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RESEARCH MEMORANDUM

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NOTES ON THE MARKOWITZ PORTFOLIO SELECTION METHOD

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Notes on the Markowitz portfolio selection method.

by

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A proof of the validity of Markowitz's critical line method is given for a more general situation than discussed by Markowitz. Next for the Markowitz case with a positive definite covariance matrix explicit expressions are derived for all efficient portfolios. Using these expressions it can be shown that the critical line in the (μ,σ^2) plane is a representation of a function which is not necessarily differentiable everywhere.

Key Words and Phrases : finance, parametric programming.

1. Introduction.

Markowitz developed the critical line method for the following portfolio selection problem cf. Markowitz (1956),(1959). Suppose an investor wants to invest an amount b in the securities 1,...,n. He invests an amount x_j (≥ 0) in security j, so

$$\sum_{j=1}^{n} x_j = b \tag{1}$$

The yearly revenue of a portfolio $X' = (x_1,\ldots,x_n)$ is a random variable $\underline{r}(X)$ with expected value $E\underline{r}(X)=\mu(X)$ and variance $\sigma^2(\underline{r}(X)) = \sigma^2(X)$. Besides the constraint (1) other constraints may exist, restricting the feasible options to a set $\mathcal{X} \subset \mathcal{R}^n$.

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A feasible portfolio is efficient if:

a) no feasible portfolio exists with larger or equal expectation and smaller variance of the revenue,

and

b) no feasible portfolio exists with smaller or equal variance and larger expectation of the revenue.

This means that a portfolio $X=\bar{X}$ is efficient if and only if it is a solution of both

$$\min_X \{\sigma^2(X) \,|\, \mu(X) \geq \mu(\bar{X}) \wedge X \in \mathcal{X}\} \tag{2}$$

and

$$\max_X \{\mu(X) \,|\, \sigma^2(X) \leq \sigma^2(\bar{X}) \wedge X \in \mathcal{X}\}. \tag{3}$$

Markowitz derived an algorithm to compute all efficient portfolios and the corresponding efficient (μ,σ^2) points, assuming $\mu(X)$ linear, and $\sigma^2(X)$ quadratic and all constraints linear. In section 2 we show that the theorem on which this algorithm is based can be reformulated for a much more general situation. Furthermore Markowitz derived some properties of the curve of efficient points, but his remarks on differentiability properties of this curve are not very explicit. In section 3 we derive explicit expressions for all efficient portfolios and give more precise statements on differentiability properties.

2. A general theorem for the computation of efficient portfolios.

Theorem.

Let

i. the set of feasible portfolios be defined by $\mathcal{X} = \{X \mid h_i(X) \geq 0,\ i \in \mathcal{J}\}$, with $\mathcal{J}$ an index set, $h_i(x)$ concave and continuously differentiable,[1] $\mathcal{X}$ compact with non empty interior,

ii. the expected value $\mu(X)$ of the revenue be concave, continuously differentiable on $\mathcal{X}$,

iii. the variance of X be continuously differentiable on $\mathcal{X}$,

then $X=\bar{X}$ is efficient if and only if,

either

a) there exists a $\bar{\lambda} > 0$, such that

$$\min_X\{\sigma^2(X) - \bar{\lambda}\,\mu(X) \mid X \in \mathcal{X}\} = \sigma^2(\bar{X}) - \bar{\lambda}\mu(\bar{X}), \tag{4}$$

or

b)

$$\max_X[\mu(X) \mid \sigma^2(X) = \min_Y\{\sigma^2(Y) \mid Y \in \mathcal{X}\}] = \mu(\bar{X}), \tag{5}$$

or

c)

$$\min_X[\sigma^2(X) \mid \mu(X) = \max_Y\{\mu(Y) \mid Y \in \mathcal{X}\}] = \sigma^2(\bar{X}). \tag{6}$$

Proof.

We first show the sufficiency property.

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1) By continuously differentiable we mean that all partial derivatives exist and are continuous. Strictly speaking, these conditions and the concavity conditions can be somewhat weakened.

