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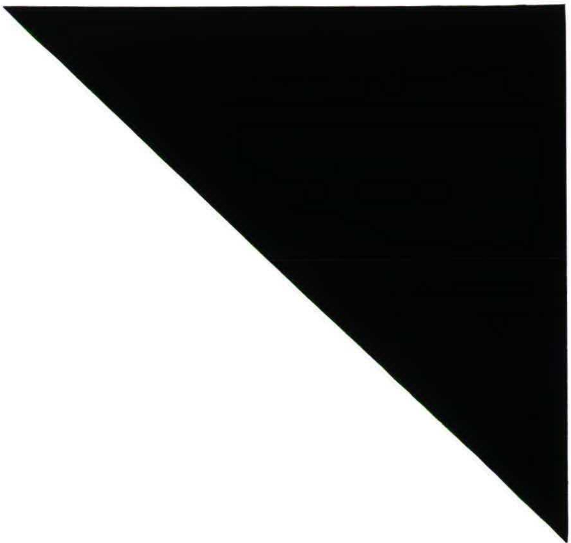

On the Representation of Admissible Rationing Schemes by Rationing Functions

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On the Representation of Admissible Rationing Schemes by Rationing Functions [‡] [§]

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Abstract

In the literature on general equilibrium models with price rigidities and rationing, usually not all rationing schemes are allowed. Examples given in the literature concern uniform rationing, proportional rationing, rationing determined by market share, or rationing determined by priority. The rationing system specifies all admissible rationing schemes. In the literature it is usually assumed that the rationing system can be specified by means of a rationing function. In this paper necessary and sufficient conditions are given for the representation of the rationing system by means of a rationing function.

1 Introduction

In the literature concerning general equilibrium models with price rigidities and rationing, it is assumed that the market mechanism specifies the price system and the rationing scheme of every consumer. The rationing scheme determines the maximal amount a consumer is allowed to supply and to demand of every commodity. However, in general not all rationing schemes are generated by the market mechanism in the economy. Sometimes rationing schemes are required to be uniform for all consumers, sometimes they depend on the amount of initial endowments owned by the various consumers, in other cases they are determined according to some priority system. The rationing system is defined as the set of all admissible rationing schemes. Usually, the rationing system is modelled in the literature as being the range of a so-called rationing function. Very general specifications of rationing functions have been given for instance in Laroque and Polemarchakis (1978), Weddepohl (1987), Herings (1992), and Movshovich (1994). Modelling the set of admissible rationing schemes by means of the range of a rationing function is usually much more difficult than directly specifying the set of admissible rationing schemes, but is much easier when giving a proof of the existence of an equilibrium. Therefore, necessary and sufficient conditions are given in this paper for the representation of a rationing system by means of a rationing function.

This paper is organized as follows. In Section 2 some examples of rationing systems frequently used in the literature are described by specifying directly the set of admissible rationing schemes. This shows the convenience of this approach to model the rationing schemes allowed in the economy. Furthermore, some interesting properties of rationing systems are given, like flexibility, market independence, connectedness, closedness, weak monotonicity, and monotonicity. Moreover, it is specified when two rationing schemes should be considered as being equivalent. In Section 3 the same examples as given in Section 2 are treated, but now the set of admissible rationing schemes is specified as being the range of a rationing function. Furthermore, some interesting properties of rationing functions are given, like flexibility, market independence, continuity, weak monotonicity, and monotonicity. Finally, in Section 4 some representation results are given, specifying necessary and sufficient conditions for representing rationing systems by means of rationing functions.

2 Rationing Systems

In the following, for $k \in \mathbb{N}$, define $I_k = \{1, \ldots, k\}$, $Q^k = \{q \in \mathbb{R}^k \mid 0 \leq q_j \leq 1, \forall j \in I_k\}$, let 0^k be a k-dimensional vector of zeroes, let 1^k be a k-dimensional vector of ones, let $-\infty^k$ be a k-dimensional vector with every component equal to $-\infty$, and let ∞^k be a k-dimensional vector with every component equal to $+\infty$. The set of extended real numbers is denoted by $\mathbb{R}^*$ and the k-dimensional Cartesian product of this set by $\mathbb{R}^{*k}$. If $x^1, x^2 \in \mathbb{R}^{*k}$, then $x^1 \leq x^2$ means $x^1_j \leq x^2_j$, $\forall j \in I_k$, $x^1 < x^2$ means $x^1 \leq x^2$ and there exists $j \in I_k$ such that $x^1_j < x^2_j$, and $x^1 \ll x^2$ means $x^1_j < x^2_j$, $\forall j \in I_k$. Similarly, $\geq, >$, and $\gg$ are defined. The set $\{x \in \mathbb{R}^{*k} \mid x \geq 0^k\}$ is denoted by $\mathbb{R}^{*k}_+$ and the set $\{x \in \mathbb{R}^{*k} \mid x \gg 0^k\}$ is denoted by $\mathbb{R}^{*k}_{++}$. The sets $\mathbb{R}^k_+$ and $\mathbb{R}^k_{++}$ are defined similarly.

An exchange economy with price rigidities is defined by $\mathcal{E} = ((X^i, \preceq^i, \omega^i)_{i \in I_M}, P)$. There is a finite number of consumers, say M, indexed by $i \in I_M$, and a finite number of commodities, say N, indexed by $j \in I_N$. For every $i \in I_M$, consumer i is characterized by a consumption set $X^i \subset \mathbb{R}^N$, a preference ordering $\preceq^i$ on X^i, and an initial endowment $\omega^i \in \mathbb{R}^N$. The set of admissible price systems is denoted by P. The total initial endowments, denoted by $\tilde{\omega}$, are defined by $\tilde{\omega} = \sum_{i \in I_M} \omega^i$. Given consumption sets X^i, $\forall i \in I_M$, the set $\prod_{i \in I_M} X^i$ is denoted by X, and if $x = (x^1, \ldots, x^M)$ is an element of X, then $x_j = (x^1_j, \ldots, x^M_j)^\top$, $\forall j \in I_N$. Given initial endowments ω^i, $\forall i \in I_M$, $\omega = (\omega^1, \ldots, \omega^M)$ and $\omega_j = (\omega^1_j, \ldots, \omega^M_j)^\top$, $\forall j \in I_N$.

The set of admissible price systems may exclude every Walrasian equilibrium price system, so it is possible that at every $p \in P$ supply is not equal to demand. Therefore, following Drèze (1975), the description of the market mechanism has to be extended in the sense that the information transmitted by the market mechanism is not only the price system, but also the maximal amount a consumer is allowed to supply of every commodity, called the rationing scheme on supply, and the maximal amount a consumer is allowed to demand of every commodity, called the rationing scheme on demand. The rationing scheme on supply is denoted by $l = (l^1, \ldots, l^M) \in \prod_{i \in I_M} -\mathbb{R}^{*N}_+$ and the rationing scheme on demand by $L = (L^1, \ldots, L^M) \in \prod_{i \in I_M} \mathbb{R}^{*N}_+$. The pair (l, L) is called the rationing scheme. For every consumer $i \in I_M$, l^i is called the rationing scheme on the supply of consumer i, L^i is called the rationing scheme on the demand of consumer i, and the pair (l^i, L^i) is called the rationing scheme of consumer i. For every $j \in I_N$, $l_j = (l^1_j, \ldots, l^M_j)^\top$ is called the rationing scheme on supply on the market of commodity $j \in I_N$, $L_j = (L^1_j, \ldots, L^M_j)^\top$ is called the rationing scheme on demand on the market of commodity j, and the pair (l_j, L_j) is called the rationing scheme on the market of commodity j.

For every consumer $i \in I_M$, define the budget set of consumer i at a price system $p \in P$ and a rationing scheme $(l^i, L^i) \in -\mathbb{R}^{*N}_+ \times \mathbb{R}^{*N}_+$, denoted by $\beta^i(p, l^i, L^i)$, as the set of consumption bundles in the consumption set of consumer i satisfying the constraints imposed by the rationing scheme of consumer i and being such that the value of these consumption bundles does not exceed $p \cdot \omega^i$, so

$$\beta^i(p, l^i, L^i) = \left\{ x^i \in X^i \,\middle|\, p \cdot x^i \leq p \cdot \omega^i \text{ and } l^i \leq x^i - \omega^i \leq L^i \right\}.$$

Notice that the requirement that $l^i \in -\mathbb{R}^{*N}_+$ and $L^i \in \mathbb{R}^{*N}_+$ implies that only voluntary trading takes place. No consumer can be forced to supply or to demand at least a certain amount of a commodity. In case $l^i = -\infty^N$ and $L^i = +\infty^N$ the definition of the budget set is equal to the usual one. Therefore, the description of the market mechanism given here is more general than the usual description.

The description of the economic system is extended in this section by the specification of a set of admissible rationing schemes, called the rationing system. The rationing system is given by the pair of sets $(\dot{l}, \dot{L})$, where $\dot{l} \subset -\mathbb{R}_+^{*MN}$ and $\dot{L} \subset \mathbb{R}_+^{*MN}$, specifying all admissible rationing schemes. The set $\dot{l}$, called the rationing system on supply, specifies all admissible rationing schemes on supply and the set $\dot{L}$, called the rationing system on demand, specifies all admissible rationing schemes on demand. The market mechanism is assumed to specify a price system $p \in P$ and a rationing scheme $(l, L) \in \dot{l} \times \dot{L}$.

