Differentiability Properties of the Efficient (μ, σ^2)-Set in the Markowitz Portfolio Selection Method

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1 Introduction

The standard portfolio selection problem with linear constraints may be formulated as follows. An investor wants to invest an amount of one unit in the securities $1, \ldots, n$. If he invests an amount x_j in security $j (j = 1, \ldots, n)$ the x_j should satisfy the conditions

$$\mathcal{A}X = B, \tag{1.1}$$

$$X \geq \mathcal{O} \tag{1.2}$$

with $\mathcal{A}$ an $(m \times n)$- matrix with full rank, B an m-vector and $X' = (x_1, \ldots, x_n)$; (1.1) includes the condition

$$\sum_{j=1}^{n} x_j = 1. \tag{1.3}$$

The yearly return on one dollar invested in security j equals $\underline{r}_j$ with $\mu_j = \mathcal{E}\underline{r}_j$; the covariance matrix of the random variables $\underline{r}_j$ is $\mathcal{C}$. The yearly return $\underline{r}(X)$ on a portfolio X equals

$$\underline{r}(X) = \sum_{j=1}^{n} x_j \underline{r}_j; \tag{1.4}$$

with $M' = (\mu_1, \ldots, \mu_n)$, the expected yearly return $\mathcal{E}\underline{r}(X)$ equals $M'X$ and will be denoted by $\mu(X)$, so

$$\mu(X) = M'X; \tag{1.5}$$

the variance $\sigma^2(\underline{r}(X))$ equals $X'\mathcal{C}X$ and will be denoted by $\sigma^2(X)$, so

$$\sigma^2(X) = X'\mathcal{C}X. \tag{1.6}$$

For equivalent formulations of the conditions (1.1), (1.2) cf. H.M. Markowitz (1987) p. 24-27, for nonlinear constraints J. Kriens and J.Th. van Lieshout (1988).

A feasible portfolio $\bar{X}$ is called efficient if it is a solution of both

$$\min_X \{\sigma^2(X) | \mu(X) \geq \mu(\bar{X}) \wedge \mathcal{A}\bar{X} = B \wedge \bar{X} \geq \mathcal{O}\} \tag{1.7}$$

and

$$\max_X \{\mu(X) | \sigma^2(X) \leq \sigma^2(\bar{X}) \wedge \mathcal{A}\bar{X} = B \wedge \bar{X} \geq \mathcal{O}\}. \tag{1.8}$$

All efficient portfolios can be derived by computing

$$\min_X \{X'\mathcal{C}X - \lambda M'X | \mathcal{A}\bar{X} = B \wedge \bar{X} \geq \mathcal{O}\} \tag{1.9}$$

for all $\lambda \geq 0$; cf. H.M. Markowitz (1959) p. 315-316, or for a precise and more general statement of the theorem underlying the algorithm J. Kriens and J.Th. van Lieshout (1988).

With $U' = (u_1, \ldots, u_m)$ and $V' = (v_1, \ldots, v_n)$ as Lagrange multipliers of (1.1) and (1.2), respectively, the Kuhn-Tucker conditions of (1.9) run

$$-2\mathcal{C}X - \mathcal{A}'U + V = -\lambda M \tag{1.10}$$

$$\mathcal{A}X \qquad\qquad = B \tag{1.11}$$

$$V'X = 0, X \geq \mathcal{O}, V \geq \mathcal{O}, U \text{ free.} \tag{1.12}$$

Loosely speaking we can describe an algorithm to solve this system for all $\lambda \geq 0$ as follows. Start with choosing $\lambda = 0$, thus with determining the minimum possible variance, and next raise λ to get (new) efficient portfolios. For specific values of λ there is a change in the basis. Let these values be $\bar{\lambda}_1, \ldots, \bar{\lambda}_k$, the corresponding efficient solutions be $\bar{X}_1, \ldots, \bar{X}_k$ with mean-variance combinations $(\mu(\bar{X}_1), \sigma^2(\bar{X}_1)), \ldots, (\mu(\bar{X}_k), \sigma^2(\bar{X}_k))$. The sequence $\bar{X}_1, \ldots, \bar{X}_k$ is called the set of **corner portfolios**, the set of all $(\mu(\bar{X}), \sigma^2(\bar{X}))$ points in the (μ, σ^2)-plane corresponding to efficient portfolios $\bar{X}$ is **the set of efficient** $(\bar{\mu}, \bar{\sigma}^2)$ **combinations** of the problem, or **the efficient frontier.**

