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Simplicial algorithms for solving the non-linear complementarity problem on the simplotope*)

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Simplicial algorithms for solving the nonlinear complementarity problem on the simplotope

by

T.M. Doup, A.H. van den Elzen and A.J.J. Talman

Abstract

Interesting problems like the search for a Nash equilibrium vector in a noncooperative game or a price equilibrium vector in an international trade model can be formulated as a nonlinear complementarity problem on the simplotope. In this paper we present three variable dimension simplicial algorithms for solving this problem. All these algorithms can start anywhere and find an approximate solution by generating a sequence of simplices of varying dimension. The algorithms presented here differ from each other in the number of rays along which the starting point can be left. First we present the already known sum- and product-ray algorithm in case the simplotope is subdivided by the so-called V-triangulation. We remark that the presentation of the sum-ray algorithm applied to that triangulation is new. The main part of the paper deals with a new algorithm on the simplotope, the so-called exponent-ray algorithm. Again the underlying triangulation is the V-triangulation. Furthermore, the interpretation of the three algorithms as adjustment processes is discussed. This interpretation further explains the difference between the three algorithms. The paper is concluded with computational results. These results show that for problems on the simplotope the sum-ray algorithm is inferior to the exponent-ray algorithm and the product-ray algorithm.

Keywords: triangulation, simplicial algorithm, equilibrium, complementarity

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1. Introduction

The simplotope S, being the product space of N unit simplices, is defined by

$$S = \prod_{j=1}^{N} S^{n_j},$$

where, for j=1,...,N,

$$S^{n_j} = \{x_j \in \mathbb{R}_+^{n_j+1} \mid \sum_{k=1}^{n_j+1} x_{jk} = 1\}$$

is the n_j-dimensional unit simplex. An element x in S can be denoted by $(x_1^T, x_2^T, \ldots, x_N^T)^T \in S$ with $x_j = (x_{j1}, \ldots, x_{jn_j+1})^T \in S^{n_j}$ for all $j \in I_N = \{1, \ldots, N\}$. Let n be equal to $\sum_{j=1}^{N} n_j$. The nonlinear complementarity problem (NLCP) on S consists of finding a vector x^* in S such that $z(x^*) \leq 0$, where $z = (z_1^T, \ldots, z_N^T)^T : S \to R^{N+n}$ with $z_j : S \to R^{n_j+1}$ for all $j \in I_N$. The function z is continuous and $x_j^T z_j(x) = 0$ for all $j \in I_N$ and for all $x \in S$.

Some interesting problems can be stated in such a form. First we mention the problem of finding a Nash equilibrium in a noncooperative N-person game. In this context a vector $x \in S$ is interpreted as a strategy vector in the strategy space S and a solution to the NLCP gives a Nash equilibrium strategy vector of the game. Another example concerns the search for equilibrium prices in an international trade model with domestic goods, traded within only one country, and internationally traded common goods. The formulation of such a model as an NLCP on S is shown in van der Laan [4]. Again the set of solutions to that NLCP on S induces the set of equilibrium prices in the economy. A special case of the NLCP on S is the case when N=1. The NLCP on one unit simplex can be used to compute equilibrium prices in so-called Walrasian economies. The unit simplex is then interpreted as the price space of the economy.

Both for the NLCP on S and the NLCP on S^n so-called variable dimension simplicial restart algorithms have been developed to approximate a solution. These algorithms generate in a simplicial subdivision of S (or S^n), starting from an arbitrary grid point, a sequence of adjacent simplices of varying dimension which terminates with a simplex that approximates a solution. Improvements of the approximate solution are obtained by decreasing the mesh of the underlying triangulation and restarting the algorithm in the just found approximation.

For solving the NLCP on S^n there are the algorithms of van der Laan, Talman and Van der Heyden [8] (see also [5]), Doup and Talman [1], and Doup, van der Laan and Talman [2]. These algorithms differ from each other in the number and the direction of the rays along which the starting point can be left. In the algorithm on S^n of [8] there are n+1 rays, one to each facet of S^n. The algorithm in [1] possesses n+1 rays pointing to each vertex of S^n while the so-called exponent-ray algorithm in [2] has $2^{n+1}-2$ rays, one to each face of the unit simplex.

In van der Laan and Talman [7] several convergent adjustment processes for solving the NLCP on the unit simplex were described. Each of these processes is related to one of the algorithms mentioned above in the sense that the path generated by each process can be followed arbitrary close with the corresponding algorithm by taking the mesh of the triangulation small enough. In [7] it is shown that these processes have an attractive economic interpretation when applied to the problem of finding equilibrium prices in a pure exchange economy.

The algorithms on S^n of both van der Laan and Talman [5] and Doup and Talman [1] were generalized for application on the simplotope. The algorithm in [5] on S^n was generalized to a simplicial variable dimension algorithm on S with $\sum_{j=1}^{N} (n_j+1)$ rays to leave the arbitrary starting point. This so-called sum-ray algorithm on S was introduced in [6] and was adapted in [8] for a more general applicability. The algorithm in [1] was generalized to the product-ray algorithm on S with $\prod_{j=1}^{N} (n_j+1)$ rays. The names of the algorithms are derived from the respective number of rays along which one can leave the starting point. The adjustment processes induced by the sum- and product-ray algorithm on S were described in van den Elzen, van der Laan and Talman [3]. In that paper also a third process which can be considered as the generalization of the exponent-process on S^n, was given. This exponent-process on S possesses $\prod_{j=1}^{N} (2^{n_j+1}-2)$ rays to leave an arbitrarily chosen (interior) initial point. As argued in [3], the latter process has a very attractive interpretation as a price- or strategy-adjustment process when applied to find equilibria in an economy and a noncooperative N-person game respectively.

In this paper we describe a new simplicial variable dimension restart algorithm on S which can start anywhere and terminates within a finite number of iterations with an approximate solution of the NLCP on S. Moreover, the sequence of adjacent simplices of varying dimension generated by the algorithm follows approximately the path of points of the exponent-process on S. Therefore we call this algorithm the exponent-ray algorithm. The so-called V-triangulation of S developed in [1] underlying the product-ray algorithm on S will also underly the new algorithm having an exponential number of rays. Furthermore, we will adapt the sum-ray algorithm on S to the V-triangulation. This latter algorithm follows approximately the path of the sum-process on S as described in [3]. In [6] and [8] the sum-ray algorithm has only been described for the well-known Q-triangulation of S since the V-triangulation is of a more recent date. As argued in [1] and [2] the V-triangulation of S^n and S is much more natural than the Q-triangulation especially when the algorithm is interpreted as following the path of a corresponding adjustment process (see also [3]).

The paper is organized as follows. In section 2 both the sum-ray and the product-ray algorithm on S are described. For the first algorithm the V-triangulation will also be the underlying simplicial subdivision. The new exponent-ray algorithm on S is presented in section 3. Section 4 explains how the algorithms can be interpreted as path following discrete procedures of the processes given in [3]. Special attention will be paid to how the variables are adapted during the algorithms. Computational results are presented in section 5. The examples used concern both international economies and noncooperative games.

2. The sum- and the product-ray algorithm on the simplotope

Let S again be as defined in the previous section. For $j \in I_N = \{1,\ldots,N\}$, the index set $I(j)$ denotes the set $\{(j,1),(j,2),\ldots,(j,n_j+1)\}$ and $I = \bigcup_{j=1}^{N} I(j)$. The number n equals $\sum_{j=1}^{N} n_j$. In the algorithms the set S is subdivided by the V-triangulation originally developed in [1]. This triangulation is completely determined by the starting point (of the algorithm) and its projection on each of the faces of S. Let v be the (arbitrarily chosen) starting point in S. For a subset K of the index set I we denote the number $\sum_{(j,h) \in K \cap I(j)} v_{jh}$ by $S_j(K)$, for all $j \in I_N$. Then the (N+n)-vector p(K) in S is defined by

$$p_{jh}(K) = \begin{cases} (1-S_j(K))/(S_j(K)+|K_j^o|) & , (j,h) \in K^o \\ (v_{jh}(1+|K_j^o|))/(S_j(K)+|K_j^o|) & , (j,h) \in K \backslash K^o \\ 0 & , (j,h) \notin K , \end{cases}$$

if $S_j(K) < 1$ and by (2.1)

$$p_{jh}(K) = \begin{cases} 1/(|K_j^o|+S_j(K)) & , (j,h) \in K^o \\ v_{jh}/(|K_j^o|+S_j(K)) & , (j,h) \in K \backslash K^o \\ 0 & , (j,h) \notin K, \end{cases}$$

if $S_j(K) = 1$. Here $K_j^o = \{(j,h) \in K \cap I(j) | v_{jh} = 0\}$ and $|K_j^o|$ is the cardinality of K_j^o. If $K \cap I(j) = \emptyset$ we define $p_j(K) = v_j$. In particular, $p(\emptyset) = v$. We call p(K) the (relative) projection of v on the boundary set $S(K) = \{x \in S | x_{jh} = 0 \text{ for all } (j,h) \notin K\}$. Although the V-triangulation is completely determined by v and all p(K)'s, $K \subset I$, the description of its simplices depends on the specific algorithm used.

We will first describe the sum- or $\sum_{j=1}^{N} (n_j+1)$-ray algorithm. As mentioned in section 1, this algorithm was already described in [6] for the case in which each S^{n_j}, $j \in I_N$, is triangulated by the well-known Q-triangulation. Here this algorithm is adapted to the recently developed and more natural V-triangulation of S (see [1]).

Let T be a subset of I such that for each $j \in I_N$ the set $T_j = T \cap I(j)$ is a proper subset of $I(j)$. Furthermore, t denotes the number of elements in T and $t(j) = |T_j|$, $j=1,\ldots,N$.

Definition 2.1 Let $\gamma_j(T_j) = ((j,k_1^j),\ldots,(j,k_{t(j)}^j))$ be a permutation of the $t(j)$ elements of the proper subset T_j of $I(j)$, $j \in I_N$, and let the permutation vector $\gamma(T)$ be given by

$$\gamma(T) = (\gamma_1(T_1),\ldots,\gamma_N(T_N)).$$

Then the set $A^1(\gamma(T))$ is defined by

$$A^1(\gamma(T)) = \{x \in S \mid x = v + \sum_{j=1}^{N} \sum_{h=1}^{t(j)} \alpha(j,k_h^j)q^1(j,k_h^j) \text{ with } \\ 0 < \alpha(j,k_{t(j)}^j) < \ldots < \alpha(j,k_1^j) < 1,\ j \in I_N\}, \tag{2.2}$$

where the (N+n)-vectors $q^1(j,k_i^j)$, $i=1,\ldots,t(j)$, $j \in I_N$, are given by

$$q_h^1(j,k_i^j) = \begin{cases} p_j(\{(j,k_1^j),\ldots,(j,k_i^j)\})-p_j(\{(j,k_1^j),\ldots,(j,k_{i-1}^j)\}) & , h=j \\ \underline{0} & , h\neq j. \end{cases}$$

Further $A^1(T) = \bigcup_{\gamma(T)} A^1(\gamma(T))$, where the union is over all permutation vectors $\gamma(T)$ of T. Observe that $A^1(\emptyset)$ is equal to $\{v\}$.

The dimension of the set $A^1(T)$ is equal to t if and only if for each permutation vector $\gamma(T)$ of T, the rank of the $(N+n)\times t$-matrix $Q^1(\gamma(T))$ with $(\sum_{h=1}^{j-1} t(h)+i)$-th column equal to $q^1(j,k_i^j)$, $i=1,\ldots,t(j)$, $j \in I_N$, is maximal and therefore equal to t.

Lemma 2.2 The rank of the matrix $Q^1(\gamma(T))$ is less than t if and only if for some $j \in I_N$, $v_{jh} = 0$ for all $(j,h) \notin T_j$ holds. We allow T_j to be equal to $I(j)$. Moreover, the rank of $Q^1(\gamma(T))$ is independent of the permutation vector $\gamma(T)$ of T.

In the sequel we only consider sets $A^1(T)$ and $A^1(\gamma(T))$ of dimension $t = |T|$. The set of subsets T of I for which this holds is denoted by τ^1. We remark that $\emptyset \in \tau^1$. The boundary of $A^1(\gamma(T))$, denoted by bd $A^1(\gamma(T))$, consists of (t-1)-dimensional subsets which are obtained by setting exactly one of the inequalities in (2.2) to an equality, i.e. for some $j \in I_N$ either $\alpha(j,k^j_{t(j)}) = 0$ or $\alpha(j,k^j_i) = \alpha(j,k^j_{i-1})$ for some i, $2 \leq i \leq t(j)$, or $\alpha(j,k^j_1) = 1$. If $\alpha(j,k^j_i) = \alpha(j,k^j_{i-1})$ for some $j \in I_N$ and some i, $2 \leq i \leq t(j)$, the corresponding boundary set of $A^1(\gamma(T))$ is also a boundary set of another area $A^1(\bar{\gamma}(T))$. The boundary set belonging to $\alpha(j,k^j_{t(j)}) = 0$ equals $A^1(\gamma(T\setminus\{(j,k^j_{t(j)})\}))$ whereas the boundary set corresponding to $\alpha(j,k^j_1) = 1$ is equal to $S^j(T) \cap A^1(\gamma(T))$ with $S^j(T) = \{x \in S \,|\, x_{jh} = 0 \text{ for all } h, (j,h) \notin T_j\}$. Now it is straightforward to derive that the boundary of $A^1(T)$ is equal to

$$\text{bd } A^1(T) = (\bigcup_{(i,h) \in T} A^1(T\setminus\{(i,h)\})) \cup (\bigcup_{j=1}^{N} (S^j(T) \cap A^1(T))).$$

This description of bd $A^1(T)$ will be of use when explaining the algorithm.

Next we describe how each t-dimensional set $A^1(T)$, $T \in \tau^1$, is triangulated into t-dimensional simplices or t-simplices and how all these triangulations form the V-triangulation of S. The number m^{-1} is the grid size of the triangulation with m some positive integer.

Definition 2.3 Let T be an element of τ^1. The set $G^1(\gamma(T))$ is the set of t-simplices $\sigma(y^1,\pi(T))$ with vertices $y^1,\ldots,y^{t+1}$ such that

(i) $y^1 = v + \sum_{j=1}^{N} \sum_{h=1}^{t(j)} a(j,k^j_h)m^{-1}q^1(j,k^j_h)$ for integers $a(j,k^j_h)$, such that $0 \leq a(j,k^j_{t(j)}) \leq \ldots \leq a(j,k^j_1) \leq m-1$, $j=1,\ldots,N$

(ii) $\pi(T) = (\pi_1,\ldots,\pi_t)$ is a permutation of the t elements of T such that for all $i=2,\ldots,t(j)$: $p>p'$ if $a(j,k^j_i) = a(j,k^j_{i-1})$ where $\pi_p = (j,k^j_i)$ and $\pi_{p'} = (j,k^j_{i-1})$, $j \in I_N$

(iii) $y^{i+1} = y^i + m^{-1} q^1(\pi_i)$, $i=1,\ldots,t$, where $q^1(j,k_h^j)$, $(j,k_h^j) \in T$, are defined as in definition 2.1.

The set $G^1(\gamma(T))$ is a triangulation of $A^1(\gamma(T))$ whereas the union $G^1(T)$ of $G^1(\gamma(T))$ over all permutation vectors of T triangulates $A^1(T)$. The set $G^1 = \bigcup_{T \in \tau^1} G^1(T)$ yields the V-triangulation of S with grid size m^{-1}. The notion of a triangulation implies that each (t-1)-face of a t-simplex, called a facet, in $G^1(\gamma(T))$ is either a facet of exactly one other t-simplex of $G^1(\gamma(T))$ or lies in bd $A^1(\gamma(T))$. A t-simplex has t+1 facets one opposite to each vertex. Two different simplices are adjacent if they share a common facet or if one of them is a facet of the other. Since the algorithm moves from one simplex to an adjacent one we will first describe how the representation of the latter one can be obtained from the representation of the first simplex if they share a common facet. So, let $\sigma(y^1,\pi(T))$ and $\bar{\sigma} = \sigma(\bar{y}^1,\bar{\pi}(T))$ be elements of $G^1(\gamma(T))$ with common facet τ opposite, say vertex y^p of σ, $1 \leq p \leq t+1$, then $\bar{\sigma}$ can be obtained from σ as given in table 1 where e(j,k) is the $(\sum_{i=1}^{j-1}(n_i+1)+k)$-th unit vector in $\mathbb{R}^{N+n}$, $k=1,\ldots,n_j+1$ and $j=1,\ldots,N$. Further, the vector a is the (N+n)-vector with $(\sum_{i=1}^{j-1}(n_i+1)+k)$-th element a_{jk} equal to a(j,k) if $(j,k) \in T$ and zero otherwise. When going from the t-simplex $\sigma(y^1,\pi(T))$ to $\bar{\sigma}$ we say that the vertex y^p has to be replaced.