A consumer $i \in I_M$ is assumed to take the price system $p \in P$ and the rationing scheme $(l^i, L^i) \in -\mathbb{R}_+^{*N} \times \mathbb{R}_+^{*N}$ as given, and to choose a best element of $\beta^i(p, l^i, L^i)$ for $\preceq^i$. Define, for every consumer $i \in I_M$, for every $(p, l^i, L^i) \in P \times -\mathbb{R}_+^{*N} \times \mathbb{R}_+^{*N}$, the set $\delta^i(p, l^i, L^i)$ as the set of consumption bundles being best elements of $\beta^i(p, l^i, L^i)$ for $\preceq^i$, i.e.,

$$\delta^i(p, l^i, L^i) = \left\{ \overline{x}^i \in \beta^i(p, l^i, L^i) \,\middle|\, \overline{x}^i \succeq^i x^i,\ \forall x^i \in \beta^i(p, l^i, L^i) \right\}.$$

The following definition of a constrained equilibrium generalizes the definitions used in the literature.

Definition 2.1 (Constrained equilibrium)
A constrained equilibrium of the economy $\mathcal{E} = ((X^i, \preceq^i, \omega^i)_{i \in I_M}, P, (\dot{l}, \dot{L}))$ *is an element*

$$(p^*, l^*, L^*, x^*) \in P \times \dot{l} \times \dot{L} \times X$$

satisfying

1. *for every consumer* $i \in I_M$, $x^{*i} \in \delta^i(p^*, l^{*i}, L^{*i})$,
2. $\sum_{i \in I_M} x^{*i} - \sum_{i \in I_M} \omega^i = 0^N$,
3. *for every commodity* $j \in I_N$, $x_j^{*i'} - \omega_j^{i'} = l_j^{*i'}$ *for some consumer* $i' \in I_M$ *implies* $x_j^{*i} - \omega_j^i < L_j^{*i}$, $\forall i \in I_M$, *and* $x_j^{*i'} - \omega_j^{i'} = L_j^{*i'}$ *for some consumer* $i' \in I_M$ *implies* $x_j^{*i} - \omega_j^i > l_j^{*i}$, $\forall i \in I_M$.

Moreover, some additional requirements are usually specified in the literature, like the assumption that there is no rationing on the market of a numeraire commodity and the exclusion of certain combinations of rationing schemes and price systems. For instance, demand rationing on a market is usually not allowed if it is possible to raise the price on this market. For a discussion of the equilibrium concept given above, the reader is referred to Drèze (1975).

Now some examples of rationing systems are given.

Example 2.2 (Unrestricted rationing system)
In the unrestricted rationing system every rationing scheme is allowed. The unrestricted rationing system on supply is defined by

$$\dot{l} = -\mathbb{R}_+^{*MN}.$$

The unrestricted rationing system on demand is defined by

$$\dot{L} = \mathbb{R}_+^{*MN}.$$

Example 2.3 (Uniform rationing system)
The uniform rationing system is used in Drèze (1975). The uniform rationing system on supply is defined by

$$\dot{l} = \left\{l \in -\mathbb{R}_+^{*MN} \middle| l^1 = \cdots = l^M\right\}.$$

The uniform rationing system on demand is defined by

$$\dot{L} = \left\{L \in \mathbb{R}_+^{*MN} \middle| L^1 = \cdots = L^M\right\}.$$

Example 2.4 (Proportional rationing system)
The proportional rationing system is used in Kurz (1982). Assume that $\omega_j^i > 0$, $\forall i \in I_M$, $\forall j \in I_N$. The proportional rationing system on supply is defined by

$$\dot{l} = \left\{l \in -\mathbb{R}_+^{*MN} \middle| \forall j \in I_N,\ \exists \lambda_j \in \mathbb{R}_+^*,\ \forall i \in I_M,\ l_j^i = -\lambda_j \omega_j^i\right\}.$$

The proportional rationing system on demand is defined by

$$\dot{L} = \left\{L \in \mathbb{R}_+^{*MN} \middle| \forall j \in I_N,\ \exists \lambda_j \in \mathbb{R}_+^*,\ \forall i \in I_M,\ L_j^i = \lambda_j \omega_j^i\right\}.$$

Example 2.5 (Market share rationing system)
The market share rationing system is used in Weddepohl (1983). For every $j \in I_N$, let numbers $\underline{\alpha}_j^i > 0$, $\forall i \in I_M$, be given such that $\sum_{i \in I_M} \underline{\alpha}_j^i = 1$. The market share rationing system on supply with respect to $\underline{\alpha} = (\underline{\alpha}_1^1, \ldots, \underline{\alpha}_N^M)^\top$ is defined by

$$\dot{l} = \left\{l \in -\mathbb{R}_+^{*MN} \middle| \forall j \in I_N,\ \exists \lambda_j \in \mathbb{R}_+^*,\ \forall i \in I_M,\ l_j^i = -\lambda_j \underline{\alpha}_j^i\right\}.$$

For every $j \in I_N$, let numbers $\overline{\alpha}_j^i > 0$, $\forall i \in I_M$, be given such that $\sum_{i \in I_M} \overline{\alpha}_j^i = 1$. The market share rationing system on demand with respect to $\overline{\alpha} = (\overline{\alpha}_1^1, \ldots, \overline{\alpha}_N^M)^\top$ is defined by

$$\dot{L} = \left\{L \in \mathbb{R}_+^{*MN} \middle| \forall j \in I_N,\ \exists \lambda_j \in \mathbb{R}_+^*,\ \forall i \in I_M,\ L_j^i = \lambda_j \overline{\alpha}_j^i\right\}.$$

Example 2.6 (Priority rationing system)

Among other rationing systems, the priority rationing system is considered in Weddepohl (1987). For every $j \in I_N$, let $\underline{\pi}_j : I_M \to I_M$ be a permutation specifying the order in which consumers are rationed on their supply on the market of commodity j, so, for every $k \in I_M$, if consumer $\underline{\pi}_j(k)$ is rationed on his supply on the market of of commodity j, then the consumers $\underline{\pi}_j(1), \ldots, \underline{\pi}_j(k-1)$ are fully rationed on their supply. The priority rationing system on supply with respect to $\underline{\pi} = (\underline{\pi}_1, \ldots, \underline{\pi}_N)$ is defined by

$$\begin{aligned} \dot{l} \ = \ \Big\{ l \in -\mathbb{R}_+^{*MN} \Big| & \forall k \in I_M \setminus \{1\},\ \forall j \in I_N,\ l_j^{\underline{\pi}_j(k)} > -\infty \Rightarrow l_j^{\underline{\pi}_j(k-1)} = 0, \\ & \forall k \in I_{M-1},\ \forall j \in I_N,\ l_j^{\underline{\pi}_j(k)} < 0 \Rightarrow l_j^{\underline{\pi}_j(k+1)} = -\infty \Big\}. \end{aligned}$$

For every $j \in I_N$, let $\overline{\pi}_j : I_M \to I_M$ be a permutation specifying the order in which consumers are rationed on their demand on the market of commodity j, so, for every $k \in I_M$, if consumer $\overline{\pi}_j(k)$ is rationed on his demand on the market of commodity j, then the consumers $\overline{\pi}_j(1), \ldots, \overline{\pi}_j(k-1)$ are fully rationed on their demand. The priority rationing system on demand with respect to $\overline{\pi} = (\overline{\pi}_1, \ldots, \overline{\pi}_N)$ is defined by

$$\begin{aligned} \dot{L} \ = \ \Big\{ L \in \mathbb{R}_+^{*MN} \Big| & \forall k \in I_M \setminus \{1\},\ \forall j \in I_N,\ L_j^{\overline{\pi}_j(k)} < +\infty \Rightarrow L_j^{\overline{\pi}_j(k-1)} = 0, \\ & \forall k \in I_{M-1},\ \forall j \in I_N,\ L_j^{\overline{\pi}_j(k)} > 0 \Rightarrow L_j^{\overline{\pi}_j(k+1)} = +\infty \Big\}. \end{aligned}$$

The following assumptions are often made with respect to the rationing system $(\dot{l}, \dot{L})$.

- No rationing on supply is admissible, i.e., $-\infty^{MN} \in \dot{l}$, no rationing on demand is admissible, i.e., $+\infty^{MN} \in \dot{L}$, full rationing on supply is admissible, i.e., $0^{MN} \in \dot{l}$, and full rationing on demand is admissible, i.e., $0^{MN} \in \dot{L}$. The rationing system on supply is flexible, i.e., $\{-\infty^{MN}, 0^{MN}\} \subset \dot{l}$. The rationing system on demand is flexible, i.e., $\{0^{MN}, +\infty^{MN}\} \subset \dot{L}$. The rationing system is flexible, i.e., both the rationing system on supply and the rationing system on demand is flexible. Notice that the rationing systems of Example 2.2, Example 2.3, Example 2.4, Example 2.5, and Example 2.6 are all flexible.

- The rationing system on supply is market independent, i.e., there exist subsets $\dot{l}_j$, $\forall j \in I_N$, of $-\mathbb{R}_+^{*M}$ such that $l \in \dot{l}$ if and only if $l_j \in \dot{l}_j$, $\forall j \in I_N$. The rationing system on demand is market independent, i.e., there exist subsets $\dot{L}_j$, $\forall j \in I_N$, of $\mathbb{R}_+^{*M}$ such that $L \in \dot{L}$ if and only if $L_j \in \dot{L}_j$, $\forall j \in I_N$. The rationing system is market independent, i.e., both the rationing system on supply and the rationing system on demand is market independent. Notice that the rationing systems of Example 2.2, Example 2.3, Example 2.4, Example 2.5, and Example 2.6 are all market independent.

- The rationing system on supply is connected, i.e., the set $\dot{l}$ is connected in $-\mathbb{R}_+^{*MN}$, the rationing system on demand is connected, i.e., the set $\dot{L}$ is connected in $\mathbb{R}_+^{*MN}$.

The rationing system is connected, i.e., both the rationing system on supply and the rationing system on demand is connected. Notice that the rationing systems of Example 2.2, Example 2.3, Example 2.4, Example 2.5, and Example 2.6 are all connected.