Case a) . Suppose $\bar{X}$ is not efficient; this implies the existence of a portfolio $X^* \in \mathcal{X}$, $X^* \neq \bar{X}$, such that

$$\{\mu(X^*) \geq \mu(\bar{X}) \wedge \sigma^2(X^*) < \sigma^2(\bar{X})\} \vee \{\sigma^2(X^*) \leq \sigma^2(\bar{X}) \wedge \mu(X^*) > \mu(\bar{X})\},$$

hence

$$\sigma^2(X^*) - \bar{\lambda}\mu(X^*) < \sigma^2(\bar{X}) - \bar{\lambda}\mu(\bar{X}),$$

for all $\bar{\lambda} > 0$, contradicting a). So $\bar{X}$ must be efficient.

Next define

$$\sigma^2_{min} := \min_X \{\sigma^2(X) \mid X \in \mathcal{X}\}$$

and

$$\mu_{max} := \max_X \{\mu(x) \mid X \in \mathcal{X}\}.$$

Case b) If $X=\bar{X}$ suffices (5), then

$$\sigma^2(\bar{X}) = \sigma^2_{min}$$

and

$$\mu(\bar{X}) = \max_X \{\mu(X) \mid \sigma^2(X) = \sigma^2_{min} \wedge X \in \mathcal{X}\}.$$

Thus $X=\bar{X}$ is efficient with minimum variance on $\mathcal{X}$.

Case c) In the same way $X=\bar{X}$ sufficing (6) implies

$$\mu(\bar{X}) = \mu_{max},$$

$$\sigma^2(\bar{X}) = \min \{\sigma^2(X) \mid \mu(X) = \mu_{max} \wedge X \in \mathcal{X}\}.$$

In other words $X = \bar{X}$ is efficient with maximum expected value on $\mathcal{X}$.

Secondly we prove that the conditions are necessary. If $X = \bar{X}$ is efficient, it solves both (2) and (3), so it is a solution of

$$\max \{-\sigma^2(X) | \mu(X)-\mu(\bar{X}) \geq 0 \wedge X \in \mathcal{X} \}, \tag{7}$$

and of

$$\max\{\mu(X) | \sigma^2(\bar{X})-\sigma^2(X) \geq 0 \wedge X \in \mathcal{X} \} \tag{8}$$

To both problems we apply the Kuhn-Tucker theorem which gives sufficient optimality conditions for the problem, maximize

$$y = f(X)$$

subject to

$$h_i(X) \geq 0 \qquad (i=1,...,l) \tag{9}$$

with $f(X)$ and $h_i(X)$ concave continuously differentiable functions.
These conditions run: $f(X)$ has its global maximum in $X=\bar{X}$ if there exist numbers $\bar{t}_i$ $(i=1,...,l)$ such that

$$\nabla f(\bar{X}) + \sum_{i=1}^{l} \bar{t}_i \nabla h_i(\bar{X}) = 0 \tag{10}$$

$$h_i(\bar{X}) \geq 0 \qquad (i = 1,...,l) \tag{11}$$

$$\bar{t}_i \geq 0 \qquad (i=1,...,l) \tag{12}$$

$$\sum_{i=1}^{l} \bar{t}_i h_i(\bar{X}) = 0 \tag{13}$$

The conditions are also necessary if a certain regularity condition is satisfied. We take Slater's condition, stating: the set defined by the conditions (9) has a non empty interior.

We now differentiate between two situations:
1) Slater's condition is satisfied, and
2) Slater's condition is not satisfied.

1) If Slater's condition is satisfied, in the case of problem (7) there exist numbers $\bar{\lambda}_1$ and $\bar{t}_{i1}$ $(i \in \mathcal{J})$ such that (10) ... (13) are fulfilled.
In the same way there exist numbers $\bar{\lambda}_2$ and $\bar{t}_{i2}$ $(i \in \mathcal{J})$ for problem (8) such that (10) ...(13) are satisfied.

