$. Next, we can concentrate on the $n + 1$ columns related to the derivatives to p. We have to show that the first $|U| - 1$ components of b are zero. Then it follows from Herings (1994) that $b = \underline{0}$ and we are done. We consider three cases:

1. $|U| = 1$. This case is already dealt with by Herings (1994).

2. $k^-(s) = k^+(s) = 1$. Then $b_k = 0,\ k \in I_{|U|-1}$, because the vectors $(\hat{y}^{k+1} - \hat{y}^1),\ k \in I_{|U|-1}$, and e are independent. Assume on the contrary that $\sum_{k=1}^{|U|-1} b_k(\hat{y}^{k+1} - \hat{y}^1)^\top + b_{n+|U|}e^\top = \underline{0}^\top$ whereas $(b_1, \ldots, b_{|U|-1}, b_{n+|U|})^\top \neq \underline{0}^\top$. However, post-multiplying the equation with p leads to $\sum_{k=1}^{|U|-1}(b_k \times 0) + b_{n+|U|} \times 1 = 0$. Thus, $b_{n+|U|} = 0$, whereas the other coefficients are zero from the independentness of the vectors $(\hat{y}^{k+1} - \hat{y}^1),\ k \in I_{|U|-1}$.

3. Other cases. Let us denote by $\tilde{M}$ the submatrix of $\hat{M}$ related to the derivatives to p of the equations (3.2), (3.5), (3.6) and (3.7). Thus, $\tilde{M}$ is of dimension $(n + |U| - (k^0(s) + 1)) \times (n + 1)$. Consider $\tilde{b}$ to be the corresponding $(n + |U| - (k^0(s) + 1))$ - subvector of b. Furthermore, assume that $\tilde{b}^\top \tilde{M} = \underline{0}^\top$ such that $\tilde{b} \neq \underline{0}$. Again, we want to obtain a contradiction and establish that $\tilde{b} = \underline{0}$. For convenience we denote the components of $\tilde{b}$ related to the $k^-(s) - 1$ rows of $\bar{M}^1$ by $\gamma_k,\ k \in I_{k^-(s)-1}$, and the components of $\tilde{b}$ related to the $k^+(s) - 1$ rows of $\bar{M}^2$ by $\delta_k,\ k \in I_{k^+(s)-1}$. The last component of $\tilde{b}$, i.e., the component related to $e^\top$, is denoted by $\bar{\delta}$. Furthermore, row k of $\tilde{M}$ is indicated by $\tilde{M}_k,\ k \in I_{n+|U|-(k^0(s)+1)}$.

   First we post-multiply both sides of the system $\tilde{b}^\top \tilde{M} = \underline{0}^\top$ with p. This leads to

$$\sum_{k=1}^{|U|-1} \tilde{b}_k(\hat{y}^{k+1} - \hat{y}^1)^\top p + \sum_{k=|U|}^{\hat{k}} \gamma_k \tilde{M}_k p + \sum_{k=\hat{k}+1}^{\bar{k}} \delta_k \tilde{M}_k p + \bar{\delta} e^\top p = 0, \tag{3.15}$$

   where $\hat{k} = |U| + k^-(s) - 1$ and $\bar{k} = |U| + k^-(s) + k^+(s) - 1$. From this system we easily derive that $\bar{\delta} = 0$. This because the other three terms are all zero. The first term equals zero because of the profit condition (3.2). Furthermore, we have that for a price vector p generated by the process, $p_\ell = ap^0_\ell,\ \ell \in I^-(s)$, and $p_\ell = \bar{a}p^0_\ell,\ \ell \in I^+(s)$. Now consider the second term of the equation above. For each row $\tilde{M}_k,\ k \in I_{|U|+k^-(s)-1} \backslash I_{|U|-1}$ it holds that $\tilde{M}_k p^0 = 0$. Besides, the only nonzero components of $\tilde{M}_k,\ k \in I_{|U|+k^-(s)-1} \backslash I_{|U|-1}$ are related to components of $I^-(s)$, and these components of p are equal to some common scalar times the corresponding components of p^0. This makes that the second term in (3.15) is equal to zero. A similar reasoning holds for the third term. To derive that all other components of $\tilde{b}$ are also zero we proceed as follows. First, we denote the term $\sum_{k=1}^{|U|-1} \tilde{b}_k(\hat{y}^{k+1} - \hat{y}^1)^\top$ by $\bar{y}^\top$. If $\bar{y} = \underline{0}$ then $\tilde{b}_k = 0,\ k \in I_{|U|-1}$, because the vectors $(\hat{y}^{k+1} - \hat{y}^1)^\top,\ k \in I_{|U|-1}$, are independent. In that case all other components of $\tilde{b}$ are zero because the rows of $\tilde{M}$ related to (3.5) and (3.6) are easily seen to be independent. Thus, let us assume that $\bar{y}_{k^1} \neq 0$ for some $k^1 \in I_{n+1}$. We are done if we can find an $\bar{x} \in \mathbf{R}^{n+1}$ such that $\bar{y}^\top \bar{x} \neq 0$ whereas $\tilde{M}_k \bar{x} = 0$, $\forall k \in I_{|U|+k^-(s)+k^+(s)-1} \backslash I_{|U|-1}$. This because by post-multiplying the system $\tilde{b}^\top \tilde{M} = \underline{0}$ with $\bar{x}$ we then obtain that $\bar{y}^\top \bar{x} = 0$. Thus, $\bar{y} = \underline{0}$ and $\tilde{b} = \underline{0}$.

In the sequel we have to consider several cases. If $k^1 \in I^0(s)$ we take $\bar{x}$ equal to $e(k^1)$. This vector clearly satisfies the demands. Thus, let us assume that $\bar{y}_k = 0$, $\forall k \in I^0(s)$. Next, consider the case in which $k^1 \in I^-(s)$. We denote $\sum_{k=|U|}^{|U|+k^-(s)-1} \gamma_k \tilde{M}_k$ by $\hat{m}^\top$ and $\sum_{k=|U|+k^-(s)}^{|U|+k^-(s)+k^+(s)-1} \delta_k \tilde{M}_k$ by $\bar{m}^\top$. We now want to construct an $\bar{x} \neq \underline{0}$ such that $\bar{y}^\top \bar{x} = 0$, $\hat{m}^\top \bar{x} \neq 0$, and $\bar{m}^\top \bar{x} \neq 0$. From this we derive by post-multiplying $\tilde{b}^\top \tilde{M} = \underline{0}$ with $\bar{x}$, and with the independentness of the rows of $\tilde{M}$ related to (3.5) and (3.6), that all γ_k and δ_k are zero. On its turn this leads to $\tilde{b}_k = 0$, $\forall k \in I_{|U|-1}$, as before and we are done.

Concerning the construction of $\bar{x}$, we observe that this is easy if also $\bar{y}_{k^2} \neq 0$ for some $k^2 \in I^+(s)$. It is easily verified that in this case we can take $\bar{x}$ such that $\bar{x}_{k^1} = -\bar{y}_{k^2}$, $\bar{x}_{k^2} = \bar{y}_{k^1}$, and $\bar{x}_k = 0$ otherwise. Next, consider the case in which $\bar{y}_k = 0$, $\forall k \in I^+(s)$. Here we distinguish two cases. In case $\bar{y}_k = 0$, $\forall k \in I^-(s)\backslash\{k^1\}$, we have to consider two subcases, i.e., $k^-(s) = 1$ and $k^-(s) > 1$. If $k^-(s) = 1$ then the second term in the equation given before disappears. We can take $\bar{x}$ equal to $e(k^1)$ such that $\bar{y}^\top \bar{x} \neq 0$, whereas $\bar{m}^\top \bar{x} = 0$. In case $k^-(s) > 1$ we may take $\bar{x} = e(\bar{k})$, with $\bar{k} \in I^-(s)\backslash\{k^1\}$.
Next, we consider the case in which there is a pair $\{k^1, k^2\} \subset I^-(s)$ such that $\bar{y}_{k^1} \neq 0$ and $\bar{y}_{k^2} \neq 0$. As above, we can take $\bar{x}$ such that $\bar{x}_{k^1} = -\bar{y}_{k^2}$ and $\bar{x}_{k^2} = \bar{y}_{k^1}$, whereas $\bar{x}_k = 0$ otherwise. However, we may have a problem if $k^-(s) = 2$ while $-\bar{y}_{k^2} = ap^0_{k^1}$ and $\bar{y}_{k^1} = ap^0_{k^2}$ for some $a \in \mathbf{R}$. This because then not only $\bar{y}^\top \bar{x} = 0$ but also $\hat{m}^\top \bar{x} = 0$. But then we obtain the desired result by considering the system $\tilde{b}^\top \tilde{M} = \underline{0}^\top$. Now, from this system we derive $\bar{y}_{k^1} = -\gamma_{|U|} p^0_{k^2}$ and $\bar{y}_{k^2} = \gamma_{|U|} p^0_{k^1}$, where without loss of generality $k^1 = k^2 - 1$. Substituting $\bar{y}_{k^1} = ap^0_{k^2}$ and $\bar{y}_{k^2} = -ap^0_{k^1}$, gives $\gamma_{|U|} = a = 0$, i.e., $\bar{y} = \underline{0}$ and we are done.