- The rationing system on supply is closed, i.e., the set $\dot{l}$ is closed in $-\mathbb{R}_+^{*MN}$, the rationing system on demand is closed, i.e., the set $\dot{L}$ is closed in $\mathbb{R}_+^{*MN}$. The rationing system is closed, i.e., both the rationing system on supply and the rationing system on demand is closed. Notice that the rationing systems of Example 2.2, Example 2.3, Example 2.4, Example 2.5, and Example 2.6 are all closed.

- For every rationing scheme on supply $l \in \dot{l}$, for every commodity $j \in I_N$, let the set $I_j^{-\infty}(l)$ be defined by $I_j^{-\infty}(l) = \{i \in I_M \mid l_j^i = -\infty\}$ and let the integer $i_j^{-\infty}(l)$ be defined by $i_j^{-\infty}(l) = \# I_j^{-\infty}(l)$. The rationing system on supply is weakly monotonic, i.e., if $\bar{l}, \hat{l} \in \dot{l}$, then, for every $j \in I_N$, $\bar{l}_j = \hat{l}_j$, or $I_j^{-\infty}(\bar{l}) = I_j^{-\infty}(\hat{l})$ and $\sum_{i \in I_M \setminus I_j^{-\infty}(\bar{l})} \bar{l}_j^i \neq \sum_{i \in I_M \setminus I_j^{-\infty}(\hat{l})} \hat{l}_j^i$, or $I_j^{-\infty}(\bar{l})$ is a proper subset of $I_j^{-\infty}(\hat{l})$, or $I_j^{-\infty}(\hat{l})$ is a proper subset of $I_j^{-\infty}(\bar{l})$. Moreover, some limit property is needed for weak monotonicity. Let $((l)^n)_{n \in \mathbb{N}}$ be a sequence in $\dot{l}$ converging to some $\bar{l} \in \dot{l}$, where, for every $j \in I_N$, for every $n \in \mathbb{N}$, $I_j^{-\infty}((l)^n)$ is a proper subset of $I_j^{-\infty}(\bar{l})$. Then, for every $j \in I_N$, for every $l \in \dot{l}$, $i_j^{-\infty}(l) \leq i_j^{-\infty}((l)^n)$ for some $n \in \mathbb{N}$, or $I_j^{-\infty}(l) = I_j^{-\infty}(\bar{l})$ and $\sum_{i \in I_M \setminus I_j^{-\infty}(l)} l_j^i \leq \sum_{i \in I_M \setminus I_j^{-\infty}(\bar{l})} \bar{l}_j^i$, or $i_j^{-\infty}(l) > i_j^{-\infty}(\bar{l})$. For every rationing scheme on demand $L \in \dot{L}$, for every commodity $j \in I_N$, define the set $I_j^{+\infty}(L)$ by $I_j^{+\infty}(L) = \{i \in I_M \mid L_j^i = +\infty\}$ and define the integer $i_j^{+\infty}(L)$ by $i_j^{+\infty}(L) = \# I_j^{+\infty}(L)$. The rationing system on demand is weakly monotonic, i.e., if $\bar{L}, \hat{L} \in \dot{L}$, then, for every $j \in I_N$, $\bar{L}_j = \hat{L}_j$, or $I_j^{+\infty}(\bar{L}) = I_j^{+\infty}(\hat{L})$ and $\sum_{i \in I_M \setminus I_j^{+\infty}(\bar{L})} \bar{L}_j^i \neq \sum_{i \in I_M \setminus I_j^{+\infty}(\hat{L})} \hat{L}_j^i$, or $I_j^{+\infty}(\bar{L})$ is a proper subset of $I_j^{+\infty}(\hat{L})$, or $I_j^{+\infty}(\hat{L})$ is a proper subset of $I_j^{+\infty}(\bar{L})$. Moreover, let $((L)^n)_{n \in \mathbb{N}}$ be a sequence in $\dot{L}$ converging to some $\bar{L} \in \dot{L}$, where, for every $j \in I_N$, for every $n \in \mathbb{N}$, $I_j^{+\infty}((L)^n)$ is a proper subset of $I_j^{+\infty}(\bar{L})$. Then, for every $j \in I_N$, for every $L \in \dot{L}$, $i_j^{+\infty}(L) \leq i_j^{+\infty}((L)^n)$ for some $n \in \mathbb{N}$, or $I_j^{+\infty}(L) = I_j^{+\infty}(\bar{L})$ and $\sum_{i \in I_M \setminus I_j^{+\infty}(L)} L_j^i \geq \sum_{i \in I_M \setminus I_j^{+\infty}(\bar{L})} \bar{L}_j^i$, or $i_j^{+\infty}(L) > i_j^{+\infty}(\bar{L})$. The rationing system is weakly monotonic, i.e., both the rationing system on supply and the rationing system on demand is weakly monotonic. The rationing system of Example 2.2 is not weakly monotonic, but the rationing systems of Example 2.3, Example 2.4, Example 2.5, and Example 2.6 are weakly monotonic.

- The rationing system on supply is monotonic, i.e., if $\bar{l}, \hat{l} \in \dot{l}$, then, for every $j \in I_N$, $\bar{l}_j \leq \hat{l}_j$ or $\bar{l}_j \geq \hat{l}_j$. The rationing system on demand is monotonic, i.e., if $\bar{L}, \hat{L} \in \dot{L}$, then, for every $j \in I_N$, $\bar{L}_j \leq \hat{L}_j$ or $\bar{L}_j \geq \hat{L}_j$. The rationing system is monotonic, i.e., both the rationing system on supply and the rationing system on demand is monotonic.

It is easily verified that a monotonic rationing system is weakly monotonic, so the rationing system of Example 2.2 is not monotonic. The rationing systems of Example 2.3, Example 2.4, Example 2.5, and Example 2.6 are monotonic.

The assumptions that no rationing on supply and no rationing on demand is admissible and that the rationing system is market independent are so basic, that they hardly can be considered as assumptions. This is also true for the assumption of connectedness. The weak monotonicity assumption is also reasonable. This assumption corresponds to the idea that if two rationing schemes on supply on a market $j \in I_N$ are given, say $\bar{l}_j$ and $\hat{l}_j$, then the consumers together are allowed to supply less at $\bar{l}_j$ than at $\hat{l}_j$, or $\bar{l}_j = \hat{l}_j$, or the consumers together are allowed to supply more at $\bar{l}_j$ than at $\hat{l}_j$ on market j. Similarly, if two rationing schemes on demand on a market $j \in I_N$ are given, say $\bar{L}_j$ and $\hat{L}_j$, then the consumers together are allowed to demand less at $\bar{L}_j$ than at $\hat{L}_j$, or $\bar{L}_j = \hat{L}_j$, or the consumers together are allowed to supply more at $\bar{L}_j$ than at $\hat{L}_j$ on market j. The monotonicity assumption is similar, but then for every rationing scheme on demand on a market $j \in I_n$ every consumer is allowed to supply less on market j at $\bar{l}_j$ than at $\hat{l}_j$, or $\bar{l}_j = \hat{l}_j$, or every consumer is allowed to supply more on market j at $\bar{l}_j$ than at $\hat{l}_j$. Similar remarks apply to monotonic rationing systems on demand.

Not all different rationing schemes are different from the point of view of the consumer. The following definition captures this idea.

Definition 2.7 (Equivalent rationing schemes)
The rationing scheme on supply $\bar{l}$ is equivalent to the rationing scheme on supply $\hat{l}$, denoted by $\bar{l} \sim \hat{l}$, if, for every $i \in I_M$, for every $j \in I_N$, $\hat{l}^i_j \geq -\omega^i_j$ implies $\bar{l}^i_j = \hat{l}^i_j$, and $\hat{l}^i_j < -\omega^i_j$ implies $\bar{l}^i_j < -\omega^i_j$. The rationing scheme on demand $\bar{L}$ is equivalent to the rationing scheme on demand $\hat{L}$, denoted by $\bar{L} \sim \hat{L}$, if, for every $i \in I_M$, for every $j \in I_N$, $\hat{L}^i_j \leq \tilde{\omega}_j - \omega^i_j$ implies $\bar{L}^i_j = \hat{L}^i_j$, and $\hat{L}^i_j > \tilde{\omega}_j - \omega^i_j$ implies $\bar{L}^i_j > \tilde{\omega}_j - \omega^i_j$. The rationing scheme $(\bar{l}, \bar{L})$ is equivalent to the rationing scheme $(\hat{l}, \hat{L})$, denoted by $(\bar{l}, \bar{L}) \sim (\hat{l}, \hat{L})$, if $\bar{l} \sim \hat{l}$ and $\bar{L} \sim \hat{L}$.

It is easily verified that the binary relation on the set of all possible rationing schemes on supply, $-\mathbb{R}^{*MN}_+$, induced by $\sim$, is an equivalence relation, the binary relation on the set of all possible rationing schemes on demand, $\mathbb{R}^{*MN}_+$, induced by $\sim$, is an equivalence relation, and the binary relation on the set of all possible rationing schemes, $-\mathbb{R}^{*MN}_+ \times \mathbb{R}^{*MN}_+$, induced by $\sim$, is an equivalence relation.

Two equivalent rationing schemes may induce different consumer behaviour. However, if, for every consumer $i \in I_M$, $\preceq^i$ is complete, transitive, and convex, and the consumption set X^i is convex, then this is not the case in a constrained equilibrium as is easily verified.

