subfaculteit der econometrie

RESEARCH MEMORANDUM

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THE PRICE-DEPENDENCE OF THE VARIANCE OF DEMAND FUNCTIONS

by

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1. INTRODUCTION

In several papers and books (e.g. Hempenius [2], Iwai [3], Mills [5], Smith [8], Zabel [10]) the classical monopoly model has been analysed for the case of stochastic demand. For the mathematical expectation of the random variable demand it is obvious that one requires the usual properties of a market demand function, in particular that it is a decreasing function of price. For the way in which the dispersion of demand (around expected demand) depends on the price, there is not such an obvious requirement. However, something must be postulated, e.g. that there is no dependence on price; see e.g. Mills [5].

In the present note the properties of the dispersion as a function of price will be investigated. Two types of goods will be distinguished: goods demanded in very small quantities, to be called durable goods, and other goods, to be called non-durable goods.

The quantity demanded will be denoted by q, with q a non-negative random variable with expectation $h(p)$, the expected demand at price p. By assumption $h(p)$ is a decreasing function of p. The quantity demanded may be written as

$$(1) \qquad q = h(p) + v$$

where $v = q-h(p)$ will be interpreted as disturbance term. In general the distributions of q and v depend on the price p, implying that the variances of q and v depend on p, i.e. one has heteroscedastic demand and disturbance. In the sequel it is analysed what type of heteroscedasticity may be expected for durable goods (Section 2) and non-durable goods (Section 3).

2. DURABLE GOODS

Durable goods are bought in very small quantities at a time. Suppose that, within the time unit used for measuring demand, an individual consumer either buys one unit of a certain durable good or he does not buy the good. Denote the quantity demanded for consumer $i (i = 1, \ldots, N)$ by X_i and the probability of buying by π_i, with π_i depending on the market price p. Then it holds:

(2) $$\mu_i = E(X_i) = \pi_i \qquad (0 \leq \pi_i \leq 1)$$

(3) $$\sigma_i^2 = \mathrm{var}(X_i) = \pi_i(1-\pi_i)$$

Market demand is defined as $q = \Sigma X_i$. The expected value of market demand, $E(q)$, is denoted by $h(p)$. As $h(p)$ only depends on p[1], π_i will be made dependent on p alone. Expected market demand thus is:

(4) $$h(p) = \sum_{i=1}^{N} \pi_i(p)$$

As it is reasonable to assume that the probability of buying, π_i, is a decreasing function of p, it holds that expected individual and market demand, μ_i and $h(p)$, respectively, are decreasing functions of p.

In order to analyze heteroscedasticity one has to know the variance of q, to be denoted by σ^2. Assuming independent buying behaviour, i.e. assumming that buyers do not react on each other's behaviour but almost exclusively on the price, one has at price p:

(5) $$\sigma^2(p) = \mathrm{var}(q)$$

$$= \sum_{i=1}^{N} \sigma_i^2 = \sum_{i=1}^{N} \pi_i(1-\pi_i)$$

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1) Market demand is considered under cet. par. conditions, viz. for given incomes, etc.

Whether σ^2 decreases or increases if p increases, depends on

$$(6) \qquad \frac{d\sigma_i^2}{dp} = (1 - 2\pi_i) \frac{d\pi_i}{dp} < 0 \text{ for } \pi_i < \frac{1}{2}$$

$$> 0 \text{ for } \pi_i > \frac{1}{2}$$

for $i = 1,\dots,N$. It thus depends on the frequency distribution of the $\pi_i (i = 1,\dots,N)$ on the interval $[0,1]$ what the sign of $d\sigma^2/dp$ is.

Assuming that the number of buyers N is large enough to approximate the frequency distribution of π_i at price p by a continuous frequency distribution over the intervall $[0,1]$, having distribution function $F(\pi)$, it is possible to write formula's (4) and (5) as

$$(7) \qquad h(p) = N \int_0^1 \pi \, dF(\pi)$$

$$(8) \qquad \sigma^2(p) = N \int_0^1 \pi(1-\pi) \, dF(\pi)$$

A distribution which is much used for the interval $[0,1]$ because of its great flexibility, is the standard beta distribution (see e.g. Johnson and Kotz [4] and Raiffa and Schlaifer [7]) with parameters r and s, having density function:

$$(9) \qquad f(\pi) = \frac{1}{B(r,s)} \pi^{r-1}(1-\pi)^{s-1} \quad (0 \leq \pi \leq 1)$$