	$\bar{y}^1$	$\bar{\pi}(T)$	$\bar{a}$
p=1	$y^1+m^{-1}q^1(\pi_1)$	$(\pi_2,\ldots,\pi_t,\pi_1)$	$a+e(\pi_1)$
1<p<t+1	y^1	$(\pi_1,\ldots,\pi_{p-2},\pi_p,\pi_{p-1},\ldots,\pi_t)$	a
p=t+1	$y^1-m^{-1}q^1(\pi_t)$	$(\pi_t,\pi_1,\ldots,\pi_{t-1})$	$a-e(\pi_t)$

Table 1 p is the index of the vertex of $\sigma(y^1,\pi(T))$ to be replaced.

In lemma 2.4. we describe the cases in which a facet lies in the boundary of $A^1(\gamma(T))$.

Lemma 2.4 Let $\sigma(y^1,\pi(T))$ be in $G^1(\gamma(T))$ and let τ be the facet of σ opposite vertex y^p, $1 \leq p \leq t+1$. Then τ lies in the boundary of $A^1(\gamma(T))$ iff one of the following cases holds:

a) $p=1$: $\pi_1=(j,k_1^j)$ for some $j \in I_N$ and $a(j,k_1^j)=m-1$

b) $1<p<t+1$: $\pi_p=(j,k_i^j),\pi_{p-1}=(j,k_{i-1}^j)$ for certain $j \in I_N$ and $1<i \leq t(j)$, and $a(j,k_i^j)=a(j,k_{i-1}^j)$

c) $p=t+1$: $\pi_t=(j,k_{t(j)}^j)$ for some $j \in I_N$ and $a(j,k_{t(j)}^j)=0$.

The lemma follows immediately from the definitions of $G^1(\gamma(T))$ and $A^1(\gamma(T))$. If a facet τ in bd $A^1(\gamma(T))$ does not lie in bd $A^1(T)$, then τ is a facet of exactly one other t-simplex $\bar{\sigma}$ in $G^1(T)$ but $\bar{\sigma}$ lies in an area $A^1(\bar{\gamma}(T))$ different from $A^1(\gamma(T))$. If τ lies in bd $A^1(T)$, then either τ lies in $S^j(T) \cap A^1(T)$ for some $j \in I_N$ or τ is a (t-1)-simplex in $G^1(T\setminus\{(i,h)\})$ for some $(i,h) \in T$. These three different cases are described in the following lemma.

Lemma 2.5 Let $\sigma(y^1,\pi(T))$ be in $G^1(\gamma(T))$ with a facet τ in bd $A^1(\gamma(T))$. If τ is the facet opposite vertex y^1, then τ is a (t-1)-simplex in $S^j(T)$ where j as given in lemma 2.4.a. When τ lies opposite the vertex y^p, $1<p<t+1$, then τ is a facet of the t-simplex $\sigma(y^1,\bar{\pi}(T))$ in $G^1(\bar{\gamma}(T))$ with $\bar{\gamma}_j(T_j) = ((j,k_1^j),\ldots,(j,k_{i-2}^j),(j,k_i^j),(j,k_{i-1}^j),\ldots,(j,k_{t(j)}^j)),\bar{\gamma}_h(T_h) = \gamma_h(T_h)$ for all $h \neq j$, and $\bar{\pi}(T) = (\pi_1,\ldots,\pi_{p-2},\pi_p,\pi_{p-1},\ldots,\pi_t)$, where j and i as in lemma 2.4.b. In the case that τ lies opposite vertex y^{t+1}, τ is the (t-1)-simplex $\sigma(y^1,\pi(\bar{T}))$ in $G^1(\gamma(\bar{T}))$ with $\bar{T} = T\setminus\{(j,k_{t(j)}^j)\}$, $\gamma_j(\bar{T}_j) = ((j,k_1^j),\ldots,(j,k_{t(j)-1}^j))$, $\gamma_h(\bar{T}_h) = \gamma_h(T_h)$ for all $h \neq j$, and $\pi(\bar{T}) = (\pi_1,\ldots,\pi_{t-1})$, with j as in lemma 2.4.c.

In the foregoing we described the subdivision of S in t-dimensional subsets $A^1(T)$, $T \in \tau^1$, and the way in which each such subset is subdivided by the V-triangulation with grid size m^{-1} in t-dimensional simplices. Moreover the steps of moving from one simplex to an adjacent one were given. Now we are ready to describe the sum-ray algorithm with underlying V-triangulation in order to solve the NLCP on S. For varying

T in τ^1 and starting with $T=\emptyset$, the algorithm generates a sequence of adjacent t-simplices in $A^1(T)$ having so-called T-complete facets in common.

Definition 2.6 For $g=t-1,t$, where $t=|T|$ and $T \subset I$, a g-simplex $\sigma(y^1,\ldots,y^{g+1})$ is T-complete if the system of linear equations

$$\sum_{i=1}^{g+1} \lambda_i \binom{z(y^i)}{1} + \sum_{(j,k)\notin T} \mu_{jk} \binom{e(j,k)}{0} - \beta \binom{e}{0} = \binom{0}{1} \tag{2.3}$$

with $\underline{0}$ the (N+n)-zero vector and e the (N+n)-vector of ones, has a solution $\bar{\lambda}_i^* \geq 0$, $i=1,\ldots,g+1$, $\mu_{jk}^* \geq 0$, $(j,k) \notin T$, and β^*. A solution of (2.3) is denoted by $(\lambda^*,\mu^*,\beta^*)$.

We need a nondegeneracy assumption on this system to guarantee convergence of the algorithm.

Nondegeneracy assumption. For $g=t-1$ the system (2.3) has a unique solution $(\lambda^*,\mu^*,\beta^*)$ with $\lambda_i^* > 0$, $i=1,\ldots,t$, and $\mu_{jk}^* > 0$, $(j,k) \notin T$, while for $g=t$ at most one variable of (λ^*,μ^*) is equal to zero.

The algorithm terminates as soon as a complete simplex is found. This notion is defined in definition 2.7 and we show in lemma 2.8 that such a simplex yields an approximate solution to the NLCP on S. Lemma 2.9 states when a simplex in some area $A^1(T)$, $T \in \tau^1$, is complete.

Definition 2.7 For $T \subset I$, a T-complete (t-1)-simplex $\sigma(y^1,\ldots,y^t)$ is complete if there is an index $j \in I_N$ such that for all x in σ, $x_{jk} = 0$ if $(j,k) \notin T_j$. We allow T_j to be equal to $I(j)$.

Lemma 2.8 Let $\varepsilon > 0$ be such that $\max_{(i,h)\in I} |z_{ih}(x)-z_{ih}(y)| < \varepsilon$ for all x and y in a simplex σ of the V-triangulation of S and let σ^* be a complete (t-1)-simplex, for some $T \subset I$, with a solution $(\lambda^*,\mu^*,\beta^*)$ of (2.3) such that $\lambda_i^* > 0, i = 1,\ldots,t$, and $\mu_{jk}^* > 0, (j,k) \notin T$. Then $x^* = \sum_{i=1}^{t} \lambda_i^* y^i$ lies in σ^* and $\beta^* \in (-\varepsilon,+\varepsilon)$. Furthermore, $|z_{ih}(x^*)-\beta^*| < \varepsilon$ if $(i,h) \in T$ and $z_{ih}(x^*) < \beta^* + \varepsilon$ if $(i,h) \notin T$.

Lemma 2.9 The 0-simplex $\sigma(v)$ is complete iff for some $(j,k) \in I$ both $v_{jk} = 1$ and $\sigma(v)$ is $\{(j,k)\}$-complete. A T-complete facet τ of a T-complete t-simplex $\sigma(y^1,\pi(T))$ in $A^1(T)$ is complete iff τ lies opposite y^1 in bd $A^1(T)$. If $\sigma(y^1,\pi(T))$ is a $(T \cup \{(j,k)\})$-complete simplex in $A^1(T)$ for some $(j,k) \notin T$ then σ is complete iff $v_{jh} = 0$ for all $(j,h) \notin T_j \cup \{(j,k)\}$.

Assuming nondegeneracy, the T-complete t-simplices $\sigma(y^1,\pi(T))$ in $A^1(\gamma(T))$ for given $T \in \tau^1$ form sequences of adjacent t-simplices having T-complete common facets. Each sequence not being a loop has two end simplices. When an end simplex is not complete it is either a $(T \cup \{(j,k)\})$-complete t-simplex or a t-simplex having a T-complete facet τ in bd $A^1(\gamma(T))$. The first case is described in lemma 2.10, while the latter case was treated in lemma 2.5.

Lemma 2.10 Let $\sigma(y^1,\pi(T))$ be a $(T \cup \{(j,k)\})$-complete t-simplex in $A^1(\gamma(T))$ for some $(j,k) \notin T$. If σ is not complete then σ is a $(T \cup \{(j,k)\})$-complete facet of exactly one (t+1)-simplex $\bar{\sigma}(y^1,\pi(T \cup \{(j,k)\}))$ in $G^1(T \cup \{(j,k)\})$. More precisely, $\bar{\sigma}$ is the (t+1)-simplex $\sigma(y^1,\pi(T \cup \{(j,k)\}))$ in $G^1(\gamma(T \cup \{(j,k)\}))$ with $\gamma_j(T_j \cup \{(j,k)\}) = ((j,k_1^j),\ldots,(j,k_{t(j)}^j),(j,k))$, and $\pi(T \cup \{(j,k)\}) = (\pi_1,\ldots,\pi_t,(j,k))$.

Combining the foregoing we see that a sequence of adjacent t-simplices in $A^1(T)$ having T-complete common facets is either a loop or a path with two end simplices. Each end simplex which is not complete or equal to $\sigma(v)$ can be connected with an end simplex of either a sequence of (t-1)-simplices in $A^1(T\backslash\{(i,h)\})$ for some $(i,h) \in T$ or a sequence of (t+1)-simplices in $A^1(T \cup \{(j,k)\})$ for some $(j,k) \notin T$. Connecting the sequences in this way we can form sequences of adjacent T-complete t-simplices with T-complete common facets in areas $A^1(T)$, $T \in \tau^1$, where t varies between 0 and n. Among these sequences there is one connecting v and a complete simplex, while all the other sequences not being a loop connect two complete simplices. The first sequence is generated by the sum-ray algorithm. Because the total number of simplices in S is finite, the number of simplices in the sequence is also finite. The solutions to (2.3) for the simplices in this sequence determine a piecewise linear path of points in S from v to x^*, with x^* as described in lemma 2.8. This path is in fact followed by the sum-ray algorithm by performing alternating linear programming pivot steps in system (2.3) and replacement steps according to table 1 and lemma 2.5. If, for some T, in $A^1(T)$ a T-complete facet τ in $A^1(T\setminus\{(i,h)\})$ is generated then $(e^T(i,h),0)^T$ is reintroduced in the system (2.3) with respect to τ. On the other hand, when a T-complete t-simplex σ in $A^1(T)$ is also $(T \cup \{(j,k)\})$-complete for some $(j,k) \notin T$ while σ is not complete, a linear programming pivot step in the system (2.3) with respect to σ is made with $(z^T(y^{t+2}),1)^T$, where y^{t+2} is the new vertex of the unique (t+1)-simplex $\bar{\sigma}$ in $A^1(T \cup \{(j,k)\})$ having σ as facet (see lemma 2.10). In section 4 it is shown that this piecewise linear path approximately follows the sum-process described in van den Elzen, van der Laan and Talman [3]. We conclude the treatment of the sum-ray algorithm with a presentation of the steps of the algorithm.

<u>Step 0.</u> Let (j,k_1^j) be the unique index for which $z_{jk_1^j}(v) = \max_{(i,h)\in I} z_{ih}(v)$. If $v_{jk_1^j} = 1$ then $\sigma(v)$ is complete and the algorithm stops, else set $T = \{(j,k_1^j)\}$, t=1, $y^1=v$, $\pi(T) = ((j,k_1^j))$, $\sigma = \sigma(y^1,\pi(T))$, $\bar{p}=2$, $a_{ih}=0$ for all $(i,h) \in I$, $\mu_{ih} = z_{jk_1^j}(v)-z_{ih}(v)$ for all $(i,h) \neq (j,k_1^j)$, $\beta = z_{jk_1^j}(v)$ and $\lambda_1=1$.

<u>Step 1.</u> Calculate $z(y^{\bar{p}})$. Perform a pivot step by bringing $(z^T(y^{\bar{p}}),1)^T$ in the linear system

$$\sum_{\substack{i=1\\ i\neq\bar{p}}}^{t+1} \lambda_i\binom{z(y^i)}{1} + \sum_{(j,k)\notin T} \mu_{jk}\binom{e(j,k)}{0} - \beta\binom{e}{0} = \binom{0}{1}.$$

If μ_{jk} becomes zero for some $(j,k) \notin T$ then go to step 3. Else λ_p is eliminated for exactly one $p \neq \bar{p}$ and the facet $\tau(y^1,\ldots,y^{p-1},y^{p+1},\ldots,y^{t+1})$ is T-complete.

<u>Step 2.</u> If p=1, $\pi_1 = (j,k_1^j)$ and $a(j,k_1^j) = m-1$, then τ is a complete simplex and the algorithm stops.

If $1<p<t+1$, and if for some $i\geq 2$, $\pi_{p-1} = (j,k^j_{i-1})$, $\pi_p = (j,k^j_i)$ and $a(j,k^j_{i-1}) = a(j,k^j_i)$, then $\sigma(y^1,\pi(T))$ and $\gamma(T)$ are adapted according to lemma 2.5; return to step 1 with $\bar{p}$ equal to p.

If $p=t+1$, $\pi_t = (j,k^j_{t(j)})$ and $a(j,k^j_{t(j)}) = 0$, then the dimension is decreased; set $t=t-1$, $T = T\setminus\{(j,k^j_{t(j)})\}$, $(i,h) = (j,k^j_{t(j)})$ while $\sigma(y^1,\pi(T))$ and $\gamma(T)$ are adapted according to lemma 2.5; go to step 4.

In all other cases $\sigma(y^1,\pi(T))$ is adapted according to table 1; return to step 1 with $\bar{p}$ equal to the index of the new vertex of σ.

<u>Step 3.</u> If $v_{jh} = 0$ for all h, $(j,h) \notin T_j \cup \{(j,k)\}$, then σ is a complete simplex and the algorithm stops. In all other cases the dimension is increased, $\sigma(y^1,\pi(T))$ and $\gamma(T)$ are adapted according to lemma 2.10, set $t=t+1$ and $T = T \cup \{(j,k)\}$, and return to step 1 with $\bar{p}$ the index of the new vertex of σ.

<u>Step 4.</u> Perform a pivot step by bringing $(e^T(i,h),0)^T$ in the linear system

$$\sum_{i=1}^{t+1} \lambda_i \binom{z(y^i)}{1} + \sum_{\substack{(j,k)\notin T\\(j,k)\neq(i,h)}} \mu_{jk}\binom{e(j,k)}{0} - \beta\binom{e}{0} = \binom{0}{1}.$$

If for some $(j,k) \notin T$, $(j,k) \neq (i,h)$, μ_{jk} becomes zero, go to step 3. Otherwise return to step 2 with p the index of the vertex whose corresponding variable λ_p is eliminated.

We now continue with the product-ray or $\prod_{j=1}^{N} (n_j+1)$-ray algorithm on S. This algorithm was already described in Doup and Talman [1]. We remark that the projections used in this paper sligthly differ from the ones which underly the V-triangulation in [1].