Definition 2.8 (Equivalent rationing systems)
The rationing system on supply $\bar{\bar{l}}$ is equivalent to the rationing system on supply $\hat{\hat{l}}$, denoted

by $\bar{\dot{l}} \sim \hat{\dot{l}}$*, if, for every* $\bar{l} \in \bar{\dot{l}}$*, there exists* $\hat{l} \in \hat{\dot{l}}$ *such that* $\bar{l} \sim \hat{l}$*, and for every* $\hat{l} \in \hat{\dot{l}}$ *there exists* $\bar{l} \in \bar{\dot{l}}$ *such that* $\hat{l} \sim \bar{l}$*. The rationing system on demand* $\bar{\dot{L}}$ *is equivalent to the rationing system on demand* $\hat{\dot{L}}$*, denoted by* $\bar{\dot{L}} \sim \hat{\dot{L}}$*, if, for every* $\bar{L} \in \bar{\dot{L}}$*, there exists* $\hat{L} \in \hat{\dot{L}}$ *such that* $\bar{L} \sim \hat{L}$*, and, for every* $\hat{L} \in \hat{\dot{L}}$*, there exists* $\bar{L} \in \bar{\dot{L}}$ *such that* $\hat{L} \sim \bar{L}$*. The rationing system* $(\bar{\dot{l}}, \bar{\dot{L}})$ *is equivalent to the rationing system* $(\hat{\dot{l}}, \hat{\dot{L}})$*, denoted by* $(\bar{\dot{l}}, \bar{\dot{L}}) \sim (\hat{\dot{l}}, \hat{\dot{L}})$*, if both* $\bar{\dot{l}} \sim \hat{\dot{l}}$ *and* $\bar{\dot{L}} \sim \hat{\dot{L}}$*.*

It is easily verified that the binary relation on the set of all possible rationing systems on supply, $2^{-\mathbf{R}_{+}^{MN}}$, induced by $\sim$, is an equivalence relation, the binary relation on the set of all possible rationing systems on demand, $2^{\mathbf{R}_{+}^{MN}}$, induced by $\sim$, is an equivalence relation, and the binary relation on the set of all possible rationing systems, $2^{-\mathbf{R}_{+}^{MN}} \times 2^{\mathbf{R}_{+}^{MN}}$, induced by $\sim$, is an equivalence relation.

3 Rationing Functions

The rationing system on supply is often defined as being the range of a function $\tilde{l} : S \to -\mathbb{R}_{+}^{MN}$ defined on some subset S of $\mathbb{R}^{N}$. Often, $S = \mathbb{R}_{+}^{N}$ or $S = Q^{N}$. From now it will be assumed in this chapter that $S = Q^{N}$. The function $\tilde{l}$ is called the rationing function on supply. For every $i \in I_M$, for every $j \in I_N$, component $(i-1)N + j$ of $\tilde{l}$ is denoted by $\tilde{l}_j^i$. Moreover, $\tilde{l}^i = (\tilde{l}_1^i, \ldots, \tilde{l}_N^i)^\top$, $\forall i \in I_M$, and $\tilde{l}_j = (\tilde{l}_j^1, \ldots, \tilde{l}_j^M)^\top$, $\forall j \in I_N$. Given $q \in Q^N$, the vector $\tilde{l}^i(q)$ yields a rationing scheme on the supply of consumer $i \in I_M$. Similarly, the rationing system on demand is often defined as being the range of a function $\tilde{L} : Q^N \to \mathbb{R}_{+}^{MN}$, called the rationing function on demand. For every $i \in I_M$, for every $j \in I_N$, component $(i-1)N + j$ of $\tilde{L}$ is denoted by $\tilde{L}_j^i$. Moreover, $\tilde{L}^i = (\tilde{L}_1^i, \ldots, \tilde{L}_N^i)^\top$ and $\tilde{L}_j = (\tilde{L}_j^1, \ldots, \tilde{L}_j^M)^\top$. Given $q \in Q^N$, the vector $\tilde{L}^i(q)$ yields a rationing scheme on the demand of consumer $i \in I_M$. The pair $(\tilde{l}, \tilde{L})$ is called a rationing function. Notice that the image of the rationing function $(\tilde{l}, \tilde{L})$ is a subset of $-\mathbf{R}_{+}^{MN} \times \mathbb{R}_{+}^{MN}$.

Definition 3.1 (Representation by a rationing function)
The rationing system on supply $\dot{l}$ *is represented by a rationing function on supply* $\tilde{l}$ *if* $\dot{l} \sim \tilde{l}(Q^N)$*. The rationing system on demand* $\dot{L}$ *is represented by a rationing function on demand* $\tilde{L}$ *if* $\dot{L} \sim \tilde{L}(Q^N)$*. The rationing system* $(\dot{l}, \dot{L})$ *is represented by a rationing function* $(\tilde{l}, \tilde{L})$ *if* $(\dot{l}, \dot{L}) \sim (\tilde{l}(Q^N), \tilde{L}(Q^N))$*.*

Now some examples of rationing functions are given, representing the rationing systems of Example 2.2, Example 2.3, Example 2.4, Example 2.5, and Example 2.6, respectively. In the examples it is assumed that $\omega_j^i \geq 0$, $\forall i \in I_M$, $\forall j \in I_N$, and $\tilde{\omega} \in \mathbb{R}_{++}^{N}$.

Example 3.2 (Unrestricted rationing function)
First, a continuous function $h^M : [0,1] \to Q^M$ will be constructed. Let f^1 be a path

from $[0,1]$ into $\{q \in \mathbb{R}^3_+ \mid \sum_{j \in I_3} q_j = 1\}$ being surjective. The existence of such a path f^1 is guaranteed by a theorem of Peano about the existence of a space-filling curve, see Armstrong (1983), Section 2.3. It is not difficult to construct a continuous function $f^2 : \{q \in \mathbb{R}^3_+ \mid \sum_{j \in I_3} q_j = 1\} \rightarrow Q^2$ being surjective. Then the function $f^2 \circ f^1 : [0,1] \rightarrow Q^2$ is a continuous function being surjective. Using the function $f^2 \circ f^1$ it is not difficult to construct a continuous function $g^2 : [0,1] \rightarrow Q^2$ being surjective and having the additional property that $g^2(0) = (0,0)^\top$ and $g^2(1) = (1,1)^\top$. For every $n \in \mathbb{N} \setminus \{1,2\}$, define the function $g^n : Q^{n-1} \rightarrow Q^n$ by $g^n(q) = (g^{n-1}(q_1, \ldots, q_{n-2})^\top, q_{n-1})^\top$, $\forall q \in Q^{n-1}$. Clearly, the function g^n is continuous and surjective for every $n \geq 3$. Define the function $h^1 : [0,1] \rightarrow Q^1$ by $h^1(q) = q$, $\forall q \in Q^1$. For every $n \in \mathbb{N} \setminus \{1\}$, define the function $h^n : [0,1] \rightarrow Q^n$ by $h^n(q) = g^n(h^{n-1}(q))$, $\forall q \in [0,1]$. Then the function h^n is continuous and surjective for every $n \in \mathbb{N}$.

Let some $\varepsilon \in \mathbb{R}_{++}$ be given. Notice that if $\dot{l}$ is the unrestricted rationing system on supply, then $\dot{l} \sim \prod_{i \in I_M} \prod_{j \in I_N} [\min(\{-\omega^1_j, \ldots, -\omega^M_j\}) - \varepsilon, 0]$. Therefore, the unrestricted rationing system on supply is represented by the rationing function on supply $\tilde{l} : Q^N \rightarrow -\mathbb{R}^{MN}_+$ obtained by defining, for every $i \in I_M$, for every $j \in I_N$,

$$\tilde{l}^i_j(q) = \left(\min(\{-\omega^1_j, \ldots, -\omega^M_j\}) - \varepsilon\right) h^M_i(q_j), \ \forall q \in Q^N.$$

If $\dot{L}$ is the unrestricted rationing system on demand, then $\dot{L} \sim \prod_{i \in I_M} \prod_{j \in I_N} [0, \tilde{\omega}_j + \varepsilon]$. Therefore, the unrestricted rationing system on demand is represented by a rationing function on demand $\tilde{L} : Q^N \rightarrow \mathbb{R}^{MN}_+$ obtained by defining, for every $i \in I_M$, for every $j \in I_N$,

$$\tilde{L}^i_j(q) = (\tilde{\omega}_j + \varepsilon)\, h^M_i(q_j), \ \forall q \in Q^N.$$

Example 3.3 (Uniform rationing function)

Let some $\varepsilon \in \mathbb{R}_{++}$ be given. The uniform rationing system on supply is represented by the rationing function on supply $\tilde{l} : Q^N \rightarrow -\mathbb{R}^{MN}_+$ obtained by defining, for every $i \in I_M$, for every $j \in I_N$,

$$\tilde{l}^i_j(q) = \left(\min(\{-\omega^1_j, \ldots, -\omega^M_j\}) - \varepsilon\right) q_j, \ \forall q \in Q^N.$$

The uniform rationing system on demand is represented by the rationing function on demand $\tilde{L} : Q^N \rightarrow \mathbb{R}^{MN}_+$ obtained by defining, for every $i \in I_M$, for every $j \in I_N$,

$$\tilde{L}^i_j(q) = (\tilde{\omega}_j + \varepsilon)\, q_j, \ \forall q \in Q^N.$$

Example 3.4 (Proportional rationing function)

Assume that $\omega^i_j > 0$, $\forall i \in I_M$, $\forall j \in I_N$. Let some $\varepsilon \in \mathbb{R}_{++}$ be given. The proportional rationing system on supply is represented by the rationing function on supply $\tilde{l} : Q^N \rightarrow -\mathbb{R}^{MN}_+$ obtained by defining, for every $i \in I_M$, for every $j \in I_N$,

$$\tilde{l}^i_j(q) = -(1 + \varepsilon)\omega^i_j q_j, \ \forall q \in Q^N.$$

The proportional rationing system on demand is represented by the rationing function on demand $\tilde{L} : Q^N \to \mathbb{R}_+^{MN}$ obtained by defining, for every $i \in I_M$, for every $j \in I_N$,

$$\tilde{L}_j^i(q) = \frac{\tilde{\omega}_j}{\min(\{\omega_j^1, \ldots, \omega_j^M\})} \omega_j^i q_j, \ \forall q \in Q^N.$$