Finally, the case in which $\bar{y}_k \neq 0$, for some $k \in I^+(s)$, whereas $\bar{y}_k = 0$, $\forall k \in I^0(s) \cup I^-(s)$, can be treated similarly as the case above in which $\bar{y}_k = 0$, $\forall k \in I^0(s) \cup I^+(s)$.

Thus, in all cases we derive that $b = \underline{0}$ and $\hat{M}$ has full rank. From this we conclude that $\Psi^{s,U}$ intersects $\underline{0}$ transversally. Because $\Psi^{s,U} : \mathbf{R}^{n+1}_{++} \times \mathbf{R}^{|U|}_{-\epsilon} \times \Omega \mapsto \mathbf{R}^{n+|U|}$ maps from a smooth manifold into a smooth manifold, we may conclude using standard arguments that the vectors (p, α) satisfying (3.2) - (3.7) constitute a C^2 1-dimensional manifold for all $w \in \Omega^*$, with $\Omega\backslash\Omega^*$ having Lebesque measure zero.

Till thusfar we considered the Jacobian related to the system (3.2) - (3.7). Next, we need successively to consider the Jacobian related to the previous system extended with one of the inequalities in (3.8) - (3.14) satisfied with equality. It is easily verified that the Jacobian related to these extended systems is also of full rank. In fact, binding restrictions in (3.8) - (3.12) are already considered by Herings (1994). The case in which one inequality in (3.13) becomes binding can be treated as before. Finally, in case (3.14) becomes binding for some k, a unit vector is added to the Jacobian. This vector is clearly independent from the vector related to (3.3). From all this it follows that the set of vectors (p, α) satisfying (3.2) - (3.7) and for which also one other constraint is binding forms a 0-dimensional set. In case more restrictions are binding the set is empty.

Finally, we need to check that the conditions on the starting vector are generically satisfied. The generic uniqueness of $S(p^0)$ follows from Lemma 3.3 below. The lemma states that the price space is subdivided into polytopes of which the full-dimensional ones correspond to price vectors with unique supply. The genericity of the condition stating that $z_\ell(p^0) \neq 0$, i.e., $\tilde{z}_\ell(p^0) \neq y^0_\ell$, $\forall \ell \in I_{n+1}$, has already been proved by Herings (1994). With all this we can follow the reasoning given by Herings (1994) for showing that for generic $w \in \Omega$, the set of tuples (p, y) satisfying (2.3) is a C^2 1-dimensional manifold, consisting of disjoint paths connecting two equilibria, loops, and with one path connecting (p^0, y^0) and an equilibrium (p^*, y^*). As a corollary it follows that the number of equilibria is generically odd. Finally, it is argued that the set of $w \in \Omega$ not having this property is closed and of measure zero. Also this can be shown as in Herings (1994). Roughly speaking, this follows from the fact that finitely many cases have to be considered which all hold generically, and because the union of the sets $B^w(s, U)$ over all $(s, U) \in \overline{\mathcal{S} \times \mathcal{U}}$ is compact. First it is shown that each set $B^w(s, U)$ is a compact C^2 1-dimensional manifold with boundary, i.e., if not empty it contains a subset diffeomorphic to the unit interval. That the sets $B^w(s, U)$ for different $(s, U) \in \overline{\mathcal{S} \times \mathcal{U}}$ can be linked, is shown by using the results obtained from adding equalities in (3.8) - (3.14). An equality in (3.8) - (3.12) corresponds to a change in s, whereas an equality in (3.13) - (3.14) corresponds to a change in U. □

The next lemma deals with the supply correspondence. Later on it will appear to be useful for the representation of the supply in the price space.

Lemma 3.3. Let be given a standard Arrow/Debreu economy with $n + 1$ commodities and a conical production structure. Furthermore, let the compactified production set $\hat{Y}$ be convex, compact and satisfy $\hat{Y} \cap (-\hat{Y}) = \{\underline{0}\}$, as for example the set $\hat{Y}$ given by (2.2). Then the following statements hold:

a) $\forall p \in S^n,\ \ S(p)$ is a polytope.
b) $S^- : \mathrm{Eff}(\hat{Y}) \to S^n$, with $S^-(y) = \{p \in S^n | y \in S(p)\}$, is u.h.c. with nonempty, convex, compact values being polytopes.
c) If $y \in S(p)$ then $\dim(S(p)) + \dim(S^-(y)) = n$.

Proof. a) By definition $S(p) = \mathrm{Eff}(\hat{Y}) \cap \{y \in \mathbf{R}^{n+1} | p^\top y = \pi(p)\}$, being the nonempty intersection of a polytope and a plane. This gives a polytope.
b) $S^-(y)$ is the intersection of S^n and the cone of normals to $\hat{Y}$ at y. The other properties of S^- can be derived in a similar way as done for S (see Bonnisseau and Cornet (1988)).
c) Let $y \in S(p)$ with $S(p)$ a face of $\mathrm{Eff}(\hat{Y})$ of say dimension k. The related cone of normals has dimension $n + 1 - k$. Intersection with S^n gives $\dim(S^-(y)) = n - k$. □

In order to illustrate the foregoing we consider the working of the adjustment process with an example. When going through this example the economics behind the process will become clear.

Example 3.1. In our example there are three commodities. Concerning the production side there are five activities, namely $a^1 = (-6, -\frac{1}{6}, 1)^\top$, $a^2 = (-2, -5, 2)^\top$, $a^3 = (1, -\frac{5}{2}, -\frac{2}{5})^\top$, $a^4 = (2, 2, -5)^\top$, and $a^5 = (-6, 1, 0)^\top$. So, for example activity a^1 uses commodities 1 and 2 as inputs, whereas commodity 3 is the output. The vector of initial endowments in the economy equals $W = (3, 3, 3)^\top$. Furthermore, the behaviour of the consumers is represented by the aggregate consumers excess demand function $\tilde{z} : S^2 \to \mathbf{R}^3$ given by $\tilde{z}(p) = \left(\frac{3+\pi(p)}{4p_1} - 3, \frac{3+\pi(p)}{2p_2} - 3, \frac{3+\pi(p)}{4p_3} - 3\right)^\top$, where $\pi(p)$ denotes the profits at p. The function $\tilde{z}$ can be thought of as being derived for an aggregate consumption sector having a Cobb-Douglas utility function with utility weights equal to $\frac{1}{4}, \frac{1}{2}$, and $\frac{1}{4}$. It is easily checked that indeed $p^\top \tilde{z}(p) = \pi(p)$, $\forall p \in S^2$.

It is somewhat more difficult to verify if the aggregate production set generated by the activities, satisfies (2.1). All the other conditions being trivially fulfilled we only have to investigate the irreversibility property (2.1.b). In Theorem 3.1 this is shown to be equivalent with the no production without input condition, i.e., from $Ax \geq \underline{0}$ and $x \geq \underline{0}$ it follows that $x = \underline{0}$, where $A = [a^1, \ldots, a^5]$. Via an almost straightforward application of Farkas' lemma it can be shown that this on its turn is equivalent to the existence of a strictly positive price vector at which all activities make losses, i.e., $\exists p \in \mathrm{int}(S^n)$ such that $A^\top p \ll \underline{0}$. It is easily seen that $p = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^\top$ satisfies the latter.

Next, we compactify Y according to (2.2) by restricting ourselves to amounts of inputs equal to maximal twice their endowment. Thus, we assume $\bar{e} = 2W$. However, to keep the analysis more tractable we consider as economically relevant extreme vectors of $\hat{Y}$ the vectors in the set $\{\underline{0}, \hat{y}^1, \ldots, \hat{y}^5\}$, where $\hat{y}^j$ is the intersection point of the ray through a^j and the restrictions formed by twice the endowments. More concretely, $\hat{y}^1 = (-6, -\frac{1}{6}, 1)^\top$, $\hat{y}^2 = (-2.4, -6, 2.4)^\top$, $\hat{y}^3 = (2.4, -6, -0.96)^\top$, $\hat{y}^4 = (2.4, 2.4, -6)^\top$ and $\hat{y}^5 = (-6, 1, 0)^\top$. By making this restriction we in fact consider the standard model with linear activities and exclude intermediate production and mergers.

In Figure 3.1 we depict the supply correspondence of the production sector and the aggregate excess demand of the consumers. Let us first consider the supply. The line segment $[a, b]$ corresponds to activity 1 and consists of price vectors at which that activity makes zero profit. The supply equals $[\underline{0}, \hat{y}^1]$. Similarly, $[b, c]$ denotes zero profit price vectors for activity 2 etc., till $[e, a]$ for activity 5. At price vectors in the polytope with extreme point set $\{a, b, c, d, e\}$ the profit is less than or equal to zero and not producing is profit maximizing. At the segment $[a, f]$ the profits of activities 1 and 5 are equal, positive and maximal. The corresponding supply set equals $[\hat{y}^1, \hat{y}^5]$. Similarly, on $[b, g]$ the supply is given by $[\hat{y}^1, \hat{y}^2]$. Thus, for price vectors in the polytope formed by the extreme point set $\{a, f, g, b\}$ the profit for activity 1 is optimal and positive. Therefore, this activity is applied at maximal scale, i.e., the production sector operates at $\hat{y}^1$. Similarly, we have the supply vector $\hat{y}^4$ at prices in $\mathrm{co}\{d, e, q, e(2), e(1), k\}$.

