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RESEARCH MEMORANDUM

UNIFORM LIMIT THEOREMS FOR MARKED POINT PROCESSES

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Uniform Limit Theorems for Marked Point Processes

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Let P be the distribution of a stationary marked point process on $\mathbf{R}$ and let P_L^0 be its Palm distribution with respect to a set L of marks. Starting from P, the probability measures $P_{i,L}$, $i \in \mathbf{Z}$, arise by shifting the origin to the i'th occurrence with mark in L. In Nieuwenhuis (1994) it is proved that $n^{-1}\sum_{i=1}^{n} P_{i,L}(B)$, B a set of realizations, tends uniformly to $Q_L^0(B)$. Here Q_L^0 is a probability measure which equals P_L^0 under a weak ergodicity condition. In the present research this uniform limit theorem is generalized by replacing 1_B by functions f with $|f|$ uniformly bounded by a fixed function g. It is also proved that similar results hold if the starting point P is replaced by $P_{L'}^0$, where L' is another set of marks with $L \cap L' = \emptyset$. As a preliminary a theorem is proved which implies an easy way to express $P_{L'}^0$-expectations in terms of P_L^0-expectations. In a "dual" theorem the roles of P and P_L^0 are interchanged. Starting from P_L^0, similar uniform limit theorems are derived for Cesaro averaged functionals. The limits can be expressed as expectations under a probability measure Q_L which equals P under a weak ergodicity condition. In a final section it is shown that uniform approximation of P_L^0 and P is still possible without ergodicity restraints.

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1 Introduction

Many problems in queueing theory concern the relationship between the arrival-stationary model and the time-stationary model. One way to compare the two models is to approximate the first when starting from the second, and vice versa. Some approximations of this type are treated in this research.

The theory of stochastic processes with an embedded marked point process ($PEMP$; see Franken, König, Arndt and Schmidt (1982) and Brandt, Franken and Lisek (1990)) seems to be the natural tool for treating such problems. Since, however, a $PEMP$ is nothing but a marked point process (MPP) with special marks, we will use the theory of MPP's on $\mathbf{R}$ to consider approximations of the type mentioned above. All results will be stated for MPP's.

Let P be the distribution of a stationary MPP Φ on $\mathbf{R}$ and let P_L^0 be its Palm distribution with respect to a set L of marks. A formal definition follows below, but intuitively P_L^0 is the conditional distribution of Φ given the occurrence of an "L-point" (an occurrence having its mark in L) in the origin. This intuitive definition is motivated by the local characterization of the Palm distribution as a limit of conditional probability measures. See Franken et al. (1982; Th. 1.3.7) or Nieuwenhuis (1994; Th. 10). Inspired by the definition of P_L^0 in (1.3) and the inversion formula in (1.5), the relationship between P and P_L^0 could (as in the unmarked case, see Nieuwenhuis (1994)) also be described by the following intuitive formulations:

P arises from P_L^0 by shifting the origin to a time point in $(-\infty, +\infty)$ chosen at random. (1.1)

P_L^0 arises from P by shifting the origin to an L -point chosen at random. (1.2)

A formalization of the intuitive random procedure in (1.1) is used for the *length-biased sampling (LBS)* procedure mentioned in Cox and Lewis (1966) to derive relations between P and the Palm distribution. In the present context of MPP's this formalization would go like this. Starting from an origin in a randomly chosen L-point (i.e. P_L^0 is the ruling probability measure), the interval up to the r'th L-point is considered. Here r is very large. In this interval a time point is chosen at random and the origin is moved to it. It is assumed that (as $r \to \infty$) the situation seen from this new position of the origin is described by P. The heuristic arguments used on page 61 of the last reference depend, however, heavily on whether a strong law of large numbers with degenerate limit holds for the sequence of interval lengths between the occurrences. The question arises if the formalization of (1.1) used in the LBS procedure is also applicable if the limit of the strong law is **non**degerate.

One of the objectives of this research is to clarify the intuitive random procedures (1.1) and (1.2) for generating P and P_L^0 by choosing obvious formalizations. The formalizations of (1.1) and (1.2) are in terms of limit results for Cesaro means. Note that the LBS procedure motivates the use of such means for (1.1) because of the shift of the origin to a time point which is chosen **at random**. In Nieuwenhuis (1994) it is proved that for (unmarked) point processes a formalization of (1.2) with Cesaro means only leads to the Palm distribution if a weak ergodicity condition is satisfied. The generalization to marked point processes is, however, straightforward. Relation (54) and Theorem 7 in the above reference can be generalized and read as follows: When starting from P the distribution of the MPP seen from an L-point, chosen at random among the first n L-points, tends (as $n \to \infty$) uniformly to a probability measure Q_L^0 which equals P_L^0 under a weak ergodicity condition. See Theorem 1.2 below. Since this theorem can also be formulated as a uniform limit result over all functions f with $|f| \leq 1$, it is natural to consider the more general problem of uniform convergence for functions f with $|f| \leq g$. In Section 4 necessary (and sufficient) conditions on g are derived for this uniform convergence to hold. See Theorem 4.2 and Corollary 4.3. In Section 5 it is proved that a similar generalization is valid if the distribution P, the starting point, is replaced by a Palm distribution $P_{L'}^0$, where L' is another nonempty set of marks with $L \cap L' = \emptyset$. When starting from $P_{L'}^0$ the distribution of the MPP seen from an L-point, chosen at random among the first n L-points, tends uniformly to P_L^0 provided that some weak ergodicity condition is satisfied.

In Section 3 a formalization of (1.1) is considered, so the roles of P and P_L^0 in Theorem 1.2 are interchanged: When starting from P_L^0 the distribution of the MPP seen from a position chosen at random between 0 and t tends uniformly to a probability measure Q_L (as $t \to \infty$) which equals P if a weak ergodicity condition is satisfied. Things can again be generalized by replacing the indicator functions by more general functions f with $|f|$ bounded by a fixed function g. Necessary (and sufficient) conditions on g are formulated for the corresponding uniform limit result, see Theorem 3.2 and Corollary 3.3. Relations between Q_L and P, and between Q_L and Q_L^0 are derived.

In Section 6 the theorems of Sections 3, 4 and 5 are applied. It is proved that, when starting from P_L^0 and P (or $P_{L'}^0$) respectively, P and P_L^0 can still be approximated uniformly by Cesaro means without assuming any ergodicity condition. Only the weights of the realizations of Φ have to be changed.

Our treatment involves conditioning on invariant σ-fields. Some preliminary lemmas are proved in Section 2. In our proofs we have to go from P_L^0 to P or from P to $P_{L'}^0$ several times. The method used to bridge these gaps (the"Radon-Nikodym approach", see Nieuwenhuis (1994; Section 1)), is a consequence of Theorem 1.1.

A theorem closely related to Theorem 3.2 is proved in Glynn and Sigman (1992). In this paper synchronous processes are considered which are associated with a point process on $[0,\infty)$. In the present research the approach is quite different from the approach in the above reference. The conditions (and their necessity) are more analyzed, the limits are characterized.

We next formalize some of the notions mentioned above and give some other definitions and notations. Let K be a complete and separable metric space. A *marked point process on* $\mathbf{R}$ *with mark space* K is a random element Φ in the set of all integer-valued measures φ on the σ-field $\text{Bor}\,\mathbf{R} \times \text{Bor}\,K$ such that:

$$\varphi(A \times K) < \infty \text{ for all bounded } A \in \text{Bor}\,\mathbf{R}.$$

Let M_K be this set and endow it with the σ-field $\mathcal{M}_K$ generated by the sets $[\varphi(A \times L) = k] := \{\varphi \in M_K : \varphi(A \times L) = k\}$, $k \in \mathbf{N}_0$, $L \in \text{Bor}\,K$ and $A \in \text{Bor}\,\mathbf{R}$. The distribution of Φ will be denoted by P, a probability measure on $(M_K, \mathcal{M}_K)$.

For $\varphi \in M_K$ and $L \in \text{Bor}\,K$ we define the counting measure φ_L on $\text{Bor}\,\mathbf{R}$ by $\varphi_L(A) := \varphi(A \times L)$, $A \in \text{Bor}\,\mathbf{R}$, and write $\Phi_L := \Phi(\cdot \times L)$, a point process on $\mathbf{R}$. Set

$$\begin{aligned} M_L^\infty &:= \{\varphi \in M_K : \varphi_L(-\infty,0) = \varphi_L(0,\infty) = \infty;\ \varphi_K(\{s\}) \le 1 \text{ for all } s \in \mathbf{R}\}, \\ M_L^0 &:= \{\varphi \in M_L^\infty : \varphi_L(\{0\}) = 1\}, \\ \mathcal{M}_L^\infty &:= M_L^\infty \cap \mathcal{M}_K \text{ and } \mathcal{M}_L^0 := M_L^0 \cap \mathcal{M}_K, \end{aligned}$$

$L \in \text{Bor}\,K$. Define $\lambda(L) := \mathbf{E}\Phi_L(0,1]$, the intensity of the L-points. It will always be assumed that $P(M_K^\infty) = 1$, and that the *intensity* $\lambda(K)$ is finite. We will only consider $L \in \text{Bor}\,K$ with $P(M_L^\infty) = 1$. The atoms of $\varphi \in M_K^\infty$ are denoted by $(X_i(\varphi), k_i(\varphi)) \in \mathbf{R} \times K$, $i \in \mathbf{Z}$, with the convention

$$\ldots < X_{-1}(\varphi) < X_0(\varphi) \le 0 < X_1(\varphi) < X_2(\varphi) < \ldots.$$

$X_i(\varphi)$ is interpreted as the i'th *occurrence* (or *point*) of φ, $k_i(\varphi)$ as the accessory *mark*. For $\varphi \in M_L^\infty$ we write $X_i^L(\varphi) := X_i(\varphi_L)$, the "$i$'th L-point of φ", and $\alpha_i^L(\varphi) := X_{i+1}^L(\varphi) - X_i^L(\varphi)$. For a realization $\varphi \in M_K^\infty$ and a scalar $t \in \mathbf{R}$ the element $T_t\varphi = \varphi(t + \cdot)$ of M_K^∞ arises from φ by shifting the origin to t and considering the realization from this new position. So, $T_t\varphi$ can be represented by the set $\{(X_j(\varphi) - t, k_j(\varphi)) : j \in \mathbf{Z}\}$ containing its atoms. The corresponding MPP is denoted by $T_t\Phi = \Phi(t + \cdot)$. We assume that $\Phi(t + \cdot) =_d \Phi$ for all $t \in \mathbf{R}$, i.e. that Φ is *stationary.*

Two types of shifts will be considered. The *time shifts* $T_t : M_K^\infty \to M_K^\infty$, $t \in \mathbf{R}$, are defined above. For fixed $L \in \operatorname{Bor} K$ with $P(M_L^\infty) = 1$ the *point shift* $\vartheta_{n,L} : M_L^\infty \to M_L^\infty$, $n \in \mathbf{Z}$, moves the origin to the n'th L-point. It is defined by $\vartheta_{n,L}\varphi := \varphi(X_n^L(\varphi) + \cdot)$. The probability measure $P_{n,L} := P\vartheta_{n,L}^{-1}$, $n \in \mathbf{Z}$, on $(M_L^\infty, \mathcal{M}_L^\infty)$ arises from P by shifting the origin to the n'th L-point. To illustrate our notation we point out that $[\vartheta_{n,L}\varphi \in B] = \{\varphi \in M_L^\infty : \vartheta_{n,L}\varphi \in B\}$, $B \in \mathcal{M}_L^\infty$ and $n \in \mathbf{Z}$.