Example 3.5 (Market share rationing function)
Let some $\varepsilon \in \mathbb{R}_{++}$ be given. For every $j \in I_N$, let numbers $\underline{\alpha}_j^i > 0$, $\forall i \in I_M$, be given such that $\sum_{i \in I_M} \underline{\alpha}_j^i = 1$. The proportional rationing system on supply with respect to $\underline{\alpha} = (\underline{\alpha}_1^1, \ldots, \underline{\alpha}_N^M)^\top$ is represented by the rationing function on supply $\tilde{l} : Q^N \to -\mathbb{R}_+^{MN}$ obtained by defining, for every $i \in I_M$, for every $j \in I_N$,

$$\tilde{l}_j^i(q) = \underline{\alpha}_j^i \left(\min \left(\left\{ -\frac{\omega_j^1}{\underline{\alpha}_j^1}, \ldots, -\frac{\omega_j^M}{\underline{\alpha}_j^M} \right\} \right) - \varepsilon \right) q_j, \ \forall q \in Q^N.$$

For every $j \in I_N$, let numbers $\overline{\alpha}_j^i > 0$, $\forall i \in I_M$, be given such that $\sum_{i \in I_M} \overline{\alpha}_j^i = 1$. The proportional rationing system on demand with respect to $\overline{\alpha} = (\overline{\alpha}_1^1, \ldots, \overline{\alpha}_N^M)^\top$ is represented by the rationing function on demand $\tilde{L} : Q^N \to \mathbb{R}_+^{MN}$ obtained by defining, for every $i \in I_M$, for every $j \in I_N$,

$$\tilde{L}_j^i(q) = \overline{\alpha}_j^i \frac{\tilde{\omega}_j + \varepsilon}{\min \left(\left\{ \overline{\alpha}_j^1, \ldots, \overline{\alpha}_j^M \right\} \right)} q_j, \ \forall q \in Q^N.$$

Example 3.6 (Priority rationing function)
Let some $\varepsilon \in \mathbb{R}_{++}$ be given. For every $j \in I_N$, let $\underline{\pi}_j : I_M \to I_M$ be a permutation specifying the order in which consumers are rationed on their supply on the market of commodity j. The priority rationing system on supply with respect to $\underline{\pi} = (\underline{\pi}_1, \ldots, \underline{\pi}_N)$ is represented by the rationing function on supply $\tilde{l} : Q^N \to -\mathbb{R}_+^{MN}$ obtained by defining, for every $i \in I_M$, for every $j \in I_N$,

$$\tilde{l}_j^i(q) = \left(\min(\{-\omega_j^1, \ldots, -\omega_j^M\}) - \varepsilon \right) \max \left(\left\{ \underline{\pi}_j^{-1}(i) - M + M q_j, 0 \right\} \right), \ \forall q \in Q^N.$$

For every $j \in I_N$, let $\overline{\pi}_j : I_M \to I_M$ be a permutation specifying the order in which consumers are rationed on their demand on the market of commodity j. The priority rationing system on demand with respect to $\overline{\pi} = (\overline{\pi}_1, \ldots, \overline{\pi}_N)$ is represented by the rationing function on demand $\tilde{L} : Q^N \to \mathbb{R}_+^{MN}$ obtained by defining, for every $i \in I_M$, for every $j \in I_N$,

$$\tilde{L}_j^i(q) = (\tilde{\omega}_j + \varepsilon) \max \left(\left\{ \overline{\pi}_j^{-1}(i) - M + M q_j, 0 \right\} \right), \ \forall q \in Q^N.$$

The following assumptions are often made with respect to the rationing function on supply $\tilde{l} : Q^N \to -\mathbb{R}_+^{MN}$ and the rationing function on demand $\tilde{L} : Q^N \to \mathbb{R}_+^{MN}$.

- It holds that $\tilde{l}(1^N) \ll -\omega$, so there is a rationing scheme on supply equivalent to $-\infty^{MN}$. For every $i \in I_M$, $\tilde{L}^i(1^N) \gg \tilde{\omega} - \omega^i$, so there is a rationing scheme on demand equivalent to $+\infty^{MN}$. It holds that $\tilde{l}(0^N) = 0^{MN}$, so full rationing on supply is admissible, and it holds that $\tilde{L}(0^N) = 0^{MN}$, so full rationing on demand is admissible. The rationing function on supply is flexible, i.e., $\tilde{l}(0^N) = 0^{MN}$ and $\tilde{l}(1^N) \ll -\omega$. The rationing function on demand is flexible, i.e., $\tilde{L}(0^N) = 0^{MN}$ and $\tilde{L}^i(1^N) \gg \tilde{\omega} - \omega^i$, $\forall i \in I_M$.

- The rationing function on supply is market independent, i.e., for every $j \in I_N$, for every $\overline{q}, \hat{q} \in Q^N$ it holds that $\tilde{l}_j(\overline{q}) = \tilde{l}_j(\hat{q})$ if $\overline{q}_j = \hat{q}_j$. The rationing function on demand is market independent, i.e., for every $\overline{q}, \hat{q} \in Q^N$, for every $j \in I_N$, it holds that $\tilde{L}_j(q) = \tilde{L}_j(\overline{q})$ if $q_j = \overline{q}_j$.

- The rationing function on supply is continuous, i.e., the function $\tilde{l}$ is continuous. The rationing function on demand is continuous, i.e., the function $\tilde{L}$ is continuous. Sometimes, the continuity assumption will be replaced by the stronger assumption of differentiability, which is clearly of a similar nature.

- The rationing function on supply is weakly monotonic, i.e., for every $q^1, q^2 \in Q^N$, for every $j \in I_N$, if $q^1_j < q^2_j$, then $\sum_{i \in I_M} \tilde{l}^i_j(q^1) > \sum_{i \in I_M} \tilde{l}^i_j(q^2)$. The rationing function on demand is weakly monotonic, i.e., for every $q^1, q^2 \in Q^N$, for every $j \in I_N$, if $q^1_j < q^2_j$, then $\sum_{i \in I_M} \tilde{L}^i_j(q^1) < \sum_{i \in I_M} \tilde{L}^i_j(q^2)$. Sometimes the weak monotonicity assumption will be needed in differentiable form. Then it is required that, for every $j \in I_N$, $\sum_{i \in I_M} \partial_{q_j} \tilde{l}^i_j(\overline{q}) < 0$, $\forall \overline{q} \in Q^N$, and $\sum_{i \in I_M} \partial_{q_j} \tilde{L}^i_j(\overline{q}) > 0$, $\forall \overline{q} \in Q^N$.

- The rationing function on supply is monotonic, i.e., for every $q^1, q^2 \in Q^N$, for every $j \in I_N$, if $q^1_j < q^2_j$, then $\tilde{l}^i_j(q^1) \geq \tilde{l}^i_j(q^2)$, $\forall i \in I_M$, and there exists $i' \in I_M$ such that $\tilde{l}^{i'}_j(q^1) > \tilde{l}^{i'}_j(q^2)$. The rationing function on demand is monotonic, i.e., for every $q^1, q^2 \in Q^N$, for every $j \in I_N$, if $q^1_j < q^2_j$, then $\tilde{L}^i_j(q^1) \leq \tilde{L}^i_j(q^2)$, $\forall i \in I_M$, and there exists $i' \in I_M$ such that $\tilde{L}^{i'}_j(q^1) < \tilde{L}^{i'}_j(q^2)$. Sometimes the monotonicity assumption will be needed in differentiable form. Then it is required that, for every $j \in I_N$, $\partial_{q_j} \tilde{l}_j(\overline{q}) < 0^M$, $\forall \overline{q} \in Q^N$, and $\partial_{q_j} \tilde{L}_j(\overline{q}) > 0^M$, $\forall \overline{q} \in Q^N$.

If an assumption is said to be made with respect to the rationing function $(\tilde{l}, \tilde{L})$, then it is meant that this assumption is made both with respect to $\tilde{l}$ and with respect to $\tilde{L}$.

It is easily verified that all the assumptions mentioned above are satisfied in Example 3.3, Example 3.4, Example 3.5, and Example 3.6, and that all assumptions mentioned above, except the assumptions concerning the weak monotonicity and the monotonicity of the rationing function, are satisfied in Example 3.2.

4 Representation Theorems

The results of this section show that there is an interesting relationship between the assumptions made with respect to the rationing system and the assumptions made with respect to the rationing function.

Theorem 4.1

Let the rationing system on supply $\dot{l}$ be represented by the rationing function on supply $\tilde{l} : Q^N \to -\mathbb{R}_+^{MN}$. If $\tilde{l}$ is flexible, then $\dot{l}$ is equivalent to a flexible rationing system on supply, if $\tilde{l}$ is market independent, then $\dot{l}$ is equivalent to a market independent rationing system on supply, if $\tilde{l}$ is continuous, then $\dot{l}$ is equivalent to a closed and connected rationing system on supply, if $\tilde{l}$ is market independent and weakly monotonic, then $\dot{l}$ is equivalent to a weakly monotonic rationing system on supply, and if $\tilde{l}$ is market independent and monotonic, then $\dot{l}$ is equivalent to a monotonic rationing system on supply.

Proof

Let $\tilde{l}$ be flexible. Then $\tilde{l}(0^N) = 0^{MN} \sim l$ for some $l \in \dot{l}$ since $\dot{l} \sim \tilde{l}(Q^N)$. Moreover, $\tilde{l}(1^N) \ll -\omega$, so $-\infty^{MN} \sim \tilde{l}(1^N) \sim l$ for some $l \in \dot{l}$, using that $\dot{l} \sim \tilde{l}(Q^N)$.