Let T be a proper subset of I containing for each $j \in I_N$ at least one element of I(j) and let T^o be a subset of T containing exactly one element, say (j,k^j_0), of each I(j). By T^1 we denote the complementary part of T^o in T. For $j=1,\ldots,N$, let $T^1_j = T^1 \cap I(j)$ consist of $t^1(j)$ elements and let $\gamma_j(T^1_j)$ be some permutation of the elements of T^1_j. The vector $\gamma(T^1)$ denotes the permutation vector $\gamma(T^1) = (\gamma_1(T^1_1),\ldots,\gamma_N(T^1_N))$.

Definition 2.11 Let T be a subset of I for which $|T_j| \geq 1$, $j \in I_N$, and let T^o and $\gamma(T^1)$ be as given above. Then the set $A^2(T^o,\gamma(T^1))$ is equal to

$$A^2(T^o,\gamma(T^1)) = \{x \in S \mid x=v + \beta q^2(T^o) + \sum_{(i,h) \in T^1} \alpha(i,h)q^2(i,h), \text{ with} \tag{2.4}$$

$$0 \leq \alpha(j,k^j_{t^1(j)}) \leq \ldots \leq \alpha(j,k^j_1) \leq \beta \leq 1,\ j \in I_N\},$$

where the (N+n)-vector $q^2(T^o)$ is given by $q^2(T^o) = e(T^o)-v$, with $e(T^o)$ the vertex of S for which $e_{jh}(T^o) = 1$ if $h = k^j_0$, $j \in I_N$, and zero otherwise. The (N+n)-vectors $q^2(j,k^j_i)$, $(j,k^j_i) \in T^1_j$, $j \in I_N$, are equal to

$$q^2_h(j,k^j_i) = \begin{cases} p_j(\{(j,k^j_0),\ldots,(j,k^j_i)\}) - p_j(\{(j,k^j_0),\ldots,(j,k^j_{i-1})\}), & h=j \\ \underline{0} & , h\neq j. \end{cases}$$

Also here we have to investigate when the rank of the matrix $Q^2(T^o,\gamma(T^1))$ with first column $q^2(T^o)$ and $(1 + \sum_{h=1}^{j-1} t^1(h) + i)$-th column $q^2(j,k^j_i)$, $i=1,\ldots,t^1(j)$, $j \in I_N$, is maximal, i.e. when $r(Q^2(T^o,\gamma(T^1))) = t = 1 + \sum_{j=1}^{N} t^1(j)$. Observe that $t = |T| - N + 1$.

Lemma 2.12 The rank of the matrix $Q^2(T^o,\gamma(T^1))$ is not equal to t iff $v_{jk}=0$ for all $(j,k) \notin T$, where we allow T_j to be equal to I(j) for any $j \in I_N$.

Clearly, the dimension of $A^2(T^o,\gamma(T^1))$ is independent of T^o and $\gamma(T^1)$ and equal to t iff $r(Q^2(T^o,\gamma(T^1))) = t$. In the sequel we only consider the t-dimensional regions $A^2(T^o,\gamma(T^1))$. The (t-1)-dimensional subsets forming the boundary of $A^2(T^o,\gamma(T^1))$ are obtained by setting exactly one inequality in (2.4) to an equality. The union of the $A^2(T^o,\gamma(T^1))$'s over all sets T^o and permutation vectors $\gamma(T^1)$ is denoted by $A^2(T)$. The dimension of $A^2(T)$ is equal to t if $r(Q^2(T^o,\gamma(T^1))) = t$ for any T^o and T^1. The set of subsets T of I for which the dimension of $A^2(T)$ equals t is denoted by τ^2. Now let S be triangulated by the V-triangulation with grid size m^{-1}, where m is some positive integer.

Definition 2.13 The set $G^2(T^o,\gamma(T^1))$ is the set of t-simplices $\sigma(y^1,\pi(T))$ with vertices $y^1,\ldots,y^{t+1}$ such that

(i) $y^1 = v + bm^{-1}q^2(T^o) + \sum_{(i,h) \in T^1} a(i,h)m^{-1}q^2(i,h)$ for integers b and $a(i,h)$, $(i,h) \in T^1$, such that for all $j \in I_N$, $0 \leq a(j,k^j_{t^1(j)}) \leq \ldots \leq a(j,k^j_1) \leq b \leq m-1$

(ii) $\pi(T) = (\pi_1,\ldots,\pi_t)$ is a permutation of the t elements consisting of T^o and the t-1 elements of T^1 such that for all $j \in I_N$ and $i=1,\ldots,t^1(j)$ if $a(j,k^j_i) = a(j,k^j_{i-1})$, $\pi_p = (j,k^j_i)$ and $\pi_{p'} = (j,k^j_{i-1})$ then $p > p'$. In the case i=1 we have $a(j,k^j_0) = b$ and $\pi_{p'} = T^o$

(iii) $y^{i+1} = y^i + m^{-1}q^2(\pi_i)$, $i=1,\ldots,t$.

The set $G^2(T^o,\gamma(T^1))$ is the triangulation of $A^2(T^o,\gamma(T^1))$ induced by the V-triangulation of S. The triangulation of $A^2(T)$ is the union of the $G^2(T^o,\gamma(T^1))$'s over all T^o and T^1 and will be denoted by $G^2(T)$. The relation between two adjacent t-simplices in the same set $G^2(T^o,\gamma(T^1))$ is again as given in table 1 where now $a_{jk} = a(j,k)$ if $(j,k) \in T^1$, $a_{jk} = b$ for all $(j,k) \in T^o$, and $a_{jk} = 0$ otherwise. Observe also that π_p might be equal to the set T^o. The remainder of this section is a review from Doup and Talman [1] and only gives the main results. For a further insight and interpretation of the replacement steps and the steps of the algorithm we refer the reader to [1].

Lemma 2.14 Let $\sigma(y^1,\pi(T))$ be a t-simplex in $G^2(T^o,\gamma(T^1))$ and τ the facet opposite vertex y^p, $1 \leq p \leq t+1$. Then τ lies in the boundary of $A^2(T^o,\gamma(T^1))$ iff one of the following cases holds:

a) $p = 1$: $\pi_1 = T^o$ and $b = m-1$;

b) $1 < p < t+1$: $\pi_p = (j,k^j_i)$, $\pi_{p-1} = (j,k^j_{i-1})$ for certain $j \in I_N$ and $2 \leq i \leq t^1(j)$ while $a(j,k^j_i) = a(j,k^j_{i-1})$, or $\pi_{p-1} = T^o$, $\pi_p = (j,k^j_1)$ and $a(j,k^j_1) = b$ for some $j \in I_N$;

c) $p = t+1$: $\pi_t = (j,k^j_{t^1(j)})$ and $a(j,k^j_{t^1(j)}) = 0$ for some $j \in I_N$.

In case a) τ is a $(t-1)$-simplex in bd S. More precisely, τ lies in $S(T)$, where $S(T)$ is given by $S(T) = \{x \in S | x_{jk} = 0 \text{ for all } (j,k) \notin T\}$. In case b) with $i > 1$, τ is a facet of the t-simplex $\sigma(y^1,\bar{\pi}(T))$ in $G^2(T^o,\bar{\gamma}(T^1))$ with $\bar{\gamma}_j(T^1_j) = ((j,k^j_1),\ldots,(j,k^j_{i-2}), (j,k^j_i), (j,k^j_{i-1}),(j,k^j_{i+1}),\ldots,(j,k^j_{t^1(j)}))$, $\bar{\gamma}_h(T^1_j) = \gamma_h(T^1_j)$ if $h \neq j$, and $\bar{\pi}(T) = (\pi_1,\ldots,\pi_{p-2},\pi_p,\pi_{p-1},\pi_{p+1},\ldots,\pi_t)$. If $i = 1$ then τ is a facet of the t-simplex $\sigma(y^1,\bar{\pi}(T))$ in $G^2(\bar{T}^o,\gamma(\bar{T}^1))$ with $\bar{T}^o_j = \{(j,k^j_1)\}$, $\bar{T}^o_h = T^o_h$ if $h \neq j$, $\bar{T}^1_j = (T^1_j \backslash \{(j,k^j_1)\}) \cup \{(j,k^j_0)\}$, $\bar{T}^1_h = T^1_h$ if $h \neq j$, $\gamma_j(\bar{T}^1_j) = ((j,k^j_0),(j,k^j_2),\ldots,(j,k^j_{t^1(j)}))$, $\gamma_h(\bar{T}^1_h) = \gamma_h(T^1_h)$ if $h \neq j$, and $\bar{\pi}(T) = (\pi_1,\ldots,\pi_{p-2},\bar{T}^o,(j,k^j_0),\pi_{p+1},\ldots,\pi_t)$. In case c) τ is the $(t-1)$-simplex $\sigma(y^1,\pi(T\backslash\{(j,k^j_{t^1(j)})\}))$ in the subset $A^2(T^o,\gamma(\bar{T}^1))$ of $A^2(T\backslash\{(j,k^j_{t^1(j)})\})$ with $\bar{T}^1_j = T^1_j\backslash\{(j,k^j_{t^1(j)})\}$, $\bar{T}^1_h = T^1_h$ if $h \neq j$, $\gamma_j(\bar{T}^1_j) = ((j,k^j_1),\ldots,(j,k^j_{t^1(j)-1}))$, $\gamma_h(\bar{T}^1_h) = \gamma_h(T^1_h)$ if $h \neq j$, and $\pi(T\backslash\{(j,k^j_{t^1(j)})\}) = (\pi_1,\ldots,\pi_{t-1})$.

For varying T, the product-ray algorithm generates in $A^2(T)$ a sequence of adjacent t-simplices with T-complete common facets.

Definition 2.15 For $g = t-1,t$, where $t = |T|-N+1$, $T \subset I$, a g-simplex $\sigma(y^1,\ldots,y^{g+1})$ is T-complete if the system of linear equations

$$\sum_{i=1}^{g+1} \lambda_i \binom{z(y^i)}{1} + \sum_{(j,k) \notin T} \mu_{jk} \binom{e(j,k)}{0} - \sum_{j=1}^{N} \beta_j \binom{\bar{e}(j)}{0} = \binom{0}{1} \qquad (2.5)$$

where $\bar{e}(j)$ denotes the (N+n)-vector with $\bar{e}_{ih}(j) = 1$ for all $(i,h) \in I(j)$ and $\bar{e}_{ih}(j) = 0$ otherwise, has a solution $\lambda_i^* \geq 0$, $i=1,\ldots,g+1$, $\mu_{jk}^* \geq 0$ for all $(j,k) \notin T$, and β_j^* for all $j \in I_N$. A solution of (2.5) is denoted by $(\lambda^*,\mu^*,\beta^*)$.

<u>Nondegeneracy assumption</u>. For g=t-1 the system (2.5) has a unique solution $(\lambda^*,\mu^*,\beta^*)$ with $\lambda_i^* > 0$, $i=1,\ldots,t$, and $\mu_{jk}^* > 0$ for all $(j,k) \notin T$, whereas for g=t at most one variable of (λ^*,μ^*) is equal to zero.

<u>Definition 2.16</u> A T-complete (t-1)-simplex $\sigma(y^1,\ldots,y^t)$ is complete if for each $x \in \sigma$ we have $x_{jk} = 0$ for all $(j,k) \notin T$.

The next lemma gives an estimate of the accuracy of an approximate solution obtained from a complete simplex.

<u>Lemma 2.17</u> Let $\varepsilon > 0$ be such that $\max_{(j,k) \in I} |z_{jk}(x)-z_{jk}(y)| < \varepsilon$ for all x and y in a simplex of the V-triangulation of S and let $\sigma^*(y^1,\ldots,y^t)$ be a complete simplex with solution $(\lambda^*,\mu^*,\beta^*)$. Then $x^* = \sum_{i=1}^{t} \lambda_i^* y^i$ lies in σ^*, $\beta_j^* \in (-\varepsilon,+\varepsilon)$ for all $j \in I_N$, $|z_{jk}(x^*) - \beta_j^*| < \varepsilon$ when $x_{jk}^* > 0$, and $z_{jk}(x^*) < \beta_j^* + \varepsilon$ if $x_{jk}^* = 0$.

<u>Lemma 2.18</u> The 0-dimensional simplex $\sigma(v)$ is complete iff for some $T \in \tau^2$, $\sigma(v)$ is T^o-complete and v is equal to the vertex $e(T^o)$ of S. A T-complete facet τ of a T-complete t-simplex $\sigma(y^1,\pi(T))$ in $A^2(T^o,\gamma(T^1))$ is complete iff τ lies opposite the vertex y^1 of σ in the subset $S(T) \cap A^2(T^o,\gamma(T^1))$ of bd S. If $\sigma(y^1,\pi(T))$ is a $(T \cup \{(j,k)\})$-complete t-simplex in $A^2(T)$ for some $(j,k) \notin T$, then σ is complete iff $v_{ih} = 0$ for all $(i,h) \notin T \cup \{(j,k)\}$.

Again, for given $T \in \tau^2$, the T-complete t-simplices $\sigma(y^1,\pi(T))$ in $A^2(T)$ form sequences of adjacent simplices with common T-complete facets. An end simplex of a sequence not being a loop is either a $(T \cup \{(j,k)\})$-complete t-simplex or a t-simplex with a T-complete facet in bd $A^2(T)$. In the latter case this facet is either a complete (t-1)-simplex or $\sigma(v)$ or an end simplex of a sequence of adjacent $(T\setminus\{(i,h)\})$-complete (t-1)-simplices in $A^2(T\setminus\{(i,h)\})$ for some $(i,h) \in T$. In the first case σ is either complete or σ is a $(T \cup \{(j,k)\})$-complete facet of the uniquely determined (t+1)-simplex $\bar{\sigma} = \sigma(y^1,\pi(T \cup \{(j,k)\})) \in G^2(T^o,\gamma(\bar{T}^1))$ in $A^2(T \cup \{(j,k)\})$, where $\bar{T}^1_j = T^1_j \cup \{(j,k)\}$, $\bar{T}^1_h = T^1_h$ if $h \neq j$, $\gamma_j(\bar{T}^1_j) = ((j,k^j_1),\ldots,(j,k^j_{t^1(j)}),(j,k))$, $\gamma_h(\bar{T}^1_h) = \gamma_h(T^1_h)$ if $h \neq j$ and $\pi(T \cup \{(j,k)\}) = (\pi_1,\ldots,\pi_t,(j,k))$. Moreover, $\bar{\sigma}$ is an end simplex of a sequence of adjacent $(T \cup \{(j,k)\})$-complete (t+1)-simplices in $A^2(T \cup \{(j,k)\})$. Linking of the sequences of adjacent T-complete t-simplices in $A^2(T)$ over all T in τ^2 yields one sequence connecting $\sigma(v)$ with a complete simplex whereas all other sequences not being a loop connect two complete simplices. The product-ray algorithm follows the first sequence by starting from $\sigma(v)$. More precisely, the product-ray algorithm follows a piecewise linear path of points from v to an approximate solution x^* by alternating linear programming pivot steps in system (2.5) and corresponding replacement steps in the triangulation. When the algorithm generates in $A^2(T)$ a T-complete facet in $A^2(T\setminus\{(i,h)\})$ for some $(i,h) \in T$, then the (i,h)-th unit vector column is reintroduced in system (2.5). On the other hand if the (j,k)-th unit vector column in (2.5) is eliminated by a linear programming pivot step, then the current σ is $(T \cup \{(j,k)\})$-complete and the algorithm continues in $A^2(T \cup \{(j,k)\})$ as described above when σ is not complete. If the approximate solution x^* is not accurate enough the algorithm can be restarted in x^* with a smaller grid size.