For $L \in \operatorname{Bor} K$ with $P(M_L^\infty) = 1$ the *Palm distribution* P_L^0 of Φ (or rather P) *with respect to* L is defined by

$$P_L^0(A) := \frac{1}{\lambda(L)}\mathbf{E}\left[\sum_{i=1}^{\Phi((0,1]\times L)} 1_A(\vartheta_{i,L}\Phi)\right], \qquad A \in \mathcal{M}_L^\infty. \tag{1.3}$$

Note the difference between P_L^0 and $P_{0,L}$, in notation as well as in interpretation. Several probability measures on $(M_L^\infty, \mathcal{M}_L^\infty)$ have been defined so far: P, P_L^0, $P_{n,L}$. In this research expectations with respect to these measures are denoted by E, E_L^0, $E_{n,L}$, respectively. When another probability measure Q on $(M_L^\infty, \mathcal{M}_L^\infty)$ is considered, we will write E_Q for the corresponding expectation. Expectation with respect to a universal probability space $(\Omega, \mathcal{F}, P)$ is (as in (1.3)) denoted by $\mathbf{E}$. Note that $P_L^0(M_L^0) = 1$. The probability measure P_L^0 has the following properties:

$$P_L^0\vartheta_{n,L}^{-1} = P_L^0 \qquad \text{for all } n \in \mathbf{Z}, \tag{1.4}$$

$$P(A) = \lambda(L)\int_0^\infty P_L^0[X_1^L(\varphi) > u;\ \varphi(u + \cdot) \in A]du, \qquad A \in \mathcal{M}_L^\infty. \tag{1.5}$$

With the choice $A = M_L^\infty$ we obtain $E_L^0\alpha_0^L = 1/\lambda(L)$. See Franken et al. (1982), Matthes, Kerstan and Mecke (1978), Kallenberg (1983), and Brandt, Franken and Lisek (1990) for more information.

The *inversion formula* (1.5) expresses P in terms of P_L^0; the definition in (1.3) expresses P_L^0 in terms of P. There is another way of going from P_L^0 to P (and vice versa).

The essence of the approach is contained in the next theorem. It is proved in Nieuwenhuis (1989); the extension to **marked** point processes is straightforward. First some notations. Let Q_1 and Q_2 be probability measures on a common measurable space. Q_1 is *dominated by* Q_2 (notation $Q_1 << Q_2$) if all Q_2-null-sets are also Q_1-null-sets; a Radon-Nikodym derivative is denoted by $\frac{dQ_1}{dQ_2}$. The measures Q_1 and Q_2 are *equivalent* (notation $Q_1 \sim Q_2$) if they have the same null-sets.

Theorem 1.1 *Let $n \in \mathbf{Z}$ and let $L \in \operatorname{Bor} K$ be such that $P(M_L^\infty) = 1$. Then*
(i) $P_{n,L} \sim P_L^0$,
(ii) $\frac{dP_{n,L}}{dP_L^0} = \lambda(L)\alpha_{-n}^L \quad P_L^0$-*a.s.*

Suppose that $f : M_L^0 \to \mathbf{R}$ is P_L^0-integrable. Since $E_L^0 f = E_{0,L}(f/\alpha_0^L)/\lambda(L)$ by part (ii), we obtain:

$$E_L^0 f = \frac{1}{\lambda(L)} E\left(\frac{1}{\alpha_0^L} f \circ \vartheta_{0,L}\right). \tag{1.6}$$

This relation expresses a transition from P to P_L^0 where $P_{0,L}$ is used as a bridge. At first the origin is shifted to the last L-point on its left, to X_0^L. Then the importance of the realizations is changed by way of the weight function $(\lambda(L)\alpha_0^L)^{-1}$. See Sections 1 and 2 of Nieuwenhuis (1994) for more information about two-step transitions of this type.

Reversely, if $g : M_L^\infty \to \mathbf{R}$ is P-integrable with $Eg = Eg \circ \vartheta_{0,L}$, then the P-expectation of g can be transformed into a P_L^0-expectation:

$$Eg = E_{0,L} g = \lambda(L) E_L^0(\alpha_0^L g). \tag{1.7}$$

For more applications of Theorem 1.1 we refer to Nieuwenhuis (1994). The approach in (1.6) and (1.7), where $P_{0,L}$ is used as a bridge between P_L^0 and P, is very common in the present research.

Consider the following invariant σ-fields:

$$\begin{aligned} \mathcal{I}'_L &:= \{A \in \mathcal{M}_L^\infty : T_t^{-1} A = A \text{ for all } t \in \mathbf{R}\} \text{ and} \\ \mathcal{I}_L &:= \{A \in \mathcal{M}_L^\infty : \vartheta_{1,L}^{-1} A = A\}. \end{aligned} \tag{1.8}$$

It is well-known that the sequence (α_i^L) is P_L^0-stationary and that

$$\frac{1}{n}\sum_{i=1}^{n}\alpha_i^L \to \bar{\alpha}_0^L := E_L^0(\alpha_0^L|\mathcal{I}_L) \quad P_L^0\text{- and } P\text{- a.s.} \tag{1.9}$$

See also Nieuwenhuis (1994; Th. 3). Φ is called *pseudo-L-ergodic* if

$$\bar{\alpha}_0^L = \frac{1}{\lambda(L)} \quad P_L^0\text{- a.s..} \tag{1.10}$$

P (or Φ) is *ergodic* if $P(A) \in \{0,1\}$ for all $A \in \mathcal{I}_K'$, and P_L^0 is *ergodic* if $P_L^0(A) \in \{0,1\}$ for all $A \in \mathcal{I}_L$.

We need more probability measures. Let Q_L^0 on $(M_L^\infty, \mathcal{M}_L^\infty)$ be defined by

$$Q_L^0(B) := E(E_L^0(1_B|\mathcal{I}_L)), \quad B \in \mathcal{M}_L^\infty. \tag{1.11}$$

This probability measure seems to be more in accordance with the intuitive definition (1.2) of P_L^0 than P_L^0 itself. This is expressed in the following theorem, which has been the inspiration and motivation for the present research. In this result Q_L^0 is approximated when starting from P. For unmarked point processes it is proved in Nieuwenhuis (1994; Section 4); the generalization to MPP's is straightforward.

Theorem 1.2 *Let $L \in \operatorname{Bor} K$ be such that $P(M_L^\infty) = 1$. Then Q_L^0 is equivalent to P_L^0 and*

$$\frac{dQ_L^0}{dP_L^0} = \lambda(L)\bar{\alpha}_0^L \quad P_L^0\text{- a.s.} \tag{1.12}$$

Q_L^0 and P_L^0 are equal iff Φ is pseudo- L -ergodic. The supremum

$$\sup_{B\in\mathcal{M}_L^\infty}\left|\frac{1}{n}\sum_{i=1}^{n}P[\vartheta_{i,L}\varphi \in B] - Q_L^0(B)\right| = \frac{\lambda(L)}{2}E_L^0\left|\frac{1}{n}\sum_{i=1}^{n}\alpha_{-i}^L - \bar{\alpha}_0^L\right|$$

tends to 0 as $n \to \infty$.

Note that $\alpha_i^L = \alpha_0^L \circ \vartheta_{i,L}$. In view of the intuitive definition (1.2) of P_L^0 we might at first sight expect that

$$\frac{1}{n}\sum_{i=1}^{n} E\alpha_i^L \to E_L^0\alpha_0^L = \frac{1}{\lambda(L)}. \tag{1.13}$$

However, since the limit result in (1.9) holds P- a.s., $n^{-1}\sum_{i=1}^n E\alpha_i^L$ will (under weak additional conditions) tend to $E\bar{\alpha}_0^L$. By (1.7) and conditioning on $\mathcal{I}_L$ we have:

$$\begin{aligned} E\bar{\alpha}_0^L &= \lambda(L)E_L^0\left(\alpha_0^L\bar{\alpha}_0^L\right) &= \lambda(L)E_L^0\left(\bar{\alpha}_0^L\right)^2 \\ &\geq \lambda(L)\left(E_L^0\bar{\alpha}_0^L\right)^2 &= \frac{1}{\lambda(L)}. \end{aligned}$$

Equality holds iff $\bar{\alpha}_0^L = 1/\lambda(L) \quad P_L^0$ -a.s., i.e. iff Φ is pseudo-L-ergodic. So, the intuitive limit in (1.13) is not necessarily correct. Note, however, that by (1.7) and (1.12) $E\bar{\alpha}_0^L = E_{Q_L^0}\alpha_0^L$. All these arguments make Theorem 1.2 less surprising.

A family $(Y_t)_{t\in I}$ of integrable random variables is called *uniformly integrable* if $\sup_{t\in I} E|Y_t|1_{|Y_t|\geq b} \to 0$ as $b \to \infty$, or, equivalently, if

$$\sup_{t\in I}\mathbf{E}|Y_t| = M < \infty \text{ and for every } \varepsilon > 0 \text{ there exists } \delta > 0 \tag{1.14}$$
such that for all events A with $\mathbf{P}(A) < \delta$ we have: $\sup_{t\in I}\mathbf{E}|Y_t|1_A < \varepsilon$.

For a probability measure Q we will abbreviate "uniformly Q-integrable" to "u.i. under Q". The following lemma will be applied in Sections 3, 4, and 5. It follows immediately from Theorem 5.4 in Billingsley (1968).

Lemma 1.3 *Let $Y, Y_1, Y_2, \ldots$ be nonnegative, real-valued, r.v.'s with $Y_n \xrightarrow{d} Y$. Then $(Y_n)_{n\geq 1}$ is uniformly integrable if and only if*

$$\mathbf{E}Y < \infty,\ \ \mathbf{E}Y_n < \infty \text{ for all } n \in \mathbf{N}, \quad \text{and } \mathbf{E}Y_n \to \mathbf{E}Y.$$

Let Q_1 and Q_2 be probability measures on a common measurable space, both dominated by a σ-finite measure μ and having densities h_1 and h_2 respectively. The *total variation distance* between Q_1 and Q_2 is defined by

$$d(Q_1, Q_2) := \int |h_1 - h_2|d\mu.$$

It is well-known that

$$d(Q_1, Q_2) = 2 \sup_A |Q_1(A) - Q_2(A)| = 2(Q_1[h_1 \geq h_2] - Q_2[h_1 \geq h_2]). \quad (1.15)$$

Some final remarks. When talking about Radon-Nikodym derivatives, the attribute a.s. (almost surely) is often suppressed. Lebesgue measure on $(0, \infty)$ is denoted by ν_+; a.e. means almost everywhere. We will often make use of the time parameters t, n, k, i, and j. The first is a continuous-time parameter, the others are discrete-time parameters.