This last set satisfies the following properties:

a. between the $(\bar{\mu}, \bar{\sigma}^2)$ points of two adjacent corner portfolios $\bar{X}_i$ and $\bar{X}_{i+1} (\neq \bar{X}_i)$ it is part of a strictly convex parabola;

b. on the interior of the segments mentioned in a, the relation

$$\left(\frac{d\sigma^2}{d\mu}\right)_{(\bar{\mu}, \bar{\sigma}^2)} = \bar{\lambda} \tag{1.13}$$

holds; it is strictly increasing as a function of μ;

c. in the $(\bar{\mu}, \bar{\sigma}^2)$ points corresponding to corner portfolios, the left hand derivative $\left(\frac{d\sigma^2}{d\mu}\right)_L$ and the right hand derivative $\left(\frac{d\sigma^2}{d\mu}\right)_R$ exist and satisfy

$$\left(\frac{d\sigma^2}{d\mu}\right)_L \leq \left(\frac{d\sigma^2}{d\mu}\right)_R . \tag{1.14}$$

From b it follows that on those segments there is a one to one correspondence between the values of $\bar{\lambda}$ and $\bar{\mu}$. In corner portfolios this is only true if $\left(\frac{d\sigma^2}{d\mu}\right)_L = \left(\frac{d\sigma^2}{d\mu}\right)_R$, which implies differentiability of the (μ, σ^2) curve. For proofs cf. H.M. Markowitz (1987), p. 176 and J. Kriens and J.Th. van Lieshout (1988).

Section 2 of this paper contains a more precise discussion of the algorithm to solve (1.10),...,(1.12) for every $\lambda \geq 0$, section 4 necessary and sufficient conditions for the equality sign in (1.14). In preparation for the second topic we present a slightly adapted form of the explicit formulae for $\bar{X}, \mu(\bar{X})$ and $\sigma^2(\bar{X})$ as derived by J. Kriens and J.TH. van Lieshout (1988) in section 3.

Section 5 compares with other literature and section 6 considers the standard portfolio case supplied with one riskless asset. Throughout the whole paper we assume $\mathcal{C}$ positive definite.

2 The algorithm

In order to present a more precise discussion of the algorithm we first prove the following lemma.

Lemma 2.1
If in a portfolio selection problem
1) $\forall_j \sigma^2(\underline{r}_j) > 0$
2) there are no linear relations between the returns $\underline{r}_j$,
portfolios $\bar{X}_1$ and $\bar{X}_2(\neq \bar{X}_1)$ with $\mu(\bar{X}_1) = \mu(\bar{X}_2)$ and $\sigma^2(\bar{X}_1) = \sigma^2(\bar{X}_2)$ cannot be efficient.

Proof
Let

$$\bar{X} = \alpha\bar{X}_1 + (1-\alpha)\bar{X}_2 \quad (0 < \alpha < 1),$$

then

$$\begin{aligned}
&\underline{r}(\bar{X}) = \alpha\underline{r}(\bar{X}_1) + (1-\alpha)\underline{r}(\bar{X}_2)\\
&\mu(\bar{X}) = \mu(\bar{X}_1) = \mu(\bar{X}_2)\\
&\sigma^2(\underline{r}(\bar{X})) = \alpha^2\sigma^2(\underline{r}(\bar{X}_1)) + 2\alpha(1-\alpha)\rho\sigma(\underline{r}(\bar{X}_1))\sigma(\underline{r}(\bar{X}_2)) +\\
&(1-\alpha)^2\sigma^2(\underline{r}(\bar{X}_2)) = \sigma^2(\bar{X}_1)[\alpha^2 + 2\alpha(1-\alpha)\rho + (1-\alpha)^2] = \sigma^2(\bar{X}_1)f(\alpha).
\end{aligned}$$

For $\rho \neq 1, f(\alpha) < 1$ for $0 < \alpha < 1$, so $\sigma^2(\bar{X}) < \sigma^2(\bar{X}_1)$ and $\bar{X}_1$ and $\bar{X}_2$ are not efficient. $\rho(\underline{r}(\bar{X}_1), \underline{r}(\bar{X}_2)) = 1$ iff all realizations of $(\underline{r}(\bar{X}_1), \underline{r}(\bar{X}_2))$ are situated on a straight line, so all points $\left(\sum_{j=1}^n x_{j1}r_j, \sum_{j=1}^n x_{j2}r_j\right)$ are on a straight line. This means

$$\exists_a \exists_d \forall_R \left(\sum_{j=1}^n x_{j2}r_j\right) = a + d\left(\sum_{j=1}^n x_{j1}r_j\right).$$

Let

$$\forall_j a_j = dx_{j1} - x_{j2},$$

then

$$\forall_R \;\; a + \sum_{j=1}^{n} a_j r_j = 0.$$

We discern four cases:

a) $\forall_j a_j = 0 \Rightarrow \forall_j d x_{j1} = x_{j2}$, leading with (1.3) to $d = 1$ and $\bar{X}_1 = \bar{X}_2$, which contradicts $\bar{X}_1 \neq \bar{X}_2$;

b) $a_i \neq 0, \forall_{j \neq i} \;\; a_j = 0 \Rightarrow \forall_R \;\; a + a_i r_i = 0$ and r_i is fixed, so $\sigma^2(\underline{r}_i) = 0$, which contradicts condition 1);

c) $a_i \neq 0, a_k \neq 0, \forall_{j \neq i,k} a_j = 0 \Rightarrow \forall_R \;\; a + a_i r_i + a_k r_k = 0$, which contradicts condition 2);

d) More than two $a_i \neq 0$; conclusion as under c).

