Axiomatic Characterizations of Solutions for Bayesian Games

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Abstract

Bayesian equilibria are characterized by means of consistency and one-person rationality in combination with non-emptiness or converse consistency. Moreover, strong and coalition-proof Bayesian equilibria of extended Bayesian games are introduced and it is seen that these notions can be characterized by means of consistency, one-person rationality, a version of Pareto optimality and a modification of converse consistency. It is shown that, in case of the strong Bayesian equilibrium correspondence, converse consistency can be replaced by non-emptiness. As examples we treat Bayesian potential games and Bayesian congestion games.

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1 Introduction

In Peleg & Tijs [1992] axiomatic characterizations are given for the Nash equilibrium correspondence on closed families of strategic games using consistency, converse consistency and one-person rationality. Also some refinements of the Nash correspondence are characterized and an indication is given that some of the results can be extended to Bayesian games.
In a subsequent paper Peleg, Potters & Tijs [1993] study the question under which conditions converse consistency can be replaced by the non-emptiness property. In this connection see also Norde, Potters, Reijnierse & Vermeulen [1993].
The purpose of this paper is to make a systematic study of axiomatizations for solutions of extended Bayesian games. Bayesian games were introduced by Harsanyi [1967], and extended Bayesian games by Einy & Peleg [1991]. In section 2 we give the necessary definitions. In section 3 it is shown that the Section 4 introduces strong and coalition-proof Bayesian equilibria and both concepts are axiomatized by consistency, one-person rationality and modifications of Par There is also a discussion on the definition of strong Bayesian equilibria and it is shown that, in order to characterize strong Bayesian equilibria, converse consis Finally, a modification of the strong Bayesian equilibrium correspondence is given. Section 5 extends the notion of potential game of Monderer & Shapley [1992] to Bayesian games and considers the existence of pure Bayesian equilibria. Also a congestion situation in the spirit of Rosenthal [1973] is considered, which gives rise to a Bayesian potential game.

2 Extended Bayesian games

In this section we formally describe the class of extended Bayesian games. This generalized form of ordinary Bayesian games enables us to define reduced Bayesian games.

2.1 Definition (Einy & Peleg [1991])

An *extended Bayesian game (EBG)* is a system

$$G = \langle N, (A_i)_{i\in N}, (T_i)_{i\in N^+}, (p_i)_{i\in N}, (u_i)_{i\in N}\rangle$$

where

(i) N is the (finite) set of *players*,

(ii) N^+ is a finite set with $N^+ \supseteq N$ and $N^+ \setminus N$ is the set of *outside players,*

(iii) for every $i \in N, A_i$ is the set of *actions* of player i,

(iv) for every $i \in N^+, T_i$ is the finite set of possible *types* of player i,

(v) for every $i \in N, p_i$ is a probability distribution on $T := \prod_{k \in N^+} T_k$ which represents the *prior* of player i,

(vi) for every $i \in N, u_i : A \times T \to \mathbb{R}$ is the *utility-function* of player i, where $A := \prod_{k \in N} A_k$.

Note that, in case $N^+ = N$, we have an ordinary Bayesian game. If $N^+ \neq N$ then, intuitively, one may consider the outside players as those players who have already chosen their strategies (in a larger Bayesian game). So an extended Bayesian game can be considered as a reduction of an ordinary Bayesian game.

From now on, if we mention an arbitrary Bayesian game G without further specification, we assume that G is the game $\langle N, (A_i)_{i \in N}, (T_i)_{i \in N^+}, (p_i)_{i \in N}, (u_i)_{i \in N} \rangle$. Sometimes we will refer to N_G as the player set belonging to the game G, to avoid confusion.

Let G be an EBG, and $i \in N$.
A *strategy* of player i is a function $x_i : T_i \to A_i$. By X_i we denote the set of strategies of player i. If $S \subseteq N, S \neq \emptyset$, then $X_S := \prod_{k \in S} X_k$ and $X := X_N$.

Let Γ be a set of EBG's. A *solution* on Γ is a function ϕ that assigns to each game $G \in \Gamma$ a subset $\phi(G)$ of the space X of strategy profiles.