We already noted that the set $\mathrm{Eff}(\hat{Y})$ is homeomorphic to a simplex. That representation is given in Figure 3.2. The subdivision of $\mathrm{Eff}(\hat{Y})$ given in this figure, satisfies the conditions as discussed before Theorem 3.2. From both figures together we can easily deduce the properties of S and S^- as stated in Lemma 3.3. For example, if $p \in [a, f]$ and $y \in S(p)$, then $S(p) = [\hat{y}^1, \hat{y}^5]$ and $S^-(y) = [a, f]$. Both are polytopes and the sum of their dimensions equals 2.

Finally, the dashed piecewise linear curves in Figure 3.1 indicate the price vectors at which for some commodity $\ell \in I_3$, the consumers excess demand $\tilde{z}_\ell(p) = d_\ell(p) - W_\ell$ is equal to zero. The curves related to $\ell = 1, 2, 3$ are denoted by I, II, and III respectively. These curves are piecewise linear because of the piecewise linearity of the profit function. Observe that these curves do not intersect inside the polytope related to zero production. This means that the model has no equilibrium without production. Outside $\mathrm{co}\{a, b, c, d, e\}$ the curves do not intersect because of the relation $p^\top \tilde{z}(p) = \pi(p)$.

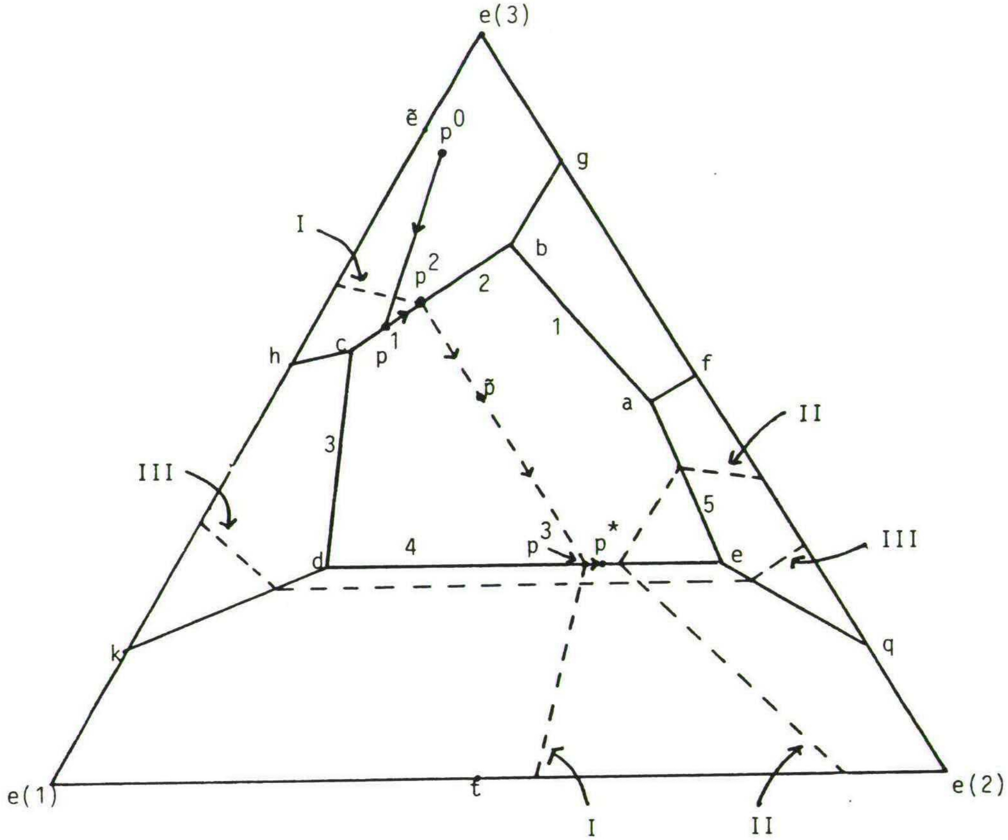

Figure 3.1. All economically relevant information of the model is represented in the price space S^2.

Let us now consider how the process operates when starting from p^0 (see Figure 3.1). By fixing p^0 we determine a subdivision of the price space S^n. To verify this consider the sets $A(s,\{0\})$ for different $s \in \mathbf{R}^3$, i.e., we take the production equal to zero at all prices. If $s = (+1,-1,-1)^\top$ then $A(s,\{0\})$ equals $[p^0, e(1)] \times \underline{0}$, and for $s = (0,-1,+1)^\top$ we obtain that $A(s,\{0\})$ equals $\mathrm{co}\{p^0, e(3), \tilde{e}\}$, with $\tilde{e}$ on the line through $e(2)$ and p^0. Similarly, we obtain the complete subdivision. At p^0 we have $\mathrm{sgn}(\tilde{z}(p^0)) = (+1,+1,-1)^\top$. Furthermore, $S(p^0) = \{\hat{y}^2\} = \{(-2.4,-6,2.4)^\top\}$. This gives $\mathrm{sgn}(z(p^0)) = (+1,+1,-1)^\top$. Thus, from p^0 the prices of the first two commodities are relatively equally increased whereas the price of commodity 3 is necessarily decreased. Graphically, the process leaves p^0 in the direction opposite to $e(3)$. It continues in this manner till it reaches p^1 where still $\mathrm{sgn}(z(p^1)) = (+1,+1,-1)^\top$. However, at p^1, $S(p^1)$ equals the segment $[\underline{0}, \hat{y}^2]$. The supply is now decreased from $\hat{y}^2$ along this ray till for

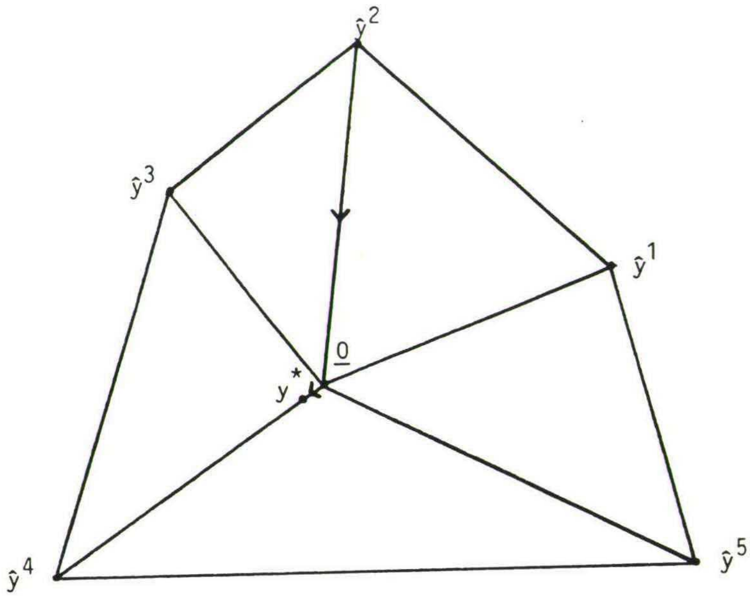

Figure 3.2. The producer behaviour along the path represented in the set $\mathrm{Eff}(\hat{Y})$ being homeomorphic to a simplex.

a certain y^2, $z_1(p^1|y^2) = 0$. Thus, at (p^1, y^2) the market of commodity 1 becomes in equilibrium. Now, it is kept in equilibrium by decreasing the relative price of commodity 1 below that of commodity 2, i.e., we move into the direction of the boundary with zero prices for commodity 1. This movement occurs along the segment of prices at which the supply equals $[0, \hat{y}^2]$, in order to satisfy optimal behaviour on the producer side. The equilibrium at the market of commodity 1 is preserved by supply adaptations of the producer. But at p^2 the production vector necessary to maintain that equilibrium becomes $\underline{0}$, i.e., $\tilde{z}_1(p^2) = 0$. From p^2 onwards the process generates price vectors p at which the consumers excess demand for commodity 1 equals zero whereas the optimal production vector is $\underline{0}$, i.e., the process moves towards p^3. At p^3 the supply set equals $[0, \hat{y}^4]$, where $\hat{y}^4$ is $(2.4, 2.4, -6)^\top$. As soon as the production is increased from $\underline{0}$ along that segment, the market for commodity 1 reveals a surplus. In order to cope with that, the process moves into the area at which $\tilde{z}_1$ is positive, while staying at the segment of price vectors for which the supply equals $[\underline{0}, \hat{y}^4]$. Thus, prices p and quantities y are adjusted such that $\mathrm{sgn}(z(p|y)) = (0, +1, -1)^\top$ while the price vectors move into the direction of e. At $p^* = \left(\frac{5}{21}, \frac{10}{21}, \frac{2}{7}\right)^\top$ also the other markets become in equilibrium with corresponding

$y^* = \frac{3}{40}\cdot(2,2,-5)^\top$. The movements on the production side are represented in Figure 3.2.