Let $\tilde{l}$ be market independent. Since $\dot{l} \sim \tilde{l}(Q^N)$, it is sufficient to show that the rationing system $\tilde{l}(Q^N)$ is market independent. It will be shown that $l \in \tilde{l}(Q^N)$ if and only if $l_j \in \tilde{l}_j(Q^N)$, $\forall j \in I_N$, thereby showing the market independence of $\tilde{l}(Q^N)$. Clearly, $l \in \tilde{l}(Q^N)$ implies $l_j \in \tilde{l}_j(Q^N)$, $\forall j \in I_N$. Let $l_j \in \tilde{l}_j(Q^N)$, $\forall j \in I_N$, be given. For every $j \in I_N$, there exists $q^j \in Q^N$ such that $l_j = \tilde{l}_j(q^j)$. Define the element q of Q^N by $q_j = q_j^j$, $\forall j \in I_N$. Then $\tilde{l}(q) = l$ by the market independence of $\tilde{l}$, so $l \in \tilde{l}(Q^N)$.

Let $\tilde{l}$ be continuous. Since $\dot{l} \sim \tilde{l}(Q^N)$, it is sufficient to show that the rationing system $\tilde{l}(Q^N)$ is closed and connected. Since Q^N is compact and connected it follows that $\tilde{l}(Q^N)$ is compact and connected in $-\mathbb{R}_+^{MN}$. Therefore, $\tilde{l}(Q^N)$ is also compact and connected in $\mathbb{R}^{*MN}$. Since $\tilde{l}(Q^N)$ is a compact subset of the Hausdorff space $\mathbb{R}^{*MN}$ it is closed in $\mathbb{R}^{*MN}$.

Let $\tilde{l}$ be market independent and weakly monotonic. Since $\dot{l} \sim \tilde{l}(Q^N)$, it is sufficient to show that $\tilde{l}(Q^N)$ is weakly monotonic. Let $\bar{l}, \hat{l} \in \tilde{l}(Q^N)$ be given and let $\bar{q}, \hat{q} \in Q^N$ be such that $\bar{l} = \tilde{l}(\bar{q})$ and $\hat{l} = \tilde{l}(\hat{q})$. Clearly, for every $j \in I_N$, $\bar{l}_j \gg -\infty^M$ and $\hat{l}_j \gg -\infty^M$. For every $j \in I_N$, if $\bar{q}_j = \hat{q}_j$, then $\bar{l}_j = \hat{l}_j$ since $\tilde{l}$ is market independent, and if, without loss of generality, $\bar{q}_j < \hat{q}_j$, then $\sum_{i \in I_M} \bar{l}_j^i > \sum_{i \in I_M} \hat{l}_j^i$ by the weak monotonicity of $\tilde{l}$.

Let $\tilde{l}$ be market independent and monotonic. Since $\dot{l} \sim \tilde{l}(Q^N)$, it is sufficient to show that $\tilde{l}(Q^N)$ is monotonic. Let $\bar{l}, \hat{l} \in \tilde{l}(Q^N)$ be given and let $\bar{q}, \hat{q} \in Q^N$ be such that $\bar{l} = \tilde{l}(\bar{q})$ and $\hat{l} = \tilde{l}(\hat{q})$. For every $j \in I_N$, if $\bar{q}_j = \hat{q}_j$, then $\bar{l}_j = \hat{l}_j$ since $\tilde{l}$ is market independent, and if, without loss of generality, $\bar{q}_j < \hat{q}_j$, then $\bar{l}_j > \hat{l}_j$ by the monotonicity of $\tilde{l}$. Q.E.D.

The proof for the results given in Theorem 4.2 concerning the rationing system on demand is similar to the proof of the corresponding results concerning the rationing system on supply given in Theorem 4.1.

Theorem 4.2

Let the rationing system on demand $\dot{L}$ be represented by the rationing function on demand $\tilde{L} : Q^N \to \mathbb{R}_+^{MN}$. If $\tilde{L}$ is flexible, then $\dot{L}$ is equivalent to a flexible rationing system on demand, if $\tilde{L}$ is market independent, then $\dot{L}$ is equivalent to a market independent rationing system on demand, if $\tilde{L}$ is continuous, then $\dot{L}$ is equivalent to a closed and connected rationing system on demand, if $\tilde{L}$ is market independent and weakly monotonic, then $\dot{L}$ is equivalent to a weakly monotonic rationing system on demand, and if $\tilde{L}$ is market independent and monotonic, then $\dot{L}$ is equivalent to a monotonic rationing system on demand.

The following two results give a converse of Theorem 4.1 and Theorem 4.2.

Theorem 4.3

Let the rationing system on supply $\dot{l}$ be flexible, market independent, closed, and connected. If the rationing system on supply $\dot{l}$ is weakly monotonic, then $\dot{l}$ can be represented by a flexible, market independent, continuous, and weakly monotonic rationing function on supply. If the rationing system on supply $\dot{l}$ is monotonic, then $\dot{l}$ can be represented by a flexible, market independent, continuous, and monotonic rationing function on supply.

Proof

Let $\dot{l}$ be weakly monotonic. A rationing function on supply $\tilde{l} : Q^N \to -\mathbb{R}_+^{MN}$ with the desired properties will be constructed. Since $\dot{l}$ is market independent it holds that there exist subsets $\dot{l}_j$, $\forall j \in I_N$, of $-\mathbb{R}_+^{*MN}$ such that $l \in \dot{l}$ if and only if $l_j \in \dot{l}_j$, $\forall j \in I_N$. Let some $j' \in I_N$ be given. For every $l_{j'} \in \dot{l}_{j'}$, let the set $I^{-\infty}(l_{j'})$ be defined by $I^{-\infty}(l_{j'}) = \{i \in I_M \mid l^i_{j'} = -\infty\}$ and let the integer $i^{-\infty}(l_{j'})$ be defined by $i^{-\infty}(l_{j'}) = \#I^{-\infty}(l_{j'})$. Let the set K be given by $K = \{k \in I^0_M \mid \exists l_{j'} \in \dot{l}_{j'}, i^{-\infty}(l_{j'}) = k\}$. Since $\dot{l}$ is flexible, it holds that $0 \in K$ and $M \in K$. For every $k \in K \setminus \{M\}$, let $\alpha(k) \in -\mathbb{R}_+$ be defined by

$$\alpha(k) = \sup\left(\left\{\textstyle\sum_{i \in I_M \setminus I^{-\infty}(l_{j'})} l^i_{j'} \middle| \, l_{j'} \in \dot{l}_{j'} \text{ and } i^{-\infty}(l_{j'}) = k\right\}\right).$$

Clearly, $\alpha(0) = 0$. Let π be an increasing function from $I_{\#K}$ into K. Notice that π is uniquely determined, $\pi(1) = 0$, and $\pi(\#K) = M$.

Let some $\alpha \in \mathbb{R}$ be given. Let the function $f^\alpha : \{s \in \mathbb{R}^* \mid s \leq \alpha\} \to [0, 1]$ be defined by

$$\begin{aligned} f^\alpha(-\infty) &= 1, \\ f^\alpha(s) &= \tfrac{\alpha - s}{1 + \alpha - s}, \ \forall s \in (\leftarrow, \alpha]. \end{aligned}$$

Notice that $f^\alpha(\alpha) = 0$. Obviously, the inverse of f^α, $(f^\alpha)^{-1} : [0, 1] \to \{s \in \mathbb{R}^* \mid s \leq \alpha\}$, is defined by

$$\begin{aligned} (f^\alpha)^{-1}(t) &= \tfrac{t + \alpha(t-1)}{t-1}, \ \forall t \in [0, 1), \\ (f^\alpha)^{-1}(1) &= -\infty. \end{aligned}$$

Clearly, if $(s^n)_{n\in\mathbb{N}}$ is a sequence in $\{s \in \mathbb{R}^* \mid s \leq \alpha\}$ and s^n converges to some $\bar{s} \in \{s \in \mathbb{R}^* \mid s \leq \alpha\}$, then the sequence $(f^\alpha(s^n))_{n\in\mathbb{N}}$ converges to $f^\alpha(\bar{s})$. Therefore, f^α is a continuous function. Similarly, it can be shown that $(f^\alpha)^{-1}$ is a continuous function.

Now a continuous, injective, and surjective function g is constructed such that with every $l_{j'} \in \dot{l}_{j'}$ a real number of $[0,1]$ is associated. This is achieved by subdividing the unit interval in $\#K = \pi^{-1}(M)$ pieces and constructing the function g such that, for every $l_{j'} \in \dot{l}_{j'}$ with $l_{j'} \neq -\infty^M$, $g(l_{j'}) \in [\frac{\pi^{-1}(i^{-\infty}(l_{j'}))-1}{\pi^{-1}(M)-1}, \frac{\pi^{-1}(i^{-\infty}(l_{j'}))}{\pi^{-1}(M)-1})$, while $\bar{l}_{j'}, \hat{l}_{j'} \in \dot{l}_{j'}$ with $\sum_{i\in I_M\setminus I^{-\infty}(\bar{l}_{j'})} \bar{l}^i_{j'} < \sum_{i\in I_M\setminus I^{-\infty}(\hat{l}_{j'})} \hat{l}^i_{j'}$ implies $g(\bar{l}_{j'}) > g(\hat{l}_{j'})$. Let the function $g : \dot{l}_{j'} \to [0,1]$ be defined by

$$\begin{aligned} g(-\infty^M) &= 1, \\ g(l_{j'}) &= \frac{\pi^{-1}(i^{-\infty}(l_{j'}))-1}{\pi^{-1}(M)-1} + \frac{f^{\alpha(i^{-\infty}(l_{j'}))}\left(\sum_{i\in I_M\setminus I^{-\infty}(l_{j'})} l^i_{j'}\right)}{\pi^{-1}(M)-1}, \ \forall l_{j'} \in \dot{l}_{j'} \setminus \{-\infty^M\}. \end{aligned}$$