2.2 Definition

Let G be an EBG and $S \subseteq N, S \neq \emptyset, x \in X$.
The *reduced* Bayesian game of G with respect to S and x is given by

$$G^{S,x} = \langle S, (A_i)_{i \in S}, (T_i)_{i \in N^+}, (p_i)_{i \in S}, (u_i^x)_{i \in S} \rangle$$

where, for every $i \in S, a_S \in A_S$ and $t \in T$

$$u_i^x(a_S, t) := u_i(((x_k(t_k))_{k \in N \setminus S}, a_S), t).$$

(Note that $A_S = \prod_{k \in S} A_k, a_S = (a_k)_{k \in S}$ etc.)

A class Γ of EBG's is *closed* if for every $G \in \Gamma$ and for every $S \subseteq N, S \neq \emptyset$ and $x \in X$ it holds that $G^{S,x} \in \Gamma$. It is easy to see that the class of all EBG's is closed.

Now we consider an example of a Bayesian game.

2.3 Example

Let a two-person Bayesian game be given by $G = \langle \{1,2\}, A_1, A_2, T_1, T_2, p_1, p_2, u_1, u_2 \rangle$ where $A_1 = \{T, B\}, \;\; A_2 = \{L, R\}, \;\; T_1 = \{\alpha, \beta\}, \;\; T_2 = \{\gamma, \delta\}$ and $p_1(\alpha, \gamma) = p_1(\alpha, \delta) = p_1(\beta, \gamma) = p_1(\beta, \delta) = \frac{1}{4}, \;\; p_2(\alpha, \gamma) = p_2(\beta, \delta) = \frac{1}{2}$ and $p_2(\alpha, \delta) = p_2(\beta, \gamma) = 0$. The payoff-functions are denoted in table 1.

		$\boxed{\gamma}$		$\boxed{\delta}$	
		L	R	L	R
$\boxed{\alpha}$	T	$0,0$	$1,2$	$1,0$	$1,0$
	B	$2,1$	$0,0$	$1,0$	$1,0$
$\boxed{\beta}$	T	$1,0$	$1,0$	$1,1$	$3,2$
	B	$0,1$	$0,1$	$3,4$	$2,2$

table 1.

So if player 1 is of type α and player 2 is of type γ, they play the upper left game. We denote a strategy of a player by a pair of actions. So

$$X_1 = \{TT, TB, BT, BB\}, \;\; X_2 = \{LL, LR, RL, RR\},$$

where for example TB is the strategy of player 1 in which he plays T if he is of type α and B if he is of type β.

Now this Bayesian game is an example of a so-called Bayesian potential game (for the definition we refer to section 5), which means that for every pair of types the corresponding bimatrix game is an ordinary potential game, where a corresponding potential is described by:

		γ: L	γ: R	δ: L	δ: R
α	T	0	2	0	0
	B	2	1	0	0
β	T	0	0	0	1
	B	-1	-1	2	0

table 2.

We denote the given (potential) function by q, so for example,

$$q((T,R),(\alpha,\gamma)) = 2, \quad q((B,L),(\beta,\gamma)) = -1.$$

If just one player deviates, then the difference in the payoff for that player is indicated by the difference in the potential function. For example

$$u_1((T,L),(\beta,\gamma)) - u_1((B,L),(\beta,\gamma)) = q((T,L),(\beta,\gamma)) - q((B,L),(\beta,\gamma)) = 0-(-1) = 1.$$

We will elaborate on potential games in section 5.

2.4 Remark

In the definition of extended Bayesian games, p_i is a probability distribution on T, for every $i \in N$. In the following definition we use, for every $i \in N$ and $t_i \in T_i$, the related probability distribution $p_i(.|t_i)$ on $T^{-i} := \prod_{k \neq i} T_k$, defined by

$$p_i(t^{-i}|t_i) := \frac{p_i(t)}{\sum_{s^{-i} \in T^{-i}} p_i(t_i, s^{-i})}$$

for every $t^{-i} \in T^{-i}$. Note that $t = (t_i, t^{-i})$.

Of course this definition is meaningful only in the case that $\sum_{s^{-i}} p_i(t_i, s^{-i}) \neq 0$, for every $t_i \in T_i$, which means that every player puts positive probability on the occurence of each of his types. In the sequel we shall assume that this is indeed the case.