2 Conditioning on invariant σ-fields

One of the objectives of the present research is to obtain approximations of the stationary distribution and the Palm distribution of a marked point process, without assuming ergodicity. To realize this in this general setting we will condition on invariant σ-fields. The results in this section are rather technical. They will be applied several times in Sections 3 to 5.

Recall the definitions of $\mathcal{I}_L$ and $\mathcal{I}'_L$ in (1.8). The following lemma is a straightforward generalization of Lemma 2 in Nieuwenhuis (1994).

Lemma 2.1 *Let* $L \in \mathrm{Bor}\, K$. *Then:*
(a) *If* $A \in \mathcal{I}_L$, *then* $\vartheta_{i,L}^{-1} A = A$ *for all* $i \in \mathbf{Z}$.
(b) $\mathcal{I}_L = \mathcal{I}'_L$.

Note that as a consequence of Lemma 2.1 every $\mathcal{I}_L$-measurable function $f : M_L^\infty \to [0, \infty)$ satisfies

$$f \circ \vartheta_{i,L}(\varphi) = f(\varphi) \quad \text{and} \quad f \circ T_t(\varphi) = f(\varphi) \quad (2.1)$$

for all $\varphi \in M_L^\infty$, $i \in \mathbf{Z}$, and $t \in \mathbf{R}$.

In view of Section 5 we next consider two disjoint, nonempty sets of marks. So, let L, $L' \in \mathrm{Bor}\, K$ and $L \cap L' = \emptyset$. Furthermore, set

$$\begin{aligned} M_{L,L'}^\infty &:= M_L^\infty \cap M_{L'}^\infty \text{ and } \mathcal{M}_{L,L'}^\infty := M_{L,L'}^\infty \cap \mathcal{M}_K, \\ \mathcal{I}_{L,L'} &:= \{A \in \mathcal{M}_{L,L'}^\infty : \vartheta_{1,L}^{-1} A = A\}, \\ \mathcal{I}'_{L,L'} &:= \{A \in \mathcal{M}_{L,L'}^\infty : T_t^{-1} A = A \text{ for all } t \in \mathbf{R}\}. \end{aligned}$$

In the presence of two sets of marks L and L', the mappings $\vartheta_{i,L}, \vartheta_{i,L'}$, and T_t will always be restricted to $M^{\infty}_{L,L'}$. The following relations can easily be proved:

$$\begin{array}{lll} \mathcal{I}'_L \cap M^{\infty}_{L'} = \mathcal{I}'_{L,L'} & \text{and} & \mathcal{I}_L \cap M^{\infty}_{L'} = \mathcal{I}_{L,L'}; \\ \mathcal{I}'_{L,L'} \subset \mathcal{I}'_L & \text{and} & \mathcal{I}_{L,L'} \subset \mathcal{I}_L. \end{array} \tag{2.2}$$

At first sight the second equality in part (b) of the next lemma seems rather surprising.

Lemma 2.2 *Let* $L, L' \in \operatorname{Bor} K$ *with* $L \cap L' = \emptyset$. *Then:*
(a) *If* $A \in \mathcal{I}_{L,L'}$, *then* $\vartheta^{-1}_{i,L} A = A$ *for all* $i \in \mathbb{Z}$;
(b) $\mathcal{I}'_{L,L'} = \mathcal{I}_{L,L'} = \mathcal{I}_{L',L}$.

Proof. Since $\mathcal{I}_{L,L'} \subset \mathcal{I}_L$, part (a) follows from Lemma 2.1(a). Part (b) is an immediate consequence of Lemma 2.1(b) and (2.2) since

$$\begin{aligned} \mathcal{I}_{L,L'} &= \mathcal{I}_L \cap M^{\infty}_{L'} = \mathcal{I}'_L \cap M^{\infty}_{L'} = \mathcal{I}'_{L,L'} = \mathcal{I}'_{L',L} \\ &= \mathcal{I}'_{L'} \cap M^{\infty}_{L} = \mathcal{I}_{L'} \cap M^{\infty}_{L} = \mathcal{I}_{L',L}. \end{aligned}$$
□

As a consequence of Lemma 2.2 every $\mathcal{I}_{L,L'}$-measurable function $f: M^{\infty}_{L,L'} \to [0,\infty)$ satisfies

$$f \circ \vartheta_{i,L}(\varphi) = f(\varphi), \quad f \circ \vartheta_{i,L'}(\varphi) = f(\varphi), \quad \text{and } f \circ T_t(\varphi) = f(\varphi) \tag{2.3}$$

for all $\varphi \in M^{\infty}_{L,L'}$, $i \in \mathbb{Z}$, and $t \in \mathbb{R}$.

Next a stationary point process Φ with distribution P is put upon the stage. Suppose that $P(M^{\infty}_L) = 1$. Since $\mathcal{I}'_L \subset \mathcal{I}'_K$ and $\mathcal{I}'_L = \mathcal{I}'_K \cap \mathcal{M}^{\infty}_L$, the σ-field $\mathcal{I}'_K$ in the definition of ergodicity of P in Section 1 may equivalently be replaced by $\mathcal{I}'_L$. As a consequence of Lemma 2.1(b) we obtain:

$$\begin{array}{llll} P & \text{is ergodic} & \Longleftrightarrow & P^0_L \ \text{ is ergodic,} \\ P & \text{is ergodic} & \Longrightarrow & P \ \text{ is pseudo-}L\text{-ergodic.} \end{array} \tag{2.4}$$

If P is pseudo-L-ergodic, then it is not necessarily ergodic. See Nieuwenhuis (1994; Example 2).

In the following lemma some special conditional expectations are compared. For $t \geq 0$ the random variable $N_L(t) : M_L^\infty \to \mathbf{N}_0$ is defined by $N_L(t, \varphi) := \varphi_L(0, t]$. Recall that $E_L^0(\alpha_0^L | \mathcal{I}_L) = \bar{\alpha}_0^L$.

Lemma 2.3 *Let L, $L' \in$ Bor K be nonempty, $L \cap L' = \emptyset$, and $P(M_{L,L'}^\infty) = 1$. The following relations hold P-a.s. as well as P_L^0 - a.s.*

(a) $E(\frac{1}{\alpha_0^L} | \mathcal{I}_L) = E(N_L(1) | \mathcal{I}_L)$,
(b) $E_L^0(\alpha_0^L | \mathcal{I}_L) > 0$,
(c) $E(\frac{1}{\alpha_0^L} | \mathcal{I}_L) = \frac{1}{E_L^0(\alpha_0^L | \mathcal{I}_L)}$.

Parts (a), (b), *and* (c) *remain valid if $\mathcal{I}_L$ is replaced by $\mathcal{I}_{L,L'}$. The resulting relations hold $P_{L'}^0$-a.s. as well.*

Proof. Let $A \in \mathcal{I}_L$. Note that $\alpha_0^L = \alpha_0^L \circ \vartheta_{0,L}$. By (2.1), (1.7), and (1.3) we have

$$E(1_A E(\frac{1}{\alpha_0^L} | \mathcal{I}_L)) = E(1_A \frac{1}{\alpha_0^L}) = E_{0,L}(1_A \frac{1}{\alpha_0^L}) = \lambda(L) P_L^0(A) = E(1_A N_L(1)).$$

So, part (a) holds P-a.s., $P_{0,L}$-a.s., and hence P_L^0-a.s. Set $B := [E_L^0(\alpha_0^L | \mathcal{I}_L) \leq 0]$. Then

$$0 \geq E_L^0(1_B E_L^0(\alpha_0^L | \mathcal{I}_L)) = E_L^0(1_B \alpha_0^L).$$

Since $P_L^0[\alpha_0^L > 0] = 1$, we obtain

$$P_L^0(B^c) = 1 \qquad \text{and} \qquad P(B^c) = E(1_{B^c} \circ \vartheta_{0,L}) = P_{0,L}(B^c) = 1.$$

Part (b) follows. Let again $A \in \mathcal{I}_L$. By (2.1) and (1.7) we have

$$\begin{aligned} E\left(1_A \frac{1}{E_L^0(\alpha_0^L | \mathcal{I}_L)}\right) &= E\left(1_A \circ \vartheta_{0,L} \frac{1}{E_L^0(\alpha_0^L | \mathcal{I}_L) \circ \vartheta_{0,L}}\right) \\ &= \lambda(L) E_L^0\left(\alpha_0^L 1_A \frac{1}{E_L^0(\alpha_0^L | \mathcal{I}_L)}\right) \\ &= \lambda(L) P_L^0(A) = E\left(1_A \frac{1}{\alpha_0^L}\right) = E\left(1_A E\left(\frac{1}{\alpha_0^L} | \mathcal{I}_L\right)\right). \end{aligned}$$

In the third equality we conditioned on $\mathcal{I}_L$. Consequently, part (c) holds P-a.s., and by (2.1) also P_L^0-a.s. Since $\mathcal{I}_{L,L'} = \mathcal{I}_L \cap M_{L'}^\infty$ and $P(M_{L,L'}^\infty) = 1$, it is obvious that (a), (b) and (c) remain valid if $\mathcal{I}_L$ is replaced by $\mathcal{I}_{L,L'}$. By (2.3) the resulting expressions also hold under $P_{0,L'}$, and hence under $P_{L'}^0$. □

In view of Section 5 we need another lemma for the case that two nonempty, disjoint sets $L, L' \in \operatorname{Bor} K$ are involved. For $i \in \mathbf{Z}$ the random variable $\xi_i : M_{L,L'}^\infty \to [0, \infty)$ is defined by

$$\xi_i(\varphi) := \varphi_{L'}(X_i^L(\varphi), X_{i+1}^L(\varphi)], \quad \varphi \in M_{L,L'}^\infty. \tag{2.5}$$

So, $\xi_i(\varphi)$ is the number of L'-points in the interval $(X_i^L(\varphi), X_{i+1}^L(\varphi)]$. Note that $\xi_i(\vartheta_{1,L}\varphi) = \xi_{i+1}(\varphi)$ for all $\varphi \in M_{L,L'}^\infty$. Hence, (ξ_j) is P_L^0-stationary. The following lemma is a generalization of Baccelli and Brémaud (1987; (3.4.2)). Recall the definition of $N_L(t)$ preceding Lemma 2.3, and note that $E(N_L(1)|\mathcal{I}_{L,L'}) > 0$ P-a.s. since (by (1.3)) $B := [E(N_L(1)|\mathcal{I}_{L,L'}) \leq 0]$ satisfies

$$0 \geq E\left(1_B E(N_L(1)|\mathcal{I}_{L,L'})\right) = E(1_B N_L(1)) = \lambda(L) P_L^0(B).$$

Lemma 2.4 *Let $L, L' \in \operatorname{Bor} K$ be nonempty, $L \cap L' = \emptyset$, and $P(M_{L,L'}^\infty) = 1$. Then*

$$E_L^0\left(\xi_0|\mathcal{I}_{L,L'}\right) = \frac{E\left(N_{L'}(1)|\mathcal{I}_{L,L'}\right)}{E(N_L(1)|\mathcal{I}_{L,L'})} = \frac{E_L^0\left(\alpha_0^L|\mathcal{I}_{L,L'}\right)}{E_{L'}^0(\alpha_0^{L'}|\mathcal{I}_{L,L'})} \qquad P_L^0\text{-}, \ P_{L'}^0\text{-}, \ \textit{and} \ \ P\textit{-a.s.}.$$