where r and s are positive functions of p and where B(r,s) is the beta-function.[2)] As it is reasonable to assume that the cases $\pi = 0$ and $\pi = 1$ do not occur, one has to choose r and s in such a way that $r > 1$ and $s > 1$, as then $f(\pi)$ vanishes for $\pi = 0$ and $\pi = 1$ and the curve of $f(\pi)$ then decreases monotonously to both sides from the mode $(r-1)/(r+s-2)$.
If $r = s$, then the distribution is symmetric around $\pi = 1/2$. The traditional measure of skewness α_3 is $c(s-r)$, with c a positive function of r and s. If $r > s$ or $r < s$, then the distribution has negative skewness or positive skewness, respectively, and the mode is larger or smaller, res-

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2) See e.g. Johnson and Kotz [4] for the now following properties.

pectively, than $\pi = 1/2$.

Introduction of the price is best done by fixing one of the parameters r or s and by making the other one dependent on p: assume r a constant and s a function of p. Assuming that the mode decreases with increasing price, it is required that $ds/dp > 0$ for all $p > 0$. Depending on the value of p, one has values for s smaller or greater than r and the distribution of π is negatively or positively skewed, respectively.

For the beta-distribution of (9) one has for h(p) in (7) and $\sigma^2(p)$ in (8):

$$(10) \qquad h(p) = \frac{Nr}{r+s}$$

$$(11) \qquad \sigma^2(p) = \frac{N\ rs}{(r+s)(r+s+1)}$$

For dh/dp a negative sign is found:[3]

$$(12) \qquad \frac{dh}{dp} = \frac{-Nr}{(r+s)^2} \cdot \frac{ds}{dp} < 0$$

For $d\sigma^2/dp$, one has

$$(13) \qquad \frac{d\sigma^2}{dp} = \frac{Nr\{r(r+1)-s^2\}}{\{r(r+1)+(2r+1)s+s^2\}^2} \cdot \frac{ds}{dp} \begin{array}{l} < 0 \text{ iff } s^2 > r(r+1) \\ > 0 \text{ iff } s^2 < r(r+1) \end{array}$$

From (13) follows that, for prices implying negative skewness of the π-distribution ($s < r$) one has $d\sigma^2/dp > 0$. For prices for which the distribution has more or less a tail to the right, i.e. has positive skewness ($s > r$), one has in general $d\sigma^2/dp < 0$, except for those prices for which it holds that $r < s(p) < \sqrt{r(r+1)}$.

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3) Alternatively, the assumption that $dh/dp < 0$, implies $ds/dp > 0$ and thus a decreasing mode with increasing price.

So assuming that the individual probabilities of buying follow a standard beta-distribution, which is hardly restrictive at all, Figure 1 shows the way in which the dispersion of demand depends on the price. For "low" prices, i.e. prices for which $s(p) > \sqrt{r(r+1)}$, one has $d\sigma^2/dp > 0$ and otherwise one has $d\sigma^2/dp < 0$.

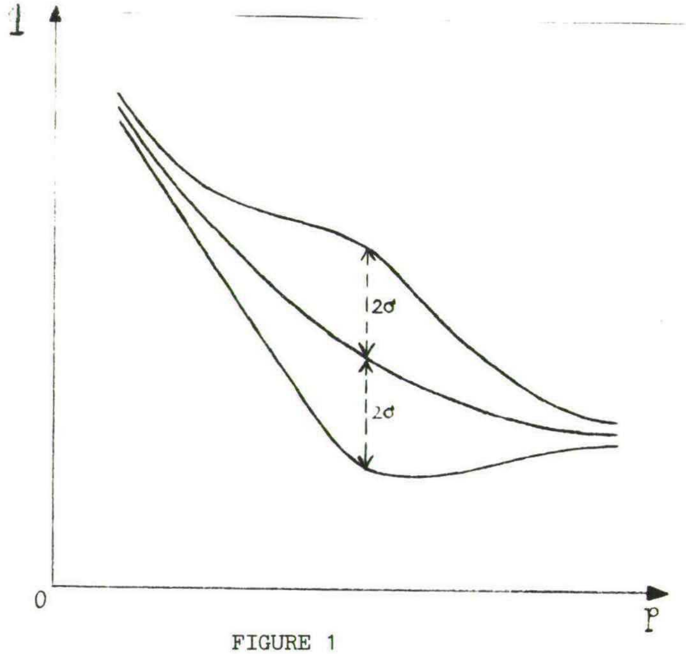

FIGURE 1

3. NON-DURABLE GOODS

For non-durable goods one must use the more realistic assumption that the number of quantity units bought during a time unit, again denoted by X_i for consumer i, is a random variable that assumes the values 0,1, 2,... which has expectation μ_i and variance σ_i^2. Further the assumption that $d\mu_i/dp < 0$, leads directly to $dh/dp < 0$, as $h(p) = E(q)$, where again $q = \Sigma X_i$.