Next define

$$\bar{\lambda} := \frac{1+\bar{\lambda}_1}{1+\bar{\lambda}_2}, \quad \bar{t}_i := \frac{1}{1+\bar{\lambda}_2}(\bar{t}_{i1}+\bar{t}_{i2}) \qquad (i \in \mathcal{J}),$$

then the two sets of conditions can be combined and rewritten as

$$-\nabla\sigma^2(\bar{X}) + \bar{\lambda}\nabla\mu(\bar{X}) + \sum_{i \in \mathcal{J}} \bar{t}_i \nabla h_i(\bar{X}) = 0. \tag{14}$$

$$h_i(\bar{X}) \geq 0 \qquad (i \in \mathcal{J}) \tag{15}$$

$$\bar{\lambda} > 0, \; \bar{t}_i \geq 0 \qquad (i \in \mathcal{J}) \tag{16}$$

$$\sum_{i \in \mathcal{J}} \bar{t}_i h_i(\bar{X}) = 0, \tag{17}$$

But this means that there exists a $\bar{\lambda}>0$, such that $X=\bar{X}$ solves the problem

$$\max_X \{-\sigma^2(X) + \bar{\lambda}\,\mu(X) \mid X \in \mathcal{X}\},$$

which is identical to (4).

2) If Slater's condition is not satisfied, this means that either $\mu(X)-\mu(\bar{X})\geq 0$ or $\sigma^2(\bar{X})-\sigma^2(X)\geq 0$ doesn't have an interior point because $\mathcal{X}$ has a non empty interior. In the first case $\mu(\bar{X})$ equals the maximum μ_{max} of $\mu(X)$ on $\mathcal{X}$ and the efficient portfolio $\bar{X}$ solves (6); in the second case $\sigma^2(\bar{X})$ equals the minimum σ^2_{min} of $\sigma^2(X)$ on $\mathcal{X}$ and the efficient portfolio $\bar{X}$ solves (5). If $\sigma^2(X) = \sigma^2_{min}$ has an unique solution, finding the corresponding efficient portfolio is equivalent to solving (4) for $\bar{\lambda} = 0$. Analogous if $\mu(X) = \mu_{max}$ has an unique solution, finding the corresponding efficient portfolio is equivalent to solving (4) for a sufficiently large value of $\bar{\lambda}$.

Remark 1.

The theorem implies that Markowitz's method for computing the efficient portfolios can also be applied if the return $\underline{r}(X)$ is a nonlinear function of X. An example of this is the case of a capital budgeting decision in which the revenue of the investment is a concave function of the investment amount (diminishing returns). This is especially the case if the capital budgeting problem is combined with liquidity constraints. Then both $\underline{r}(X)$ and the conditions $h_i(X)$ resulting from the liquidity constraints are non linear functions of X.

3. The set of efficient (μ,σ^2) points in the Markowitz' case.

We now specialize to the original portfolio selection problem of Markowitz. Suppose the yearly revenue of one dollar invested in security j equals $\underline{r}_j$ with $E\underline{r}_j=\mu_j$; the covariance matrix of the $\underline{r}_j$ is $\mathcal{C}$. If $M'= (\mu_1,\ldots,\mu_n)$, then

$$\mu(X) = M'X, \tag{18}$$

$$\sigma^2(X) = X' \mathcal{C} X. \tag{19}$$

The constraints are

$$\mathcal{A} X \le B, \tag{20}$$

$$X \ge 0. \tag{21}$$

If the feasible set $\mathcal{X}$ has a non empty interior, the efficient portfolios can be found by applying the theorem of section 2 in which the left hand side of (4) now reduces to

$$\min_X \{X' \mathcal{C} X - \bar{\lambda} M'X \mid \mathcal{A} X \le B \wedge X \ge 0\}.$$