Remark 3.4. Finally, we make a clarifying remark concerning the choice of p^0. In Theorem 3.2 we gave two restrictions on p^0, i.e., $S(p^0)$ has to consist of a unique element and $z_\ell(p^0) \neq 0$, $\forall \ell$. To illustrate the first restriction consider any starting vector on $[b,c]$ in Figure 3.1 with $y^0 = \hat{y}^2$. But then $\text{sgn}(z(p^0|y^0)) = (+1,+1,-1)^\top$ and we have to decrease y from $\hat{y}^2$ along the ray to $\underline{0}$. In case p^0 lies on the segment $(p^2, b]$ this goes fine; y is decreased to $\underline{0}$, $\text{sgn}(z(p^0|\underline{0})) = (+1,+1,-1)^\top$ and p is adapted into the direction opposite to $e(3)$. Thus, sometimes it is no problem if $S(p^0)$ is multi-valued. However, if $p^0 \in [c, p^2)$ then $\text{sgn}(z(p^0, \underline{0})) = (-1,+1,-1)^\top$ and by decreasing y from $\hat{y}^2$ we reach a production $\bar{y}$ at which $\text{sgn}(z(p^0,\bar{y})) = (0,+1,-1)^\top$ and get stuck.

Concerning the second condition on p^0, i.e., $z_\ell(p^0) \neq 0$, $\forall \ell$, we indicate that also this condition is merely sufficient. What we need is that there is a unique pair (s^0, U^0) such that $(p^0, y^0) \in \text{bd}(B(s^0, U^0))$. If this would not be the case then cycling could occur. In case $z_\ell(p^0) \neq 0$, $\forall \ell$, and $z(p^0)$ is unique, then s^0 is uniquely determined by z whereas U is determined by $\{y^0\}$. Of course, $(p^0, y^0) \in \text{bd}(B(s^0, U^0))$ due to the p^0-component. However, it may happen that $(p^0, y^0) \in \text{bd}(B(s^0, U^0))$, whereas $\text{sgn}(z_\ell(p^0)) = 0$ for some $\ell \in I_{n+1}$. Consider for example $\tilde{p}$ in Figure 3.1. Note that $\tilde{p}$ lies in the interior of $\text{co}\{a,b,c,d,e\}$, i.e., the corresponding production equals $\underline{0}$ and $z(\tilde{p})$ is equal to $\tilde{z}(\tilde{p})$. We easily derive that $\text{sgn}(z(\tilde{p})) = (0,+1,-1)^\top$. Furthermore, $A((0,+1,-1)^\top, \{0\})$ equals $\text{co}\{\tilde{p}, e(2), t\} \times \underline{0}$, where t lies at the line through $\tilde{p}$ and $e(3)$. Again, we denote by $U = \{0\}$ that there is no production. Furthermore, $[\tilde{p}, p^3] \times \{\underline{0}\} \subset B((0,+1,-1)^\top, \{0\})$, i.e., $(\tilde{p}, \underline{0}) \in \text{bd}(B((0,+1,-1)^\top, \{0\}))$. In addition there is no other $\bar{s} \neq (0,+1,-1)^\top$ such that $(\tilde{p}, \underline{0}) \in \text{bd}(B(\bar{s}, \{0\}))$. The two sign patterns that have to be considered are $\bar{s} = (+1,+1,-1)^\top$ and $\bar{s} = (-1,+1,-1)^\top$. Let us examine the first case. Then $A(\bar{s}, \{0\}) = [\tilde{p}, t] \times \underline{0}$. However, at price vectors p near $\tilde{p}$ at $[\tilde{p}, t]$, we have that $\text{sgn}(z(p)) = (-1,+1,-1)^\top$. Thus, the part of $[\tilde{p}, t]$ near $\tilde{p}$ is no subset of $B(\bar{s}, \{0\})$ and $(\tilde{p}, \underline{0}) \notin \text{bd}(B(\bar{s}, \{0\}))$. The case with $\bar{s} = (-1,+1,-1)^\top$ can be treated similarly.

In conclusion, this remark shows that although in Theorem 3.2 the generic existence and convergence has been proved, in practice even more cases can be handled.

We mentioned already in Section 2 that our process can be generated arbitrarily accurate by the simplicial algorithm of Doup, van der Laan and Talman (1987). An earlier algorithm that is suited for an economy with linear production activities is the algorithm of Scarf (1973). Within the region of interest, i.e., for prices at which no activity makes profit, the limiting behaviour of that algorithm equals our procedure. However, outside that region Scarf's algorithm is somewhat artificial. Besides, the algorithm has very restrictive starting possibilities.

Another simplicial algorithm is given by Talman, Yamamoto, and Yang (1993). This algorithm can also be applied to more general problems. However, its application to this specific model has drawbacks related to its economic interpretation. More precisely, along the path traced by their procedure the producers do not behave optimally.

4 Economies with other convex production structures

In this section, we apply the process defined by (2.3) on models with a production structure out of the following four classes; the polyhedral production structure, the generalized linear activity model, the production structure defined by convex C^1 functions, and the strictly convex production structure. We argue that the process converges generically in the sense of Theorem 3.2 within these classes of economies. Concerning the proofs we confine ourselves here to stipulating the differences in relation to the proof of Theorem 3.2. Furthermore, we argue that the process can also be applied to models with a demand correspondence obeying certain regularities.

In Section 3 is dealt with the case in which the total production set Y is a cone. But the process also works in case Y is a polyhedral set, of which a cone is a special case. The main economic difference is that in this more general model the equilibrium profit may be positive. The analysis of the process is similar to the one given in the previous section because in both cases the compactified total production set is a polytope. Because any convex set can be approximated arbitrarily close by a polyhedral set, our process can serve as an "approximating" adjustment process for any convex economy. At the end of this section we provide an illustration of the process for an economy with a polyhedral production set.

Next, we consider the generalized linear activity model as discussed for example in Mas-Colell (1985), Chapter 3. Here we also have constant returns to scale, but now the technologies are not fixed and allowance for input substitution with respect to price changes is made. Van der Laan and Kremers (1993) speak about nonlinear constant returns to scale. In this setting each generalized linear activity j is thought of as to be described by a C^1, homogeneous and convex profit function π^j (see for example Varian (1992) for the convexity argument). Now, Y is written as cl$\{y \in \mathbf{R}^{n+1} | y \leq$

$\sum_{j=1}^{m} \alpha_j \partial\pi^j(p) \, : \, \alpha_j \geq 0, \; p \gg \underline{0}\}$, where $\partial\pi^j(p)$ equals $y^j(p)$, i.e., the technique used by activity j at prices p. The latter follows from Hotelling's lemma. In this model the production set is dependent on p. More precisely, for each price vector p the set $Y(p)$ is a cone spanned by the $y^j(p)$'s. Thus, compactifying the supply set at prices p as in Section 3 leads to a polyhedral set.

To apply our process to this model we have to assume that $\forall p \gg \underline{0}$, the production cone $Y(p)$ satisfies (2.1). Furthermore, like in Section 3, we can subdivide $\forall p \gg \underline{0}$ the set $\mathrm{Eff}(\hat{Y}(p))$ into subsets $\mathrm{Eff}(\hat{Y}(U))$, with U related to a subset of the extreme vectors of $\mathrm{Eff}(\hat{Y}(p))$ and $\dim(\mathrm{Eff}(\hat{Y}(U)))$ equal to $|U|-1$. We assume that some fixed subdivision can be used for all $p \gg \underline{0}$. This is a regularity condition stating that the structure of production does not change due to changes in p. For example, no techniques coincide at some p. From all this we obtain a slighty adjusted definition of the sets $A(s,U)$, i.e.,

$$
\begin{aligned}
A(s,U) = \{(p,y) \in S^n \times \mathrm{Eff}(\hat{Y}(p)) | \; & p^\top \hat{y}^k = \max_{\bar{y} \in \hat{Y}(p)} p^\top \bar{y}, \; k \in U \\
& y = \sum_{k \in U} \alpha_k \hat{y}^k(p) \text{ with } \alpha_k \geq 0 \text{ and } \sum_{k \in U} \alpha_k = 1 \\
& p_\ell / p_\ell^0 = \max\nolimits_r p_r / p_r^0 \text{ if } s_\ell = +1 \\
& p_\ell / p_\ell^0 = \min\nolimits_r p_r / p_r^0 \text{ if } s_\ell = -1\}.
\end{aligned}
$$

In principal we can apply the techniques used in Section 3 to prove the generic existence and convergence of the path for this model.