Notice that

$$g(0^M) = \frac{\pi^{-1}(0)-1}{\pi^{-1}(M)-1} + \frac{0}{\pi^{-1}(M)-1} = 0.$$

Now it is shown that g is continuous. Let $((l_{j'})^n)_{n\in\mathbb{N}}$ be a sequence in $\dot{l}_{j'}$ converging to some $\bar{l}_{j'} \in \dot{l}_{j'}$. Suppose the sequence $(g((l_{j'})^n))_{n\in\mathbb{N}}$ does not converge to $g(\bar{l}_{j'})$. From the continuity of $f^{\alpha(k)}$, $\forall k \in K \setminus \{M\}$, and since $i^{-\infty}(\bar{l}_{j'}) = M$ implies $\bar{l}_{j'} = -\infty^M$, it follows that if $i^{-\infty}((l_{j'})^n) = i^{-\infty}(\bar{l}_{j'})$, $\forall n \in \mathbb{N}$, then $g((l_{j'})^n) \to g(\bar{l}_{j'})$, a contradiction with the supposition that $(g((l_{j'})^n))_{n\in\mathbb{N}}$ does not converge to $g(\bar{l}_{j'})$. Consequently, without loss of generality, $i^{-\infty}((l_{j'})^n) \neq i^{-\infty}(\bar{l}_{j'})$, $\forall n \in \mathbb{N}$. From the weak monotonicity of $\dot{l}$ it follows that, for every $n \in \mathbb{N}$, $I^{-\infty}((l_{j'})^n)$ is a proper subset of $I^{-\infty}(\bar{l}_{j'})$ or $I^{-\infty}(\bar{l}_{j'})$ is a proper subset of $I^{-\infty}((l_{j'})^n)$. Without loss of generality, since $(l_{j'})^n \to \bar{l}_{j'}$, it holds that, for every $n \in \mathbb{N}$, $I^{-\infty}((l_{j'})^n)$ is a proper subset of $I^{-\infty}(\bar{l}_{j'})$, and, moreover, $\sum_{i\in I_M\setminus I^{-\infty}((l_{j'})^n)} l^{in}_{j'} \to -\infty$. If $\pi^{-1}(i^{-\infty}((l_{j'})^n)) = \pi^{-1}(i^{-\infty}(\bar{l}_{j'})) - 1$, $\forall n \in \mathbb{N}$, then, since $\dot{l}$ is market independent and weakly monotonic, it follows that $\sum_{i\in I_M\setminus I^{-\infty}(\bar{l}_{j'})} \bar{l}^i_{j'} = \alpha(i^{-\infty}(\bar{l}_{j'}))$ and, since $\sum_{i\in I_M\setminus I^{-\infty}((l_{j'})^n)} l^{in}_{j'} \to -\infty$, it follows that

$$\begin{aligned} g((l_{j'})^n) &= \frac{\pi^{-1}(i^{-\infty}(\bar{l}_{j'}))-2}{\pi^{-1}(M)-1} + \frac{f^{\alpha(i^{-\infty}((l_{j'})^n))}(\sum_{i\in I_M\setminus I^{-\infty}((l_{j'})^n)} l^{in}_{j'})}{\pi^{-1}(M)-1} \\ &\to \frac{\pi^{-1}(i^{-\infty}(\bar{l}_{j'}))-1}{\pi^{-1}(M)-1} \\ &= \frac{\pi^{-1}(i^{-\infty}(\bar{l}_{j'}))-1}{\pi^{-1}(M)-1} + \frac{f^{\alpha(i^{-\infty}(\bar{l}_{j'}))}(\alpha(i^{-\infty}(\bar{l}_{j'})))}{\pi^{-1}(M)-1} = g(\bar{l}_{j'}), \end{aligned}$$

a contradiction with the supposition that $(g((l_{j'})^n))_{n\in\mathbb{N}}$ does not converge to $g(\bar{l}_{j'})$. Consequently, without loss of generality, for every $n \in \mathbb{N}$, $I^{-\infty}((l_{j'})^n) \subset I^{-\infty}(\bar{l}_{j'})$ and $\pi^{-1}(i^{-\infty}((l_{j'})^n)) < \pi^{-1}(i^{-\infty}(\bar{l}_{j'})) - 1$. So, there exists an element $\hat{l}_{j'} \in \dot{l}_{j'}$ with $i^{-\infty}(\hat{l}_{j'}) > i^{-\infty}((l_{j'})^n)$, $\forall n \in \mathbb{N}$, and $i^{-\infty}(\hat{l}_{j'}) < i^{-\infty}(\bar{l}_{j'})$. Using that $\dot{l}$ is market independent, this leads to a contradiction to the weak monotonicity of $\dot{l}$. Consequently, g is a continuous function.

It will be shown that $\dot{l}_{j'}$ is connected in $\mathbb{R}^{*M}$. Suppose $\dot{l}_{j'}$ is not connected in $\mathbb{R}^{*M}$, then there exist two disjoint, non-empty subsets of $\dot{l}_{j'}$, say $\bar{\dot{l}}_{j'}$ and $\hat{\dot{l}}_{j'}$, both being open in $\dot{l}_{j'}$ and whose union equals $\dot{l}_{j'}$. Obviously, the sets $\dot{l}_1 \times \cdots \times \dot{l}_{j'-1} \times \bar{\dot{l}}_{j'} \times \dot{l}_{j'+1} \times \cdots \times \dot{l}_N$ and $\dot{l}_1 \times \cdots \times \dot{l}_{j'-1} \times \hat{\dot{l}}_{j'} \times \dot{l}_{j'+1} \times \cdots \times \dot{l}_N$ are two disjoint, non-empty subsets of $\prod_{j \in I_N} \dot{l}_j$, both being open in $\prod_{j \in I_N} \dot{l}_j$, contradicting the connectedness of $\dot{l}$. Consequently, $\dot{l}_{j'}$ is connected in $\mathbb{R}^{*M}$.

Since the function g is continuous and $\dot{l}_{j'}$ is connected in $\mathbb{R}^{*M}$ it follows that $g(\dot{l}_{j'})$ is connected in $[0,1]$ and hence an interval. Clearly, $0^M \in \dot{l}_{j'}$ and $-\infty^M \in \dot{l}_{j'}$ since $\dot{l}$ is flexible and market independent. Since $g(0^M) = 0$ and $g(-\infty^M) = 1$ it follows that g is surjective.

Next it is shown that g is injective. Suppose g is not injective, then there exists $\bar{l}_{j'}, \hat{l}_{j'} \in \dot{l}_{j'}$ such that $\bar{l}_{j'} \neq \hat{l}_{j'}$ and $g(\bar{l}_{j'}) = g(\hat{l}_{j'})$. From the definition of g it follows that $i^{-\infty}(\bar{l}_{j'}) = i^{-\infty}(\hat{l}_{j'})$ and $\sum_{i \in I_M \setminus I^{-\infty}(\bar{l}_{j'})} \bar{l}^i_{j'} = \sum_{i \in I_M \setminus I^{-\infty}(\hat{l}_{j'})} \hat{l}^i_{j'}$. This yields a contradiction to the weak monotonicity of $\dot{l}$. Consequently g is injective.

It is easily verified that the topological spaces Q^M and $\mathbb{R}^{*M}$ are homeomorphic. Therefore, a set closed in $\mathbb{R}^{*M}$ is also compact in $\mathbb{R}^{*M}$. Since $\dot{l}$ is closed in $\mathbb{R}^{*MN}$, it follows easily from the market independence that $\dot{l}_{j'}$ is closed in $\mathbb{R}^{*M}$, hence compact in $\mathbb{R}^{*M}$. Since $\dot{l}_{j'}$ is a compact topological space, $[0,1]$ is a Hausdorff space, and $g : \dot{l}_{j'} \to [0,1]$ is a continuous, injective, and surjective function, it follows that g is a homeomorphism, so $g^{-1} : [0,1] \to \dot{l}_{j'}$ is a continuous function. Let the function $h : [0,1] \to \dot{l}_{j'}$ be defined by $h = g^{-1}$.

The function h would represent $\dot{l}_{j'}$ if its image is a subset of $-\mathbb{R}^M_+$. The function h will now be modified in order to guarantee this.

When $K = \{0, M\}$, then there exists $\bar{t} \in (0,1)$ such that, for every $t \in [\bar{t}, 1]$, $h_i(\bar{t}) < -\omega^i_{j'}$, $\forall i \in I_M$. Let the function $\tilde{l}_{j'} : Q^N \to -\mathbb{R}^M_+$ be defined by

$$\tilde{l}_{j'}(q) = h(\bar{t} q_{j'}), \ \forall q \in Q^N.$$

Now consider the case that $\{0, M\}$ is a proper subset of K. By the weak monotonicity of $\dot{l}$ there exists a uniquely determined proper subset $\bar{I}$ of I_M such that $i^{-\infty}(l_{j'}) = \pi(2)$ for some $l_{j'} \in \dot{l}_{j'}$ implies $I^{-\infty}(l_{j'}) = \bar{I}$. Since the set $[0, \frac{1}{\pi^{-1}(M)-1}]$ is compact and $h_i(t) > -\infty$, $\forall i \in I_M \setminus \bar{I}$, $\forall t \in [0, \frac{1}{\pi^{-1}(M)-1}]$, it holds that, for every $i \in I_M \setminus \bar{I}$, the continuous function h_i has a minimum on $[0, \frac{1}{\pi^{-1}(M)-1}]$, say α^i. For every $i \in I_M \setminus \bar{I}$, let the real number $\overline{\alpha}^i$ be defined by $\overline{\alpha}^i = \min(\{\alpha^i, -\omega^i_{j'}\}) - 1$. For every $i \in I_M \setminus \bar{I}$, let the function $\tilde{l}^i_{j'} : Q^N \to -\mathbb{R}_+$ be defined by

$$\tilde{l}^i_{j'}(q) = \max\left(\{\overline{\alpha}^i, h_i(q_{j'})\}\right), \ \forall q \in Q^N. \tag{1}$$