The points $(\bar{\mu},\bar{\sigma}^2)$ corresponding to efficient portfolios constitute the efficient points in the (μ,σ^2) plane, sometimes called the critical line of the problem. If we start with $\lambda=0$ and next raise λ, we get different efficient portfolios. For specific values of λ, there is a change in the basis; suppose these values are $\bar{\lambda}_1,\ldots,\bar{\lambda}_k$ and corresponding efficient solutions are $\bar{X}_1,\ldots,\bar{X}_k$. We form the (sub)sequence $\bar{X}_{j_1},\ldots,\bar{X}_{j_h}$ from $\bar{X}_1,\ldots,\bar{X}_k$ for which the $(\bar{\mu},\bar{\sigma}^2)$ combinations are different. This (sub)sequence is called the set of corner portfolios. We have

$$M'\bar{X}_{j_i} < M'\bar{X}_{j_{i+1}} \tag{22}$$

and

$$\bar{X}'_{j_i}\,\mathscr{C}\,\bar{X}_{j_i} < \bar{X}'_{j_{i+1}}\,\mathscr{C}\,\bar{X}_{j_{i+1}}. \tag{23}$$

The critical line in the (μ,σ^2) plane has the following properties.

a. Between the (μ,σ^2) points of two adjacent corner portfolios, it is part of a strictly convex parabola.

b. On the segments mentioned in a, the relation

$$\left[\frac{d\sigma^2}{d\mu}\right]_{(\bar{\mu},\bar{\sigma}^2)} = \bar{\lambda} \tag{24}$$

holds.

c. For $\mathscr{C}$ positive definite every point of the critical line (= every efficient portfolio $\bar{X}_b$) satisfies

$$\bar{X}_b = A + D\bar{\lambda} \tag{25}$$

with A and D constants which can be explicitly computed; moreover $\mu(\bar{X}_b)$ is a linear function of $\bar{\lambda}$ with coefficient $\neq 0$.

Only property b is well known from literature, cf. Markowitz (1956) p. 16, or Zangwill (1969) p. 66-68. We shall now prove properties a and c.

<u>Proof of property a</u>.

We consider a part of the critical line between two adjacent corner portfolios, so the efficient portfolios that are convex combinations of these corner portfolios. For simplicity we note these corner portfolios not as X_{j_i} and $X_{j_{i+1}}$ but as X_i and X_{i+1}.
The efficient portfolios of this part of the critical line can be written as:

$$\bar{X} = \alpha(X_i - X_{i+1}) + X_{i+1} \qquad \alpha \in [0,1].$$

With (18) and (19) it follows:

$$\mu(\bar{X}) = \alpha M'(X_i - X_{i+1}) + M'X_{i+1} \tag{26}$$

and

$$\sigma^2(\bar{X}) = \alpha^2(X_i - X_{i+1})' \mathscr{C} (X_i - X_{i+1}) + 2\alpha(X_i - X_{i+1})' \mathscr{C} X_{i+1} + X'_{i+1} \mathscr{C} X_{i+1}. \tag{27}$$

Elimination of α from (26) and substitution in (27) gives a quadratic expression of $\sigma^2(\bar{X})$ as a function of $\mu(\bar{X})$ with as a coefficient of $\mu(\bar{X})^2$

$$\frac{(X_i - X_{i+1})' \mathscr{C} (X_i - X_{i+1})}{\{M'(X_i - X_{i+1})\}^2}.$$

This coefficient is positive, because (22) gives

$$\{M'(X_i - X_{i+1})\}^2 > 0,$$

and (23) leads to

$$(X_i - X_{i+1})' \mathscr{C} (X_i - X_{i+1}) = \sigma^2(X_i - X_{i+1}) = \sigma^2(\underline{r}(X_i) - \underline{r}(X_{i+1})) \geq$$

$$\geq (\sigma(\underline{r}(X_i)) - \sigma(\underline{r}(X_{i+1})))^2 > 0.$$

So it follows directly that $\sigma^2(\bar{X})$ is a strictly convex function of $\mu(\bar{X})$.

Proof of property c.