Proof. If $t_1, t_2 \geq 0$ with $t_1 \leq t_2$, we write $N_{L'}(t_1, t_2] := N_{L'}(t_2) - N_{L'}(t_1)$. Note that, with this notation, $\xi_i = N_{L'}(X_i^L, X_{i+1}^L]$. Since (ξ_i) is P_L^0-stationary, we obtain

$$\frac{1}{n} N_{L'}(0, X_n^L] \to E_L^0(\xi_0|\mathcal{I}_{L,L'}) \qquad P_L^0\text{- a.s.}. \tag{2.6}$$

(Note that $N_{L'}(0, X_n^L] = \sum_{i=0}^{n-1} \xi_i \quad P_L^0$-a.s..) Since $\xi_i = \xi_i \circ \vartheta_{0,L}$, Relation (2.6) holds as well with P instead of P_L^0; cf. Theorem 1.1 (i). As

$$\frac{N_L(t)}{t} \frac{N_{L'}(0, X_{N_L(t)}^L]}{N_L(t)} \leq \frac{1}{t} N_{L'}(0, t] \leq \frac{N_{L'}(0, X_{N_L(t)+1}^L]}{N_L(t)+1} \frac{N_L(t)+1}{t}$$

on $[N_L(t) > 0]$, and

$$\frac{N_L(t)}{t} \to E(N_L(1)|\mathcal{I}_{L,L'}) \quad \text{and} \quad N_L(t) \to \infty \ \ P\text{-a.s.}, \tag{2.7}$$

we obtain

$$\frac{N_{L'}(t)}{t} \to E(N_L(1)|\mathcal{I}_{L,L'}) E_L^0(\xi_0|\mathcal{I}_{L,L'}) \ \ P\text{-a.s.} \tag{2.8}$$

Replacing L by L' in (2.7) yields $E(N_{L'}(1)|\mathcal{I}_{L,L'})$ as another limit of $t^{-1}N_{L'}(t)$, P-a.s. Hence,

$$E_L^0(\xi_0|\mathcal{I}_{L,L'}) = \frac{E\left(N_{L'}(1)|\mathcal{I}_{L,L'}\right)}{E(N_L(1)|\mathcal{I}_{L,L'})} \ \ P\text{-a.s..} \tag{2.9}$$

By (2.3), Relation (2.9) holds under $P_{0,L}$ or $P_{0,L'}$ as well. By Theorem 1.1 it also holds with P_L^0 or $P_{L'}^0$ instead of P. Lemma 2.3 yields

$$E(N_L(1)|\mathcal{I}_{L,L'}) = \frac{1}{E_L^0\left(\alpha_0^L|\mathcal{I}_{L,L'}\right)} \quad \text{and} \quad E(N_{L'}(1)|\mathcal{I}_{L,L'}) = \frac{1}{E_{L'}^0(\alpha_0^{L'}|\mathcal{I}_{L,L'})}$$

P_L^0-, $P_{L'}^0$-, and P-a.s.. Combining the above observations completes the proof. □

$P_{L'}^0$-expectations can directly be expressed in terms of P_L^0-expectations by Neveu's *exchange formula* (or *cycle formula*)

$$E_{L'}^0 f = \frac{\lambda(L)}{\lambda(L')} E_L^0 \left[\sum_{i=1}^{\xi_0} f \circ \vartheta_{i,L'}\right], \tag{2.10}$$

where $f : M_{L,L'}^{\infty} \to [0,\infty)$ is $P_{L'}^0$-integrable. This can be proved by replacing 1_A in (1.3) by $\sum_{i=1}^{\xi_0} f \circ \vartheta_{i,L'}$; see also Neveu (1977).

3 Approximation of P starting from P_L^0

In Glynn and Sigman (1992) convergence is considered for Cesaro means, uniform over functions f with $|f|$ bounded by a fixed function g. In the context of synchronous processes associated with a point process on $[0,\infty)$ sufficient conditions are formulated in Theorem 3.1 of this reference. In the present section we derive necessary and sufficient conditions for similar results within the framework of marked point processes on $\mathbf{R}$, using techniques which follow from Theorem 1.1. The Cesaro means $t^{-1}\int_0^t E_L^0(f \circ T_x)dx$ and $t^{-1}\int_0^t P_L^0[T_x\varphi \in B]dx$ will be considered. The limit $Q_L(B)$ of the latter is equal to $P(B)$ under a weak ergodicity condition. The relationship between Q_L and P, and between Q_L and Q_L^0 in (1.11) is investigated.

By a generalization to *marked* point processes of Theorem 3 in Nieuwenhuis (1994) we have

$$\frac{1}{t}\int_0^t f \circ T_x dx \to E(f|\mathcal{I}_L) \quad P\text{- and } P_L^0\text{-a.s.} \tag{3.1}$$

for all functions $f : M_L^\infty \to \mathbf{R}$ with $E|f| < \infty$. The limit $E(f|\mathcal{I}_L)$ equals Ef if Φ is ergodic. If $(t^{-1}\int_0^t f \circ T_x dx)_{t\geq 1}$ is u.i. under P_L^0, then

$$\frac{1}{t}\int_0^t E_L^0(f \circ T_x)dx \to E_L^0(E(f|\mathcal{I}_L)). \tag{3.2}$$

In this case we obtain for the choice $f(\varphi) = \varphi_L(0,1]$:

$$\frac{1}{t}\int_0^t E_L^0 N_L(x, x+1]dx \to E_L^0(E(N_L(1)|\mathcal{I}_L)). \tag{3.3}$$

Note that $N_L(x, x+1] = N_L(1) \circ T_x$. By the intuitive definition (1.1) of P it might be expected that the limit in (3.3) is equal to $EN_L(1) = \lambda(L)$. However, by (1.6), conditioning on $\mathcal{I}_L$, and Lemma 2.3 we obtain:

$$\begin{aligned} E_L^0(E(N_L(1)|\mathcal{I}_L)) &= \frac{1}{\lambda(L)}E\left(\frac{1}{\alpha_0^L}E(N_L(1)|\mathcal{I}_L)\right) = \frac{1}{\lambda(L)}E(E(N_L(1)|\mathcal{I}_L))^2 \\ &\geq \frac{1}{\lambda(L)}(EN_L(1))^2 = \lambda(L). \end{aligned}$$

Equality holds iff Φ is pseudo-L-ergodic. We conclude that for a formalization of (1.1) without any ergodicity restraint, we have to be careful because $E_L^0(E(f|\mathcal{I}_L))$ is not necessarily equal to Ef. It is, however, possible to write $E_L^0(E(f|\mathcal{I}_L))$ as an expectation of f. Let the probability measure Q_L on $(M_L^\infty, \mathcal{M}_L^\infty)$ be defined by

$$Q_L(B) := E_L^0[E(1_B|\mathcal{I}_L)], \quad B \in \mathcal{M}_L^\infty.$$

By Theorem 1.1(ii) and conditioning on $\mathcal{I}_L$ we obtain

$$Q_L(B) = \frac{1}{\lambda(L)} E\left[\frac{1}{\alpha_0^L} E(1_B|\mathcal{I}_L)\right] = \frac{1}{\lambda(L)} E\left[1_B E\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right)\right].$$

Since $E(1/\alpha_0^L|\mathcal{I}_L) > 0$ P-a.s.,

$$Q_L \sim P \qquad \text{and} \qquad \frac{dQ_L}{dP} = \frac{1}{\lambda(L)} E\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right) \qquad P\text{-a.s.} \tag{3.4}$$

Consequently, $E_{Q_L} f = E(fE(1/\alpha_0^L|\mathcal{I}_L))/\lambda(L) = E_L^0(Ef|\mathcal{I}_L))$. So, the limit in (3.2) is equal to $E_{Q_L} f$.

Uniform integrability will be the main condition to obtain limit results as in (3.2). For nonnegative functions f we can transform uniform P_L^0-integrability of the family $(t^{-1}\int_0^t f \circ T_x dx)_{t\geq 1}$ into uniform P-integrability for a similar family of r.v.'s.

Lemma 3.1 *Let $g : M_L^\infty \to [0,\infty)$ be P-integrable. Then:*

$$\left(\frac{1}{t}\int_0^t g \circ T_x dx\right)_{t\geq 1} \textit{u.i. under} \quad P_L^0 \iff \left(\frac{1}{\alpha_0^L}\frac{1}{t}\int_0^t g \circ T_x dx\right)_{t\geq 1} \textit{u.i. under} \quad P$$
$$\iff \left(g\frac{1}{t}\int_0^t \frac{1}{\alpha_0^L \circ T_{-x}} dx\right)_{t\geq 1} \textit{u.i. under} \quad P.$$

Proof. It is an easy exercise to prove that under P_L^0 uniform integrability of the family $\left(t^{-1}\int_0^t g \circ T_x dx\right)_{t\geq 1}$ is equivalent to uniform integrability of the sequence $(n^{-1}\int_0^n g \circ T_x dx)_{n\in\mathbb{N}}$. By Lemma 1.3, (3.1) with f replaced by g, and (1.6) we obtain:

$$\left(\tfrac{1}{t}\int_0^t g \circ T_x dx\right)_{t\geq 1} \text{ u.i. under } P_L^0$$

$$\iff \begin{cases} E_L^0(E(g|\mathcal{I}_L)) < \infty, \;\; E\left(\frac{1}{n\alpha_0^L}\int_{X_0^L}^{X_0^L+n} g \circ T_x dx\right) < \infty \text{ for all } n \in \mathbb{N}, \\ \frac{1}{n\lambda(L)} E\left(\frac{1}{\alpha_0^L}\int_{X_0^L}^{X_0^L+n} g \circ T_x dx\right) \to E_L^0(E(g|\mathcal{I}_L)). \end{cases}$$

Note that

$$
\begin{aligned}
&\frac{1}{n\lambda(L)} E\left|\frac{1}{\alpha_0^L}\int_{X_0^L}^{X_0^L+n} g\circ T_x dx - \frac{1}{\alpha_0^L}\int_0^n g\circ T_x dx\right| \le \\
&\le \frac{1}{n\lambda(L)}\left\{E\left[\frac{1}{\alpha_0^L}\int_{X_0^L}^0 g\circ T_x dx 1_{[X_0^L+n<0]}\right] + E\left[\frac{1}{\alpha_0^L}\int_{X_0^L+n}^n g\circ T_x dx 1_{[X_0^L+n<0]}\right]\right. \\
&\qquad \left. + E\left[\frac{1}{\alpha_0^L}\int_{X_0^L}^0 g\circ T_x dx 1_{[X_0^L+n\ge 0]}\right] + E\left[\frac{1}{\alpha_0^L}\int_{X_0^L+n}^n g\circ T_x dx 1_{[X_0^L+n\ge 0]}\right]\right\} \\
&\le \frac{1}{n\lambda(L)}\left\{E\left[\frac{1}{\alpha_0^L}\int_{1_{X_0^L}}^{X_1^L} g\circ T_x dx\right] + E\left[\frac{1}{\alpha_0^L}\int_{X_0^L+n}^{X_1^L+n} g\circ T_x dx\right]\right\} \\
&= \frac{1}{n}\left\{E_L^0\int_0^{\alpha_0^L} g\circ T_x dx + E_L^0\int_0^{\alpha_0^L} g\circ T_n\circ T_x dx\right\} = \frac{1}{n\lambda(L)}\{Eg + Eg\circ T_n\} \\
&= \frac{2}{n\lambda(L)} Eg.
\end{aligned}
$$