Theorem 4.1. Let the production sector represent a generalized linear activity model described by C^3, homogeneous and convex profit functions π^j, $j \in I_m$, and satisfying (2.1). Under the conditions stated in Theorem 3.2 the process defined by (2.3) converges generically in the sense of Theorem 3.2.

Sketch of proof. As in the proof of Theorem 3.2 we define the excess demand function $\hat{z}^w : A(s,U) \mapsto \mathbf{R}^{n+1}$ by $\hat{z}^w(p,y) = \tilde{z}^w(p) - y$. For $\hat{z}^w$ to be C^2 we impose $\hat{y}^k(p)$, $k \in U$, to be C^2. This is clearly the case if the profit functions π^j, $j \in I_m$, are C^3. The rest of the proof goes along the same lines followed for proving Theorem 3.2. First, we obtain a system similar to (3.2)-(3.14). Only now, in equations (3.2) and (3.13), we get $\hat{y}^k(p)$ instead of $\hat{y}^k$, $\forall k \in I_{|U|}$. This leads to some changes in that part of the Jacobian matrix related to the derivatives to p. However, also now the rank of $\hat{M}$ is full. The changes concerning the differentiation of (3.4) are not relevant because these rows of $\hat{M}$ are shown to be independent via the derivatives to w that do not change. Concerning the derivative of (3.2) to p we obtain $\forall k \in I_{|U|-1}$, $(\hat{y}^k(p) - \hat{y}^1(p))^\top + p^\top(\partial(\hat{y}^k(p) - \hat{y}^1(p)))$ instead of $\hat{y}^k - \hat{y}^1$. By the imposed regularity condition these vectors are independent. That the rank of $\hat{M}$ is full can be shown in almost the same way as in the proof of Theorem 3.2. The difference concerns the first part of the proof. First, we obtain a system similar to (3.15). However, now we are left with $\sum_{k=1}^{|U|-1} \tilde{b}_k p^\top \partial(\hat{y}^k(p) - \hat{y}^1(p))p + \bar{\delta} = 0$. Due to the convexity of π^k, $k \in I_{|U|-1}$, we obtain $|U|$ nonnegative terms, and thus $\bar{\delta} = 0$. The rest of the proof mimics that of Theorem 3.2. □

Graphically, the difference with Section 3 can be indicated with Figure 3.1. For the model sketched here all line segments related to supply and consumers excess demand would become curves. We remark that our process generalizes the process of van der Laan and Kremers (1993) in two respects. Firstly, we allow for joint production, i.e., a firm may produce more outputs. Secondly, again we do not restrict ourselves to starting vectors leading to nonpositive profits. In case we start from a price vector with nonpositive profits, whereas there is no joint production, then our process coincides with the process of van der Laan and Kremers (1993).

Another standard production set is formed by means of a finite number of C^1 convex functions $\eta^k : \mathbf{R}^{n+1} \mapsto \mathbf{R}$, $k \in K$. More precisely, $Y = \{y \in \mathbf{R}^{n+1} | \eta^k(y) \leq 0,\ \forall k \in K\}$. Furthermore, we assume that $\forall y \in \mathbf{R}^{n+1}$ the collection $\{\partial\eta^k(y) | k \in K(y)\}$ is linearly independent, where $K(y) = \{k \in K | \eta^k(y) = 0\}$. This implies that for any $K' \subset K$ the set $\eta(K') = \{y \in \mathbf{R}^{n+1} | \eta^k(y) = 0,\ k \in K'\}$ is a C^1 manifold of dimension $n + 1 - |K'|$. In Mas-Colell (1985), Section 3.7, the relation between this production structure and the polyhedral structure is discussed. To compactify Y as in Section 2 we have to add some linear constraints on the productions, which can be viewed as additional η's.

Again we have to adapt the definition of $A(s, U)$. We now define sets $A(s, U)$, $U \subset K$, by

$$
\begin{aligned}
A(s,U) = \{(p,y) \in S^n \times \mathrm{Eff}(\hat{Y}) |\quad & p^\top y = \max_{\bar{y} \in \hat{Y}} p^\top \bar{y} \\
& \eta^k(y) = 0,\ \forall k \in U \\
& p_\ell / p_\ell^0 = \max\nolimits_r p_r / p_r^0 \text{ if } s_\ell = +1 \\
& p_\ell / p_\ell^0 = \min\nolimits_r p_r / p_r^0 \text{ if } s_\ell = -1\}.
\end{aligned}
$$

Now, we can rewrite the profit maximizing relation between p and y as $p = \sum_{k \in U} \lambda_k \partial\eta^k(y)$, with $\lambda_k \geq 0$. The λ_k's are nonnegative because the vectors p and

$\partial\eta^k(y),\ k \in I_{|U|}$, point into the same halfspace away from $\hat{Y}$. The relation between p and $\partial\eta^k(y),\ k \in I_{|U|}$, follows by differentiating the Langrangian. This condition is necessary and sufficient because the vectors $\eta^k(y),\ k \in I_{|U|}$, are differentiable and convex.

Theorem 4.2. Let the total production set be defined by a finite set of convex C^2 functions as above, and let it satisfy (2.1). Then the conclusions of Theorem 3.2 hold if all other conditions stated in that theorem are satisfied.

Sketch of proof. Again, we consider the function $\hat{z}^w : A(s,U) \mapsto \mathbf{R}^{n+1}$, with $\hat{z}^w(p,y) = \hat{z}^w(p) - y$, which is C^2 because the functions $\eta^k,\ k \in U$, are C^2. We now represent the pair $(p,y) \in B^w(s,U)$ by the tuple $(p,\lambda,y,w) \in \mathbf{R}^{n+1}_{++} \times \mathbf{R}^{|U|}_{-\epsilon} \times \eta(U) \times \Omega$. From this we obtain a system of equations describing a $(p,y) \in B^w(s,U)$, similar as (3.2) - (3.14) for the cone production structure. However, some changes occur in the (in)equalities corresponding to production. More precisely, instead of (3.2), (3.3), (3.13), and (3.14) we obtain

$$\begin{aligned} p - \sum_{k\in U} \lambda_k \partial\eta^k(y) &= \underline{0} && (4.1)\\ \eta^k(y) &= 0, \quad \forall k \in U && (4.2)\\ \lambda_k &\geq 0, \quad \forall k \in U && (4.3)\\ -\eta^k(y) &\geq 0, \quad \forall k \notin U. && (4.4) \end{aligned}$$

Furthermore, we now view $\hat{z}$ as a function of (p,y,w). Let us consider the Jacobian matrix $\bar{M}$ related to the system (4.1), (4.2), and (3.4)-(3.7). In this case $\bar{M}$ is of dimension $(2n+|U|+1) \times ((2+c)(n+1)+|U|)$. Consider now a vector $b \in \mathbf{R}^{2n+|U|+1}$ such that $b^\top \bar{M} = \underline{0}^\top$. The components of b related to (3.4) are zero by reasoning as in Section 3. Next, we consider the submatrix of $\bar{M}$ related to the differentiation of (4.1) and (4.2) to λ and y. By the independentness of $\{\partial\eta^k(y) | k \in U\}$ the components of b related to (4.1) are zero. This can be derived from the columns of $\bar{M}$ related to λ. Next, it follows from the columns of $\bar{M}$ related to the differentiation to y, that the components of b related to (4.2) are zero. Finally, as in Section 3 we derive that all other components of b are zero. The rest of the analysis is again similar to that in the proof of Theorem 3.2. $\square$

Finally, we consider a model with a strictly convex aggregated production set. This means that if $y^1, y^2 \in Y$ then $[y^1, y^2] \subset \text{int}(Y)$. In that case the supply at each price vector is unique and we obtain a supply function, also because of our compactness construction. In fact this model can be treated similarly as a pure exchange model with the total excess demand instead of the consumers excess demand. The expression (2.3) remains valid, whereas the generic existence and convergence has already been proved by Herings (1994).

Concerning the consumer side we note that our process in some cases also works if the consumer demand is represented by a correspondence. However, the problems encountered here are more complicated than for the production side. First of all we restrict our scope to models of consumer behaviour that can be considered as the behaviour of one representative consumer. We assume the total consumption set X to be $\mathbf{R}^{n+1}_{++}$. For each price vector $p \in \mathbf{R}^{n+1}_{++}$ we then obtain a set of economic possibilities, i.e., the budget set $B(p) = \{x \in \mathbf{R}^{n+1}_{++} | p^\top x \leq p^\top W\}$. In fact we may compactify $B(p)$, because zero amounts are never demanded. This compactified budget set will be denoted by $\hat{B}(p)$. The relevant part of $\hat{B}(p)$, i.e., $\{x \in \mathbf{R}^{n+1}_{+} | p^\top x = p^\top W\}$, can be thought of as corresponding to the set $\text{Eff}(\hat{Y}(p))$ in the generalized linear activity model. More precisely, because $W \in \mathbf{R}^{n+1}_{++}$ and $p \in \mathbf{R}^{n+1}_{++}$, $\hat{B}(p)$ is an n-dimensional simplex in $\mathbf{R}^{n+1}_{+}$.