3. The exponent-ray algorithm on the product space of unit simplices

In this section we present the generalization of the algorithm of Doup, van der Laan and Talman [2] on S^n, although the underlying projections again differ slightly. To describe the appropriate subdivision of S in regions we need the notion of a sign vector and some further notation. For s being a sign vector in $\mathbb{R}^{N+n}$, i.e. $s_{jh} \in \{-1, 0, 1\}$ for all $(j,h) \in I$, we define the subsets $I_j^+(s)$, $I_j^o(s)$ and $I_j^-(s)$ of $I(j)$ by $I_j^+(s) = \{(j,h) \in I(j) | s_{jh} = +1\}$, $I_j^o(s) = \{(j,h) \in I(j) | s_{jh} = 0\}$, $I_j^-(s) = \{(j,h) \in I(j) | s_{jh} = -1\}$ and accordingly $I^+(s)$, $I^o(s)$ and $I^-(s)$ as their respective union over all $j \in I_N$. Let the sets $J^+(s)$, $J^o(s)$ and $J^-(s)$ be given by $J^+(s) = \{j \in I_N | I_j^+(s) \neq \emptyset\}$, $J^o(s) = \{j \in I_N | I_j^+(s) = \emptyset,$ $I_j^o(s) \neq \emptyset\}$ and $J^-(s) = \{j \in I_N | I_j^+(s) = \emptyset,\ I_j^o(s) = \emptyset\}$.
Furthermore, we define the set Ω of so-called allowed sign vectors as

$$\Omega = \{s \in \mathbb{R}^{N+n} | s \text{ is a sign vector such that } I_j^+(s) \neq \emptyset \text{ or } I_j^-(s) \neq \emptyset \text{ for all } j \in I_N \text{ while for at least one } k \in I_N,\ I_k^+(s) \neq \emptyset \text{ and } I_k^-(s) \neq \emptyset\}.$$

With $z(j) \geq 0$ the number of elements in $I_j^o(s)$, a permutation of these elements is denoted by $\gamma_j(s)$, i.e. $\gamma_j(s) = ((j,k_1^j),\dots,(j,k_{z(j)}^j))$, $j \in I_N$, while $\gamma(s)$ denotes the vector of permutations $(\gamma_1(s),\dots,\gamma_N(s))$. For each $\gamma(s)$, $s \in \Omega$, and all $j \in I_N$ we now define the index sets $Z_j^+(s)$, $Z_j^o(s)$ and $Z_j^-(s)$ as follows. When $j \in J^+(s)$ then $Z_j^+(s) = I_j^+(s)$, $Z_j^o(s) = I_j^o(s)$ and $Z_j^-(s) = \emptyset$. For all $j \in J^o(s)$, $Z_j^+(s) = \{(j,k_1^j)\}$, $Z_j^o(s) = I_j^o(s) \setminus \{(j,k_1^j)\}$ and $Z_j^-(s) = I_j^-(s)$, while if $j \in J^-(s)$ then $Z_j^+(s) = I_j^-(s)$ and $Z_j^o(s) = Z_j^-(s) = \emptyset$. Furthermore, let $Z_j(s) = Z_j^+(s) \cup Z_j^o(s) \cup Z_j^-(s)$ and accordingly let $Z(s)$, $Z^+(s)$, $Z^o(s)$ and $Z^-(s)$ be the union of the corresponding sets over all $j \in I_N$. Now we can define sets $A^3(s,\gamma(s))$, $s \in \Omega$, forming a subdivision of S.

Definition 3.1 Let s be some sign vector in Ω and $\gamma(s)$ a permutation vector as defined above. The set $A^3(s,\gamma(s))$ is given by

$$A^3(s,\gamma(s))=\{x \in S \mid x = v + \alpha(Z^+(s))q^3(Z^+(s)) +$$

$$\sum_{(i,h) \in Z^o(s)} \alpha(i,h)q^3(i,h) + \sum_{j \in J^o(s)} \alpha(Z_j^-(s))q^3(Z_j^-(s))$$

with $\quad 0<\alpha(j,k^j_{z(j)})<\dots<\alpha(j,k^j_1)<\alpha(Z^+(s))<1,\ j \in J^+(s)$, and

$$0<\alpha(Z_j^-(s))<\alpha(j,k^j_{z(j)})<\dots<\alpha(j,k^j_2)<\alpha(Z^+(s))<1,\ j \in J^o(s)\}, \quad (3.1)$$

where the vectors $q^3(Z^+(s))$, $q^3(j,k^j_i)$, $(j,k^j_i) \in Z^o(s)$, $q^3(Z_j^-(s))$ for $j \in J^o(s)$, are given by

$$q^3(Z^+(s)) = p(Z^+(s)) - v$$

$$q^3(j,k^j_i) = p(Z^+(s) \cup \{(j,k^j_1),\dots,(j,k^j_i)\}) - p(Z^+(s) \cup \{(j,k^j_1),\dots,(j,k^j_{i-1})\})$$

and

$$q^3(Z_j^-(s)) = p(Z^+(s) \cup Z_j^o(s) \cup Z_j^-(s)) - p(Z^+(s) \cup Z_j^o(s)).$$

Here $p(\cdot)$ is again the projection of v as defined in section 2.

The following lemma describes when the rank of the matrix $Q^3(s,\gamma(s))$ containing the columns $q^3(Z^+(s))$, $q^3(j,k^j_i)$ for (j,k^j_i) in $Z^o(s)$, and $q^3(Z_j^-(s))$ for j in $J^o(s)$, has maximal rank, i.e. when $r(Q^3(s,\gamma(s)))=t$, where $t = 1 + \sum_{j=1}^{N} z(j)$.

Lemma 3.2 The matrix $Q^3(s,\gamma(s))$ described above has rank less than t, $t = 1 + \sum_{j=1}^{N} z(j)$, iff for all $j \in J^+(s)$ holds that $v_{jk} = 0$ for all $(j,k) \notin Z_j(s)$.

The dimension of $A^3(s,\gamma(s))$ is t iff $r(Q^3(s,\gamma(s))) = t$. Observe that the rank of the matrix $Q^3(s,\gamma(s))$ is independent of $\gamma(s)$. The set τ^3 will denote the set of sign vectors s for which the dimension of $A^3(s,\gamma(s))$ equals t. In the sequel we consider only these areas. The boundary of $A^3(s,\gamma(s))$, $s \in \tau^3$, consists of a number of (t-1)-dimensional subsets with one of the inequalities in (3.1) set to an equality, i.e. either $\alpha(Z^+(s)) = 1$ or, for some $j \in J^+(s)$, $\alpha(Z^+(s)) = \alpha(j,k^j_1)$, $\alpha(j,k^j_{i-1}) = \alpha(j,k^j_i)$ with $i \in \{2,\dots,z(j)\}$ or $\alpha(j,k^j_{z(j)}) = 0$, or, for some $j \in J^o(s)$, $\alpha(Z^+(s)) = \alpha(j,k^j_2)$, $\alpha(j,k^j_{i-1}) = \alpha(j,k^j_i)$ with $i \in \{3,\dots,z(j)\}$, $\alpha(j,k^j_{z(j)}) = \alpha(Z^-_j(s))$, or $\alpha(Z^-_j(s)) = 0$.

Let $A^3(s)$, $s \in \tau^3$, be the union of $A^3(s,\gamma(s))$ over all permutation vectors $\gamma(s)$ of $I^o(s)$, then S is subdivided into t-dimensional areas $A^3(s)$ with $s \in \tau^3$. For the case N=2, $n_1=2$, $n_2=1$ and the case N=3, $n_1=n_2=n_3=1$ some regions are illustrated in figure 3.1.a and 3.1.b respectively.

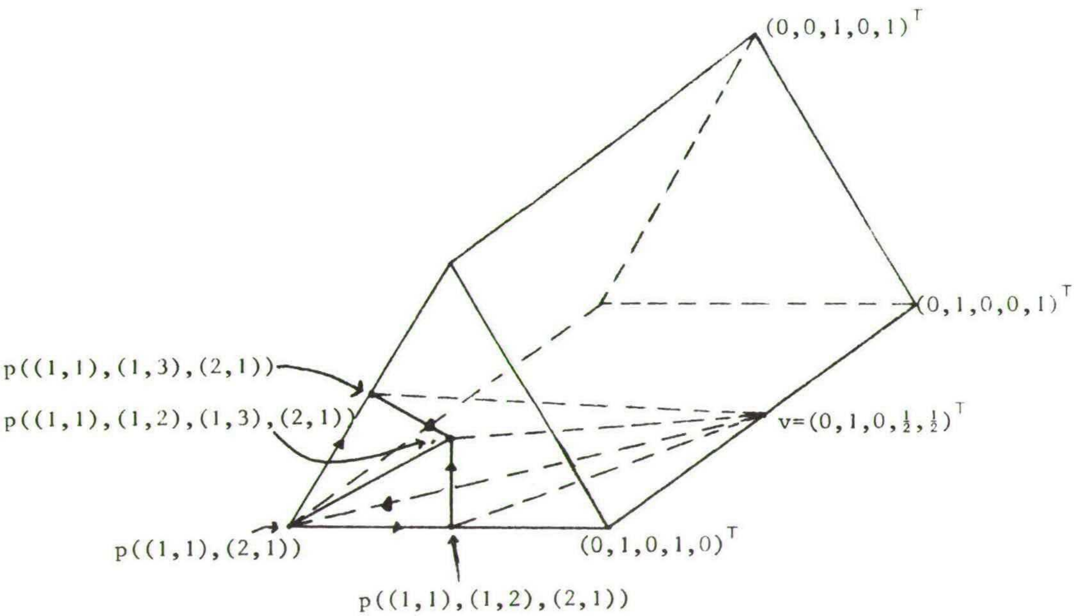

Figure 3.1.a Illustration of $A^3(s)$, $s=(+1,0,0,+1,-1)^T$, which is subdivided into $A^3(s,((1,2),(1,3)))$ and $A^3(s,((1,3),(1,2)))$; $\dim A^3(s) = \sum_{j=1}^{2} z(j) + 1 = 3$

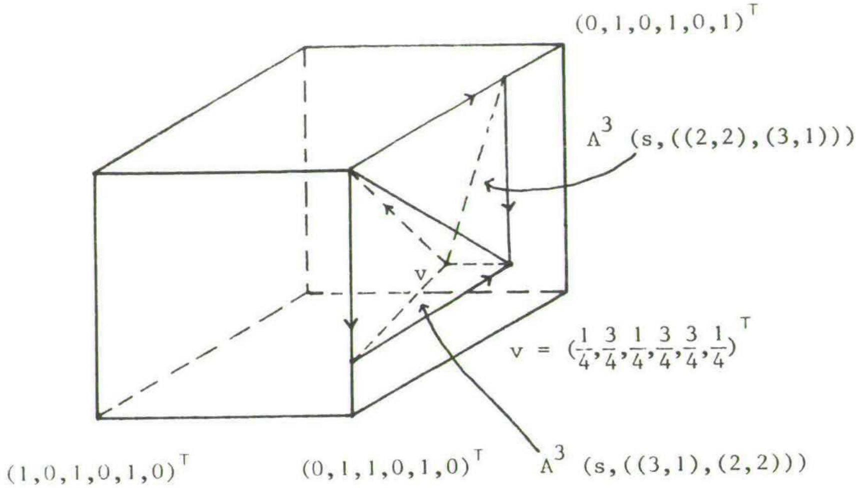

Figure 3.1.b Illustration of $A^3(s)$, $s = (-1,+1,0,-1,0,+1)^T$. The dimension of $A^3(s)$ equals $\sum_{j=1}^{3} z(j) + 1 = 3$

Let S be triangulated by the V-triangulation with grid size m^{-1}, where m is some positive integer, and with projection vectors (2.1), then each region $A^3(s)$, $s \in \tau^3$, is triangulated by this triangulation in t-simplices. In fact, each subset $A^3(s,\gamma(s))$ is triangulated by the set $G^3(s,\gamma(s))$ of t-simplices defined as follows.

Definition 3.3 Let $s \in \tau^3$ and $\gamma(s)$ be a permutation vector of $I^o(s)$. The set $G^3(s,\gamma(s))$ is the collection of t-simplices $\sigma(y^1,\pi(s))$ with vertices $y^1,\ldots,y^{t+1}$ such that

$$\text{(i)}\quad y^1 = v + a(Z^+(s))m^{-1}q^3(Z^+(s)) + \sum_{(i,h) \in Z^o(s)} a(i,h)m^{-1}q^3(i,h) + \sum_{j \in J^o(s)} a(Z_j^-(s))m^{-1}q^3(Z_j^-(s)),$$

for integers $a(Z^+(s))$, $a(i,h)$, $(i,h) \in Z^o(s)$, and $a(Z_j^-(s))$, $j \in J^o(s)$, such that $0 \leq a(j,k_{z(j)}^j) \leq \ldots \leq a(j,k_1^j) \leq a(Z^+(s)) \leq m-1$ for all $j \in J^+(s)$ and $0 \leq a(Z_j^-(s) \leq a(j,k_{z(j)}^j) \leq \ldots \leq a(j,k_2^j) \leq a(Z^+(s)) \leq m-1$ if $j \in J^o(s)$

(ii) $\pi(s) = (\pi_1,\ldots,\pi_t)$ is a permutation of the t elements consisting of $Z^+(s)$, the $(t-1) - |J^o(s)|$ elements of $Z^o(s)$, and the $|J^o(s)|$ elements $Z_j^-(s)$, $j \in J^o(s)$, such that the following holds: if $a(j,k_1^j) = a(Z^+(s))$ or $a(j,k_i^j) = a(j,k_{i-1}^j)$ for some i in $\{2,\ldots,z(j)\}$ and $j \in J^+(s)$, this implies $p > p'$ with $\pi_p = (j,k_1^j)$ and $\pi_{p'} = Z^+(s)$, or $\pi_p = (j,k_i^j)$ and $\pi_{p'} = (j,k_{i-1}^j)$ respectively; if for some $j \in J^o(s)$, $a(j,k_2^j) = a(Z^+(s))$ or $a(j,k_i^j) = a(j,k_{i-1}^j)$ for some i in $\{3,\ldots,z(j)\}$ or $a(Z_j^-(s)) = a(j,k_{z(j)}^j)$, this implies $p > p'$ with $\pi_p = (j,k_2^j)$ and $\pi_{p'} = Z^+(s)$, $\pi_p = (j,k_i^j)$ and $\pi_{p'} = (j,k_{i-1}^j)$ or $\pi_p = Z_j^-(s)$ and $\pi_{p'} = (j,k_{z(j)}^j)$ respectively

(iii) $y^{i+1} = y^i + m^{-1}q^3(\pi_i) \quad ,i=1,\ldots,t.$

Now the union $G^3(s)$ of $G^3(s,\gamma(s))$ over all $\gamma(s)$ triangulates $A^3(s)$, whereas $G^3 = \bigcup_{s \in \tau^3} G^3(s)$ triangulates S according to the V-triangulation with grid size m^{-1}. Since this algorithm also moves from one simplex in G^3 to an adjacent one, we describe in table 2 how $\bar\sigma = \sigma(\bar y^1,\bar\pi(s))$ can be obtained from $\sigma(y^1,\pi(s))$ when σ and $\bar\sigma$ are two t-simplices in $G^3(s,\gamma(s))$ having a common facet opposite vertex y^p of σ, $1 \leq p \leq t+1$. In this table $e(Z^+(s))$, $e(i,h)$ for $(i,h) \in Z^o(s)$, and $e(Z_j^-(s))$ for $j \in J^o(s)$, are given by $e_{jk}(Z^+(s)) = 1$ if $(j,k) \in Z^+(s)$ and zero otherwise, $e_{jk}(i,h) = 1$ if $(j,k) = (i,h)$ and zero otherwise while $e_{ih}(Z_j^-(s)) = 1$ if $(i,h) \in Z_j^-(s)$ and zero otherwise. Further, the (N+n)-vector a is defined by $a_{jk} = a(Z^+(s))$ if $(j,k) \in Z^+(s)$, $a_{jk} = a(j,k)$ for $(j,k) \in Z_j^o(s)$, $a_{jk} = a(Z_j^-(s))$ if $(j,k) \in Z_j^-(s)$, and $a_{jk} = 0$ for all $(j,k) \notin Z_j(s)$, $j=1,2,\ldots,N$.

	$\bar{y}^1$	$\bar{\pi}(s)$	$\bar{a}$
$p=1$	$y^1+m^{-1}q^3(\pi_1)$	$(\pi_2,\dots,\pi_t,\pi_1)$	$a+e(\pi_1)$
$1<p<t+1$	y^1	$(\pi_1,\dots,\pi_{p-2},\pi_p,\pi_{p-1},\dots,\pi_t)$	a
$p=t+1$	$y^1-m^{-1}q^3(\pi_t)$	$(\pi_t,\pi_1,\dots,\pi_{t-1})$	$a-e(\pi_t)$

Table 2. p is the index of the vertex to be replaced.

We will now consider the case in which a facet of a t-simplex in $G^3(s,\gamma(s))$ lies in bd $A^3(s,\gamma(s))$.