For efficient portfolios $X = \bar{X}$ with $\mu_{min} < \mu(\bar{X}) < \mu_{max}$ there exist numbers $\bar{\lambda}$ and $\bar{t}_i$ $(i \in \mathcal{J})$ satisfying (14) ... (17). Specializing to the problem of this section, combining the Lagrange multipliers of the conditions (20) in $U' = (u_1,\dots,u_m)$, those of (21) in $V' = (v_1,\dots,v_n)$ and adding slack variables $y_1,\dots,y_m$ to (20), (14) and (15) reduce to

$$-2\,\mathcal{C}\,\bar{X} - \mathcal{A}'\bar{U} + \bar{V} = -\bar{\lambda}M \tag{28}$$

and

$$\mathcal{A}\,\bar{X} + \bar{Y} = B \tag{29}$$

$$\bar{X} \geq 0.$$

An expression which holds for every efficient portfolio can be derived as follows. Denote the basic variables of X by X_b and the corresponding parts of M, $\mathcal{C}$ and $\mathcal{A}$ by M_{b_1}, $\mathcal{C}_{b_1}$ and $\mathcal{A}_{b_1}$, then as will be shown in appendix A, $\bar{X}_b$ can be written as

$$\bar{X}_b = A + D\,\bar{\lambda} \tag{25}$$

with

$$A = \mathcal{C}_{b_1}^{-1}\,\mathcal{A}_{b_1}'(\,\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\,\mathcal{A}_{b_1}')^{-1}B_{b_1} \tag{30}$$

and

$$D = \tfrac{1}{2}\,[\,\mathcal{C}_{b_1}^{-1} - \mathcal{C}_{b_1}^{-1}\mathcal{A}_{b_1}'(\,\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}_{b_1}')^{-1}\,\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}]M_{b_1}. \tag{31}$$

Substituting (25) into (18) and (19), we get

$$\mu(\bar{X}_b) = M_{b_1}'\,A + M_{b_1}'\,D\,\bar{\lambda} \tag{32}$$

$$\sigma^2(\bar{X}_b) = A'\,\mathcal{C}_{b_1}A + 2\,A'\,\mathcal{C}_{b_1}D\,\bar{\lambda} + D'\,\mathcal{C}_{b_1}D\,\bar{\lambda}^2. \tag{33}$$

Furthermore in appendix B it will be shown that

$$M'_{b_1} D \neq 0. \tag{34}$$

for every efficient portfolio.

Remark 2.

Using the formulae (32) and (33) it can be shown that if $\mathscr{C}$ is positive definite, the critical line needs not to be differentiable everywhere on the open interval (μ_{min}, μ_{max}). Dr. J. Vörös from Pěcz University (Hungary) provided us the following example.

$$M = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \ \mathscr{C} = \begin{bmatrix} 3 & 3 & -1 \\ 3 & 11 & 23 \\ -1 & 23 & 75 \end{bmatrix}, \ \mathscr{A} = (1 \quad 1 \quad 1), \ B = (1).$$

The critical line of this problem is not differentiable in the point corresponding to the corner portfolio $X' = (0 \quad 1 \quad 0)$. On the parabola to the left of this point the variables x_1 and x_2 are in the basis, on the righthand side the variables x_2 and x_3. For the point $\mu = 3$, $\sigma^2 = 11$ the lefthand side and righthand side derivative can be computed as follows. Substitute the corresponding parts of M, $\mathscr{C}^{-1}$, $\mathscr{A}$ and B into (32) and (33) and next eliminate λ. It then turns out that $\lim\limits_{\mu\uparrow 3} \frac{d\sigma^2}{d\mu} = 8$ and $\lim\limits_{\mu\downarrow 3} \frac{d\sigma^2}{d\mu} = 12$.

Appendix A.

Proof of the formulae (25), (30) and (31).