So $\rho(\underline{r}(\bar{X}_1), \underline{r}(\bar{X}_2)) \neq 1$ and the lemma is proved.

Remark 2.1. From the proof it follows that conditions 1) and 2) are also necessary. Moreover the conditions 1) and 2) hold iff $\mathcal{C}$ is positive definite.

From lemma 2.1 it is clear that for $\mathcal{C}$ positive definite the corner portfolios $\bar{X}_1, \ldots, \bar{X}_k$ are uniquely determined. However, there are not always as many different corner portfolios as there are different bases during the computations; different bases may yield the same portfolio and also different values $\bar{\lambda}_i$ may yield the same portfolio. In this respect the notation in section 1 is misleading.

Starting the algorithm with $\lambda = 0$ and next raising λ, the algorithm produces a series of bases. Bases which hold for just one value of λ are dropped so that only bases corresponding to nondegenerate λ-intervals are left.
Denote for a given basis of the system (1.10) ,..., (1.12), the set of basic x-variables by $(X_b)_i$. In section 3 we will show that the values $(\bar{X}_b)_i$ of the basic x-variables satisfy

$$(\bar{X}_b)_i = A_i + D_i \bar{\lambda} \tag{2.1}$$

for all $\bar{\lambda}$ in the corresponding interval; the constants A_i and D_i will be computed explicitly. So if $\mathcal{C}$ is positive definite the whole efficient frontier is uniquely determined.

With corner portfolios there correspond at least two vectors D_i, the vector corresponding to the "old" basis and the vector corresponding to the "new" basis. But there may be more associated vectors D_i, either because there exists an equivalent basis for the "old" or for the "new" basis producing the same corner portfolio $(\bar{X}_b)_i$, or because the series of vectors D_i contains one or more vectors $D = \mathcal{O}$. In the latter case the same vector $(\bar{X}_b)_i$ is produced for different values of λ. If the "new" basis is uniquely determined, then for efficient portfolios which are not corner portfolios the vector D_i is uniquely determined.

3 Explicit expressions for efficient portfolios

Starting from the Kuhn-Tucker conditions for the solution of (1.9), J. Kriens and J.TH. van Lieshout (1988) derive an expression for the basic variables which, if $\mathcal{C}$ is positive definite, holds for every efficient portfolio. We present their results in a slightly adapted form.

For a fixed value $\bar{\lambda}$ of λ (1.10) ,..., (1.12) run

$$-2\mathcal{C}X - \mathcal{A}'U + V = -\bar{\lambda}M \tag{3.1}$$

$$\mathcal{A}X \qquad\qquad = B \tag{3.2}$$

$$V'X = 0, X \geq \mathcal{O}, V \geq \mathcal{O}, U \text{ free.} \tag{3.3}$$

The equations (3.1) and (3.2) can be summarized as

$$\begin{array}{|ccc|c|} \hline X' & U' & V' & \\ \hline -2\mathcal{C} & -\mathcal{A}' & \mathcal{J} & -\bar{\lambda}M \\ \mathcal{A} & \mathcal{O} & \mathcal{O} & B \\ \hline \end{array} \tag{3.4}$$

If

$$Z_b' = (X_b', U', V_b') \tag{3.5}$$

denotes a set of basic variables for a given efficient portfolio (3.4) can be partitioned into

$$\begin{array}{|ccccc|c|} \hline X_b' & X_{nb}' & U' & V_b' & V_{nb}' & \\ \hline -2\mathcal{C}_{b_1} & -2\mathcal{C}_{nb_1} & -\mathcal{A}_b' & \mathcal{O} & \mathcal{J} & -\bar{\lambda}M_b \\ -2\mathcal{C}_{b_2} & -2\mathcal{C}_{nb_2} & -\mathcal{A}_{nb}' & \mathcal{J} & \mathcal{O} & -\bar{\lambda}M_{nb} \\ \mathcal{A}_b & \mathcal{A}_{nb} & \mathcal{O} & \mathcal{O} & \mathcal{O} & B \\ \hline \end{array} \tag{3.6}$$