Since $Eg < \infty$ it follows that the right-hand part of the above equivalence is in turn equivalent to

$$
\begin{cases}
E_L^0(E(g|\mathcal{I}_L)) < \infty, \quad E\left(\frac{1}{n\alpha_0^L}\int_0^n g\circ T_x dx\right) < \infty \text{ for all } n\in\mathbf{N}, \\
\frac{1}{n\lambda(L)} E\left(\frac{1}{\alpha_0^L}\int_0^n g\circ T_x dx\right) \to E_L^0(E(g|\mathcal{I}_L)).
\end{cases}
$$

By Lemma 1.3 the first equivalence of the theorem follows immediately. Since

$$
\frac{1}{t}\int_0^t \frac{g\circ T_x}{\alpha_0^L} dx \to \frac{E(g|\mathcal{I}_L)}{\alpha_0^L} \text{ and } \frac{1}{t}\int_0^t \frac{g}{\alpha_0^L\circ T_{-x}} dx \to gE\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right) \quad P\text{- a.s.,}
$$

the second equivalence is also a consequence of Lemma 1.3 (use Fubini's theorem, stationarity of P, and conditioning on $\mathcal{I}_L$). □

In the following theorem $\sup_{|f|\le g}$ means the supremum over all measurable functions $f : M_L^\infty \to \mathbf{R}$ with $|f| \le g$. Recall the definition of pseudo-L-ergodicity in (1.10).

Theorem 3.2 *Let $g : M_L^\infty \to [0,\infty)$ be P-integrable. Then $(t^{-1}\int_0^t g\circ T_x dx)_{t\ge 1}$ is uniformly P_L^0-integrable iff $E_L^0(E(g|\mathcal{I}_L)) < \infty$, $E_L^0(g\circ T_x) < \infty$ ν_+-a.e., and*

$$
\sup_{|f|\le g}\left|\frac{1}{t}\int_0^t E_L^0(f\circ T_x)dx - E_{Q_L}f\right| \to 0 \text{ as } t\to\infty. \tag{3.5}
$$

If Φ is pseudo-L-ergodic, then the limits $E_{Q_L}f$ are equal to Ef.

Proof. First the only if-part of the iff statement. The finiteness of the expectations follows from Lemma 1.3 and Fubini's theorem. By Theorem 1.1 we have

$$\frac{1}{t}\int_0^t E_L^0(f\circ T_x)dx = \frac{1}{\lambda(L)t}\int_0^t E(\frac{1}{\alpha_0^L}f\circ T_x\circ\vartheta_{0,L})dx.$$

So, to prove (3.5) it is sufficient to prove that (3.6) and (3.7) below are satisfied:

$$\sup_{|f|\leq g}\frac{1}{\lambda(L)t}\left|\int_0^t E(\frac{1}{\alpha_0^L}f\circ T_x\circ\vartheta_{0,L})dx - \int_0^t E(\frac{1}{\alpha_0^L}f\circ T_x)dx\right|\to 0, \tag{3.6}$$

$$\sup_{|f|\leq g}\left|\frac{1}{\lambda(L)t}\int_0^t E(\frac{1}{\alpha_0^L}f\circ T_x)dx - E_{Q_L}f\right|\to 0, \tag{3.7}$$

as $t\to\infty$. By considering the expression below successively on $[X_0^L+n<0]$ and $[X_0^L+n\geq 0]$ as in the proof of Lemma 3.1, we obtain:

$$\frac{1}{\lambda(L)t\alpha_0^L}\left|\int_0^t f\circ T_x\circ\vartheta_{0,L}dx - \int_0^t f\circ T_x dx\right| \leq \frac{1}{\lambda(L)t\alpha_0^L}\left\{\int_{X_0^L}^{X_1^L} g\circ T_x dx + \int_{X_0^L+t}^{X_1^L+t} g\circ T_x dx\right\}$$

for all functions $f: M_L^\infty\to[0,\infty)$ with $|f|\leq g$. This upper bound does not depend on f. So, the supremum in (3.6) is bounded from above by

$$\frac{1}{\lambda(L)t}E(\frac{1}{\alpha_0^L}\textstyle\int_0^{\alpha_0^L} g\circ T_x\circ\vartheta_{0,L}dx) + \frac{1}{\lambda(L)t}E(\frac{1}{\alpha_0^L}\int_t^{\alpha_0^L+t} g\circ T_x\circ\vartheta_{0,L}dx) = \frac{2}{\lambda(L)t}Eg.$$

Again arguments as in the proof of Lemma 3.1 are used here. Relation (3.6) follows immediately. Next (3.7). By Theorem 1.1 and stationarity of P we have

$$\begin{aligned}\left|\frac{1}{\lambda(L)t}\textstyle\int_0^t E(\frac{1}{\alpha_0^L}f\circ T_x)dx - E_{Q_L}f\right| &= \frac{1}{\lambda(L)}\left|\frac{1}{t}\textstyle\int_0^t E(f\frac{1}{\alpha_0^L\circ T_{-x}})dx - E\left(fE(\frac{1}{\alpha_0^L}\,|\mathcal{I}_L)\right)\right|\\ &\leq \frac{1}{\lambda(L)}E\left[g\left|\frac{1}{t}\textstyle\int_0^t\frac{1}{\alpha_0^L\circ T_{-x}}dx - E(\frac{1}{\alpha_0^L}|\mathcal{I}_L)\right|\right].\end{aligned}$$

This upper bound tends to zero because of the second equivalence in Lemma 3.1. Relation (3.7) follows.

The if-part of the iff statement follows immediately from (3.1) (with f replaced by g) and Lemma 1.3. The last part of the theorem is a consequence of (3.4). □

Let $g : M_L^\infty \to [0, \infty)$ be P-integrable. By (1.14) and Lemma 3.1 the following implications are obvious:

$$\begin{array}{llll} (g \circ T_x)_{x>0} & \text{u.i. under } P_L^0 & \Longrightarrow \left(t^{-1}\int_0^t g\circ T_x dx\right)_{t\geq 1} & \text{u.i. under } P_L^0, \\ \left(\frac{1}{\alpha_0^L} g \circ T_x\right)_{x>0} & \text{u.i. under } P & \Longrightarrow \left(t^{-1}\int_0^t g\circ T_x dx\right)_{t\geq 1} & \text{u.i. under } P_L^0. \end{array} \tag{3.8}$$

Note also that

$$\begin{aligned} \sup_{x>0} E\left(\frac{1}{\alpha_0^L} g\circ T_x 1_{[\frac{1}{\alpha_0^L} g\circ T_x > b]}\right) &\leq \sup_{x>0} E\left(\frac{1}{\alpha_0^L} g\circ T_x 1_{[\frac{1}{\alpha_0^L} > \sqrt{b}]}\right) + \\ &\quad + \sup_{x>0} E\left(\frac{1}{\alpha_0^L} g\circ T_x 1_{[g\circ T_x > \sqrt{b}]}\right) \\ \leq \sqrt{E\left(\frac{1}{\alpha_0^L}\right)^2 1_{\left[\left(\frac{1}{\alpha_0^L}\right)^2 > b\right]}}\sqrt{Eg^2} &+ \sqrt{E\left(\frac{1}{\alpha_0^L}\right)^2}\sqrt{E(g^2 1_{[g^2>b]})}. \end{aligned}$$

Consequently,

Corollary 3.3 *Suppose that* $E(1/\alpha_0^L)^2 < \infty$. *Let* $g : M_L^\infty \to [0,\infty)$ *be such that* $Eg^2 < \infty$. *Then*

$$\sup_{|f|\leq g}\left|\frac{1}{t}\int_0^t E_L^0(f\circ T_x)dx - E_{Q_L} f\right| \to 0 \ \text{as } t\to\infty.$$

When starting from P_L^0, we can consider Q_L as the uniform limit (as $t \to \infty$) of the distribution of the MPP seen from a position chosen at random in the interval $(0, t)$. The limit Q_L is equal to P if $n^{-1}\sum_{i=1}^n \alpha_i^L \to 1/\lambda(L)$ $\quad P_L^0$-a.s.. These assertions are expressed in the following corollary. It is an immediate consequence of Theorem 3.2.

Corollary 3.4 *The convergence*

$$\frac{1}{t}\int_0^t P_L^0[T_x\varphi \in B]dx \to Q_L(B) \tag{3.9}$$

holds uniformly over $B \in \mathcal{M}_L^\infty$. $Q_L = P$ *iff* Φ *is pseudo-L-ergodic.*

The existence of the limit in (3.9) was already proved in Nawrotzki (1978; Satz 2.1).

Note that by stationarity of P and the right-hand part of (2.1),

$$\begin{aligned} Q_L[T_a\varphi \in B] &= \frac{1}{\lambda(L)} E\left[E\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right) 1_B \circ T_a\right] \\ &= \frac{1}{\lambda(L)} E\left[E\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right) 1_B\right] = Q_L(B) \end{aligned}$$

for all $B \in \mathcal{M}_L^\infty$ and $a \in \mathbf{R}$. Hence, Q_L is also stationary. Since $Q_L = P$ and $Q_L^0 = P_L^0$ (see (1.12)) provided that Φ is pseudo-L-ergodic, one might wonder if Q_L^0 is the Palm distribution with respect to L associated with Q_L. To prove that this is usually not the case, let $\tilde{Q}_L^0$ be this Palm distribution associated with Q_L and let $\tilde{\lambda}(L)$ be the intensity of the L-points under Q_L. Recall the definition of $N_L(1)$ preceding Lemma 2.3. By (3.4), conditioning on $\mathcal{I}_L$, Theorem 1.1, and Lemma 2.3 we have

$$\begin{aligned} \tilde{\lambda}(L) &= E_{Q_L} N_L(1) = \frac{1}{\lambda(L)} E\left(\frac{1}{\alpha_0^L} E(N_L(1)|\mathcal{I}_L)\right) = \\ &= E_L^0(E(N_L(1)|\mathcal{I}_L)) = E_L^0\left(\frac{1}{\bar{\alpha}_0^L}\right), \end{aligned}$$

provided that this expectation is finite. By applying Theorem 1.1 also to $(Q_L, \tilde{Q}_L^0)$ we obtain

$$\begin{aligned} \tilde{Q}_L^0(B) &= \frac{1}{\tilde{\lambda}(L)} E_{Q_L}\left(\frac{1}{\alpha_0^L} 1_B \circ \vartheta_{0,L}\right) &= \frac{1}{\tilde{\lambda}(L)\lambda(L)} E\left(\frac{1}{\alpha_0^L} E\left(\frac{1}{\alpha_0^L}\middle| \mathcal{I}_L\right) 1_B \circ \vartheta_{0,L}\right) \\ &= \frac{1}{\tilde{\lambda}(L)} E_L^0\left(\frac{1}{\bar{\alpha}_0^L} 1_B\right) &= \frac{E_L^0\left(1_B/\bar{\alpha}_0^L\right)}{E_L^0\left(1/\bar{\alpha}_0^L\right)} \end{aligned}$$

for all $B \in \mathcal{M}_L^\infty$. Consequently,

$$\tilde{Q}_L^0 \sim P_L^0 \quad \text{and} \quad \frac{d\tilde{Q}_L^0}{dP_L^0} = \frac{1/\bar{\alpha}_0^L}{E_L^0(1/\bar{\alpha}_0^L)}. \tag{3.10}$$

Hence (cf. (1.12)),

$$\frac{d\tilde{Q}_L^0}{dQ_L^0} = \frac{d\tilde{Q}_L^0}{dP_L^0}\frac{dP_L^0}{dQ_L^0} = \frac{1/(\bar{\alpha}_0^L)^2}{\lambda(L) E_L^0(1/\bar{\alpha}_0^L)}$$

and

$$\tilde{Q}_L^0 = Q_L^0 \qquad \text{iff} \qquad \Phi \text{ is pseudo-}L\text{-ergodic.}$$

This last result ensures that Q_L^0 is the Palm distribution with respect to L associated with Q_L iff Φ is pseudo-L-ergodic.