For the producers we have a linear profit function to maximize. The representative consumer has to maximize a utility function that might be rather complex. For the process to work the resulting representation of the demand should be like that for the supply in Figure 3.1. More precise statements are made in Lemma 3.3. One class of utility functions that satisfy these requirements are those that induce the consumption of commodities in fixed proportions. In the example below we consider a demand correspondence resulting from such a utility function. Furthermore, the production set is given by a polyhedral set.

Example 4.1. Again we consider an exchange economy with three commodities. The total endowments W equal $\left(4, \frac{22}{15}, \frac{4}{5}\right)^\top$ whereas the consumers excess demand at prices p, i.e., $\tilde{z}(p)$ equals

$$\begin{array}{ll}
\left(\frac{p^\top w+\pi(p)}{p_1+\frac{1}{2}p_2+\frac{1}{4}p_3}, \frac{p^\top w+\pi(p)}{2(p_1+\frac{1}{2}p_2+\frac{1}{4}p_3)}, \frac{p^\top w+\pi(p)}{4(p_1+\frac{1}{2}p_2+\frac{1}{4}p_3)}\right)^\top - W & \text{if } p_1 \leq p_3 \text{ and } 4p_1 \leq 1+p_2 \\
\left(\frac{p^\top w+\pi(p)}{4(\frac{1}{4}p_1+p_2+\frac{1}{2}p_3)}, \frac{p^\top w+\pi(p)}{(\frac{1}{4}p_1+p_2+\frac{1}{2}p_3)}, \frac{p^\top w+\pi(p)}{2(\frac{1}{4}p_1+p_2+\frac{1}{2}p_3)}\right)^\top - W & \text{if } p_3 \geq p_2 \text{ and } 4p_1 \geq 1+p_2 \\
\left(\frac{p^\top w+\pi(p)}{4(\frac{1}{4}p_1+\frac{1}{2}p_2+p_3)}, \frac{p^\top w+\pi(p)}{2(\frac{1}{4}p_1+\frac{1}{2}p_2+p_3)}, \frac{p^\top w+\pi(p)}{(\frac{1}{4}p_1+\frac{1}{2}p_2+p_3)}\right)^\top - W & \text{if } p_1 \geq p_3 \text{ and } p_2 \geq p_3,
\end{array}$$

where $p^\top w$ and $\pi(p)$ denote the total initial wealth and profit at prices p respectively. The demand correspondence is generated by the utility function $u : \mathbf{R}^3_{++} \mapsto \mathbf{R}$ of the representative consumer given by $u(x_1, x_2, x_3) =$

$$\max\{\min\{x_1, 2x_2, 4x_3\}, \min\{4x_1, x_2, 2x_3\}, \min\{4x_1, 2x_2, x_3\}\}.$$

This function gives piecewise linear indifference curves. The price regions at which the different expressions for the demand correspondence are relevant, are represented in Figure 4.1 by the dashed lines originating from $p^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^\top$. Thus, at price vectors in $\text{co}\{p^*, e(2), e(3), e\}$ the first expression of $\tilde{z}(p)$ is valid. Here, e denotes the vector $\left(\frac{1}{4}, 0, \frac{3}{4}\right)^\top$. This region is indicated by I. Similarly, regions II and III are indicated and refer to the corresponding expressions of $\tilde{z}$. Of course, at price vectors on for example the segment $[e(1), p^*)$, the demand vectors form also a line segment, i.e., the convex hull of demands relating to regions II and III. Thus, the demand side of this economy satisfies properties as stated in Lemma 3.3.

The efficiency frontier of the polytope $\hat{Y}$, $\text{Eff}(\hat{Y})$, is characterized by the set of economically relevant extreme vectors $\{\underline{0}, \hat{y}^1, \ldots, \hat{y}^4\}$, where $\hat{y}^1 = (-\frac{3}{2}, 1, 1)^\top$, $\hat{y}^2 = (-1, -1, \frac{3}{2})^\top$, $\hat{y}^3 = (-4, 2, 2)^\top$, and $\hat{y}^4 = (-4, -4, 4)^\top$. This production set can be seen as derived from two firms having constant returns to scale production on $[\underline{0}, \hat{y}^1]$ and $[\underline{0}, \hat{y}^2]$ respectively. Beyond $\hat{y}^1$ ($\hat{y}^2$) the firms observe less efficient techniques along $[\hat{y}^1, \hat{y}^3]$ and $[\hat{y}^2, \hat{y}^4]$ respectively. Note that the compactified production set $\hat{Y}$ somewhat differs from the set given in (2.2). However, no equilibrium production can occur outside $\text{Eff}(\hat{Y})$ because there is insufficient supply of commodity 1. In Figure 4.1 all relevant information about the supply correspondence is gathered. Let us consider Figure 4.1 more carefully. Here, a, b, c, d, denote the price vectors $\left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right)^\top$, $\left(\frac{2}{7}, \frac{1}{5}, \frac{18}{35}\right)^\top$, $\left(\frac{3}{11}, \frac{2}{11}, \frac{6}{11}\right)^\top$, and $\left(0, \frac{1}{4}, \frac{3}{4}\right)^\top$, respectively. The numbers $0, 1, \ldots, 4$ refer to the corresponding extreme points of $\text{Eff}(\hat{Y})$. Thus, at the interior of $\text{co}\left\{e(2), \left(\frac{2}{7}, \frac{5}{7}, 0\right)^\top, b, c, d\right\}$ the supply equals $\hat{y}^3$, whereas $[\hat{y}^2, \hat{y}^3]$ is the supply set at prices on $[b, c]$. The rest of Figure 4.1 can be understood similarly.

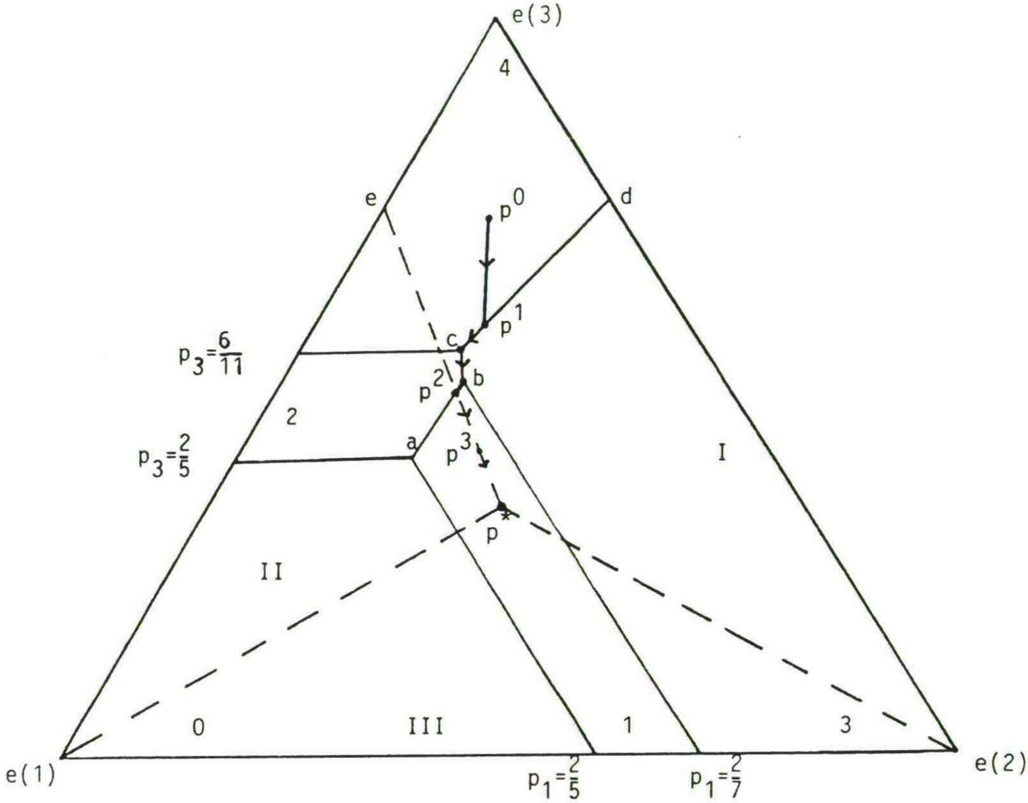

Figure 4.1. Representation of the demand and supply correspondences.

In the figure we present an example of the adjustment process bringing about an equilibrium. We consider the path starting from $p^0 = \left(\frac{1}{7}, \frac{1}{8}, \frac{41}{56}\right)^\top$. At that price vector the production technique used equals $(-4, -4, 4)^\top$. This gives $\pi(p^0) = \frac{13}{7}$, whereas $(p^0)^\top w = \frac{563}{420}$. From this we derive that $D(p^0) \approx \{(8.23, 4.12, 2.06)^\top\}$. This evidently leads to an excess demand for commodities 1 and 2, while commodity 3 is in excess supply. The process leaves p^0 by decreasing the price of commodity 3, whereas the prices of commodities 1 and 2 are increased relatively equal, i.e., it moves into the direction opposite to $e(3)$. The process continues in this manner till $p^1 = \left(\frac{8}{36}, \frac{7}{36}, \frac{21}{36}\right)^\top$ is reached at which $S(p^1) = [\hat{y}^3, \hat{y}^4]$ and $D(p^1) = \{(4.96, 2.48, 1.24)^\top\}$. Now, the producers gradually change from $\hat{y}^4$ to $\hat{y}^3$. But $\operatorname{sgn}(z)$ remains $(+1, +1, -1)^\top$ till y becomes equal to $0.16\hat{y}^4 + 0.84\hat{y}^3$, with $z_2(p^1|y) = 0$.