2.5 Definition

Let G be an EBG and $x \in X$.

x is a *Bayesian equilibrium (BE)* of G if for all $i \in N, t_i \in T_i$ and $a_i \in A_i$:

$$\sum_{t^{-i}} p_i(t^{-i}|t_i) u_i((x_j(t_j))_{j \in N}, t) \geq \sum_{t^{-i}} p_i(t^{-i}|t_i) u_i((x_j(t_j))_{j \in N \setminus \{i\}}, a_i, t).$$

We denote $BE(G) := \{x \in X \mid x$ is a BE of $G\}$.
To shorten notation we define

$$U_i(x|t_i) := \sum_{t^{-i}} p_i(t^{-i}|t_i) u_i((x_j(t_j))_{j\in N}, t)$$

for every $x \in X, t_i \in T_i$. Then, for every $x \in X$:

$$x \in BE(G) \quad \text{iff} \quad \text{for all } i \in N, t_i \in T_i, y_i \in X_i: \ \ U_i(x|t_i) \geq U_i(x^{-i}, y_i|t_i).$$

In section 5 we show that each Bayesian potential game with consistent priors has at least one Bayesian equilibrium.
In example 2.3, the strategy tuple (TB, RL) is a Bayesian equilibrium, but the players do not have consistent priors.

3 Axiomatizations of the Bayesian equilibrium correspondence

In this section we give two different characterizations of the Bayesian equilibrium correspondence. The first one is based on consistency and converse consistency (cf. Peleg & Tijs [1992]), the second one on consistency and non-emptiness and uses in its proof the ancestor property (cf. Peleg, Potters & Tijs [1993]).

3.1 Definition

Let Γ be a closed set of EBG's and ϕ a solution on Γ.

(i) ϕ satisfies *one-person rationality (OPR)* on Γ if

$$\phi(G) = \{x_i \in X_i \mid U_i(x_i|t_i) \geq U_i(y_i|t_i) \ \text{ for every } t_i \in T_i \text{ and } y_i \in X_i\}$$

for every one-player Bayesian game $G = \langle \{i\}, A_i, (T_j)_{j\in\{i\}^+}, p_i, u_i\rangle$ in Γ.

(ii) ϕ satisfies *consistency (CONS)* on Γ if for every game $G \in \Gamma$, for every coalition $S \subsetneq N, S \neq \emptyset$ and for every $x \in \phi(G)$ it holds that $x_S \in \phi(G^{S,x})$.

Defining $\tilde{\phi}(G) := \{x \in X \mid \text{ for every } \ S \subsetneq N, S \neq \emptyset : \ x_S \in \phi(G^{S,x})\}$, we have that ϕ satisfies $CONS$ iff $\phi(G) \subseteq \tilde{\phi}(G)$ for every $G \in \Gamma$.

We will show that the Bayesian equilibrium solution satisfies OPR and $CONS$. However,

OPR and $CONS$ do not axiomatize BE. In section 4 we will see that the strong Bayesian equilibrium solution (SBE) also satisfies OPR and $CONS$. To characterize BE we use the following property.

3.2 Definition

We say that a solution ϕ satisfies *converse consistency (COCONS)* on a closed set Γ of EBG's if

$$\tilde{\phi}(G) \subseteq \phi(G) \quad \text{for every } G \in \Gamma \text{ with } |N| \geq 2.$$

For a detailed discussion of consistency and converse consistency we refer to Peleg & Tijs [1992].

3.3 Lemma

Let Γ be a closed set of EBG's.

Then BE satisfies OPR, $CONS$ and $COCONS$ on Γ.

Proof.

(i) By definition BE satisfies OPR.

(ii) Let $G \in \Gamma, |N| \geq 2$ and $x \in BE(G)$.

Let $S \subsetneqq N, i \in S$ and $t_i \in T_i$. Then for every $y_i \in X_i$:

$$\begin{aligned} U_i^x(x_S|t_i) &= U_i(x|t_i) \geq U_i(x^{-i}, y_i|t_i) \\ &= U_i^x(x_{S\setminus\{i\}}, y_i|t_i) \end{aligned}$$

where $U_i^x(x_S|t_i) := \sum_{t^{-i}} p_i(t^{-i}|t_i) u_i^x((x_j(t_j))_{j \in S}, t)$.