We rewrite the equations (28) and (29), omitting the bars, to get variables X,Y,U and V, as follows

$$\begin{array}{ccccc} X' & Y' & U' & V' & \\ \hline -2\,\mathcal{C} & \mathcal{O} & -\mathcal{A}' & \mathcal{I} & -\bar{\lambda}M \\ \mathcal{A} & \mathcal{I} & \mathcal{O} & \mathcal{O} & B \end{array} \tag{35}$$

Let

$$\bar{Z}'_b = (\bar{X}'_b ,\bar{Y}'_b ,\bar{U}'_b ,\bar{V}'_b) \tag{36}$$

be the feasible basic solution belonging to the eficient portfolio, then (35) can be partitioned into

$$\begin{array}{ccccccccc} X'_b & X'_{nb} & Y'_b & Y'_{nb} & U'_b & U'_{nb} & V'_b & V'_{nb} & \\ \hline -2\,\mathcal{C}_{b_1} & -2\,\mathcal{C}_{nb_1} & \mathcal{O} & \mathcal{O} & -\mathcal{A}'_{b_1} & -\mathcal{A}'_{b_2} & \mathcal{O} & \mathcal{I} & -\bar{\lambda}M_{b_1} \\ -2\,\mathcal{C}_{b_2} & -2\,\mathcal{C}_{nb_2} & \mathcal{O} & \mathcal{O} & -\mathcal{A}'_{nb_1} & -\mathcal{A}'_{nb_2} & \mathcal{I} & \mathcal{O} & -\bar{\lambda}M_{b_2} \\ \mathcal{A}_{b_1} & \mathcal{A}_{nb_1} & \mathcal{O} & \mathcal{I} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & B_{b_1} \\ \mathcal{A}_{b_2} & \mathcal{A}_{nb_2} & \mathcal{I} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & B_{b_2} \end{array} \tag{37}$$

The matrix $-2\,\mathcal{C}$ is partitioned into the square matrices $-2\,\mathcal{C}_{b_1}$ and $-2\,\mathcal{C}_{nb_2}$ corresponding to basic and non-basic variables x_j and into $-2\,\mathcal{C}_{b_2}$ and $-2\,\mathcal{C}_{nb_1}$

with $\mathcal{C}_{b_2} = \mathcal{C}_{nb_1}' \cdot \mathcal{A}_{b_1}$ and $\mathcal{A}_{nb_1}$ represent the active constraints, $\mathcal{A}_{b_2}$ and $\mathcal{A}_{nb_2}$ the non-active constraints. Therefore we get identity matrices in the fourth place of the Y_b' column and the third place of the Y_{nb}' column. The other partitions are evident.

The matrix of basic vectors is

$$\mathcal{B} = \begin{bmatrix} -2\,\mathcal{C}_{b_1} & \mathcal{O} & -\mathcal{A}_{b_1}' & \mathcal{O} \\ -2\,\mathcal{C}_{b_2} & \mathcal{O} & -\mathcal{A}_{nb_1}' & \mathcal{I} \\ \mathcal{A}_{b_1} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{A}_{b_2} & \mathcal{I} & \mathcal{O} & \mathcal{O} \end{bmatrix} \qquad (38).$$

To facilitate computations we reshuffle rows and columns into

$$\mathcal{B}_v = \begin{bmatrix} -2\,\mathcal{C}_{b_1} & -\mathcal{A}_{b_1}' & \mathcal{O} & \mathcal{O} \\ \mathcal{A}_{b_1} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ -2\,\mathcal{C}_{b_2} & -\mathcal{A}_{nb_1}' & \mathcal{I} & \mathcal{O} \\ \mathcal{A}_{b_2} & \mathcal{O} & \mathcal{O} & \mathcal{I} \end{bmatrix} \qquad (39).$$

The values of the basic variables are

$$\bar{Z}_{bv} = \mathcal{B}_v^{-1} \begin{bmatrix} 0 \\ B_{b_1} \\ 0 \\ B_{b_2} \end{bmatrix} - \bar{\lambda}\ \mathcal{B}_v^{-1} \begin{bmatrix} M_{b_1} \\ 0 \\ M_{b_2} \\ 0 \end{bmatrix} \tag{40}$$

with $\bar{Z}'_{bv} = (\bar{X}'_b, \bar{U}'_b, \bar{V}'_b, \bar{Y}'_b)$. In order to get an explicit expression for $\bar{X}_b$ we compute $\mathcal{B}_v^{-1}$:

$$\mathcal{B}_v^{-1} = \left[\begin{array}{c|c} \begin{bmatrix} -2\ \mathcal{C}_{b_1} & -\mathcal{A}'_{b_1} \\ \mathcal{A}_{b_1} & \mathcal{O} \end{bmatrix}^{-1} & \mathcal{O} \\ \hline -\begin{bmatrix} -2\ \mathcal{C}_{b_2} & -\mathcal{A}'_{nb_1} \\ \mathcal{A}_{b_2} & \mathcal{O} \end{bmatrix} \begin{bmatrix} -2\ \mathcal{C}_{b_1} & -\mathcal{A}'_{b_1} \\ \mathcal{A}_{b_1} & \mathcal{O} \end{bmatrix}^{-1} & \begin{bmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{I} \end{bmatrix} \end{array}\right] \tag{41}.$$

Because $\mathcal{B}_v$ has an inverse, $\begin{bmatrix} -2\ \mathcal{C}_{b_1} & -\mathcal{A}'_{b_1} \\ \mathcal{A}_{b_1} & \mathcal{O} \end{bmatrix}^{-1}$ exists and since $\mathcal{C}$ is positive definite $\mathcal{C}_{b_1}^{-1}$ exists and also $(\mathcal{A}_{b_1}\ \mathcal{C}_{b_1}^{-1}\ \mathcal{A}'_{b_1})^{-1}$, cf. Hadley (1961) pp 107-109. Hence

$$\begin{bmatrix} -2\ \mathcal{C}_{b_1} & -\mathcal{A}'_{b_1} \\ \mathcal{A}_{b_1} & \mathcal{O} \end{bmatrix}^{-1} =$$

$$\left[\begin{array}{c|c} -\frac{1}{2}\,\mathscr{C}_{b_1}^{-1}+\frac{1}{2}\,\mathscr{C}_{b_1}^{-1}\,\mathscr{A}_{b_1}'\,(\,\mathscr{A}_{b_1}\,\mathscr{C}_{b_1}^{-1}\,\mathscr{A}_{b_1}')^{-1}\,\mathscr{A}_{b_1}\,\mathscr{C}_{b_1}^{-1} & \mathscr{C}_{b_1}^{-1}\,\mathscr{A}_{b_1}'(\,\mathscr{A}_{b_1}\,\mathscr{C}_{b_1}^{-1}\,\mathscr{A}_{b_1}')^{-1} \\ \hline -(\,\mathscr{A}_{b_1}\,\mathscr{C}_{b_1}^{-1}\,\mathscr{A}_{b_1}')^{-1}\,\mathscr{A}_{b_1}\,\mathscr{C}_{b_1}^{-1} & -2(\,\mathscr{A}_{b_1}\,\mathscr{C}_{b_1}^{-1}\,\mathscr{A}_{b_1}')^{-1} \end{array}\right] \qquad (42).$$

Substitution of (42) in (41) and the result into (40) gives

$$\bar{X}_b = \mathscr{C}_{b_1}^{-1}\,\mathscr{A}_{b_1}'(\,\mathscr{A}_{b_1}\,\mathscr{C}_{b_1}^{-1}\,\mathscr{A}_{b_1}')^{-1}B_{b_1}+$$

$$+\bar{\lambda}[\tfrac{1}{2}\,\mathscr{C}_{b_1}^{1} - \tfrac{1}{2}\,\mathscr{C}_{b_1}^{-1}\,\mathscr{A}_{b_1}'(\,\mathscr{A}_{b_1}\,\mathscr{C}_{b_1}^{-1}\,\mathscr{A}_{b_1}')^{-1}\,\mathscr{A}_{b_1}\,\mathscr{C}_{b_1}^{-1}]M_{b_1},$$

with

$$A = \mathscr{C}_{b_1}^{-1}\,\mathscr{A}_{b_1}'(\,\mathscr{A}_{b_1}\mathscr{C}_{b_1}^{-1}\,\mathscr{A}_{b_1}\,\mathscr{C}\,)^{-1}B_{b_1} \qquad (30)$$

and

$$D = \tfrac{1}{2}\,[\,\mathscr{C}_{b_1}^{-1} - \mathscr{C}_{b_1}^{-1}\,\mathscr{A}_{b_1}'(\,\mathscr{A}_{b_1}\,\mathscr{C}_{b_1}^{-1}\,\mathscr{A}_{b_1}')^{-1}\,\mathscr{A}_{b_1}\,\mathscr{C}_{b_1}^{-1}]M_{b_1}, \qquad (31)$$

as was to be proved.