In order to analyse the dependence of $\sigma^2 = \text{var}(q)$ on p, one has to be more specific about the $\sigma_i^2 = \text{var}\ X_i$, as one has $\sigma^2 = \Sigma\sigma_i^2$. In management science a much used distribution for a total demand q is the Poisson distribution; see e.g. Hadley and Whitin [1] . It seems reasonable to use the Poisson distribution also for the X_i, as the assumedly independent buying behaviour leads to Poisson distributed total demand. Denoting the Poisson parameters by λ_i, it follows from $E(X_i) = \mathrm{var}\,(X_i) = \lambda_i$ and $d\mu_i/dp < 0$ that $d\sigma^2/dp < 0$. It is not necessary to assume that $d\mu_i/dp < 0$: as long as $dh/dp < 0$, one also has $d\sigma^2/dp < 0$ for the case of individual Poisson distributions.

Another interesting case is the one of individual distributions with constant variation coefficients, say $\sigma_i/\mu_i = c > 0$. As then $\sigma_i^2 = c^2\,\mu_i^2$, one has from $d\mu_i/dp < 0$ that $d\sigma^2/dp < 0$. The case of constant variation coefficients and also the case of individual Poisson distributions are special cases of distributions for which $\sigma_i^2 = g_i(\mu_i)$ and $d\,g_i/d\mu_i > 0$, as one then has:

$$(14) \qquad \frac{d\sigma^2}{dp} = \Sigma\,\frac{d\sigma_i^2}{d\mu_i}\cdot\frac{d\mu_i}{dp} < 0$$

This result is restrictive in the sense that <u>all</u> individual variances are assumed to increase with individual mean demand, although the assumption seems a realistic one for the great majority of consumers; see also Prais and Houthakker [6] and Theil [9] .

From this section one may conclude that for non-durable goods it seems reasonable to expect a dispersion as in Figure 2.

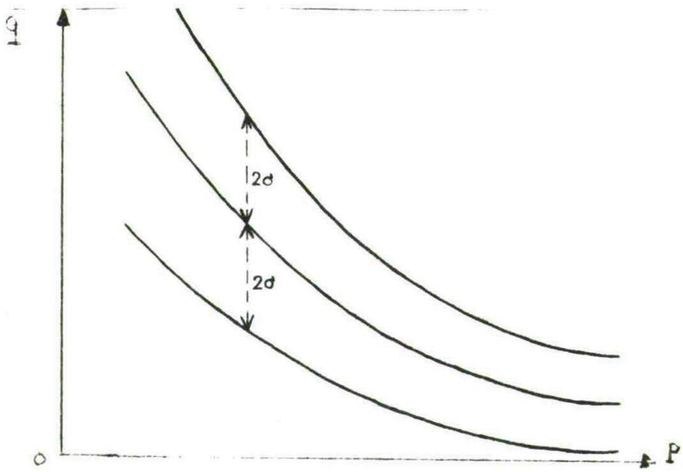

FIGURE 2.

4. CONCLUSION

The above conclusions, conveniently summarized in Figure 1 for durable goods and in Figure 2 for non-durable goods, hold under ceteris paribus conditions for other variables influencing demand. Empirical testing for a particular kind of heteroscedasticity with respect to the price will thus, for example, have to take into account shifts in demand because of changes in incomes.

A more general conclusion from the above might run as follows. Suppose one wishes to get an insight into the way in which the variance of a certain economic variable, say Y, depends on variables, say $X_1,\ldots,X_k$, that influence Y. Then one should start with a, possibly very simple, stochastic micro-theory for Y, allowing to analyse the dependence of the variance of Y on one or more of the X_i on the micro level. By aggregation one then tries to obtain conclusions at the macrolevel.
The result of this approach will always be that $E(Y|X_1,\ldots,X_k)$ and var $(Y|X_1,\ldots,X_k)$ depend on $X_1,\ldots,X_k$. (As an for estimation purposes convenient approximation one often assumes that $E(Y|X_1,\ldots,X_k)$ depends linearly on the X_i.) In the above it has been investigated, by means of this approach, in what way var$(q|p)$ depends on p in monopoly situations.

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