For $A \in \mathcal{I}_L$ we have (see (3.4) and (1.12))

$$\begin{aligned} Q_L(A) &= \frac{1}{\lambda(L)} E(1_A E(\frac{1}{\alpha_0^L}|\mathcal{I}_L)) &= \frac{1}{\lambda(L)} E(1_A \frac{1}{\alpha_0^L}), \\ Q_L^0(A) &= \lambda(L) E_L^0(1_A E_L^0(\alpha_0^L|\mathcal{I}_L)) &= \lambda(L) E_L^0(1_A \alpha_0^L). \end{aligned}$$

By Theorem 1.1(ii) we conclude,

$$Q_L|_{\mathcal{I}_L} = P_L^0\Big|_{\mathcal{I}_L} \quad \text{and} \quad Q_L^0\Big|_{\mathcal{I}_L} = P|_{\mathcal{I}_L}. \tag{3.11}$$

4 Approximation of P_L^0 starting from P

When starting from P, the distribution of Φ seen from an L-point chosen at random from the first n L-points tends uniformly to Q_L^0 as $n \to \infty$; see Theorem 1.2. In the present section we generalize this result to a uniform limit theorem for averaged functionals $(n^{-1} \sum_{i=1}^n Ef \circ \vartheta_{i,L})_{n \in \mathbf{N}}$.

For all functions $f : M_L^\infty \to \mathbf{R}$ with $E_L^0|f| < \infty$ we have

$$\frac{1}{n}\sum_{i=1}^n f \circ \vartheta_{i,L} \to E_L^0(f|\mathcal{I}_L) \quad P_L^0 \text{ - and } P \text{ -a.s.,} \tag{4.1}$$

cf. Nieuwenhuis (1994; Th. 3). Note that the limit is equal to $E_L^0 f$ if Φ is ergodic. If $(n^{-1} \sum_{i=1}^n Ef \circ \vartheta_{i,L})_{n \geq 1}$ is u.i. under P, then

$$\frac{1}{n}\sum_{i=1}^n Ef \circ \vartheta_{i,L} \to E(E_L^0(f|\mathcal{I}_L)). \tag{4.2}$$

Because of (1.12) and (1.7) it is an easy exercise to prove that the limit in (4.2) is equal to $E_{Q_L^0} f$.

The main condition in Theorem 4.2 below is about uniform P-integrability of $(n^{-1}\sum_{i=1}^n g\circ\vartheta_{i,L})_{n\geq 1}$. In the following lemma this is characterized. It will be applied in the proof of the theorem.

Lemma 4.1 *Let $g: M_L^\infty \to [0,\infty)$ be P_L^0-integrable. Then*

$$\left(\frac{1}{n}\sum_{i=1}^n g\circ\vartheta_{i,L}\right)_{n\geq 1} \textit{ u.i. under } P \iff \left(\alpha_0^L\frac{1}{n}\sum_{i=1}^n g\circ\vartheta_{i,L}\right)_{n\geq 1} \textit{ u.i. under } P_L^0$$
$$\iff \left(g\frac{1}{n}\sum_{i=1}^n \alpha_{-i}^L\right)_{n\geq 1} \textit{ u.i. under } P_L^0.$$

Proof. By (4.1) and Lemma 1.3, (1.7), and (1.4) we obtain:

$$\left(\tfrac{1}{n}\textstyle\sum_{i=1}^n g\circ\vartheta_{i,L}\right)_{n\geq 1} \quad \text{u.i. under } P$$

$$\iff \begin{cases} E(E_L^0(g|\mathcal{I}_L))<\infty,\ Eg\circ\vartheta_{i,L}<\infty \text{ for all } i\in\mathbb{N}, \text{ and} \\ \frac{1}{n}\sum_{i=1}^n Eg\circ\vartheta_{i,L}\to E(E_L^0(g|\mathcal{I}_L)) \end{cases}$$

$$\iff \begin{cases} E_L^0\left(\alpha_0^L E_L^0(g|\mathcal{I}_L)\right)<\infty,\ E_L^0\left(\alpha_0^L g\circ\vartheta_{i,L}\right)<\infty \text{ for all } i\in\mathbb{N}, \text{ and} \\ \frac{1}{n}\sum_{i=1}^n E_L^0\left(\alpha_0^L g\circ\vartheta_{i,L}\right)\to E_L^0\left(\alpha_0^L E_L^0(g|\mathcal{I}_L)\right) \end{cases}$$

$$\iff \begin{cases} E_L^0(g\bar\alpha_0^L)<\infty,\ E_L^0(g\alpha_{-i}^L)<\infty \text{ for all } i\in\mathbb{N}, \text{ and} \\ \frac{1}{n}\sum_{i=1}^n E_L^0(g\alpha_{-i}^L)\to E_L^0(g\bar\alpha_0^L). \end{cases}$$

Note that

$$\alpha_0^L\frac{1}{n}\sum_{i=1}^n g\circ\vartheta_{i,L}\to\alpha_0^L E_L^0(g|\mathcal{I}_L) \text{ and } g\frac{1}{n}\sum_{i=1}^n\alpha_{-i}^L\to g\bar\alpha_0^L \quad P_L^0\text{- a.s..} \tag{4.3}$$

So, by Lemma 1.3 the right-hand parts of the second and third equivalences above are in turn equivalent to uniform P_L^0-integrability of $\left(\alpha_0^L n^{-1}\sum_{i=1}^n g\circ\vartheta_{i,L}\right)_{n\geq 1}$ and $\left(gn^{-1}\sum_{i=1}^n\alpha_{-i}^L\right)_{n\geq 1}$, respectively. □

The following theorem is a generalization of a part of Theorem 1.2. Here $\sup_{|f|\leq g}$ means the supremum over all measurable functions $f: M_L^\infty\to\mathbf{R}$ with $|f|\leq g$.

Theorem 4.2 *Let $g : M_L^\infty \to [0,\infty)$ be P_L^0-integrable. Then $(n^{-1}\sum_{i=1}^n g \circ \vartheta_{i,L})_{n\geq 1}$ is uniformly P-integrable iff $E(E_L^0(g|\mathcal{I}_L)) < \infty$, $Eg \circ \vartheta_{i,L} < \infty$ for all $i \in \mathbf{N}$, and*

$$\sup_{|f|\leq g}\left|\frac{1}{n}\sum_{i=1}^n Ef \circ \vartheta_{i,L} - E_{Q_L^0} f\right| \to 0. \tag{4.4}$$

If Φ is pseudo-L-ergodic, then the limits $E_{Q_L^0} f$ are equal to $E_L^0 f$.

Proof. The last part follows immediately, since $E_{Q_L^0} f = \lambda(L) E_L^0 \left(\bar{\alpha}_0^L f\right)$. Suppose that $(n^{-1}\sum_{i=1}^n g \circ \vartheta_{i,L})$ is u.i. under P. By (4.1) and Lemma 1.3 the finiteness of $E(E_L^0(g|\mathcal{I}_L))$ and $Eg \circ \vartheta_{i,L}$, $i \in \mathbf{N}$, is obvious. By Theorem 1.1 we obtain

$$\begin{aligned}
\left|\frac{1}{n}\sum_{i=1}^n Ef \circ \vartheta_{i,L} - E_{Q_L^0} f\right| &= \left|\frac{1}{n}\sum_{i=1}^n E_{i,L} f - E_{Q_L^0} f\right| \\
&= \lambda(L)\left|E_L^0\Big(\frac{1}{n}\sum_{i=1}^n f\alpha_{-i}^L\Big) - E_L^0(f\bar{\alpha}_0^L)\right| \\
&\leq \lambda(L) E_L^0\left[g\Big|\frac{1}{n}\sum_{i=1}^n \alpha_{-i}^L - \bar{\alpha}_0^L\Big|\right]
\end{aligned}$$

for all measurable functions $f : M_L^\infty \to \mathbf{R}$ with $|f| \leq g$. This upper bound does not depend on f, and tends to zero because of the last equivalence in Lemma 4.1. Relation (4.4) follows. The reversed implication of the iff statement is an immediate consequence of (4.1) and Lemma 1.3. □

Remark. In view of Section 6 slight generalizations of Lemma 4.1 and Theorem 4.2 are of interest. Apart from $g : M_L^\infty \to [0,\infty)$ with $E_L^0 g < \infty$, an arbitrary (but fixed) $\mathcal{I}_L$-measurable function $\beta : M_L^\infty \to [0,\infty)$ is considered. Since $\beta n^{-1}\sum_{i=1}^n g \circ \vartheta_{i,L} \to \beta E_L^0(g|\mathcal{I}_L)$ P-a.s., it is an easy exercise to prove that the conclusions of Lemma 4.1 and Theorem 4.2 remain valid if g is replaced by βg and f by βf; $\sup_{|f|\leq g}$ remains unchanged. By these replacements (4.4) turns into (cf. (2.1)):

$$\sup_{|f|\leq g}\left|\frac{1}{n}\sum_{i=1}^n E\left(\beta f \circ \vartheta_{i,L}\right) - E_{Q_L^0}(\beta f)\right| \to 0.$$

Note that the P_L^0-integrability of g (and not of βg) remains the only condition for the validity of the equivalence in Theorem 4.2 when generalized as above.