Lemma 3.4 Let $\sigma(y^1,\pi(s))$ be a t-simplex in $G^3(s,\gamma(s))$ and τ the facet opposite vertex y^p for some p, $1 \leq p \leq t+1$. Then τ lies in the boundary of $A^3(s,\gamma(s))$ iff one of the following cases holds:

a) $p=1$: $\pi_1=Z^+(s)$ and $a(Z^+(s)) = m-1$

b) $1<p<t+1$:

1. $\pi_{p-1}=Z^+(s)$, $\pi_p=(j,k_1^j)$ for certain $j \in J^+(s)$, and $a(\pi_{p-1}) = a(\pi_p)$
2. $\pi_{p-1}=(j,k_{i-1}^j)$, $\pi_p=(j,k_i^j)$ for certain $j \in J^+(s)$ and i, $1 < i \leq z(j)$, and $a(\pi_{p-1}) = a(\pi_p)$
3. $\pi_{p-1}=Z^+(s)$, $\pi_p=(j,k_2^j)$ for certain $j \in J^o(s)$ and $a(\pi_{p-1}) = a(\pi_p)$
4. $\pi_{p-1}=(j,k_{i-1}^j)$, $\pi_p=(j,k_i^j)$ for certain $j \in J^o(s)$ and i, $2 < i \leq z(j)$, and $a(\pi_{p-1}) = a(\pi_p)$
5. $\pi_{p-1}=(j,k_{z(j)}^j)$ if $z(j) > 1$ or $\pi_{p-1}=Z^+(s)$ if $z(j)=1$, $\pi_p=Z_j^-(s)$, and $a(\pi_{p-1}) = a(\pi_p)$ for certain $j \in J^o(s)$

c) $p=t+1$:

1. $\pi_t=(j,k_{z(j)}^j)$ for certain $j \in J^+(s)$ and $a(\pi_t) = 0$
2. $\pi_t=Z_j^-(s)$ for certain $j \in J^o(s)$ and $a(\pi_t) = 0$.

The lemma follows immediately from the definitions of $G^3(s,\gamma(s))$ and $A^3(s,\gamma(s))$. In lemma 3.5 we consider more carefully the cases indicated in the foregoing lemma and it appears that a facet in bd $A^3(s,\gamma(s))$ either lies in bd S or is a facet of exactly one t-simplex in $G^3(s,\bar{\gamma}(s))$ with $\bar{\gamma}(s)$ differing from $\gamma(s)$ or is a (t-1)-simplex in $A^3(\bar{s},\gamma(\bar{s}))$ for some $\bar{s}$ with $|I^o(\bar{s})| = |I^o(s)| - 1$.

Lemma 3.5 In case a of lemma 3.4 τ lies in $S(Z(s))$, i.e. τ lies in the set $\{x \in S | x_{jk} = 0$ for all $(j,k) \in I_j^-(s)$, $j \in J^+(s)\}$. For the cases b1 - b5 of lemma 3.4 we have

b1 $\tau = \sigma(y^1,\pi(\bar{s}))$ is a (t-1)-simplex in $G^3(\bar{s},\gamma(\bar{s}))$, where $\bar{s}_{j,k_1^j} = 1$, $\bar{s}_{ih} = s_{ih}$ if $(i,h) \neq (j,k_1^j)$, $\gamma_j(\bar{s}) = ((j,k_2^j),\ldots,(j,k_{z(j)}^j))$, $\gamma_h(\bar{s}) = \gamma_h(s)$ if $h \neq j$, and $\pi(\bar{s}) = (\pi_1,\ldots,\pi_{p-2},Z^+(\bar{s}),\pi_{p+1},\ldots,\pi_t)$

b2 τ is a facet of the t-simplex $\sigma(y^1,\bar{\pi}(s))$ in $G^3(s,\bar{\gamma}(s))$, where $\bar{\gamma}_j(s) = ((j,k_1^j),\ldots,(j,k_{i-2}^j),(j,k_i^j),(j,k_{i-1}^j),\ldots,(j,k_{z(j)}^j))$, $\bar{\gamma}_h(s) = \gamma_h(s)$ if $h \neq j$, and $\bar{\pi}(s) = (\pi_1,\ldots,\pi_{p-2},\pi_p,\pi_{p-1},\pi_{p+1},\ldots,\pi_t)$

b3 τ is a facet of the t-simplex $\sigma(y^1,\bar{\pi}(s))$ in $G^3(s,\bar{\gamma}(s))$, where $\bar{\gamma}_j(s) = ((j,k_2^j),(j,k_1^j),\ldots,(j,k_{z(j)}^j))$, $\bar{\gamma}_h(s) = \gamma_h(s)$ if $h \neq j$, and $\bar{\pi}(s) = (\pi_1,\ldots,\pi_{p-1},(j,k_1^j),\pi_{p+1},\ldots,\pi_t)$

b4 this case has already been described in b2

b5 τ is the (t-1)-simplex $\sigma(y^1,\pi(\bar{s}))$ in $G^3(\bar{s},\gamma(\bar{s}))$, where $\bar{s}_{j,k_{z(j)}^j} = -1$, $\bar{s}_{ih} = s_{ih}$ if $(i,h) \neq (j,k_{z(j)}^j)$, $\gamma_j(\bar{s}) = ((j,k_1^j),\ldots,(j,k_{z(j)-1}^j))$, $\gamma_h(\bar{s}) = \gamma_h(s)$ if $h \neq j$, $\pi(\bar{s}) = (\pi_1,\ldots,\pi_{p-2},Z_j^-(\bar{s}),\pi_{p+1},\ldots,\pi_t)$ if $z(j) > 1$, and $\pi(\bar{s}) = (\pi_1,\ldots,\pi_{p-2},Z^+(\bar{s}),\pi_{p+1},\ldots,\pi_t)$ if $z(j) = 1$.

Finally for the cases c1 and c2 of lemma 3.4 holds c1 τ is the $(t-1)$-simplex $\sigma(y^1,\pi(\bar{s}))$ in $G^3(\bar{s},\gamma(\bar{s}))$, where $\bar{s}_{j,k^j_{z(j)}} = -1$, $\bar{s}_{ih} = s_{ih}$ if $(i,h) \neq (j,k^j_{z(j)})$, $\gamma_j(\bar{s}) = ((j,k^j_1),\ldots,(j,k^j_{z(j)-1}))$, $\gamma_h(\bar{s}) = \gamma_h(s)$ if $h \neq j$, and $\pi(\bar{s}) = (\pi_1,\ldots,\pi_{t-1})$

c2 τ is the $(t-1)$-simplex $\sigma(y^1,\pi(\bar{s}))$ in $G^3(\bar{s},\gamma(\bar{s}))$, where $\bar{s}_{j,k^j_1} = 1$, $\bar{s}_{ih} = s_{ih}$ if $(i,h) \neq (j,k^j_1)$, $\gamma_j(\bar{s}) = ((j,k^j_2),\ldots,(j,k^j_{z(j)}))$, $\gamma_h(\bar{s}) = \gamma_h(s)$ if $h \neq j$, and $\pi(\bar{s}) = (\pi_1,\ldots,\pi_{t-1})$.

The definition of the areas $A^3(s,\gamma(s))$, $s \in \tau^3$, implies that some sets are represented by more than one sign vector. These cases are described in the following lemma, whose proof is a direct result of the definition of the $A^3(s,\gamma(s))$'s.

Lemma 3.6 Let s^1 be a sign vector in τ^3 with permutation vector $\gamma(s^1)$. If $I^-_j(s^1) = I(j)$ for some $j \in I_N$, then $A^3(s^1,\gamma(s^1)) = A^3(s^2,\gamma(s^2))$, where $s^2_{jk} = 1$ for all $(j,k) \in I(j)$, $s^2_h = s^1_h$ if $h \neq j$, and $\gamma(s^2) = \gamma(s^1)$. If $I^-_j(s^1)=\{(j,h)\}$ for some $j \in J^o(s^1)$, then $A^3(s^1,\gamma(s^1)) = A^3(s^2,\gamma(s^2))$, where $s^2_{j,k^j_1} = 1$, $s^2_{jh} = 0$, $s^2_{i\ell} = s^1_{i\ell}$ if $(i,\ell) \notin \{(j,h),\ (j,k^j_1)\}$, $\gamma_j(s^2) = ((j,k^j_2),\ldots,(j,k^j_{z(j)}),(j,h))$, and $\gamma_h(s^2) = \gamma_h(s^1)$ if $h \neq j$.

Now we introduce the concept of an s-complete simplex, where $s \in \Omega$. This notion is comparable with the concept of a T-complete simplex in case of the sum- and the product-ray algorithms.

Definition 3.7 For $s \in \Omega$, a g-simplex $\sigma(y^1,\ldots,y^{g+1})$, $g=t-1,t$, where $t = 1 + |I^o(s)|$, is s-complete if the system of linear equations

$$\sum_{i=1}^{g+1} \lambda_i \binom{z(y^i)}{1} - \sum_{(j,k) \notin I^o(s)} \mu_{jk} s_{jk} \binom{e(j,k)}{0} = \binom{0}{1} \qquad (3.2)$$

has a nonnegative solution λ^*_i, $i=1,\ldots,g+1$, and μ^*_{jk}, $(j,k) \notin I^o(s)$. A solution of (3.2) is denoted by (λ^*,μ^*).

Also for this algorithm we need a nondegeneracy assumption to guarantee convergency.

Nondegeneracy assumption. For $g=t-1$ the system (3.2) has a unique solution (λ^*,μ^*) with $\lambda_i^* > 0$, $i=1,\ldots,t$, and $\mu_{jk}^* > 0$, $(j,k) \notin I^o(s)$. For $g=t$ at most one variable of λ_i^*, $i=1,\ldots,t+1$, and μ_{jk}^*, $(j,k) \notin I^o(s)$, is equal to zero.

Observe that if for all x in an s-complete g-simplex, $\sigma(y^1,\ldots,y^{g+1})$, $g = t-1$ or t, $x_{jk} = 1$ for some $(j,k) \notin I^o(s)$, then according to the condition that $\sum_{h=1}^{n_j+1} x_{jh}z_{jh}(x) = 0$, we have $z_{jk}(x) = 0$ so that $\mu_{jk} = 0$. To obtain a nondegenerate solution (λ^*,μ^*) in this case we perturb $z_{jk}(x)$ slightly as follows. For all vectors y in S having one or more components equal to one we define $z_{jk}(y) = +\alpha$ if both $y_{jk} = 1$ and $z_j(y) < 0$, and $z_{jk}(y) = -\alpha$ if $y_{jk} = 1$ and $z_{jh}(y)$ positive for at least one $(j,h) \in I(j)$, where α is some small positive number. Without loss of generality we assume that v does not solve the NLCP on S.

For varying $s \in \tau^3$ the exponent-ray algorithm will generate a sequence of adjacent t-simplices with s-complete common facets in $A^3(s)$. The algorithm stops whenever it reaches a complete simplex as defined below. In lemma 3.9 it is shown that such a simplex yields an approximate solution to the NLCP. Let $\bar{z}$ be the piecewise linear approximation of z with respect to the underlying triangulation. For $\bar{z}$ holds that $\bar{z}(\bar{x}) = \sum_{i=1}^{t+1} \lambda_i z(y^i)$ if $\bar{x} = \sum_{i=1}^{t+1} \lambda_i y^i$ is a point in the t-simplex $\sigma(y^1,\ldots,y^{t+1})$.

Definition 3.8 An s-complete (t-1)-simplex $\sigma(y^1,\ldots,y^t)$ for arbitrary sign vector s with $1 \leq t \leq n+1$ is complete if for each $j \in I_N$

$$s_{jk} \leq 0 \qquad \text{if } \bar{x}_{jk} = 0$$

and either

$$s_{jh} \leq 0 \qquad \text{for all } (j,h) \in I(j) \text{ for which } \bar{x}_{jh} > 0$$

or

$$s_{jh} \geq 0 \qquad \text{for all } (j,h) \in I(j) \text{ for which } \bar{x}_{jh} > 0,$$

where $\bar{x} = \Sigma_{i=1}^{t} \lambda_i^* y^i$.

Notice that s does not necessarily lie in τ^3. If a (t-1)-simplex is complete and $\bar{x} = \Sigma_{i=1}^{t} \lambda_i^* y^i$, then according to (3.2)

$$\bar{z}_{jk}(\bar{x}) \leq 0 \qquad \text{if } \bar{x}_{jk} = 0$$

and for each $j \in I_N$ either $\bar{z}_{jh}(\bar{x}) \leq 0$ for all $(j,h) \in I(j)$ for which $\bar{x}_{jh} > 0$ or $\bar{z}_{jh}(\bar{x}) \geq 0$ for all $(j,h) \in I(j)$ for which $\bar{x}_{jh} > 0$. Observe that $\bar{x}_j^T \bar{z}_j(\bar{x})$ is in general not equal to zero although $\bar{x}_j^T z_j(\bar{x}) = 0$, $j \in I_N$.

Lemma 3.9 Let $\varepsilon > 0$ be such that $\max_{(j,h) \in I} |z_{jh}(x) - z_{jh}(y)| < \varepsilon$ for all x and y in the same simplex σ of the V-triangulation of S, and let $\sigma^*(y^1,\ldots,y^t)$ be a complete simplex with solution (λ^*,μ^*). Then $\bar{x} = \sum_{i=1}^{t} \lambda_i^* y^i$ lies in σ^* and satisfies $z_{jh}(\bar{x}) \in (-\varepsilon,+\varepsilon)$ if $(j,h) \in I^o(s)$, $z_{jh}(\bar{x}) < \varepsilon$ for all $(j,h) \in I^-(s)$, and $|z_{jh}(\bar{x}) - \bar{z}_{jh}(\bar{x})| < \varepsilon$ for all $(j,h) \in I^+(s)$ where $\sum_{(j,\ell) \in I_j^+(s)} \bar{x}_{j\ell} \bar{z}_{j\ell}(\bar{x}) < \varepsilon$.

Next we describe when a simplex in $A^3(s)$ is complete. Let $V_j(x) = \{(j,h) \in I(j) | x_{jh} = 0\}$ and $V_j^c(x) = I(j) \backslash V_j(x)$, $j \in I_N$ and $x \in S$. Furthermore, for $j \in I_N$ let $c_j(s) = \min\{|I_j^+(s)|, |I_j^-(s) \cap V_j^c(v)|\}$ and $c(s) = \sum_{j \in J^+(s)} c_j(s)$.

Theorem 3.10 Let $\sigma(y^1,\pi(s))$, $s \in \tau^3$, be an s-complete t-simplex in $G^3(s,\gamma(s))$. Then σ is complete iff at a solution (λ^*,μ^*) for some (j,k) not in $I_j^o(s)$, $j \in J^+(s)$, $\mu_{jk}^* = 0$, $c(s) = 1$ and either $I_j^+(s) = \{(j,k)\}$ or $I_j^-(s) \cap V_j^c(v) = \{(j,k)\}$. A facet of σ is complete iff $\lambda_1^* = 0$, $\pi_1 = Z^+(s)$ and $a(Z^+(s)) = m-1$.