The matrix $-2\mathcal{C}$ is partitioned into the square matrices $-2\mathcal{C}_{b_1}$ and $-2\mathcal{C}_{nb_2}$ corresponding to basic and non-basic x-variables and into $-2\mathcal{C}_{b_2}$ and $-2\mathcal{C}_{nb_1}$ with $\mathcal{C}_{b_2} = \mathcal{C}'_{nb_1}.\mathcal{A}_b, M_b$ and $\mathcal{A}_{nb}, M_{nb}$ also correspond to basic and non-basic variables respectively. The matrix of coefficients of basic variables is

$$\mathcal{B} = \begin{pmatrix} -2\mathcal{C}_{b_1} & -\mathcal{A}'_b & \mathcal{O} \\ -2\mathcal{C}_{b_2} & -\mathcal{A}'_{nb} & \mathcal{J} \\ \mathcal{A}_b & \mathcal{O} & \mathcal{O} \end{pmatrix}. \tag{3.7}$$

To facilitate computations Kriens and van Lieshout reshuffle (3.7) into

$$\mathcal{B}_v = \begin{pmatrix} -2\mathcal{C}_{b_1} & -\mathcal{A}'_b & \mathcal{O} \\ \mathcal{A}_b & \mathcal{O} & \mathcal{O} \\ -2\mathcal{C}_{b_2} & -\mathcal{A}'_{nb} & \mathcal{J} \end{pmatrix}. \tag{3.8}$$

The values of the basic variables are

$$\bar{Z}_b = \begin{pmatrix} \bar{X}_b \\ \bar{U} \\ \bar{V}_b \end{pmatrix} = \mathcal{B}_v^{-1} \begin{pmatrix} \mathcal{O} \\ B \\ \mathcal{O} \end{pmatrix} - \bar{\lambda}\mathcal{B}_v^{-1} \begin{pmatrix} M_b \\ \mathcal{O} \\ M_{nb} \end{pmatrix}. \tag{3.9}$$

We find explicit expressions for these values by computing $\mathcal{B}_v^{-1}$:

$$\mathcal{B}_v^{-1} = \left(\begin{array}{c|c} \begin{pmatrix} -2\mathcal{C}_{b_1} & -\mathcal{A}'_b \\ \mathcal{A}_b & \mathcal{O} \end{pmatrix}^{-1} & \mathcal{O} \\ \hline \begin{pmatrix} 2\mathcal{C}_{b_2} & \mathcal{A}'_{nb} \end{pmatrix} \begin{pmatrix} -2\mathcal{C}_{b_1} & -\mathcal{A}'_b \\ \mathcal{A}_b & \mathcal{O} \end{pmatrix}^{-1} & \mathcal{J} \end{array}\right) \tag{3.10}$$

with

$$\begin{pmatrix} -2\mathcal{C}_{b_1} & -\mathcal{A}'_b \\ \mathcal{A}_b & \mathcal{O} \end{pmatrix}^{-1} = \tag{3.11}$$

$$\left(\begin{array}{c|c} -\frac{1}{2}\mathcal{C}_{b_1}^{-1} + \frac{1}{2}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_b(\mathcal{A}_b\mathcal{C}_{b_1}^{-1}\mathcal{A}'_b)^{-1}\mathcal{A}_b\mathcal{C}_{b_1}^{-1} & \mathcal{C}_{b_1}^{-1}\mathcal{A}'_b(\mathcal{A}_b\mathcal{C}_{b_1}^{-1}\mathcal{A}'_b)^{-1} \\ -(\mathcal{A}_b\mathcal{C}_{b_1}^{-1}\mathcal{A}'_b)^{-1}\mathcal{A}_b\mathcal{C}_{b_1}^{-1} & -2(\mathcal{A}_b\mathcal{C}_{b_1}^{-1}\mathcal{A}'_b)^{-1} \end{array}\right).$$

Substituting (3.11) into (3.10) and the result into (3.9), we find

$$\bar{X}_b = A + D\bar{\lambda} \tag{3.12}$$

with

$$A = \mathcal{C}_{b_1}^{-1}\mathcal{A}_b'(\mathcal{A}_b\mathcal{C}_{b_1}^{-1}\mathcal{A}_b')^{-1}B \tag{3.13}$$

and

$$D = \frac{1}{2}[\mathcal{C}_{b_1}^{-1} - \mathcal{C}_{b_1}^{-1}\mathcal{A}_b'(\mathcal{A}_b\mathcal{C}_{b_1}^{-1}\mathcal{A}_b')^{-1}\mathcal{A}_b\mathcal{C}_{b_1}^{-1}]M_b. \tag{3.14}$$

The corresponding values $\mu(\bar{X}_b)$ and $\sigma^2(\bar{X}_b)$ are

$$\mu(\bar{X}_b) = M_b'A + M_b'D\bar{\lambda} \tag{3.15}$$

$$\sigma^2(\bar{X}_b) = A'\mathcal{C}_{b_1}A + D'\mathcal{C}_{b_1}D\bar{\lambda}^2 \tag{3.16}$$

(note that the coefficient of $\bar{\lambda}$ equals 0).