Hence $x_S \in BE(G^{S,x})$, so $BE(G) \subseteq \tilde{BE}(G)$.

(iii) Let $G \in \Gamma, |N| \geq 2$ and $x \in X$ be such that $x_S \in BE(G^{S,x})$ for all $S \subsetneqq N, S \neq \emptyset$.

Take $i \in N$ and $t_i \in T_i$. Then $x_i \in BE(G^{\{i\},x})$ so for every $y_i \in X_i$:

$$\begin{aligned} U_i^x(x_i|t_i) &\geq U_i^x(y_i|t_i) \quad \text{hence} \\ U_i(x|t_i) &\geq U_i(x^{-i}, y_i|t_i). \end{aligned}$$

So $x \in BE(G)$ and $\tilde{BE}(G) \subseteq BE(G)$. $\square$

In fact, OPR, $CONS$ and $COCONS$ characterize BE, as the next theorem shows.

3.4 Theorem

Let ϕ be a solution on a closed set Γ of EBG's.

Then ϕ satisfies *OPR*, *CONS* and *COCONS* iff $\phi(G) = BE(G)$ for every $G \in \Gamma$.

Proof.

We give a proof of the 'only if'-part by induction on the number of players.
Suppose ϕ satisfies *OPR*, *CONS* and *COCONS*.

- Let G be a one person game in Γ. Then $\phi(G) = BE(G)$ by *OPR* of ϕ and *BE*.

- Let $k \in \{2, 3, 4, ...\}$ be such that, for every $G \in \Gamma$ with less than k players, we have that $BE(G) = \phi(G)$, and let G be a k-person game in Γ. Then we have

$$\phi(G) \underset{(CONS \text{ of } \phi)}{\subseteq} \tilde{\phi}(G) \underset{\text{(ind.hyp.)}}{=} \widetilde{BE}(G) \underset{(COCONS \text{ of } BE)}{\subseteq} BE(G)$$

and

$$BE(G) \underset{(CONS \text{ of } BE)}{\subseteq} \widetilde{BE}(G) \underset{\text{(ind.hyp.)}}{=} \tilde{\phi}(G) \underset{(COCONS \text{ of } \phi)}{\subseteq} \phi(G).$$

So $\phi(G) = BE(G)$. □

For the second characterization we introduce, for every set Γ of EBG's and every solution ϕ on Γ, a directed graph $Graph(\Gamma, \phi)$. The vertices of this graph are pairs (G, x) where $G \in \Gamma$ and $x \in \phi(G)$. There is an edge from (G, x) to (H, y) if $N_H \subsetneq N_G, H = G^{N_H,x}$ and $y = x_{N_H}$. In this case we call (G, x) an *ancestor* of (H, y).

3.5 Definition

Let Γ be a closed class of EBG's and ϕ a solution on Γ.
The graph $Graph(\Gamma, \phi)$ satisfies the *ancestor property (AP)* if for every vertex (H, y) there is a $G \in \Gamma$ such that $\phi(G) \neq \emptyset$ and (G, x) is an ancestor of (H, y) for every $x \in \phi(G)$.

3.6 Definition

(i) ϕ satisfies *non-emptiness (NEM)* on Γ if $\phi(G) \neq \emptyset$ for every $G \in \Gamma$.

(ii) ϕ is *minimal w.r.t. NEM, OPR* and *CONS* if ϕ satisfies these properties and for every solution $\bar{\phi}$ with $\bar{\phi} \subseteq \phi$ on Γ which satisfies *NEM*, *OPR* and *CONS*, we have that $\bar{\phi} = \phi$.

These definitions are due to Peleg, Potters & Tijs [1993].

3.7 Lemma

For every closed class Γ of EBG's and every solution ϕ on Γ satisfying NEM , OPR and $CONS$:

if $Graph(\Gamma, \phi)$ satisfies AP, then ϕ is minimal w.r.t. NEM, OPR and $CONS$.