<u>Proof.</u> We first prove that $I_j^+(s) \cap V_j(x) = \emptyset$, $j \in I_N$, for all x in $\sigma(y^1,\pi(s))$ with x unequal to v. Since σ lies in $G^3(s,\gamma(s))$ we have for all $(j,k) \in I_j^+(s)$

$$y_{jk}^1 = v_{jk} + a(Z^+(s))m^{-1}q_{jk}^3(Z^+(s)) + \sum_{(j,h) \in I_j^o(s)} a(j,h)m^{-1}q_{jk}^3(j,h).$$

Suppose that $a(Z^+(s)) = 0$. Then according to definition 3.3 $a(j,h) = 0$ for all $(j,h) \in I_j^o(s)$, $j \in J^+(s) \cup J^o(s)$. Hence, the vertex y^1 is equal to v and π_1 must be equal to $Z^+(s)$. Since for $(j,k) \in I_j^+(s), i=2,\ldots,t+1$, $y_{jk}^i=v_{jk} + (\sum_{p=1}^{i-1} q_{jk}^3(\pi_p))/m=(1 - \frac{1}{m})v_{jk} + \frac{1}{m}\, p_{jk}(Z^+(s) \cup \{\pi_2,\ldots,\pi_{i-1}\})$ we obtain that $y_{jk}^i > 0$. Consequently, $x_{jk} > 0$ for all $x \neq y^1 = v$. When $a(Z^+(s)) > 0$ it follows immediately that $y_{jk}^i > 0$ for all $i=1,\ldots,t+1$, so that $x_{jk} > 0$ for all x in σ and $(j,k) \in I_j^+(s)$. On the other hand for all $j \in J^+(s)$ we have that for x in σ, $I_j^-(s) \cap V_j^c(x) = I_j^-(s) \cap V_j^c(v)$, because $Z_j^-(s) = \emptyset$ and $x_{jk} = (1-\alpha(Z^+(s)))v_{jk}$, $0 < \alpha(Z^+(s)) < 1$, for all $(j,k) \in I_j^-(s)$. Therefore, if σ is an s-complete t-simplex with $c(s) = 1$, $\mu_{jk}^* = 0$ for some $(j,k) \notin I_j^o(s)$ and either $I_j^+(s) = \{(j,k)\}$ or $I_j^-(s) \cap V_j^c(v) = \{(j,k)\}$, where $j \in J^+(s)$, then σ is also an $\bar{s}$-complete simplex. For $\bar{s}$ holds that $\bar{s}_{jk} = 0$, $\bar{s}_{ih} = s_{ih}$ for all $(i,h) \neq (j,k)$ whereas $I_j^+(\bar{s}) \cap V_j(x) = \emptyset$ and either $I_j^+(\bar{s}) \cap V_j^c(x) = \emptyset$ or $I_j^-(s) \cap V_j^c(x) = \emptyset$ for all x in σ and $j \in J^+(\bar{s})$. Hence, σ is a complete simplex according to definition 3.8. The reverse implication is now straightforward to derive.

When $\lambda_1^* = 0$, $\pi_1 = Z^+(s)$ and $a(Z^+(s)) = m-1$, then according to lemma 3.5, the facet τ of σ opposite vertex y^1 lies in $S_{jk} = \{x \in S \mid x_{jk} = 0\}$, $(j,k) \in I_j^-(s)$ and $j \in J^+(s)$. Consequently, $I_j^-(s) \cap V_j^c(x) = \emptyset$ for all x in τ and $j \in J^+(s)$, so that τ is complete. The reverse implication follows along the same lines. $\square$

We remark that if $z(v) \leq 0$ then v solves the nonlinear complementarity problem. If not $z(v) \leq 0$, then the 0-dimensional simplex $\sigma(v)$ is an s^o-complete facet of the 1-dimensional simplex $\sigma(v,(Z^+(s^o)))$ in $A^3(s^o)$, where $s^o = \text{sgn } z(v)$. Recall that $s^o_{jk} = +1$ if $v_{jk} = 1$ and $z_j(v) \leq 0$ and that $s^o_{jk} = -1$ if $v_{jk} = 1$ and $z_{jh}(v) > 0$ for at least one index $(j,h) \in I(j)$ unequal to (j,k). From the nondegeneracy assumption it follows that there is no other sign vector s in τ^3 for which $\sigma(v)$ is an s-complete facet of a 1-simplex $\sigma(v,\pi(s))$ in $A^3(s)$. For given $s \in \tau^3$, the s-complete t-simplices in $A^3(s)$ now form sequences of adjacent t-simplices with common s-complete facets. A sequence which is not a loop has two end simplices. An end simplex is either an s-complete t-simplex σ in $G^3(s,\gamma(s))$ with a solution (λ^*,μ^*) such that $\mu^*_{jk} = 0$ for some (j,k) in $I^+(s) \cup I^-(s)$ or is an s-complete t-simplex with an s-complete facet τ in the boundary of $A^3(s)$. In the latter case the facet τ is either, according to theorem 3.10, a complete (t-1)-simplex or is, according to lemma 3.5, an $\bar{s}$-complete (t-1)-simplex in $A^3(\bar{s})$ for some $\bar{s} \neq s$. This simplex in $A^3(\bar{s})$ is again an end simplex of a sequence of adjacent (t-1)-simplices in $A^3(\bar{s})$ with common $\bar{s}$-complete facets, where $\bar{s}$ differs from s in only one component which is 0 in s. In the former case the s-complete t-simplex σ is complete iff the conditions of theorem 3.10 hold. The case in which σ is not complete is described in the next two lemmas. Lemma 3.11 describes the case when $\mu^*_{jk} = 0$ for some (j,k) in $I^+(s)$ and lemma 3.12 the case when $\mu^*_{jk} = 0$ for some (j,k) in $I^-(s)$.

<u>Lemma 3.11</u> If $\mu^*_{jk} = 0$ for some (j,k) in $I^+_j(s)$ and σ is not complete then the s-complete t-simplex $\sigma(y^1,\pi(s))$ is either 1) a facet of an $\bar{s}$-complete (t+1)-simplex $\bar{\sigma}$ in $G^3(\bar{s})$ with $\bar{s}_{jk} = 0$ or 2) an $\bar{s}$-complete t-simplex $\bar{\sigma}$ in $G^3(\bar{s})$, with $\bar{s}_{jk} = 0$ and $\bar{s}_{jk^j_{z(j)}} = -1$. More precisely, the following possibilities can occur.

1.i) $|I^+_j(s)| = 1$ and $I^-_j(s) \neq \emptyset$: $\sigma(y^1,\pi(s))$ is a facet of the (t+1)-simplex $\sigma(y^1,\pi(\bar{s}))$ in $G^3(\bar{s},\gamma(\bar{s}))$, where $\bar{s}_{jk} = 0$, $\bar{s}_{ih} = s_{ih}, (i,h) \neq (j,k)$, $\gamma_j(\bar{s}) = ((j,k),(j,k^j_1),\ldots,(j,k^j_{z(j)}))$, $\gamma_h(\bar{s}) = \gamma_h(s)$ for all $h \neq j$, and $\pi(\bar{s}) = (\pi_1,\ldots,\pi_t,Z^-_j(s))$

ii) $|I_j^+(s)| > 1$: $\sigma(y^1,\pi(s))$ is a facet of the (t+1)-simplex $\sigma(y^1,\pi(\bar{s}))$ in $G^3(\bar{s},\gamma(\bar{s}))$, where $\bar{s}_{jk} = 0$, $\bar{s}_{ih} = s_{ih}, (i,h) \neq (j,k)$, $\gamma_j(\bar{s}) = ((j,k),(j,k_1^j),\ldots,(j,k_{z(j)}^j))$, $\gamma_h(\bar{s}) = \gamma_h(s)$ for all $h \neq j$, and $\pi(\bar{s}) = (\pi_1,\ldots,\pi_{p-1},Z^+(\bar{s}),(j,k),\pi_{p+1},\ldots,\pi_t)$

2. $|I_j^+(s)| = 1$ and $I_j^-(s) = \emptyset$: $\sigma(y^1,\pi(s))$ is also the t-simplex $\sigma(y^1,\pi(\bar{s}))$ in $G^3(\bar{s},\gamma(\bar{s}))$, where $\bar{s}_{jk} = 0$, $\bar{s}_{jk_{z(j)}^j} = -1$, $\bar{s}_{ih} = s_{ih}$ for all other (i,h), $\gamma_j(\bar{s}) = ((j,k),(j,k_1^j),\ldots,(j,k_{z(j)-1}^j))$, $\gamma_h(\bar{s}) = \gamma_h(s)$ for all $h \neq j$, and $\pi(\bar{s}) = (\pi_1,\ldots,\pi_{p-1},Z_j^-(\bar{s}),\pi_{p+1},\ldots,\pi_t)$ where $\pi_p = (j,k_{z(j)}^j)$.

In the case 2 described in lemma 3.11 the t-simplex σ is also an $\bar{s}$-complete simplex in the area $A^3(\bar{s},\gamma(\bar{s}))$ which is equal to $A^3(s,\gamma(s))$ (see lemma 3.6). We illustrate this case in figure 3.2 where N=2 and $n_1=n_2=1$. In case 1 of lemma 3.11 σ is an $\bar{s}$-complete facet of the uniquely determined (t+1)-simplex $\sigma(y^1,\pi(\bar{s}))$ in $A^3(\bar{s})$.

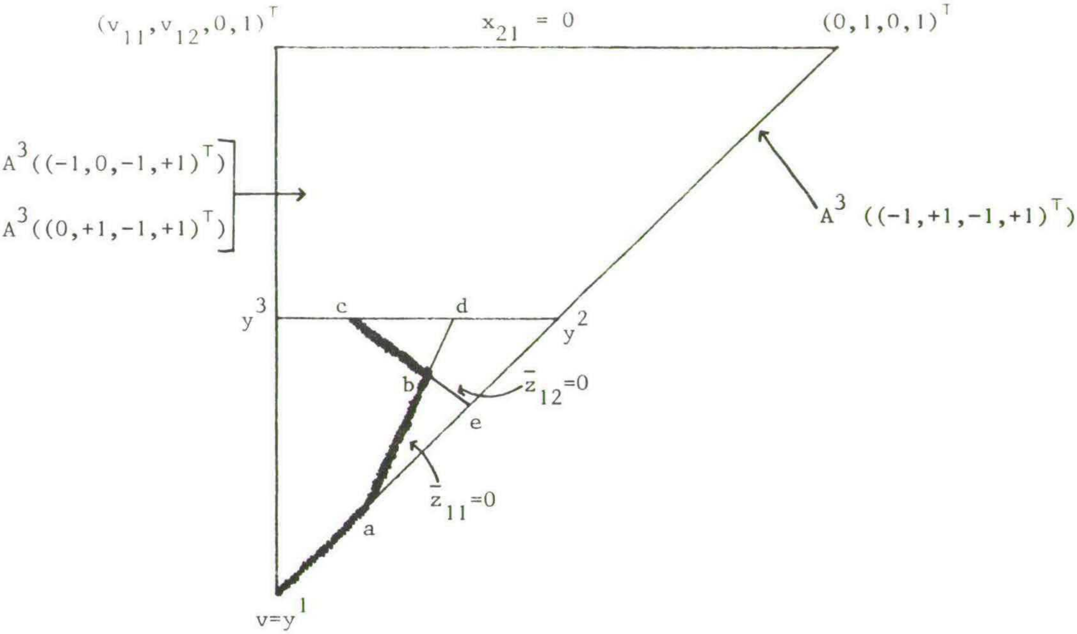

Figure 3.2 The starting point v lies in the interior of $S^1 \times S^1$. The grid size of the triangulation is $\frac{1}{2}$. Concerning the sign pattern of z we have that sgn $z(y^3)$ = sgn $z(y^1) = (-1,+1,-1,+1)^T$ and sgn $z(y^2) = (+1,-1,-1,+1)^T$. Further sgn $\bar{z}(a) = (0,+1,-1,+1)^T$, sgn $\bar{z}(b) = (0,0,-1,+1)^T$, sgn $\bar{z}(c) = (-1,0,-1,+1)^T$, sgn $\bar{z}(d) = (0,-1,-1,+1)^T$ and sgn $\bar{z}(e) = (+1,0,-1,+1)^T$. The algorithm follows the heavily drawn line $x = \Sigma_i \lambda_i y^i$ and goes from $A^3((-1,+1,-1,+1)^T)$ via $A^3((0,+1,-1,+1)^T)$ into $A^3((-1,0,-1,+1)^T)$.

Lemma 3.12 If $\mu^*_{jk} = 0$ for some (j,k) in $I_j^-(s)$ and σ is not complete then the s-complete t-simplex $\sigma(y^1,\pi(s))$ is either 1) a facet of an $\bar{s}$-complete (t+1)-simplex $\bar{\sigma}$ in $G^3(\bar{s})$ with $\bar{s}_{jk} = 0$ or 2) an $\bar{s}$-complete t-simplex in $G^3(\bar{s})$ with $\bar{s}_{jk} = 0$ and $\bar{s}_{jk_1^j} = 1$. More precisely, the following possibilities can occur.

1.i) $|I_j^-(s)| > 1$, $I_j^+(s) = \emptyset$ and $I_j^o(s) \neq \emptyset$: $\sigma(y^1,\pi(s))$ is a facet of the (t+1)-simplex $\sigma(y^1,\pi(\bar{s}))$ in $G^3(\bar{s},\gamma(\bar{s}))$, where $\bar{s}_{jk} = 0$, $\bar{s}_{ih} = s_{ih}$, $(i,h) \neq (j,k)$, $\gamma_j(\bar{s}) = ((j,k_1^j),\ldots,(j,k_{z(j)}^j),(j,k))$, $\gamma_h(\bar{s}) = \gamma_h(s)$ for all $h \neq j$, and $\pi(\bar{s}) = (\pi_1,\ldots,\pi_{p-1},(j,k),Z_j^-(\bar{s}),\ldots,\pi_t)$

ii) $|I_j^-(s)| > 1$, $I_j^+(s) = \emptyset$ and $I_j^o(s) = \emptyset$: $\sigma(y^1,\pi(s))$ is a facet of the (t+1)-simplex $\sigma(y^1,\pi(\bar{s}))$ in $G^3(\bar{s},\gamma(\bar{s}))$, where $\bar{s}_{jk} = 0$, $\bar{s}_{ih} = s_{ih}$, $(i,h) \neq (j,k)$, $\gamma_j(\bar{s}) = ((j,k))$, $\gamma_h(\bar{s}) = \gamma_h(s)$ for all $h \neq j$, and $\pi(\bar{s}) = (\pi_1,\ldots,\pi_{p-1},(j,k),Z_j^-(\bar{s}),\ldots,\pi_t)$

iii) $|I_j^-(s)| > 1$ and $I_j^+(s) \neq \emptyset$: $\sigma(y^1,\pi(s))$ is a facet of the (t+1)-simplex $\sigma(y^1,\pi(\bar{s}))$ in $G^3(\bar{s},\gamma(\bar{s}))$, where $\bar{s}_{jk} = 0$, $\bar{s}_{ih} = s_{ih}$, $(i,h) \neq (j,k)$, $\gamma_j(\bar{s}) = ((j,k_1^j),\ldots,(j,k_{z(j)}^j),(j,k))$, $\gamma_h(\bar{s}) = \gamma_h(s)$ for all $h \neq j$, and $\pi(\bar{s}) = (\pi_1,\ldots,\pi_t,(j,k))$

2. $|I_j^-(s)| = 1$ and $I_j^+(s) = \emptyset$: $\sigma(y^1,\pi(s))$ is also the t-simplex $\sigma(y^1,\pi(\bar{s}))$ in $G^3(\bar{s},\gamma(\bar{s}))$, where $\bar{s}_{jk} = 0$, $\bar{s}_{jk_1^j} = 1$, $\bar{s}_{ih} = s_{ih}$ for all other (i,h), $\gamma_j(\bar{s}) = ((j,k_2^j),\ldots,(j,k_{z(j)}^j),(j,k))$, $\gamma_h(\bar{s}) = \gamma_h(s)$ for all $h \neq j$, and $\pi(\bar{s}) = (\pi_1,\ldots,\pi_{p-1},(j,k),\pi_{p+1},\ldots,\pi_t)$ where $\pi_p = Z_j^-(s)$.

The case 3.12.2 is comparable with case 2 of lemma 3.11. Observe that each simplex defined in the two lemmas indeed exists since σ is not complete. So, the end simplex of each sequence of adjacent s-complete t-simplices in $A^3(s)$ with s-complete common facets and not being a loop can be linked with a sequence in another area $A^3(\bar{s})$ unless the end simplex is complete or equal to $\sigma(v)$. The latter sequence can be a sequence of adjacent $\bar{s}$-complete (t-1)-simplices with common $\bar{s}$-complete facets in $A^3(\bar{s})$, where for some $(j,k) \in I$, $\bar{s}_{jk} = \pm 1$ and $s_{jk} = 0$, while $\bar{s}_{\ell p} = s_{\ell p}$ for all $(\ell,p) \neq (j,k)$. Another possibility is that an end simplex in $A^3(s)$ is an $\bar{s}$-complete facet of a (t+1)-simplex $\bar{\sigma}$ in $A^3(\bar{s})$, with

$\bar{s}_{jk} = 0$, $s_{jk} = \pm 1$, $\bar{s}_{\ell p} = s_{\ell p}$ for all $(\ell,p) \neq (j,k)$. The simplex $\bar{\sigma}$ is then an end simplex of a path of adjacent $\bar{s}$-complete $(t+1)$-simplices in $A^3(\bar{s})$ with common $\bar{s}$-complete facets. The last possibility concerns the case in which an s-complete end simplex σ in $A^3(s)$ is also an $\bar{s}$-complete t-simplex in $A^3(\bar{s})$ for some $\bar{s} \in \tau^3$ (see lemma 3.11.2 and 3.12.2). The simplex σ is then also an end simplex of a sequence of $\bar{s}$-complete adjacent t-simplices in $A^3(\bar{s})$. In this way all paths can be linked. As a result there exists a path of adjacent s-complete simplices in regions $A^3(s)$, $s \in \tau^3$, connecting $\sigma(v)$ and a complete simplex. The number of simplices along this path is finite because the total number of simplices in S is finite. The exponent-ray algorithm generates this sequence of simplices starting with $\sigma(v)$ by following a piecewise linear path from v to an approximate solution x^* induced by system (3.2). The successive steps of the algorithm result from linear programming pivoting steps in system (3.2) combined with corresponding replacement steps in the triangulation. A decrease in dimension of the current simplex is followed by introducing a unit vector column in system (3.2). On the other hand the dimension is increased when such a column is eliminated by a linear programming pivoting step. We remark that the p.l. path followed by the algorithm might have more than one linear piece in a simplex (see lemma 3.11 and 3.12, case 2). This is caused by the fact that $\bar{x}_j^T \bar{z}_j(\bar{x})$ is in general unequal to zero, $j \in I_N$. A further interpretation of the algorithm is presented in section 4. Here we conclude this section with a presentation of the formal steps of the algorithm.