If the vector $\begin{pmatrix} M \\ \mathcal{O} \end{pmatrix}$ is linear independent of the basis (3.7), it can be shown that

$$M_b'.D \neq 0. \tag{3.17}$$

To prove this Kriens and van Lieshout study problem (1.7) with $\mathcal{A}X \leq B$. With obvious adaptations in the notation, the Kuhn-Tucker conditions of this problem are in our case

$$-2\mathcal{C}X \quad - \mathcal{A}'U + M\lambda + V \qquad = \mathcal{O} \tag{3.18}$$

$$\mathcal{A}X \qquad = B \tag{3.19}$$

$$M'X - y_{m+1} \qquad = \bar{\mu} \tag{3.20}$$

$$X'V = y_{m+1}.\lambda = 0, X \geq \mathcal{O}, V \geq \mathcal{O}, y_{m+1} \geq 0, \lambda \geq 0, U \text{ free.} \tag{3.21}$$

Because $\begin{pmatrix} M \\ \mathcal{O} \end{pmatrix}$ is assumed to be linear independent of $\mathcal{B}_v$, the vector Z_b (3.5) completed with λ, forms a basic solution of (3.18),...,(3.21). Reordering in the same way as in (3.8) the matrix of basic vectors changes into

$$\mathcal{B}_v^* = \begin{pmatrix} \mathcal{B}_v & K \\ L' & \mathcal{O} \end{pmatrix} \tag{3.22}$$

with

$$L' = (M_b' \;\; \mathcal{O}' \;\; \mathcal{O}') \tag{3.23}$$

and

$$K' = (M_b' \;\; \mathcal{O}' \;\; M_{nb}'). \tag{3.24}$$

Using the existence of $(\mathcal{B}_v^*)^{-1}$, (3.17) can be proved.

Remark 3.1. The condition $\begin{pmatrix} M \\ \mathcal{O} \end{pmatrix}$ linear independent of the basis (3.7) is incorrectly suppressed by J. Kriens and J.TH. van Lieshout (1988). J. Kriens (1989) provides a counter example.

4 Necessary and sufficient conditions for differentiability of the efficient frontier

Because of property b in section 1 we can restrict the discussion to the points $(\mu(\bar{X}_i), \sigma^2(\bar{X}_i))$, in the sequel to be denoted by $(\bar{\mu}_i, \bar{\sigma}_i^2)$. Furthermore we only discuss nondegenerate models.

Condition 1.
The efficient frontier (e.f., for short) is differentiable in the point $(\bar{\mu}_i,\ \bar{\sigma}_i^2)$ iff one value $\bar{\lambda}$ corresponds to it.

Proof.
Follows directly from (1.13) and (1.14).

Condition 2.
The e.f. is differentiable in the point $(\bar{\mu}_i, \bar{\sigma}_i^2)$ iff no corresponding $\bar{X}_b$-vector can be represented by (2.1) with $D = \mathcal{O}$.

Proof.
Necessary: $D = \mathcal{O}$ implies the same vector $\bar{X}_b$ and thus the same point $\bar{\mu}_i, \bar{\sigma}_i^2)$ for more than one value of $\bar{\lambda}$.
Sufficient: $D \neq \mathcal{O}; \bar{\lambda}_1 \neq \bar{\lambda}_2 \Rightarrow \bar{X}(\bar{\lambda}_1) \neq \bar{X}(\bar{\lambda}_2)$ and so different points $(\bar{\mu}, \bar{\sigma}^2)$, cf. lemma 2.1.

Condition 3.
The e.f. is differentiable in the point $(\bar{\mu}_i, \bar{\sigma}_i^2)$ iff no corresponding $\bar{X}_b$-vector can be represented by (2.1) with $M_b'.D = 0$.

Proof.
Follows from $D \neq \mathcal{O} \leftrightarrows M_b'.D \neq 0$.
$\rightarrow$ if $\bar{\lambda}$ changes, $\bar{X}_b$ changes and $\mu(\bar{X}_b)$ must change (lemma 2.1), so $M_b'.D \neq 0$ (cf. (3.15)).
$\leftarrow$ trivial.

Condition 4.

The e.f. is differentiable in the point $(\bar{\mu}_i, \bar{\sigma}_i^2)$ iff $(\mathcal{B}_v^*)^{-1}$ exists.

Proof.

Follows from $(\mathcal{B}_v^*)^{-1}$ exists $\leftrightarrows$ $M_b'.D \neq 0$.