Thus, from p^1 the process continues by relatively decreasing the price of commodity

2 below that of commodity 1, while keeping the market of commodity 2 in equilibrium. Besides, the producers behave optimally, i.e., the process moves towards c. At that price vector we have $D(c) \approx \{\frac{1}{11}(48,24,12)^\top\}$. Thus, the production y equals $0.786\hat{y}^3+0.214\hat{y}^4$ in order to keep the market of commodity 2 in equilibrium. Observe that at c also $\hat{y}^2$ becomes optimal. If the related technique is taken into production this would ceterus paribus yield an excess demand for commodity 2. In order to prevent this to happen, $\hat{y}^4$ is deleted and y changes to $0.572\hat{y}^3 + 0.428\hat{y}^2$, still at price vector c. Next, the process moves towards b. At b we have that $b^\top w = \frac{13}{7}$, $\pi(b) = \frac{2}{7}$, and $D(b) = \{\frac{1}{24}(100,50,25)^\top\}$. Because the market of commodity 2 is still in equilibrium we derive that y equals $\frac{97}{180}\hat{y}^3 + \frac{83}{180}\hat{y}^2$. The sign pattern of the excess demand vector still equals $(+1,\ 0,-1)^\top$.

At price vector b also production vector $\hat{y}^1$ becomes optimal. When this technique is used, then $\hat{y}^3$ has to be deleted in order to keep $z_2(b|y)$ equal to zero. More precisely, the production changes towards $\bar{y} = \frac{97}{120}\hat{y}^1 + \frac{23}{120}\hat{y}^2$ with $z_2(b|\bar{y}) \approx \left(\frac{377}{240}, 0, \frac{-205}{240}\right)^\top$. Now, the process moves into the direction of a, whereas the producers utilize a combination of $\hat{y}^1$ and $\hat{y}^2$. This is continued till $p^2 = \left(\frac{3}{10}, \frac{1}{5}, \frac{1}{2}\right)^\top$ is reached. At that price vector the demand becomes set-valued, i.e., $D(p^2) = \left[\left(\frac{1286}{315}, \frac{1286}{630}, \frac{1286}{1260}\right)^\top, \left(\frac{1286}{1260}, \frac{1286}{315}, \frac{1286}{630}\right)^\top\right]$. To continue the process into the direction of a, we first have to change the demand from $\left(\frac{1286}{315}, \frac{1286}{630}, \frac{1286}{1260}\right)^\top$ towards the other end point of $D(p^2)$. If we move the demand that way, the demand for commodity 2 increases, and to keep the market of that commodity in equilibrium we have to decrease the weight on $\hat{y}^2$ that uses commodity 2 as an input. Finally, we end up with only using $\hat{y}^1$, i.e., the weight on $\hat{y}^2$ becomes zero, whereas the consumers demand a mixture of $\left(\frac{1286}{315}, \frac{1286}{630}, \frac{1286}{1260}\right)^\top$ and $\left(\frac{1286}{1260}, \frac{1286}{315}, \frac{1286}{630}\right)^\top$ with weights respectively $\frac{509}{643}$ and $\frac{134}{643}$. Thus, the process moves from p^2 into the direction of p^*, i.e., the process moves on the line segment of price vectors at which the consumers demand a mixture of the bundles stated above while it enters the region at which the producers utilize $\hat{y}^1$ exclusively. For the total excess demand we obtain $z(p^2|x,\hat{y}^1) \approx \left(\frac{119}{126}, 0, \frac{-119}{210}\right)^\top$, with x as above.

However, at $p^3 = \left(\frac{8}{25}, \frac{7}{25}, \frac{10}{25}\right)^\top$, the relative price of commodity 2 being in equilibrium, becomes equal to the relative price of commodity 1 being in excess demand, i.e., $\frac{p_2^3}{p_2^0} = \frac{p_1^3}{p_1^0} = \frac{56}{25}$. According to (2.3) and the description thereafter, the process removes the equilibrium on the market for commodity 2 and brings it into a situation of excess demand. Furthermore, the relative prices of the commodities 1 and 2 are kept equal to each other. More precisely, when the process reaches p^3, the consumers demand the bundles $\left(\frac{829}{210}, \frac{829}{420}, \frac{829}{840}\right)^\top$ and $\left(\frac{829}{840}, \frac{829}{210}, \frac{829}{420}\right)^\top$ with weights $\frac{622}{829}$ and $\frac{207}{829}$ respectively, in order to keep the market of commodity 2 in equilibrium at production $y = \hat{y}^1$. Furthermore, $\text{sgn}(z(p^3, \hat{y}^1)) = (+1, 0, -1)^\top$. From p^3 onwards the consumers are forced to demand more of the second bundle given above in order to induce the sign pattern of the excess demand to change into $(+1,+1,-1)^\top$. This because in the second bundle relatively more of commodity 2 is demanded. However, when the weights become $\frac{1271}{2487}$ and $\frac{1216}{2487}$ respectively, the market of commodity 1 becomes in equilibrium. Then that market is kept in equilibrium according to (2.3) and the relative price of commodity 1 is decreased below that of commodity 2, i.e., the process moves towards p^* via price vectors at which the demand is set-valued.

At $p^* = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)^\top$, the extreme bundles related to the first two parts of the demand correspondence equal $\left(\frac{406}{105},\frac{203}{105},\frac{203}{210}\right)^\top$ and $\left(\frac{203}{210},\frac{406}{105},\frac{203}{105}\right)^\top$ respectively. The weights to keep the market of commodity 1 in equilibrium are $\frac{322}{609}$ and $\frac{287}{609}$ respectively. Thus, $\operatorname{sgn}(z(p^*,\hat{y}^1))$ still equals $(0,+1,-1)^\top$. But at p^* also the third part of the demand correspondence becomes valid with bundle $\left(\frac{203}{210},\frac{203}{105},\frac{406}{105}\right)^\top$. Now its weight is increased from zero while keeping the market of commodity 1 in equilibrium. The latter means that the weight on the second bundle decreases accordingly. When the weights equal $\frac{322}{609}$, $\frac{168}{609}$ and $\frac{119}{609}$, an equilibrium $(p^*,x^*,y^*)^\top$ is reached where $p^* = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)^\top$, $x^* = \left(\frac{315}{206},\frac{777}{315},\frac{567}{315}\right)^\top$, and $y^* = \left(\frac{-3}{2},1,1\right)^\top$.

5 The class of semi-algebraic convex production economies

As we have seen in Section 2, the basic Arrow/Debreu model can be summarized by the total excess demand correspondence $Z : \mathbf{R}^{n+1}_{++} \mapsto \mathbf{R}^{n+1}$, being u.h.c., homogeneous of degree zero and satisfying Walras' law. Furthermore, its values are nonempty, convex and compact. In this section we consider the class of Arrow/Debreu economies characterized by a semi-algebraic excess demand correspondence. The latter means that its graph can be expressed as a finite union of polynomial equalities and inequalities. Below, we will become more precise. We prove that there exists for each economy out of this class and each starting vector (p^0,x^0,y^0) a path-connected set of vectors (p,x,y) satisfying (2.3), with excess demand $z(p|x,y) = x - y - W$, where $x \in D(p)$ and $y \in S(p)$, whereas this set contains the starting vector (p^0,x^0,y^0) and an equilibrium (p^*,x^*,y^*). Thus, for any "semi-algebraic economy" and any starting vector (p^0,x^0,y^0) there exists a path of vectors (p,x,y) satisfying (2.3) and connecting the starting vector and an equilibrium. However, contrary to Sections 3 and 4, that path may be not unique. The idea of the proof is greatly inspired by Schanuel, Simon, and Zame (1991) who apply the notion of a semi-algebraic set in a game-theoretic context.

Definition 5.1. A semi-algebraic set in $\mathbf{R}^n$ is a finite union of sets of the form

$$\{x \in \mathbf{R}^n | p_1(x) = 0, \ldots, p_r(x) = 0,\ q_1(x) < 0, \ldots, q_s(x) < 0\},$$

where p_q, $q \in I_r$, and q_m, $m \in I_s$, are polynomials with real coefficients.

Definition 5.2. If A, B are semi-algebraic sets then a correspondence $F : A \mapsto B$ is semi-algebraic if its graph is a semi-algebraic set.