<u>Step 0</u>. [Initialization] Set $s_{jk} = \operatorname{sgn} z_{jk}(v)$ for all $(j,k) \in I$. If $s_{jk} \leq 0$ for all $(j,k) \in I$ then the algorithm stops with the solution v. Otherwise set $t=1$, $y^1=v$, $\pi(s) = (Z^+(s))$, $\sigma = \sigma(y^1,\pi(s))$, $\gamma_j(s) = \emptyset$ for all $j \in I_N$, $\bar{p} = 2$, $a_{jk} = 0$ for all $(j,k) \in I$, $\mu_{jk} = |z_{jk}(v)|$ for all $(j,k) \in I$, $\lambda_1 = 1$, $c_j(s) = \min \{|I_j^+(s)|\ ,\ |I_j^-(s) \cap V_j^c(v)|\}$ for all $j \in I_N$.

<u>Step 1</u>. Calculate $z(y^{\bar{p}})$ and perform an l.p. pivot step by bringing $(z^T(y^{\bar{p}}),1)^T$ in the linear system

$$\sum_{\substack{i=1\\ i\neq \bar{p}}}^{t+1} \lambda_i \binom{z(y^i)}{1} - \sum_{(j,k)\notin I^o(s)} \mu_{jk} s_{jk} \binom{e(j,k)}{0} = \binom{0}{\bar{1}} .$$

If μ_{jk} becomes zero for some $(j,k) \notin I^o(s)$ then go to step 3. Else λ_p is eliminated for exactly one $p \neq \bar{p}$ and the facet $\tau(y^1,\ldots,y^{p-1},y^{p+1},\ldots,y^{t+1})$ is s-complete.

<u>Step 2</u>. If $p = 1$, $\pi_1 = Z^+(s)$ and $a(Z^+(s)) = m-1$ then τ is complete and the algorithm stops.

In the case $1 < p < t+1$ and if

i) $\pi_{p-1} = Z^+(s)$, $\pi_p = (j,k_1^j)$ for some $j \in J^+(s)$, and $a(\pi_{p-1}) = a(\pi_p)$, then s, $\gamma(s)$ and $\sigma(y^1,\pi(s))$ are adapted according to lemma 3.5, case b1; set $t = t-1$ and $(i,h) = (j,k_1^j)$, adapt $c_j(s)$ and go to step 4

ii) $\pi_{p-1} = (j,k_{i-1}^j)$, $\pi_p = (j,k_i^j)$ for some $j \in J^+(s)$, $1<i\leq z(j)$, and $a(\pi_{p-1}) = a(\pi_p)$, then $\gamma(s)$ and $\sigma(y^1,\pi(s))$are adapted according to lemma 3.5, case b2; return to step 1 with $\bar{p}$ the index of the new vertex of σ

iii) $\pi_{p-1} = Z^+(s)$, $\pi_p = (j,k_2^j)$ for certain $j \in J^o(s)$ and $a(\pi_{p-1}) = a(\pi_p)$, then $\gamma(s)$ and $\sigma(y^1,\pi(s))$ are adapted according to lemma 3.5, case b3; return to step 1 with $\bar{p}$ the index of the new vertex of σ

iv) $\pi_{p-1} = (j,k_{i-1}^j)$, $\pi_p = (j,k_i^j)$ for certain $j \in J^o(s)$, $2<i\leq z(j)$, and $a(\pi_{p-1}) = a(\pi_p)$, then $\gamma(s)$ and $\sigma(y^1,\pi(s))$ are adapted according to case b4 of lemma 3.5; return to step 1 with $\bar{p}$ the index of the new vertex of σ

v) $\pi_{p-1} = (j,k_{z(j)}^j)$ if $z(j) > 1$ or $\pi_{p-1} = Z^+(s)$ if $z(j) = 1$, $\pi_p = Z_j^-(s)$ and $a(\pi_{p-1}) = a(\pi_p)$ for certain $j \in J^o(s)$, then s, $\gamma(s)$ and $\sigma(y^1,\pi(s))$ are adapted according to lemma 3.5, case b5; set $t=t-1$, $(i,h) = (j,k_{z(j)}^j)$, and adapt $c_j(s)$; go to step 4.

In the case $p = t+1$ and if

i) $\pi_t = (j,k^j_{z(j)})$ for certain $j \in J^+(s)$ and $a(\pi_t) = 0$, then s, $\gamma(s)$ and $\sigma(y^1,\pi(s))$ are adapted according to lemma 3.5, case c1; set $t=t-1$, $(i,h) = (j,k^j_{z(j)})$, and adapt $c_j(s)$; go to step 4

ii) $\pi_t = Z^-_j(s)$ for some $j \in J^o(s)$ and $a(\pi_t) = 0$, then s, $\gamma(s)$ and $\sigma(y^1,\pi(s))$ are adapted according to lemma 3.5, case c2; set $t=t-1$, $(i,h) = (j,k^j_1)$, and adapt $c_j(s)$; go to step 4.

In all other cases $\sigma(y^1,\pi(s))$ and a are adapted according to table 2 and return to step 1 with $\bar{p}$ the index of the new vertex of σ.

<u>Step 3.</u> [Increase dimension]

If $c(s) = 1$ and either $I^+_j(s) = \{(j,k)\}$ or both $j \in J^+(s)$ and $I^-_j(s) \cap V^c_j(v) = \{(j,k)\}$, then σ is complete and the algorithm stops.

If σ is not complete, $(j,k) \in I^+_j(s)$, and if $|I^+_j(s)| = 1$ and $I^-_j(s) = \emptyset$, then s, $\gamma(s)$ and $\sigma(y^1,\pi(s))$ are adapted according to lemma 3.11, case 2. Further, set $(i,h) = (j,k^j_{z(j)})$, adapt $c_j(s)$ and go to step 4.

If σ is not complete, $(j,k) \in I^-_j(s)$ and $|I^-_j(s)| = 1$ and $I^+_j(s) = \emptyset$, then s, $\gamma(s)$ and $\sigma(y^1,\pi(s))$ are adapted according to lemma 3.12, case 2. Further, set $(i,h) = (j,k^j_1)$, adapt $c_j(s)$ and go to step 4.

In all other cases adapt s, $\gamma(s)$ and $\sigma(y^1,\pi(s))$ according to lemma 3.11, case 1, if $(j,k) \in I^+_j(s)$ and according to lemma 3.12, case 1, if $(j,k) \in I^-_j(s)$. Further, set $t=t+1$, adapt $c_j(s)$ and return to step 1 with $\bar{p}$ the index of the new vertex of σ.

<u>Step 4</u>. Perform an l.p. pivot step by bringing $-s_{ih}(e^{T}(i,h),0)^{T}$ in the system

$$\sum_{i=1}^{t+1} \lambda_i \binom{z(y^i)}{1} - \sum_{\substack{(j,k) \notin I^0(s) \\ (j,k) \neq (i,h)}} \mu_{jk} s_{jk} \binom{e(j,k)}{0} = \binom{0}{1}.$$

If μ_{jk}, $(j,k) \notin I^0(s)$ and $(j,k) \neq (i,h)$, becomes zero then return to step 3. If λ_p becomes zero then go to step 2.

4. The paths followed by the algorithms

In the foregoing the sequences of simplices of varying dimension generated by three simplicial algorithms were described. The sum-ray algorithm generates a path of adjacent T-complete simplices in areas $A^1(T)$ for varying T in τ^1. In case of the product-ray algorithm a sequence of adjacent T-complete simplices is generated in areas $A^2(T)$ for varying T in τ^2. Finally, the exponent-ray algorithm generates a sequence of adjacent s-complete simplices in areas $A^3(s)$, s in τ^3. All the algorithms stop within a finite number of steps with a so-called complete simplex from which an approximating solution to the NLCP can be obtained. Notice that the definition of completeness is different for each algorithm.

In this section we explain what happens along the paths of the algorithms in terms of the piecewise linear approximation $\bar{z}$ of z with respect to the underlying V-triangulation. From this it will be immediately clear that these simplicial paths approximately follow the paths of the corresponding processes described in van den Elzen, van der Laan and Talman [3]. By taking the grid size of the triangulation small enough the algorithms can follow the paths of the processes as close as we want.

Let us first consider the sum-ray algorithm. For a better insight we rewrite the sets $A^1(T)$, $T \in \tau^1$, as

$$A^1(T) = \{x \in S \mid x_{jk} \geq b_j v_{jk} \text{ if } (j,k) \in T,$$

$$x_{jk} = b_j v_{jk} \text{ if } (j,k) \notin T,\ 0<b_j<1, j \in I_N\},$$

where v is the arbitrarily chosen starting point.

Regarding the definition of T-completeness we define for each $T \in \tau^1$ the set $\bar{C}^1(T)$ by

$$\bar{C}^1(T) = \{x \in S \mid \bar{z}_{jk}(x) = \max_{(i,h) \in I} \bar{z}_{ih}(x) \text{ if } (j,k) \in T\}.$$

We denote the set $A^1(T) \cap \bar{C}^1(T)$ by $\bar{B}^1(T)$, while $\bar{B}^1$ is the union of the sets $\bar{B}^1(T)$ over all T in τ^1.
In [3] it is shown that the set $B^1 = \bigcup_{T \in \tau^1} (A^1(T) \cap C^1(T))$ with

$$C^1(T) = \{x \in S \mid z_{jk}(x) = \max_{(i,h) \in I} z_{ih}(x) \text{ if } (j,k) \in T\}$$

consists of a disjoint union of piecewise smooth paths and loops. The path in B^1 connecting v and a solution x^* is the path followed by the sum-process. In the same way we can show that $\bar{B}^1$ consists of a disjoint union of piecewise linear paths and loops with one path, $\bar{P}^1$, connecting v and an approximate solution. We can show that this path is in fact followed by the sum-ray algorithm. More precisely, let $\sigma(y^1,\ldots,y^{g+1})$ be a T-complete g-simplex, g=t-1,t, in $A^1(T)$ for some T in τ^1 with solution (λ,μ,β). Then the point $\bar{x} = \sum_{i=1}^{g+1} \lambda_i y^i$ lies in σ and is an element of $\bar{B}^1(T)$. Moreover, the path $\bar{P}^1$ coincides with the piecewise linear path of points generated by the sum-ray algorithm. Because $\bar{z}$ converges to z if the grid size of the triangulation goes to zero, the piecewise linear path generated by the sum-ray algorithm can follow the piecewise smooth path of the sum-process arbitrary close by taking the grid size small enough.

When $v_j = e_j(j,k)$ where (j,k) is the index for which $z_{jk}(v) = \max_{(i,h) \in I} z_{ih}(v)$, then v solves the NLCP on S and $\bar{P}^1 = \{v\}$. If v is not a solution, the path $\bar{P}^1$ leaves v by increasing the (j,k)-th component of v for which the z-value is maximal, and by decreasing proportionally the other components of v_j in order to keep the sum of the components equal to one. This procedure is continued until a point $\bar{x}$ is reached for which $\bar{z}_{\ell p}(\bar{x})=\bar{z}_{jk}(\bar{x})$ for some $(\ell,p) \neq (j,k)$. Then the path $\bar{P}^1$ continues with points x in $A^1(\{(j,k),(\ell,p)\})$ while keeping both $\bar{z}_{\ell p}(x)$ and $\bar{z}_{jk}(x)$ maximal, by increasing the (ℓ,p)-th component of $\bar{x}$ and decreasing the other components of $\bar{x}_\ell$ if ℓ is unequal to j or by relatively increasing $\bar{x}_{\ell p}$ away from the $\bar{x}_{jh}$'s, $h \neq k$, if ℓ is equal to j.

In general the path $\bar{P}^1$ consists of points x for which there is a T in τ^1 such that for all $j \in I_N$ the components x_{jk} of x_j with $(j,k) \notin T_j$ are relatively (to v) equal to each other and relatively smaller than the components x_{jh} of x_j with $(j,h) \in T_j$, for which the $\bar{z}$-value is maximal. As soon as for some index (ℓ,p) not in T, $\bar{z}_{\ell p}(x)$ becomes equal to $\max_{(i,h) \in I} \bar{z}_{ih}(x)$, the (ℓ,p)-th component of x_ℓ is relatively increased away from the $x_{\ell h}$'s with $(\ell,h) \notin T_\ell$, while $\bar{z}_{\ell p}(x)$ is kept equal to the maximum of $\bar{z}$. In this way a piecewise linear path in $\bar{B}^1(T \cup \{(\ell,p)\})$ is followed. If, on the other hand, for some $(i,h) \in T$, x_{ih} becomes relatively equal to x_{ik}, $(i,k) \notin T_i$, then x_{ih} is not further decreased but is kept relatively equal to x_{ik}, $(i,k) \notin T_i$, while $\bar{z}_{ih}(x)$ is decreased away from the maximum of $\bar{z}$. So, $\bar{P}^1$ continues in $A^1(T\setminus\{(i,h)\})$. When $x_{jk} = 0$ for all $(j,k) \notin T_j$ and some $j \in I_N$, the path $\bar{P}^1$ stops with an approximate solution to the NLCP.

This completes the description of the sum-ray algorithm in terms of an adjustment process with respect to the p.l. approximation $\bar{z}$ of z. Observe that this interpretation of the sum-ray algorithm with the V-triangulation is more natural than in case of the Q-triangulation as used in [8] where the increase of the (j,k)-th component of x is compensated by the same decrease of an arbitrary other component of x_j instead of a proportional decrease of all other components of x_j.

Along the same lines we can describe the p.l. path of points generated by the product-ray algorithm on S as a path followed by an adjustment procedure with respect to $\bar{z}$. To do so, let us rewrite the sets $A^2(T)$, T in τ^2, by

$$A^2(T) = \{x \in S \mid x_{jk} > bv_{jk} \text{ if } (j,k) \in T,$$

$$x_{jk} = bv_{jk} \text{ if } (j,k) \notin T, \text{ where } 0 < b < 1\}.$$

Furthermore we define the sets $\bar{C}^2(T)$, T in τ^2, by

$$\bar{C}^2(T) = \{x \in S \mid \bar{z}_{jk}(x) = \max_{(j,h) \in I(j)} \bar{z}_{jh}(x),\ (j,k) \in T_j,\ j \in I_N\}.$$

Again one can show that the set $\bar{B}^2$ being the union of the sets $\bar{B}^2(T) = A^2(T) \cap \bar{C}^2(T)$ over all T in τ^2, consists of a disjoint union of piecewise linear paths and loops. Exactly one path, $\bar{P}^2$, connects the point v and an approximate solution. This path is the path generated by the product-ray algorithm. Moreover, $\bar{P}^2$ approximates the piecewise smooth path followed by the product-process on S. The latter path is the path in the set $B^2 = \bigcup_{T \in \tau^2} (A^2(T) \cap C^2(T))$ which connects v and a solution point, where

$$C^2(T) = \{x \in S \,|\, z_{jk}(x) = \max_{(j,h) \in I(j)} z_{jh}(x),\ (j,k) \in T_j,\ j \in I_N\}.$$

In the case that v is the vertex e(T) of S such that both $T_j = \{(j,k_0^j)\}$ and $z_{jk_0^j}(v) = \max_{(j,h) \in I(j)} z_{jh}(v)$ for all $j \in I_N$, the set $\bar{P}^2$ consists of the point v and v solves the NLCP. Otherwise the path $\bar{P}^2$ leaves v by increasing for all j the (j,k_0^j)-th component of v for which the z_j-value is maximal, and by decreasing proportionally the other components of v in order to keep the path in S. This procedure is continued until a point $\bar{x}$ is reached for which $\bar{z}_{jk}(\bar{x}) = \bar{z}_{jk_0^j}(\bar{x})$ for some $(j,k) \notin T$. Then the path $\bar{P}^2$ continues with points x in $A^2(T \cup \{(j,k)\})$ by increasing x_{jk} relatively away from the x_{ih}'s, $(i,h) \notin T$ and keeping $\bar{z}_{jk}(x)$ equal to $\bar{z}_{jk_0^j}(x)$.