By (1.14) it is obvious that

$$(g \circ \vartheta_{i,L})_{i\geq 1} \text{ u.i. under } P \Longrightarrow \left(\frac{1}{n}\sum_{i=1}^{n} g \circ \vartheta_{i,L}\right)_{n\geq 1} \text{ u.i. under } P. \tag{4.5}$$

Note also that

$$E(g \circ \vartheta_{i,L} 1_{[g\circ\vartheta_{i,L}>b]}) = \lambda(L) E_L^0(\alpha_{-i}^L g 1_{[g>b]}) \leq \lambda(L)\sqrt{E_L^0(\alpha_0^L)^2 E_L^0 g^2 1_{[g>b]}},$$

which tends to zero as $b \to \infty$, provided that $E_L^0(\alpha_0^L)^2$ and $E_L^0 g^2$ (or, equivalently, $E\alpha_0^L$ and $E(g^2 \circ \vartheta_{0,L}/\alpha_0^L))$ are finite. We conclude:

Corollary 4.3 *Suppose that $E_L^0(\alpha_0^L)^2 < \infty$. Let $g : M_L^\infty \to [0,\infty)$ be such that $E_L^0 g^2 < \infty$. Then*

$$\sup_{|f|\leq g} \left|\frac{1}{n}\sum_{i=1}^{n} Ef \circ \vartheta_{i,L} - E_{Q_L^0} f\right| \to 0 \quad \textit{as } n \to \infty.$$

5 Approximation of P_L^0 starting from $P_{L'}^0$

In this section two nonempty, disjoint sets of marks, L and L', are considered. For the case that P is replaced by $P_{L'}^0$ results similar to the results of Section 4 are derived.

Let $L, L' \in \text{Bor}\, K$ be such that $L \cap L' = \emptyset$ and $P(M_{L,L'}^\infty) = 1$. Since $\mathcal{I}_{L,L'} = \mathcal{I}_{L',L}$ (cf. Lemma 2.2(b)), we will omit the subscripts and write $\mathcal{I}$ for both invariant σ-fields. When two sets of marks are involved, we will always restrict $\vartheta_{i,L}, \vartheta_{i,L'}$, and T_t to $M_{L,L'}^\infty$. We will prove a theorem similar to Theorem 4.2 in the case that P is replaced by $P_{L'}^0$. Some preliminaries are needed first. Random variables $\xi_i, i \in \mathbf{Z}$, are defined by

$$\xi_i(\varphi) := \varphi_{L'}(X_i^L(\varphi), X_{i+1}^L(\varphi)], \;\; \varphi \in M_{L,L'}^\infty, \tag{5.1}$$

the number of L'-points between the i'th and the $(i+1)$'th L-point. Note that

$$\xi_i \circ \vartheta_{j,L}(\varphi) = \xi_{i+j}(\varphi) \tag{5.2}$$

for all $\varphi \in M^{\infty}_{L,L'}, i \in \mathbf{Z}$, and $j \in \mathbf{Z}$. The following theorem is the analogue of Theorem 1.1 for the case that P is replaced by $P^0_{L'}$.

Theorem 5.1 *Let $n \in \mathbf{Z}$. Then*

(i) $P^0_{L'}\vartheta^{-1}_{n,L} << P^0_L$,

(ii) $\dfrac{d(P^0_{L'}\vartheta^{-1}_{n,L})}{dP^0_L} = \dfrac{\lambda(L)}{\lambda(L')}\xi_{-n}$ *P^0_L-a.s.*

Proof. By (2.10) we obtain

$$\begin{aligned} P^0_{L'}[\vartheta_{n,L}\varphi \in A] &= \frac{\lambda(L)}{\lambda(L')}E^0_L\left[\sum_{i=1}^{\xi_0} 1_A \circ \vartheta_{n,L} \circ \vartheta_{i,L'}\right] \\ &= \frac{\lambda(L)}{\lambda(L')}E^0_L(\xi_0(1_A \circ \vartheta_{n,L})) = \frac{\lambda(L)}{\lambda(L')}E^0_L(\xi_{-n}1_A). \end{aligned}$$

The last equality is a consequence of (1.4) and (5.2). The theorem follows immediately. □

For a stationary marked point process with $\alpha_i^L = 2$ and $\alpha_i^{L'} = 6$ P-a.s. (and hence P^0_L - and $P^0_{L'}$ - a.s., cf. Theorems 1.1(i) and 5.1(i)), $i \in \mathbf{Z}$, we have

$$P^0_L[\xi_{-n} = 0] = \frac{2}{3} \quad \text{and} \quad P^0_{L'}[\vartheta_{n,L}\varphi \in [\xi_{-n} = 0]] = P^0_{L'}[\xi_0 = 0] = 0.$$

So, P^0_L and $P^0_{L'}\vartheta^{-1}_{n,L}$ are not necessarily equivalent. As an immediate consequence of Theorem 5.1 (take $A = M^{\infty}_{L,L'}$ in the proof) we obtain

$$E^0_L\xi_{-n} = \frac{\lambda(L')}{\lambda(L)}, \quad n \in \mathbf{Z}. \tag{5.3}$$

See also Baccelli and Brémaud (1987; (3.4.2)).

Recall (4.1). Since $P^0_{L'}\vartheta^{-1}_{0,L} << P^0_L$ it is obvious that the convergence holds $P^0_{L'}$-a.s. as well:

$$\frac{1}{n}\sum_{i=1}^{n} f \circ \vartheta_{i,L} \to E^0_L(f|\mathcal{I}) \quad P^0_{L'} \text{ -a.s.} \tag{5.4}$$

for all P_L^0-integrable functions $f : M_{L,L'}^\infty \to \mathbf{R}$. If $(n^{-1}\sum_{i=1}^n f \circ \vartheta_{i,L})_{n\geq 1}$ is u.i. under $P_{L'}^0$, then

$$\frac{1}{n}\sum_{i=1}^n E_{L'}^0 f \circ \vartheta_{i,L} \to E_{L'}^0(E_L^0(f|\mathcal{I})). \tag{5.5}$$

The limit in (5.5) can be written as an expectation of f. Let the probability measure $Q_{L,L'}^0$ be defined by

$$Q_{L,L'}^0(B) := E_{L'}^0(E_L^0(1_B|\mathcal{I})), \quad B \in \mathcal{M}_{L,L'}^\infty. \tag{5.6}$$

Set $M^0 := M_L^0 \cap M_{L'}^\infty$. Note that $P_L^0[E_L^0(1_{M^0}|\,\mathcal{I}) = 1] = 1$. Since $P_{L'}^0\vartheta_{0,L}^{-1} << P_L^0$, we obtain by (2.3) that $P_{L'}^0[E_L^0(1_{M^0}|\mathcal{I}) = 1] = 1$. Hence, $Q_{L,L'}^0(M^0) = 1$. By Theorem 5.1 and Lemma 2.4 we have

$$\begin{aligned} Q_{L,L'}^0(B) &= E_{L'}^0(E_L^0(1_B|\mathcal{I}) \circ \vartheta_{0,L}) = \frac{\lambda(L)}{\lambda(L')}E_L^0(E_L^0(1_B|\mathcal{I})\xi_0) \\ &= \frac{\lambda(L)}{\lambda(L')}E_L^0(1_B E_L^0(\xi_0|\mathcal{I})) = \frac{\lambda(L)}{\lambda(L')}E_L^0(1_B \frac{E_L^0(\alpha_0^L|\mathcal{I})}{E_{L'}^0(\alpha_0^{L'}|\mathcal{I})}), \end{aligned}$$

$B \in \mathcal{M}_{L,L'}^\infty$. Consequently, on $(M_{L,L'}^\infty, \mathcal{M}_{L,L'}^\infty)$,

$$Q_{L,L'}^0 \sim P_L^0 \quad \text{and} \quad \frac{dQ_{L,L'}^0}{dP_L^0} = \frac{\lambda(L)}{\lambda(L')}E_L^0(\xi_0|\mathcal{I}) = \frac{\lambda(L)E_L^0(\alpha_0^L|\mathcal{I})}{\lambda(L')E_{L'}^0(\alpha_0^{L'}|\mathcal{I})}. \tag{5.7}$$

Note also that $Q_{L,L'}^0 = Q_L^0$ if Φ is pseudo-L'-ergodic; cf. (1.12). By Theorem 5.1 and (5.7) the limit $E_{L'}^0(E_L^0(f|\mathcal{I}))$ in (5.5) is equal to

$$\frac{\lambda(L)}{\lambda(L')}E_L^0(\xi_0 E_L^0(f|\mathcal{I})) = \frac{\lambda(L)}{\lambda(L')}E_L^0(f E_L^0(\xi_0|\mathcal{I})) = E_{Q_{L,L'}^0} f.$$

Next we state the analogue of Lemma 4.1. Apart from replacing P by $P_{L'}^0$, and α_0^L by ξ_0, its proof is similar to the proof of Lemma 4.1. Theorem 5.1 and, again, Lemma 1.3 supply important ingredients.

Lemma 5.2 *Let $g : M^{\infty}_{L,L'} \to [0,\infty)$ be P^0_L-integrable. Then:*

$$\left(\frac{1}{n}\sum_{i=1}^{n} g\circ\vartheta_{i,L}\right)_{n\geq 1} \text{ u.i. under } P^0_{L'} \iff \left(\xi_0\frac{1}{n}\sum_{i=1}^{n} g\circ\vartheta_{i,L}\right)_{n\geq 1} \text{ u.i. under } P^0_L$$
$$\iff \left(g\frac{1}{n}\sum_{i=1}^{n}\xi_{-i}\right)_{n\geq 1} \text{ u.i. under } P^0_L.$$

The following theorem is the analogue of Theorem 4.2; $\sup_{|f|\leq g}$ means the supremum over all measurable functions $f : M^{\infty}_{L,L'} \to \mathbf{R}$ with $|f| \leq g$.

Theorem 5.3 *Let $g : M^{\infty}_{L,L'} \to [0,\infty)$ be P^0_L-integrable. Then $(n^{-1}\sum_{i=1}^{n} g\circ\vartheta_{i,L})$ is uniformly $P^0_{L'}$-integrable iff $E^0_{L'}(E^0_L(g|\mathcal{I})) < \infty$, $E^0_{L'} g\circ\vartheta_{i,L} < \infty$ for all $i\in\mathbf{N}$, and*

$$\sup_{|f|\leq g}\left|\frac{1}{n}\sum_{i=1}^{n} E^0_{L'} f\circ\vartheta_{i,L} - E_{Q^0_{L,L'}} f\right| \to 0. \tag{5.8}$$

If Φ is pseudo-L-ergodic and pseudo-L′-ergodic, then the limits $E_{Q^0_{L,L'}} f$ are equal to $E^0_L f$.

Proof. The last part is a consequence of (5.7). Suppose that $(n^{-1}\sum_{i=1}^{n} g\circ\vartheta_{i,L})_{n\geq 1}$ is u.i. under $P^0_{L'}$. For all measurable $f : M^{\infty}_{L,L'} \to \mathbf{R}$ with $|f| \leq g$ we have (cf. Theorem 5.1 and (2.3)),

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n} E^0_{L'} f\circ\vartheta_{i,L} - E_{Q^0_{L,L'}} f\right| &= \frac{\lambda(L)}{\lambda(L')}\left|\frac{1}{n}\sum_{i=1}^{n} E^0_L(f\xi_{-i}) - E^0_L(fE^0_L(\xi_0|\mathcal{I}))\right| \\ &\leq \frac{\lambda(L)}{\lambda(L')} E^0_L\left[g\left|\frac{1}{n}\sum_{i=1}^{n}\xi_{-i} - E^0_L(\xi_0|\mathcal{I})\right|\right].\end{aligned}$$

This upper bound does not depend on f and tends to zero (as $n\to\infty$) because of Lemma 5.2, which proves (5.8). The reversed implication follows from (5.4) and Lemma 1.3. □

Remark. Lemma 5.2 and Theorem 5.3 can be generalized slightly by considering, apart from the P^0_L-integrable, nonnegative function g, a fixed $\mathcal{I}$-measurable function $\beta : M^{\infty}_{L,L'} \to [0,\infty)$. The conclusions of the lemma and the theorem remain valid if g and f are replaced by βg and βf. Relation (5.8) turns into (cf. (2.3))

$$\sup_{|f|\leq g}\left|\frac{1}{n}\sum_{i=1}^{n} E^0_{L'}(\beta f\circ\vartheta_{i,L}) - E_{Q^0_{L,L'}}(\beta f)\right| \to 0.$$

Again $E_L^0 g < \infty$ remains the only assumption.