Notice that, for every $i \in I_M \setminus \bar{I}$, for every $q \in Q^N$,

$$\tilde{l}^i_{j'}(q) = h_i(q_{j'}), \text{ if } q_{j'} \in [0, \tfrac{1}{\pi^{-1}(M)-1}], \tag{2}$$

and, moreover,

$$h_i(q_{j'}) \geq -\omega^i_{j'} \text{ implies } \widetilde{l}^i_{j'}(q) = h_i(q_{j'}), \text{ and } h_i(q_{j'}) < -\omega^i_{j'} \text{ implies } \widetilde{l}^i_{j'}(q) < -\omega^i_{j'}. \qquad (3)$$

Let $\widehat{t} \in (0, \frac{1}{\pi^{-1}(M)-1})$ be such that, for every $i \in \overline{I}$,

$$h_i(t) \leq \tfrac{1}{\#\overline{I}} \sum_{i \in I_M \setminus \overline{I}} \overline{\alpha}^i + \min\left(\left\{0, -\omega^i_{j'}\right\}\right), \ \forall t \in [\widehat{t}, 1]. \qquad (4)$$

Notice that such a real number $\widehat{t}$ exists since h is continuous and $h_i(t) = -\infty$, $\forall i \in \overline{I}$, $\forall t \in [\frac{1}{\pi^{-1}(M)-1}, 1]$. Let $\widehat{q}$ be any element of Q^N such that $\widehat{q}_{j'} = \widehat{t}$. Notice that, for every $i \in \overline{I}$, $h_i(t) < -\omega^i_{j'}$, $\forall t \in [\widehat{t}, 1]$. For every $i \in \overline{I}$, let the function $\widetilde{l}^i_{j'} : Q^N \to -\mathbb{R}_+$ be defined by

$$\begin{aligned} \widetilde{l}^i_{j'}(q) &= h_i(q_{j'}), && \forall q \in Q^N \text{ with } q_{j'} \leq \widehat{t}, \\ \widetilde{l}^i_{j'}(q) &= h_i(\widehat{t}) + \widehat{t} - q_{j'} + \tfrac{1}{\#\overline{I}} \textstyle\sum_{i \in I_M \setminus \overline{I}} \left(\widetilde{l}^i_{j'}(\widehat{q}) - \widetilde{l}^i_{j'}(q)\right), && \forall q \in Q^N \text{ with } q_{j'} > \widehat{t}. \end{aligned}$$

Notice that, for every $i \in \overline{I}$, $\widetilde{l}^i_{j'}$ is continuous and, using (4), for every $q \in Q^N$ with $q_{j'} > \widehat{t}$,

$$\begin{aligned} \widetilde{l}^i_{j'}(q) &= h_i(\widehat{t}) + \widehat{t} - q_{j'} + \tfrac{1}{\#\overline{I}} \sum_{i \in I_M \setminus \overline{I}} \left(\widetilde{l}^i_{j'}(\widehat{q}) - \widetilde{l}^i_{j'}(q)\right) \\ &< \tfrac{1}{\#\overline{I}} \sum_{i \in I_M \setminus \overline{I}} \overline{\alpha}^i + \min\left(\left\{0, -\omega^i_{j'}\right\}\right) - \tfrac{1}{\#\overline{I}} \sum_{i \in I_M \setminus \overline{I}} \widetilde{l}^i_{j'}(q) \leq -\omega^i_{j'}, \qquad (5) \end{aligned}$$

where for the last inequality (1) is used.
Using the previous two paragraphs, a rationing function on supply $\widetilde{l} : Q^N \to -\mathbb{R}^{MN}_+$ is constructed. Using (3) and (5) it follows easily that $\dot{l}$ is represented by $\widetilde{l}$. Obviously, $\widetilde{l}$ is flexible, market independent, and continuous. Let $q^1, q^2 \in Q^N$ be such that $q^1_{j'} < q^2_{j'} \leq \widehat{t}$. Using (2) and the construction of h it follows that

$$\sum_{i \in I_M} \widetilde{l}^i_{j'}(q^1) = \sum_{i \in I_M} h_i(q^1_{j'}) > \sum_{i \in I_M} h_i(q^2_{j'}) = \sum_{i \in I_M} \widetilde{l}^i_{j'}(q^2).$$

Let $q^1, q^2 \in Q^N$ be such that $\widehat{t} \leq q^1_{j'} < q^2_{j'}$. Then

$$\begin{aligned} \sum_{i \in I_M} \widetilde{l}^i_{j'}(q^1) &= \sum_{i \in I_M \setminus \overline{I}} \widetilde{l}^i_{j'}(q^1) + \sum_{i \in \overline{I}} \left(h_i(\widehat{t}) + \widehat{t} - q^1_{j'}\right) + \sum_{i \in \overline{I}} \tfrac{1}{\#\overline{I}} \sum_{i \in I_M \setminus \overline{I}} \left(\widetilde{l}^i_{j'}(\widehat{q}) - \widetilde{l}^i_{j'}(q^1)\right) \\ &= \sum_{i \in \overline{I}} \left(h_i(\widehat{t}) + \widehat{t} - q^1_{j'}\right) + \sum_{i \in I_M \setminus \overline{I}} \widetilde{l}^i_{j'}(\widehat{q}) \\ &> \sum_{i \in \overline{I}} \left(h_i(\widehat{t}) + \widehat{t} - q^2_{j'}\right) + \sum_{i \in I_M \setminus \overline{I}} \widetilde{l}^i_{j'}(\widehat{q}) \\ &= \sum_{i \in I_M} \widetilde{l}^i_{j'}(q^2). \end{aligned}$$

Let $q^1, q^2 \in Q^N$ be such that $q^1_{j'} \leq \hat{t} \leq q^2_{j'}$ and $q^1_{j'} < q^2_{j'}$. Then the two cases considered above immediately yield that $\sum_{i \in I_M} \tilde{l}^i_{j'}(q^1) > \sum_{i \in I_M} \tilde{l}^i_{j'}(q^2)$. Therefore $\tilde{l}$ is weakly monotonic. So, $\tilde{l}$ satisfies all the desired properties.

Next, let $\dot{l}$ be a monotonic rationing system on supply. Let some $j' \in I_N$ be given. Construct the continuous, injective, and surjective function $h : [0,1] \rightarrow \dot{l}_{j'}$ as in the first part of the proof and, for $\alpha \in \mathbb{R}$, let the function $f^\alpha : \{s \in \mathbb{R}^* \mid s \leq \alpha\} \rightarrow [0,1]$ be defined as before. For every $i \in I_M$, let $\alpha^i = \min(\{-\omega^i_{j'}, 0\}) - 1$ and let the function $g^i : -\mathbb{R}^*_+ \rightarrow [\alpha^i - 1, 0]$ be defined by

$$
\begin{aligned}
g^i(s) &= \alpha^i - f^0(s - \alpha^i), && \forall s \in -\mathbb{R}^*_+ \setminus [\alpha^i, 0], \\
g^i(s) &= s, && \forall s \in [\alpha^i, 0].
\end{aligned}
$$

Let the function $\tilde{l}_{j'} : Q^N \rightarrow -\mathbb{R}^M_+$ be defined by

$$
\tilde{l}_{j'}(q) = \left(g^1\left(h_1(q_{j'})\right), \ldots, g^M\left(h_M(q_{j'})\right)\right)^\top, \ \forall q \in Q^N.
$$

In this way a rationing function on supply $\tilde{l} : Q^N \rightarrow -\mathbb{R}^{MN}_+$ is constructed. It is easily verified that $\dot{l}$ is represented by $\tilde{l}$ and that $\tilde{l}$ is flexible, market independent, continuous, and monotonic. Q.E.D.

The proof for the results concerning the rationing system on demand given in Theorem 4.4 is similar.

Theorem 4.4

Let the rationing system on demand $\dot{L}$ be flexible, market independent, closed, and connected. If the rationing system on demand $\dot{L}$ is weakly monotonic, then $\dot{L}$ can be represented by a flexible, market independent, continuous, and weakly monotonic rationing function on demand. If the rationing system on demand $\dot{L}$ is monotonic, then $\dot{L}$ can be represented by a flexible, market independent, continuous, and monotonic rationing function on demand.

The results of this section show that the set of admissible rationing schemes can be described equally well by means of a rationing system as by a rationing function and provide necessary and sufficient conditions that make a representation possible.

References

Armstrong, M.A. (1983), *Basic Topology*, Springer-Verlag, New York.

Drèze, J.H. (1975), "Existence of an Exchange Equilibrium under Price Rigidities", *International Economic Review, 16,* 301-320.

Herings, P.J.J. (1992), *On the Structure of Constrained Equilibria,* FEW Research Memorandum 587, Tilburg University, Tilburg.

Kurz, M. (1982), "Unemployment Equilibrium in an Economy with Linked Prices", *Journal of Economic Theory, 26,* 100-123.

Laroque, G., and H. Polemarchakis (1978), "On the Structure of the Set of Fixed Price Equilibria", *Journal of Mathematical Economics, 5,* 53-69.

Movshovich, S.M. (1994), "A Price Adjustment Process in a Rationed Economy", *Journal of Mathematical Economics, 23,* 305-321.

Weddepohl, C. (1983), "Fixed Price Equilibria in a Multifirm Model", *Journal of Economic Theory, 29,* 95-108.

Weddepohl, C. (1987), "Supply-Constrained Equilibria in Economies with Indexed Prices", *Journal of Economic Theory, 43,* 203-222.

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