In general the product-ray algorithm generates points x for which there is a T in τ^2 such that all components x_{jk} of x with $(j,k) \notin T$ are, relatively to v, equal to each other but relatively smaller than the components x_{ih} of x, $(i,h) \in T$, for which $\bar{z}_{ih}(x) = \max_{(i,g) \in I(i)} \bar{z}_{ig}(x)$. Notice the difference with the description of the p.l. path $\bar{P}^1$. If for some (j,p) not in T, $\bar{z}_{jp}(x)$ becomes equal to the $\max_h \bar{z}_{jh}(x)$ then the (j,p)-th component of x is relatively increased away from the x_{ih}'s with $(i,h) \notin T$. Besides $\bar{z}_{jp}(x)$ is kept equal to $\max_{(j,k) \in I(j)} \bar{z}_{jk}(x)$ so that $\bar{P}^2$ continues in $\bar{B}^2(T \cup \{(j,p)\})$. If, however, for some $(j,h) \in T$, x_{jh} becomes relatively equal to x_{ig}, $(i,g) \notin T$, then this component is not further decreased but is kept relatively equal to these x_{ig}'s while

$\bar{z}_{jh}(x)$ is decreased away from the maximal $\bar{z}_j$-value. In this way the path $\bar{P}^2$ continues in $\bar{B}^2(T\setminus\{(j,h)\})$. When $\bar{x}_{jk} = 0$ for all $(j,k) \notin T$, the path $\bar{P}^2$ stops with an approximate solution to the NLCP.

Finally we discuss how the piecewise linear path followed by the exponent-ray algorithm, can be interpreted as a path approximately generated by the exponent-process. Let $\bar{\tau}^3$ be the set of sign vectors defined by

$$\bar{\tau}^3 = \{s \in \mathbb{R}^{N+n} \mid s \text{ is a sign vector such that } I_j^+(s) = \emptyset \text{ or } I_j^-(s) \cap V_j^c \neq \emptyset \text{ for all } j \text{ while for at least one } k,\ I_k^+(s) \neq \emptyset\}.$$

Then for $s \in \bar{\tau}^3$ we define the set $A_0^3(s)$ by

$$A_0^3(s) = \{x \in S \mid \begin{array}{lll} x_{jk}=(1+\alpha_j)v_{jk} & \text{if } s_{jk}=+1 & \text{and } v_{jk}>0 \\ x_{jk}=\alpha_j & \text{if } s_{jk}=+1 & \text{and } v_{jk}=0 \\ bv_{jk}\leq x_{jk}\leq(1+\alpha_j)v_{jk} & \text{if } s_{jk}=0 & \text{and } v_{jk}>0 \\ 0\leq x_{jk}\leq\alpha_j & \text{if } s_{jk}=0 & \text{and } v_{jk}=0 \\ x_{jk}=bv_{jk} & \text{if } s_{jk}=-1 , & \end{array} \quad \text{where } 0<b<1<1+\alpha_j \text{ for all } j\}. \tag{4.1}$$

Furthermore, let the set $C^3(s)$ be defined by

$$C^3(s) = C\ell(\{x \in S \mid \operatorname{sgn} z(x) = s\}) \quad , \ s \in \bar{\tau}^3,$$

where $C\ell(W)$ denotes the closure of a set W.

Then the set $B^3 = \bigcup_{s\in\bar{\tau}^3} (A_0^3(s) \cap C^3(s))$ consists of a disjoint union of piecewise smooth loops and paths as shown in [3]. One path in B^3 connects the point v and a solution point. This path, P^3, is the path which is generated by the exponent-process on S. To clarify its relationship with the piecewise linear path followed by the exponent-ray algorithm on S, we first define areas $\bar{C}^3(s)$, $s \in \tau^3$, by

$$\bar{C}^3(s) = C\ell(\{x \in S \mid \text{sgn } \bar{z}(x) = s\}) \quad , \quad s \in \tau^3.$$

It can easily be shown that the set $\bar{B}^3 = \cup_s (A^3(s) \cap \bar{C}^3(s))$, where the union is over all $s \in \tau^3$, consists of a disjoint union of piecewise linear loops and paths. One path, $\bar{P}^3$, connects the point v and an approximate solution. From the definition of s-completeness we obtain that this path is generated by the exponent-ray algorithm.

Notice that for each point x on the piecewise smooth path P^3 the condition $x_j^T z_j(x) = 0$ for all j in I_N holds whereas for a point $\bar{x}$ on $\bar{P}^3$, $\bar{x}_j^T \bar{z}_j(\bar{x})$ is typically not equal to zero. This fact explains the different sets of sign vectors for which the sets $B^3(s) = A_0^3(s) \cap C^3(s)$ and $\bar{B}^3(s) = A^3(s) \cap \bar{C}^3(s)$ are defined.

In general we can say that the path of the exponent-process can be followed arbitrarily close by $\bar{P}^3$ by taking the grid size of the triangulation small enough. In fact, each area $A_0^3(s)$, where $I_j^+(s) = \emptyset$ for one or more indices $j \in I_N$, is subdivided by areas $A^3(\tilde{s})$ with $\tilde{s} \in \tau^3$ such that $I_j^+(\tilde{s}) \subset I(j) \backslash V_j$ or $I_j^-(\tilde{s}) \cup I_j^o(s) = I(j)$ for those j while $\tilde{s}_i = s_i$ otherwise. Areas $A^3(s)$ and $A_0^3(s)$ related to other sign vectors coincide. Furthermore by decreasing the mesh, $\bar{z}$ approaches z so that the set $\bar{C}^3(\tilde{s})$, $\tilde{s} \in \tau^3$, approximates the set $C^3(s)$, $s \in \bar{\tau}^3$, with $\tilde{s}$ and s related as above. Consequently, by taking the mesh of the triangulation of S small enough, the set $\bar{B}^3(s)$ is arbitrarily close to $B^3(s)$ if $s \in \bar{\tau}^3$ and $I_j^+(s) \neq \emptyset$ for all j. Moreover, if $s \in \bar{\tau}^3$ and $I_j^+(s) = \emptyset$ for at least one j, then $B^3(s)$ is approximated by the union of $\bar{B}^3(s')$ over all sign vectors s' in τ^3 such that $s'_j = s_j$ if $I_j^+(s) \neq \emptyset$ and $I_j^+(s') \subset I(j) \backslash V_j$ or $I_j^-(s') \cup I_j^o(s') = I(j)$ if $I_j^+(s) = \emptyset$.

Therefore the p.l. path $\bar{P}^3$ leading from v to an approximate solution can be similarly interpreted as the piecewise smooth path P^3 of the exponent-process.

The piecewise linear path $\bar{P}^3$ can be interpreted as follows. The components v_{jk} of v for which $z_{jk}(v)$ is negative are initially decreased with the same rate whereas for each j the components of v_j corresponding to positive z-values are initially increased. The rate with which the positive v_{jk}'s are increased is equal to the absolute amount with which the v_{jk}'s equal to zero are increased. If $z_j(v) \leq 0$ then v_j is initially not adapted, whereas v_{jk} is decreased if $v_{jk} = 1$ and $z_j(v) \not\leq 0$. Observe that we assume that $z_{jk}(v) \neq 0$ for all $(j,k) \in I$ since $z_{jk}(x)$ is slightly perturbed if $x_{jk} = 1$. The procedure is continued until a point $\bar{x}$ in S is reached for which $\bar{z}_{ih}(\bar{x})$ is zero for some $(i,h) \in I$. If $\bar{z}_{ih}(x)$ was negative then the algorithm proceeds by increasing the (i,h)-th component relatively away from the $x_{\ell p}$'s for which $\bar{z}_{\ell p}(x) < 0$ while $\bar{z}_{ih}(x)$ is kept equal to zero. If $\bar{z}_{ih}(x)$ was positive then x_{ih} is decreased relatively away from the components of x_i corresponding to positive z-values while $\bar{z}_{ih}(x)$ is also kept equal to zero.

In general the p.l. path $\bar{P}^3$ generates points x in S such that in principle the components x_{ih} of x for which $\bar{z}_{ih}(x)$ is negative are relatively equal to each other, but relatively smaller than the other components of x. Further, for all j the components x_{jk} of x_j for which $\bar{z}_{jk}(x)$ is positive are also relatively (absolutely) equal to each other but relatively (absolutely) larger than the other components of x_j. The rate α_j with which these components x_{jk} for which $v_{jk} > 0$ are larger than v_{jk} is equal to the value of these components x_{jk} for which v_{jk} is zero.

As soon as a vector $\bar{x}$ is reached for which $\bar{z}_{jk}(\bar{x})$ is 0 for some index (j,k) for which $\bar{z}_{jk}(x)$ was negative then the algorithm continues with points x whose (j,k)-th component is increased relatively away from the components x_{ih} of x for which $\bar{z}_{ih}(x)$ is still negative while keeping $\bar{z}_{jk}(x)$ equal to zero. When $\bar{z}_{jk}(x)$ was positive then x_{jk} is decreased relatively away from the components x_{jh} of x_j for which $\bar{z}_{jh}(x)$ is positive (if any) while $\bar{z}_{jk}(x)$ is also kept equal to zero.

If on the other hand at a point $\bar{x}$ on the path $\bar{P}^3$ the component $\bar{x}_{jk}$ for which $\bar{z}_{jk}(\bar{x}) = 0$ becomes relatively equal to the $\bar{x}_{ih}$'s for which $\bar{z}_{ih}(\bar{x})$ is negative, then $\bar{x}_{jk}$ is not further decreased. The algorithm continues with vectors x whose (j,k)-th component is kept relatively equal to the x_{ih}'s for which $\bar{z}_{ih}(x)$ is negative while $\bar{z}_{jk}(x)$ is decreased away from zero. Similarly, if for some j the component $\bar{x}_{jk}$ for which $\bar{z}_{jk}(\bar{x}) = 0$ becomes relatively (absolutely) equal to the $\bar{x}_{jh}$'s for which $\bar{z}_{jh}(\bar{x})$ is positive, then $\bar{x}_{jk}$ is not further increased. The algorithm proceeds with vectors x for which x_{jk} is kept relatively equal to the x_{jh}'s whose corresponding $\bar{z}$-value is positive while $\bar{z}_{jk}(x)$ is increased away from zero. This interpretation only holds when for each j there is an index (j,h) for which $\bar{z}_{jh}(x) > 0$ and an index (j,k) for which $\bar{z}_{jk}(x) < 0$ and $x_{jk} > 0$, i.e. when $\operatorname{sgn} \bar{z}(x) \in \tau^3 \cap \bar{\tau}^3$. Otherwise the interpretation of the path differs slightly.

As soon as a vector $\bar{x}$ is generated such that for each j, $\bar{x}_{jk} = 0$ implies $\bar{z}_{jk}(\bar{x}) \leqslant 0$ and either $\bar{z}_{jh}(\bar{x}) \geqslant 0$ for all (j,h) for which $\bar{x}_{jh} > 0$ or $\bar{z}_{jh}(\bar{x}) \leqslant 0$ for all these (j,h), the path $\bar{P}^3$ terminates and $\bar{x}$ is an approximate solution to the NLCP.

5. Computational results

The algorithms presented in this paper have been applied to the noncooperative N-person game and an international economy. For a description of the underlying functions $z : S \to R^{N+n}$ we refer the reader to Doup and Talman [1].

In both applications we start the algorithms in the barycentre of S. The grid size for the first application is $m^{-1} = 1$ and for the second application $m^{-1} = \frac{1}{2}$. When a complete simplex is found the grid is refined with a factor of two. In the first application we restart the algorithm in the approximate solution. However, if this solution lies close to a boundary face of S we project it on that boundary face. In the second application we restart the algorithm in the approximate solution. The grid refinement is stopped when the accuracy of the approximate solution is sufficient. The accuracy is given by $\max_{jk} \; z_{jk}(x^{\nu})$ where x^{ν} is the approximate solution in round $\nu, \nu = 1,2,\ldots$ In the first application the algorithms are stopped when we obtain an accuracy of 10^{-8} and in the second application if an accuracy of 10^{-7} is obtained. Throughout this section we will use the following notations; FE: accumulated number of function evaluations, LP: accumulated number of linear programming steps and ν: the number of rounds to obtain the required accuracy.

The data of the three games we discuss can be found in [1].

Game 1. Three players with each player having two strategies.

The solution of this game is $x^* = (1/5,4/5;\ 3/7,4/7;\ 2/3,1/3)$. The results for game 1 are given in table 1.

Algorithm	FE	LP	ν
sum-ray	54	51	7
product-ray	33	35	4
exponent-ray	59	82	7

Table 1. The results for game 1.

Game 2. Three players with each player having three strategies.

The solution of this game is $x^* = (3/7,4/7,0;\ 0,1,0;\ 0,2/3,1/3)$.
The results for game 2 are given in table 2.

Algorithm	FE	LP	ν
sum-ray	21	18	3
product-ray	15	14	1
exponent-ray	21	23	2

Table 2. The results for game 2.

Game 3. Four players with each player having two strategies

The solution of this game is $x^* = (1/5,4/5;\ 1,0;\ 1,0;\ 2/3,1/3)$.
The results for game 3 are given in table 3.

Algorithm	FE	LP	ν
sum-ray	18	14	3
product-ray	18	16	2
exponent-ray	41	56	8

Table 3. The results for game 3.

The second application concerns the international economy described in van der Laan [4]. The computational results presented in table 4 concern the same examples as described in [4] and [1]. Each country has two non-common goods, the number of common goods varies between 2 and 6, whereas the number of countries varies between 2 and 5.

number of common goods	number of countries	sum-ray alg.		product-ray alg.		exponent-ray alg.	
		FE	LP	FE	LP	FE	LP
2	2	60	53	54	47	60	53
	3	125	117	85	78	81	74
	4	178	170	97	90	116	109
	5	225	217	128	121	148	142
3	2	97	90	56	49	60	52
	3	127	119	87	80	91	84
	4	182	174	95	88	123	116
	5	261	253	109	102	151	144
4	2	113	105	67	60	68	60
	3	132	124	107	100	104	97
	4	212	204	118	111	133	126
	5	309	301	145	138	163	155
5	2	134	126	79	72	76	68
	3	157	149	97	90	120	112
	4	257	249	145	138	167	161
	5	354	346	182	175	170	162
6	2	166	158	89	82	95	87
	3	176	168	147	139	115	107
	4	346	338	195	188	188	183
	5	458	450	221	214	212	206

Table 4. The results for the international economy.

The computational results show that both the exponent-ray and the product-ray algorithm are significantly better than the sum-ray algorithm in case of the international trade economies whereas the product-ray algorithm is superior to the other two methods in case of a noncooperative game. The latter could be due to the specific properties of the underlying function z in case of games. The equilibrium positions of a player are determined by the strategies of the other players. This could cause a lot of steps in the exponent-ray algorithm when one or more players are already in equilibrium. This feature doesn't hold for an international economy, in which an equilibrium position of a certain country is determined by all prices simultaneously. A second feature concerns the accuracy of an approximate solution found by the algorithms for a given grid size. As discussed in [1] such an accuracy is in general much better for the product-ray algorithm than for the sum-ray algorithm. In [2] it is shown that the exponent-ray algorithm on S^n yields a worse accuracy than for the (n+1)-ray algorithm on S^n. Lemma 3.9 shows that the latter is also true on S.

More tests and research could clarify the different results for the two applications given above.

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