Blume and Zame (1992) relate in the context of a pure exchange economy the property of the demand correspondence being semi-algebraic to properties of the consumption set and the preference relation. They prove that in case the consumption sets and the graphs of the preference relations are semi-algebraic, the total demand correspondence is also semi-algebraic. This holds if the consumption sets are closed and convex, whereas the preference relations have to be complete, transitive and continuous. Similarly, it can be proved that if the total production set satisfies (2.1) and is semi-algebraic then also the total supply correspondence is semi-algebraic. Thus, here we provide sufficient conditions on the primitive notions of the model to guarantee semi-algebraicness of the total excess demand correspondence.

In their paper Blume and Zame also consider the class of finitely sub-analytic correspondences, which encompasses the semi-algebraic ones. For more details we refer to Blume and Zame (1992). We will not consider sub-analytic correspondences any further in this section but note that the whole analysis goes through for this class.

The class of semi-algebraic economies encompasses parts of the economies treated in Sections 3 and 4. For example, the polyhedral production economy in principle fits in the framework because one can describe such a production set in a semi-algebraic way by using linear functions. However, it is not allowed that some coefficients in the functional relations describing the production set are not rational. The same holds for the other production sets discussed in Section 4. For the latter also some sets cannot be described by polynomials. On the other hand, all semi-algebraic models fit in some category of Section 4 concerning the production structure. However, the demand structure is allowed to be rather general and includes demand correspondences as considered in Section 4, with rational coefficients.

To prove the statement made in the beginning of this section we first rewrite (2.3) for the case in which also the demand is allowed to be a correspondence. Thus, we choose a starting vector (p^0, x^0, y^0) such that $p^0 \in \mathrm{int}(S^n)$, $x^0 \in D(p^0)$ and $y^0 \in S(p^0)$. Now we are interested in vectors (p, x, y) with $p \in \mathrm{int}(S^n)$, $x \in D(p)$ and $y \in S(p)$ such that $\forall \ell \in I_{n+1}$

$$\begin{array}{rll} p_\ell/p_\ell^0 = \max_r p_r/p_r^0 & \text{if} & z_\ell(p|x,y) > 0 \\ \min_r p_r/p_r^0 \leq p_\ell/p_\ell^0 \leq \max_r p_r/p_r^0 & \text{if} & z_\ell(p|x,y) = 0 \\ \min_r p_r/p_r^0 = p_\ell/p_\ell^0 & \text{if} & z_\ell(p|x,y) < 0. \end{array} \tag{5.1}$$

We denote the set of tuples (p, x, y) satisfying (5.1) by $\mathcal{B}(p^0; Z)$.

Theorem 5.1. Let be given a convex Arrow/Debreu economy with production, characterized by a semi-algebraic excess demand correspondence Z. Then $\forall p^0 \in \mathrm{int}(S^n)$, the set $\mathcal{B}(p^0; Z)$ has a path-connected subset containing (p^0, x^0, y^0), with $x^0 \in D(p^0)$ and $y^0 \in S(p^0)$, and an equilibrium (p^*, x^*, y^*).

Proof. From Definitions 5.1 and 5.2 it is obvious that (5.1) defines a semi-algebraic set if Z is a semi-algebraic correspondence. The left-hand side of (5.1) defines a subdivision of S^n. Each subset is defined by linear inequalities. The right-hand side of (5.1) is formed by (in)-equalities related to a semi-algebraic correspondence and is therefore also semi-algebraic. The intersection of the two is then semi-algebraic.

Next, we observe that $\mathcal{B}(p^0; Z)$ can be seen as the limiting set of a sequence of piecewise linear paths related to the simplicial algorithm as given by Doup, van der Laan and Talman (1987). To apply the simplicial algorithm the price space S^n is subdivided into simplices. Related to this triangulation we take a piecewise linear approximation $\bar{z}^0$ of the correspondence Z as follows. First, observe that any $p \in S^n$ can be written uniquely as $p = \sum_{k=1}^{n+1} \lambda_k p^k$, with $\lambda_k \geq 0$ and $\sum_{k=1}^{n+1} \lambda_k = 1$. Here $p^1, \ldots, p^{n+1}$ are the vertices of an n-dimensional simplex containing p. Then $\bar{z}^0(p) = \sum_{k=1}^{n+1} \lambda_k z(p^k)$, for some $z(p^k) \in Z(p^k)$. It is obvious that $\bar{z}^0$ is well-defined and piecewise linear.

Now consider a given subdivision of S^n and a related function $\bar{z}^0$. When we apply the simplicial algorithm of Doup, van der Laan and Talman (1987) to $\bar{z}^0$ we obtain that the set (5.1) with Z replaced by $\bar{z}^0$ contains for all starting vectors a piecewise linear path connecting (p^0, x^0, y^0) and an approximate equilibrium. See for example Herings, Talman and Yang (1994), where this is illustrated by applying lexicographic pivoting.

Next, we make the simplices smaller and smaller, thus obtaining a sequence of functions $\bar{z}^t$, $t \in \mathbf{N} \cup \{0\}$. Similarly, we get a sequence of piecewise linear paths P^t obeying (5.1) with $\bar{z}^t, t = 0, 1, 2, \ldots$. Because for $t \mapsto \infty$ the function $\bar{z}^t$ becomes an approximate selection of Z, any point defined by (5.1) can be approximated arbitrary close by a sequence of vectors generated by consecutively applying the simplicial algorithm (see Herings (1993), Theorem 4.4). From this we can prove that the set of vectors given by (5.1) contains a connected set including (p^0, x^0, y^0) and (p^*, x^*, y^*). The formal proof follows Herings (1993), Theorem 4.10. From the proof it can be deduced that this subset is also semi-algebraic.

Thus, now we have derived that there exists a subset of $\mathcal{B}(p^0; Z)$ that contains (p^0, x^0, y^0) and (p^*, x^*, y^*) while being a connected semi-algebraic set. But then that subset is also path-connected. The latter follows from the triangulability property of semi-algebraic sets (see Schanuel e.a. (1990) and basic references given there). □

We stress the fact that this result holds for any economy in the given class and for all starting vectors. It is not merely a generic property as are the results given in Sections 3 and 4. We illustrate this with an example.

Example 5.1. We consider a pure exchange economy with three commodities characterized by a semi-algebraic consumers excess demand function $\tilde{z} : S^2 \mapsto \mathbf{R}^3$, and a starting price vector p^0 for which (5.1) contains more than one path connecting the starting vector and an equilibrium. Note that a pure exchange economy is a special case of a production economy, i.e., with the total production set equal to the nonpositive orthant. In Figure 5.1 we sketch the excess demand pattern of this economy. At the starting price vector p^0 we have $\mathrm{sgn}(\tilde{z}(p^0)) = (+1, 0, -1)^\top$. The price vectors obeying the conditions (5.1) for the sign vector $(+1, +1, -1)^\top$ lie on the segment $[p^0, a]$. Similarly, the price vectors related to $(+1, 0, -1)^\top$ are on the curve connecting p^0 and b, the curve connecting a and c, and on the curve connecting d and p^*. Finally, the price vectors related to the sign vector $(+1, -1, -1)^\top$ are on the segments $[p^0, b]$ and $[c, d]$. All the pieces together form the set (5.1) and contain several paths from p^0 to p^*. Observe that $p^0 \in \mathrm{bd}(B(+1, +1, -1)^\top)$ and $p^0 \in \mathrm{bd}(B(+1, 0, -1)^\top)$ and is therefore nongeneric, i.e., it does not fit in the framework of Section 3.

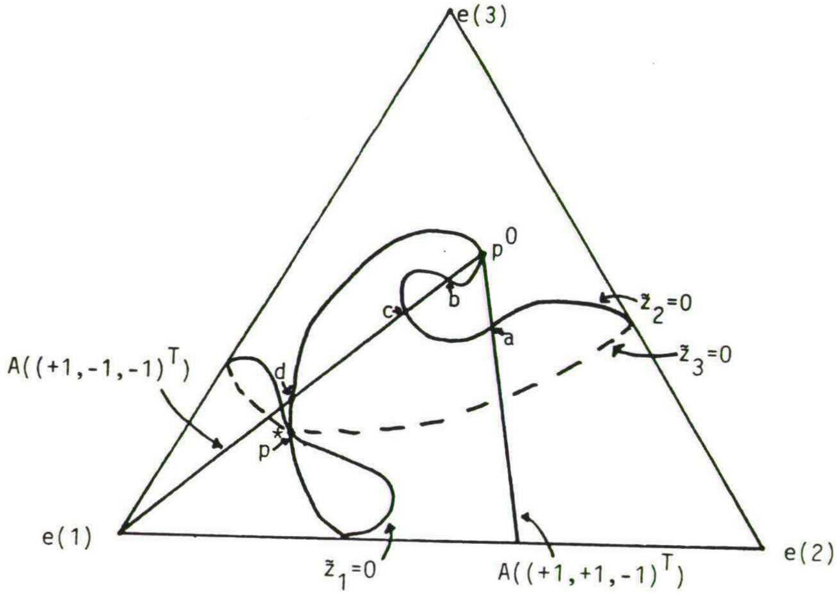

Figure 5.1. An exchange economy with nongeneric starting vector.

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