Note that

$$E_{L'}^0(g \circ \vartheta_{i,L} 1_{[g\circ\vartheta_{i,L} > b]}) = \frac{\lambda(L)}{\lambda(L')} E_L^0(\xi_{-i} g 1_{[g>b]}) \leq \frac{\lambda(L)}{\lambda(L')} \sqrt{E_L^0 \xi_0^2 E_L^0 g^2 1_{[g>b]}}$$

for all $i \in \mathbf{Z}$. The hypothesis about uniform integrability in Theorem 5.3 is satisfied if $(g \circ \vartheta_{i,L})_{i\geq 1}$ is u.i. under $P_{L'}^0$, and hence if $E_L^0(\xi_0^2) < \infty$ (or, equivalently, $E_{L'}^0 \xi_0 < \infty$) and $E_L^0(g^2) < \infty$.

In Konstantopoulos and Walrand (1988; Th.3) weak convergence of the sequence $(P_{L'}^0[\vartheta_{n,L}\varphi \in .])_{n\geq 1}$ of probability measures is considered under some additional mixing condition. See also König and Schmidt (1986). The following corollary of Theorem 5.3 concerns uniform convergence of the sequence $(n^{-1}\sum_{i=1}^n P_{L'}^0[\vartheta_{n,L}\varphi \in .])_{n\geq 1}$ without any additional condition. It expresses that starting with $P_{L'}^0$ we can (as $n \to \infty$) consider $Q_{L,L'}^0$ as the distribution of the MPP seen from an L-point chosen at random among the first n L-points.

Corollary 5.4 *Let $L, L' \in \mathrm{Bor}\, K$ be such that $L \cap L' = \emptyset$ and $P(M_{L,L'}^\infty) = 1$. Then*

$$\sup_{B \in \mathcal{M}_{L,L'}^\infty} \Big|\frac{1}{n}\sum_{i=1}^n P_{L'}^0[\vartheta_{i,L}\varphi \in B] - Q_{L,L'}^0(B)\Big| = \frac{\lambda(L)}{2\lambda(L')} E_L^0 \Big|\frac{1}{n}\sum_{i=1}^n \xi_{-i} - E_L^0(\xi_0|\mathcal{I})\Big|.$$

This supremum tends to 0 as $n \to \infty$.

Proof. By Theorem 5.1 the probability measures $n^{-1}\sum_{i=1}^n P_{L'}^0 \vartheta_{i,L}^{-1}$, $n \in \mathbf{Z}$, are all dominated by P_L^0 with Radon-Nikodym derivatives $(\lambda(L)/\lambda(L'))n^{-1}\sum_{i=1}^n \xi_{-i}$. The equality is an immediate consequence of (1.15) and (5.7). The convergence to 0 follows from Theorem 5.3 with the choice $g \equiv 1$. □

6 Approximations without ergodicity restraints

The intuitive random procedures (1.2) and (1.1) for generating P_L^0 and P were formalized in Theorem 1.2 and Corollary 3.4. For a direct approximation of these probability measures a weak ergodicity condition was needed. In this section the results of Sections 3 to 5 will be applied to derive approximations of P_L^0 and P without assuming ergodicity properties.

The limits in Theorem 1.2, Corollary 3.4, and Corollary 5.4 are not P_L^0, P, and P_L^0, but Q_L^0, Q_L, and $Q_{L,L'}^0$, respectively. The pairwise relationships between corresponding probability measures were described by Radon-Nikodym derivatives, which are repeated here:

$$\frac{dQ_L^0}{dP_L^0} = \lambda(L)\bar{\alpha}_0^L, \quad \frac{dQ_L}{dP} = \frac{1}{\lambda(L)\bar{\alpha}_0^L}, \quad \frac{dQ_{L,L'}^0}{dP_L^0} = \frac{\lambda(L)}{\lambda(L')} E_L^0(\xi_0|\mathcal{I}). \tag{6.1}$$

For approximation of P_L^0, starting from P and $P_{L'}^0$ respectively, choices for g and β in the remarks following Theorem 4.2 and 5.3 are suggested by (6.1). Choose, respectively,

$$g \equiv 1 \text{ and } \beta = \frac{1}{\lambda(L)\bar{\alpha}_0^L}, \quad g \equiv 1 \text{ and } \beta = \frac{\lambda(L')}{\lambda(L)} \frac{1}{E_L^0(\xi_0|\mathcal{I})}. \tag{6.2}$$

For g in Theorem 3.2 we take $\lambda(L)\bar{\alpha}_0^L$.

Theorem 6.1

(a) $\sup_{B \in M_L^\infty} \left| \frac{1}{n} \sum_{i=1}^{n} E\left(\frac{1}{\lambda(L)\bar{\alpha}_0^L} 1_B \circ \vartheta_{i,L} \right) - P_L^0(B) \right| \to 0$ *as* $n \to \infty$.

(b) $\sup_{B \in M_{L,L'}^\infty} \left| \frac{1}{n} \sum_{i=1}^{n} E_{L'}^0 \left(\frac{\lambda(L')}{\lambda(L)} \frac{1}{E_L^0(\xi_0|\mathcal{I})} 1_B \circ \vartheta_{i,L} \right) - P_L^0(B) \right| \to 0$ *as* $n \to \infty$.

(c) *If* $E\bar{\alpha}_0^L < \infty$, *then* $\sup_{B \in M_L^\infty} \left| \frac{1}{t} \int_0^t E_L^0(\lambda(L)\bar{\alpha}_0^L 1_B \circ T_x) dx - P(B) \right| \to 0$ *as* $t \to \infty$.

Proof. For (a) and (b) we choose g and β as suggested in (6.2). By Theorems 1.1 and 5.1 we have:

$$E\left(\frac{1}{\bar{\alpha}_0^L}\right) = \lambda(L) E_L^0 \left(\frac{\alpha_0^L}{\bar{\alpha}_0^L}\right) = \lambda(L) \text{ and } E_{L'}^0\left(\frac{1}{E_L^0 \xi_0|\mathcal{I})}\right) = \frac{\lambda(L)}{\lambda(L')} E_L^0 \left(\frac{\xi_0}{E_L^0(\xi_0|\mathcal{I})}\right) = \frac{\lambda(L)}{\lambda(L')}.$$

So, the hypotheses about uniform integrability are satisfied since the corresponding sequences contain only one integrable element. By reducing the sets of functions f to the functions 1_B with $B \in M_L^\infty$ and $B \in M_{L,L'}^\infty$, respectively, the parts (a) and (b) are immediate consequences of the remarks following Theorems 4.2 and 5.3.

For (c) we apply Theorem 3.2 with $g = \lambda(L)\bar{\alpha}_0^L$. The condition that Eg is finite causes the hypothesis in (c). □

Remarks. By (6.1) the summed expectations in (a) and the integrands in (c) are equal to $E_{Q_L}(1_B \circ \vartheta_{i,L})$ and $E_{Q_L^0}(1_B \circ T_x)$, respectively. Let ξ_0' be originated from ξ_0 in (5.1) by interchanging L and L'. By Lemmas 2.2 (b) and 2.4 it is obvious that

$$E_{L'}^0(\xi_0'|\mathcal{I}) = \frac{1}{E_L^0(\xi_0|\mathcal{I})} \quad P_{L'}^0\text{-a.s.}. \tag{6.3}$$

By interchanging L and L' in the right-hand relation in (6.1), it follows that the summed expectations in (b) are equal to $E_{Q_{L',L}^0}(1_B \circ \vartheta_{i,L})$.

The finiteness of $E\bar{\alpha}_0^L$ is equivalent to the finiteness of $E_L^0(\bar{\alpha}_0^L)^2$. By Jensen's inequality we have:

$$\left(\bar{\alpha}_0^L\right)^2 \leq E_L^0((\alpha_0^L)^2\,|\mathcal{I}_L) \quad P_L^0 \text{ - a.s. and } E_L^0\left(\bar{\alpha}_0^L\right)^2 \leq E_L^0\left(\alpha_0^L\right)^2.$$

So, the hypothesis in (c) is satisfied if $E_L^0(\alpha_0^L)^2 < \infty$. All parts of Theorem 6.1 can be generalized to uniform limit results for functions f with $|f| \leq g$, similar to Theorems 4.2, 5.3, and 3.2.

At the end of this section we give interpretations of the results in Theorem 6.1. Note that by Jensen's inequality,

$$E\left(\lambda(L)\bar{\alpha}_0^L\right) = (\lambda(L))^2\, E_L^0\left(\bar{\alpha}_0^L\right)^2 \geq 1 = E_L^0\left(\lambda(L)\bar{\alpha}_0^L\right)$$

(a strict inequality holds in the non-pseude-L-ergodic case). So, in a transition from P to P_L^0 the importance of realizations φ for which $\lambda(L)\bar{\alpha}_0^L(\varphi)$ is relatively large, should be reconsidered. We conclude that (a) and (c) in Theorem 6.1 can be interpreted as follows:

> P_L^0 arises from P by first changing the weights of the realizations by way of the weight function $1/(\lambda(L)\bar{\alpha}_0^L)$, followed by shifting the origin to an L-point chosen at random from the first n L-points and letting n tend to infinity.

> P arises from P_L^0 by first changing the weights of the realizations by way of the weight function $\lambda(L)\bar{\alpha}_0^L$, followed by shifting the origin to a time point chosen at random in $(0,t)$ and letting t tend to infinity.

By (5.3) and Jensen's inequality, we have:

$$E_{L'}^0\left(\frac{\lambda(L)}{\lambda(L')}E_L^0(\xi_0|\mathcal{I})\right) = \left(\frac{\lambda(L)}{\lambda(L')}\right)^2 E_L^0\left(E_L^0(\xi_0|\mathcal{I})\right)^2 \geq 1 = E_L^0(\frac{\lambda(L)}{\lambda(L')}E_L^0(\xi_0|\mathcal{I})).$$

A strict inequality holds if Φ is not pseudo-L-ergodic, or not pseudo-L'-ergodic. So, in a transition from $P_{L'}^0$ to P_L^0 the importance of realizations for which $\lambda(L)E_L^0(\xi_0|\mathcal{I})/\lambda(L')$ is relatively large, should be reconsidered:

> P_L^0 arises from $P_{L'}^0$ by first changing the weights of the realizations by way of the weight function $\lambda(L')/(\lambda(L)E_L^0(\xi_0|\mathcal{I}))$, followed by shifting the origin to an L-point chosen at random from the first n L-points and letting n tend to infinity.

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