Appendix B.

Proof of formula (34).

We use the fact that an efficient portfolio with expected value $\bar{\mu}$ solves problem (7), which in this case reduces to,
maximize

$$-X' \mathscr{C} X$$

subject to

$$\mathscr{A} X \le B$$

$$M'X \ge \bar{\mu}$$

$$X \ge 0.$$

The Kuhn and Tucker conditions with Lagrange multipliers $\bar{U}$, $\bar{\lambda}_1$ and $\bar{V}$ and slack variables $\bar{Y}$ and $\bar{y}_{n+1}$ are

$$-2 \mathscr{C} \bar{X} \qquad - \mathscr{A}'\bar{U} + M\bar{\lambda}_1 + \bar{V} = 0 \tag{43}$$

$$\mathscr{A} \bar{X} + \bar{Y} \qquad = B \tag{44}$$

$$M'\bar{X} \qquad -\bar{y}_{n+1} \qquad = \bar{\mu} \tag{45}$$

$$\bar{X}'\bar{V} + \bar{Y}'\bar{U} + \bar{y}_{n+1}.\bar{\lambda}_1 = 0.$$

$$\bar{X} \ge 0,\ \bar{Y} \ge 0,\ \bar{y}_{n+1} \ge 0,\ \bar{U} \ge 0,\ \bar{V} \ge 0.$$

For the equations (43), (44), (45), vector (36) completed with $\bar{\lambda}_1$, forms a basic solution. Reordering in the same way as (39), the matrix of basic vectors changes into

$$\mathscr{B}^*_v = \begin{bmatrix} \mathscr{B}_v & K \\ L' & 0 \end{bmatrix}$$

with

$$L' = (M'_{b_1} \ \ O' \ \ O' \ \ O') \tag{46}$$

and

$$K' = (M'_{b_1} \ O' \ M'_{b_2} \ O') \tag{47}.$$

$\mathscr{B}^*_v$ has an inverse, so $(\mathscr{B}^*_v)^{-1}$ exists, just as $\mathscr{B}_v^{-1}$ and $(L' \mathscr{B}_v^{-1} K)^{-1}$, cf. again Hadley (1961) pp. 107-109. Now

$$(\mathscr{B}^*_v)^{-1} = \begin{bmatrix} \mathscr{B}_v^{-1} - \mathscr{B}_v^{-1}K(L' \mathscr{B}_v^{-1}K)^{-1}L' \mathscr{B}_v^{-1} & \mathscr{B}_v^{-1}K(L' \mathscr{B}_v^{-1}K)^{-1} \\ (L' \mathscr{B}_v^{-1}K)^{-1}L' \mathscr{B}_v^{-1} & -(L' \mathscr{B}_v^{-1} K)^{-1} \end{bmatrix}.$$

Substitution of (46), (41) and (47) in $-(L' \mathscr{B}_v^{-1}K)^{-1}$ gives

$$\tfrac{1}{2} [M'_{b_1} \{ \mathscr{C}_{b_1}^{-1} - \mathscr{C}_{b_1}^{-1} \mathscr{A}'_{b_1} (\mathscr{A}_{b_1} \mathscr{C}_{b_1}^{-1} \mathscr{A}'_{b_1})^{-1} \mathscr{A}_{b_1} \mathscr{C}_{b_1}^{-1}\} M_{b_1}]^{-1},$$

which is, but for a constant, the reciprocal of the left hand side of (34).

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