Proof. By straightforwardly extending the proof of Theorem 1 of Peleg, Potters & Tijs [1993]. □

This lemma has an interesting application if we take $\phi = BE$ and $\Gamma = \Gamma^{BE}$ (the class of all EBG's which have at least one BE). We already know (see the proof of Theorem 3.4) that a solution $\bar{\phi}$ which satisfies OPR and $CONS$ on Γ^{BE} is contained in BE. If $\bar{\phi}$ also satisfies NEM and if we can prove that $Graph(\Gamma^{BE}, BE)$ has the ancestor property, then $\bar{\phi} = BE$, so BE is characterized on Γ^{BE} by NEM, OPR and $CONS$.

3.8 Theorem

Let ϕ be a solution on Γ^{BE}.
Then ϕ satisfies NEM, OPR and $CONS$ iff $\phi(G) = BE(G)$ for every $G \in \Gamma^{BE}$.

Proof.

We prove the 'only if'-part.
By lemma 3.7, it suffices to prove that $Graph(\Gamma^{BE}, BE)$ satisfies AP.
Let $G \in \Gamma^{BE}$ and $x \in BE(G)$. We construct a game $H \in \Gamma^{BE}$ with $N_H \supsetneqq N_G$ such that H has exactly one Bayesian equilibrium y with the property that $y_{N_G} = x$ and $H^{N_G,y} = G$.
Define $H := \langle N_G \cup \{0\}, (A_i^\star)_{i \in N_G \cup \{0\}}, (T_i^\star)_{i \in N_G^+ \cup \{0\}}, (p_i^\star)_{i \in N_G \cup \{0\}}, (u_i^\star)_{i \in N_G \cup \{0\}} \rangle$, where

- $A_0^\star := \{\alpha, \beta\}, \;\; A_i^\star := A_i \;\; (i \in N_G)$,
- $T_0^\star := \{t_0\}, \;\; T_i^\star := T_i \;\; (i \in N_G^+)$,
- for every $i \in N_G$:
  $p_i^\star(t, t_0) := p_i(t)$ for every $t \in T$
  and:
  $p_0^\star(s, t_0) := 1$ for a fixed $s \in T$
  $p_0^\star(t, t_0) := 0$ for every $t \in T \setminus \{s\}$,
- for every $a \in A$ and every $i \in N_G, t \in T$:
  $u_i^\star((a, \alpha), (t, t_0)) := u_i(a, t)$
  $u_i^\star((a, \beta), (t, t_0)) := -1$ if $a_i \neq x_i(t_i)$
  $u_i^\star((a, \beta), (t, t_0)) := 1$ if $a_i = x_i(t_i)$

and:

$$
\begin{array}{ll}
u_0^\star((a,\alpha),(t,t_0)) := 2 & \text{if } a = (x_1(t_1),...,x_n(t_n)) \\
u_0^\star((a,\alpha),(t,t_0)) := -1 & \text{if } a \neq (x_1(t_1),...,x_n(t_n)) \\
u_0^\star((a,\beta),(t,t_0)) := 0 \quad . &
\end{array}
$$

Since player 0 has only one type, we can identify a strategy of player 0 with an action.

Claim: (x,α) is the unique BE in H.

Proof of the claim:

Let $(y,\beta) \in X \times A_0^\star$ and $y \neq x$. Choose $i \in N_G, t_i \in T_i$ such that $y_i(t_i) \neq x_i(t_i)$. Then

$$
\begin{aligned}
U_i^\star((y,\beta)|t_i) &= \sum_{t^{-i}\in T^{-i}} p_i^\star(t^{-i},t_0|t_i) u_i^\star(((y_j(t_j))_{j\in N_G},\beta),(t,t_0)) \\
&= \sum_{t^{-i}\in T^{-i}} p_i^\star(t^{-i},t_0|t_i)\cdot -1 < \sum_{t^{-i}\in T^{-i}} p_i^\star(t^{-i},t_0|t_i) \\
&= \sum_{t^{-i}\in T^{-i}} p_i^\star(t^{-i},t_0|t_i) u_i^\star(((y_j(t_j))_{j\in N_G\setminus\{i\}},x_i(t_i),\beta),(t,t_0)) = U_i^\star((y^{-i},x_i,\beta)|t_i) \quad .
\end{aligned}
$$

So player i can profitably deviate , which shows that $(y,\beta) \notin BE(H)$.