$\rightarrow$ see J. Kriens and J.TH. van Lieshout (1988) p. 190-191.

$\leftarrow$ if $M_b'.D \neq 0$, all elements of $(\mathcal{B}_v^*)^{-1}$ exist and $\mathcal{B}_v^*.(\mathcal{B}_v^*)^{-1} = \mathcal{J}$.

Condition 5.

The e.f. is differentiable in the point $(\bar{\mu}_i, \bar{\sigma}_i^2)$ iff $\begin{pmatrix} M \\ \mathcal{O} \end{pmatrix}$ is linear independent of the vectors of $\mathcal{B}_v$.

Proof.

$\begin{pmatrix} M \\ \mathcal{O} \end{pmatrix}$ linear independent of the vectors of $\mathcal{B}_v$ $\leftrightarrows$ inverse of $\mathcal{B}_v^*$ exists.

5 Relations with statements on differentiability in the literature

The theorem stated by J. Vörös (1987) and J. Kriens (1989) are easily checked through applying the conditions of section 4. We combine these theorems in one new theorem. Define $\mu_{\min} := \min_i \mu_i, \mu_{\max} := \max_i \mu_i$,

$$\mathcal{M} = (m_{ij}) := \mathcal{C}_b^{-1} \tag{5.1}$$

$$f := \sum_{i=1}^{k} \sum_{j=1}^{k} m_{ij} \tag{5.2}$$

$$d := \sum_{i=1}^{k} (\sum_{j=1}^{k} m_{ij}\mu_j). \tag{5.3}$$

Theorem 5.1

If in the investment problem subject to (1.2) and (1.3), $\mathcal{C}$ positive definite, a corner portfolio with $\mu \in (\mu_{\min}, \mu_{\max})$ has $k(\geq 1)x$-variables in the basis, then the set of efficient $(\bar{\mu}, \bar{\sigma}^2)$ points is nondifferentiable in the corresponding $(\bar{\mu}, \bar{\sigma}^2)$ point if and only if there exists a representation of $\bar{X}_b = (\bar{x}, \ldots, \bar{x}_k)$ with $\forall_{1 \geq i.j \geq k} \mu_i = \mu_j$.

Proof.

We distinguish between $k = 1$ and $k > 1$.

Sufficient.

$k = 1$. Suppose $\bar{x}_i > 0$, then $\bar{x}_i = 1, \mathcal{C}_{b_1} = (c_{ii}), \mathcal{A}_b = (1), M_b = (\mu_i)$. Substitution of these values into (3.14) leads to

$$\begin{aligned} D &= \tfrac{1}{2}\left[\mathcal{C}_{b_1}^{-1} - \mathcal{C}_{b_1}^{-1}\mathcal{A}_b'(\mathcal{A}_b\mathcal{C}_{b_1}^{-1}\mathcal{A}_b')^{-1}\mathcal{A}_b\mathcal{C}_{b_1}^{-1}\right] M_b \\ &= \tfrac{1}{2}c_{ii}^{-1}\left[1 - (c_{ii}^{-1})^{-1}c_{ii}^{-1}\right]\mu_i = 0. \end{aligned} \tag{5.4}$$

So $D = \mathcal{O}$ and from condition 2 nondifferentiability follows.

$k > 1$. Let the representation with k variables in the basis be

$$\overline{X}_b = \begin{pmatrix} \overline{x}_1 \\ \vdots \\ \overline{x}_k \end{pmatrix}, \mathcal{C}_{b_1} = \begin{pmatrix} c_{11} & \dots & c_{1k} \\ \vdots & & \vdots \\ c_{k1} & \dots & c_{kk} \end{pmatrix}, \mathcal{A}'_b = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, M_b = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix},$$

then

$$(\mathcal{A}_b \mathcal{C}_{b_1}^{-1} \mathcal{A}'_b)^{-1} = \frac{1}{f} \tag{5.5}$$

and D can be rewritten as

$$D = \frac{1}{2}\mathcal{M}\left[\mathcal{J} - \frac{1}{f}\begin{pmatrix} \sum_i m_{i1} & \dots & \sum_i m_{ik} \\ \vdots & & \vdots \\ \sum_i m_{i1} & \dots & \sum_i m_{ik} \end{pmatrix}\right]\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix}. \tag{5.6}$$

If $\mu_1 = \ldots = \mu_k$, then $D = \mathcal{O}$ and condition 2 leads again to nondifferentiability.

Necessary.
$k = 1$. Trivial.
$k > 1$. If there is nondifferentiability then there exists a representation with $D = \mathcal{O}$. For this vector (5.6) is equivalent to

$$\left[\mathcal{J} - \frac{1}{f}\begin{pmatrix} \sum_i m_{i1} & \dots & \sum_i m_{ik} \\ \vdots & & \vdots \\ \sum_i m_{i1} & \dots & \sum_i m_{ik} \end{pmatrix}\right]\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{5.7}$$

or

$$\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} = \frac{1}{f}\begin{pmatrix} \sum_j & (\sum_i m_{ij}) & \mu_j \\ \vdots & & \vdots \\ \sum_j & (\sum_i m_{ij}) & \mu_j \end{pmatrix} = \begin{pmatrix} \frac{d}{f} \\ \vdots \\ \frac{d}{f} \end{pmatrix},$$

so nondifferentiability implies $\mu_1 = \ldots = \mu_k$.

Remark 5.1
In the case of constraint (1.1) instead of (1.3), X_b cannot contain only one x-variable if

(1.1) contains two or more independent constraints.

Remark 5.2

Theorem 5.1 combines the theorems 5.1 and 5.2 in J. Kriens (1989) and generalizes the case $k > 1$ to situations in which the basis contains x-variables with value 0. The theorem also generalizes theorem 2 by J. Vörös (1987).

Remark 5.3

The theorem can likewise be proved by directly applying condition 5 from section 4.

6 The standard portfolio selection problem with one riskless asset.

The standard portfolio selection problem with conditions (1.2) and (1.3) can also be formulated as

$$\min_X \sigma^2(X) = X'\mathcal{C}X \tag{6.1}$$

subject to

$$X \geq \mathcal{O} \tag{6.2}$$

$$\sum_{j=1}^{n} x_j = 1 \tag{6.3}$$

$$M'X = \mu, \tag{6.4}$$

using μ as a parameter; the optimal solution is denoted as $\bar{X}(\mu)$.
Now, consider the standard portfolio case with one riskless asset: minimize (6.1) subject to (6.2),

$$\sum_{j=1}^{n} x_j + y = 1 \tag{6.5}$$

$$M'X + i\ y = \mu, \tag{6.6}$$

where y is the share of capital invested in the riskless asset and i is the rate of interest; we allow y to be positive, 0 or negative.
We can easily state that for $\mu = i$ the optimal solution runs $\bar{y} = 1, \bar{X}(i) = \mathcal{O}$ with $\sigma^2(\underline{r}(\bar{X}(i))) = 0$. Thus we can restrict to the case $\mu > i$; furthermore we assume $i < \max_j\{\mu_j\}$. Let again $X_b' = (x_1, \ldots, x_k)$ represent the set of basic x-variables and

$X'_{nb} = (x_{k+1}, \ldots, x_n)$ the set of non-basic x-variables. Denote the Lagrange multipliers of (6.5) and (6.6) by u_1 and λ respectively, and let $I_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ with n elements. The Kuhn-Tucker equations for the problem (6.1), (6.2), (6.5), (6.6) are:

$$2\mathcal{C}_{b_1} X_b + I_k . u_1 - M_b \lambda = \mathcal{O} \tag{6.7}$$

$$2\mathcal{C}_{b_2} X_b + I_{n-k} . u_1 - M_{nb} \lambda \geq \mathcal{O} \tag{6.8}$$

$$-u_1 + i\lambda = 0 \tag{6.9}$$

$$X \geq \mathcal{O} \tag{6.2}$$

$$I'_k X_b + y = 1 \tag{6.10}$$

$$M'_b X_b + iy = \mu. \tag{6.11}$$

From (6.7) we have

$$X_b = -\frac{1}{2} u_1 \mathcal{C}_{b_1}^{-1} I_k + \frac{1}{2} \lambda \mathcal{C}_{b_1}^{-1} M_b. \tag{6.12}$$

With (5.2), (5.3) and

$$e := \sum_{i=1}^{k} \sum_{j=1}^{k} m_{ij} \mu_i \mu_j \tag{6.13}$$

we can derive from (6.10) and (6.11)

$$I'_k X_b = -\frac{1}{2} f u_1 + \frac{1}{2} d\lambda = 1 - y \tag{6.14}$$

$$M_b'X_b = -\frac{1}{2}du_1 + \frac{1}{2}e\lambda = \mu - iy. \tag{6.15}$$

Lemma 6.1

The expression $fi^2 - 2di + e$ is always positive, except in the case $\forall_{i\in\{1,\ldots,k\}}\mu_i = i$.

Proof

$(M_b - iI_k)'\mathcal{C}_{b_1}^{-1}(M_b - iI_k) = fi^2 - 2di + e = 0$ iff $M_b = iI_k \leftrightarrows \forall_{i\in\{1,\ldots,k\}}\mu_i = i$ (cf. also J. Vörös (1987)).

As $\forall_{i\in\{1,\ldots,k\}}\mu_i = i$ implies $\mu = i$, the case we excluded, $fi^2 - 2di + e$ is always > 0 in our model.

Lemma 6.2

For a given set of basic x- variables X_b the problem (6.1), (6.2), (6.5), (6.6) has a unique solution.

Proof

Eliminating y from (6.14), (6.15) and using (6.9) we find

$$\lambda = \frac{2(\mu - i)}{fi^2 - 2di + e} \tag{6.16}$$

and

$$u_1 = \frac{2i(\mu - i)}{fi^2 - 2di + e}. \tag{6.17}$$

From these equations and (6.12) it follows that the solution is unique.

In the remainder of this section we exploit the well-known property that in the (μ, σ)-plane the e.f. of the model (6.1), (6.2), (6.5), (6.6) is a straight line through the point $\mu = i, \sigma = 0$ which touches the e.f. of the risky assets of the model (6.1),...,(6.4) if this e.f. is differentiable (cf. e.g. Th.E. Copeland and J.F. Weston (1988) p. 179-180). This property implies that we can find the e.f. of the risky assets by using i as a parameter: with every value of i there corresponds one point of the e.f. of risky assets and so one set of basic variables X_b. Formulae (6.12) and (6.8) provide a simple procedure for deriving the corner portfolios of the risky assets. Therefore we rewrite (6.12) and (6.8) by substituting (6.16) and (6.17) into

$$\mathcal{C}_{b_1}^{-1} M_b - i\, \mathcal{C}_{b_1}^{-1} I_k \geq \mathcal{O} \tag{6.18}$$

$$\mathcal{C}_{b_2}\mathcal{C}_{b_1}^{-1} M_b - M_{nb} + i(I_{n-k} - \mathcal{C}_{b_2}\mathcal{C}_{b_1}^{-1} I_k) \geq \mathcal{O}. \tag{6.19}$$

The algorithm runs as follows.

Step 1: Determine $\max\{\mu_j\}$ and fill up the sets X_b and X_{nb}.
Step 2: Find the smallest value of i for which (6.18) and (6.19) hold.

Step 3: If $i = -\infty$ then stop. Otherwise remove the variable from X_b into X_{nb} if $X_b \geq \mathcal{O}$ gives the smallest i, or inversely. Repeat step 2.

If we apply this algorithm to the well-known Markowitz example

$$M = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} 3 & 3 & -1 \\ 3 & 11 & 23 \\ -1 & 23 & 75 \end{pmatrix},$$

then $\mu_3 = \max\{\mu_j\}, X_{b_1} = (x_3)$ and we find successively

$$X_{b_2} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}, X_{b_3} = (x_2), x_{b_4} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, X_{b_5} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, X_{b_b} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}.$$

The expressions (6.7) (or (6.12)) and (6.8) are a special form of the equations (3.1). Raising the value $\bar{\lambda}$ of λ from 0 to the largest relevant value in the standard algorithm is the same as lowering $i = \frac{u_1}{\lambda}$ from the largest relevant value to $-\infty$ in the algorithm just presented. Both algorithms produce exactly the same steps, albeit in a reverse order.

The model of this section gives us the opportunity to present another proof of theorem 5.1.

Another proof of theorem 5.1.

If the e.f. of the risky assets is nondifferentiable in a point P there are different interest rates i from where we can draw subgradients to P. For this interval of values i the return on the portfolio of risky assets is the same i.e. independent of i. Using (6.15) we get

$$M_b' X_b = -\frac{1}{2} d u_1 + \frac{1}{2} e \lambda = \mu \tag{6.20}$$

with μ the expected return of the corresponding portfolio. We substitute (6.16) and (6.17) for λ and u_1 to find

$$(d - f\mu)i - (e - d\mu) = 0. \tag{6.21}$$

This can hold for the whole interval of i-values iff

$$d - f\mu = 0 \text{ and } e - d\mu = 0 \tag{6.22}$$

from which follows

$$f\mu^2 - 2d\mu + e = 0 \tag{6.23}$$

and with lemma 6.1: $\mu_i = \mu_j = \mu$ for all $x_i, x_j \in X_b$.

Considering the case $\mu_i = \mu_j = \mu$ for all $x_i, x_j \in X_b$ we find with (5.2), (5.3), (6.13)

$$e = \mu^2 f \text{ and } d = \mu f \tag{6.24}$$

and with (6.12)

$$X_b = -\frac{1}{2}(u_1 - \mu\lambda)\mathcal{C}_{b_1}^{-1} I_k. \tag{6.25}$$

Substitution of the equations (6.16) and (6.17) leads to

$$X_b = \frac{1}{f}\mathcal{C}_{b_1}^{-1} I_k, \tag{6.26}$$

which is independent of i, so subgradients can be drawn to a whole interval of i-values.

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