Also $(x,\beta) \notin BE(H)$, because

$$
\begin{aligned}
U_0^\star((x,\beta)|t_0) &= \sum_{t\in T} p_0^\star(t|t_0) u_0^\star(((x_i(t_i))_{i\in N_G},\beta),(t,t_0)) \\
&< \sum_{t\in T} p_0^\star(t|t_0) u_0^\star(((x_i(t_i))_{i\in N_G},\alpha),(t,t_0)) = U_0^\star((x,\alpha)|t_0) \quad .
\end{aligned}
$$

Now let $(y,\alpha) \in X \times A_0^\star$ and $y \neq x$. Then

$$
\begin{aligned}
U_0^\star((y,\alpha)|t_0) &= \sum_{t\in T} p_0^\star(t|t_0) u_0^\star(((y_i(t_i))_{i\in N_G},\alpha),(t,t_0)) \\
&= \sum_{t\in T} p_0^\star(t|t_0)\cdot -1 < \sum_{t\in T} p_0^\star(t|t_0)\cdot 0 \\
&= \sum_{t\in T} p_0^\star(t|t_0) u_0^\star(((y_i(t_i))_{i\in N_G},\beta),(t,t_0)) = U_0^\star((y,\beta)|t_0) \quad .
\end{aligned}
$$

So player 0 can profitably deviate, which shows that $(y,\alpha) \notin BE(H)$.

Clearly $(x,\alpha) \in BE(H)$, which finishes the proof. □

4 Strong and coalition proof Bayesian equilibria

In this section we define and characterize strong Bayesian equilibria and coalition-proof

Bayesian equilibria which are generalizations of strong Nash equilibria (see Aumann [1959]) and coalition-proof Nash equilibria (see Bernheim, Peleg & Whinston [1987]) for games in strategic form. We also discuss the definitions and provide some modifications.

4.1 Definition

Let G be an EBG and $x \in BE(G)$.
x is a *strong Bayesian equilibrium (SBE)* if there is no coalition $S \subseteq N, S \neq \emptyset$, which has an *improvement upon* x, which means that there is no $y_S \in X_S$ such that, for all $i \in S, t_i \in T_i$:

$$U_i(x|t_i) < U_i(x_{N \setminus S}, y_S|t_i) \ .$$

By $SBE(G)$ we denote the set of strong Bayesian equilibria of G.

This definition is due to Ichiishi & Idzik [1992]. Ichiishi and Idzik investigate Bayesian societies, which are more general than Bayesian games and allow binding agreements. However, their definition, when applied to Bayesian games, is essentially the same as our definition 4.1.

We explicitly define SBE as a refinement of BE. In order to get SBE (G) as a subset of BE (G), it is not sufficient to define $SBE(G)$ as the set of all strategy combinations which cannot be improved upon, as the next example shows.

4.2 Example

Let $N = \{1, 2\}$, $A_1 = \{T\}$, $A_2 = \{L, R\}$, $|T_1| = 1$, $T_2 = \{\alpha, \beta\}$ and the payoff-functions u_1 and u_2 as denoted in table 3. Note that $u_1 = u_2$. The priors are arbitrary.

$$\begin{array}{ccc} & \boxed{\alpha} & \\ & L & R \\ T & \boxed{0} & \boxed{1} \end{array} \qquad \begin{array}{ccc} & \boxed{\beta} & \\ & L & R \\ T & \boxed{2} & \boxed{0} \end{array}$$

table 3.

The strategy (T, LL) is no BE, because $U_2(T, LL|\alpha) < U_2(T, RL|\alpha)$. However, (T, LL) cannot be improved upon: if there would be an improvement it can only be (T, RL), but in that case only type α of player 2 profits, while type β gets the same amount.

Two questions may be raised with respect to the validity of the definition of SBE.

(i) Let $x \in X$ and let y_S be an improvement of a coalition S upon x. Can S choose to play y_S without changing the beliefs of its members ? Would the beliefs change then the payoffs to the types of the players in S would also change. Therefore, some members of S might no longer prefer $(y_S, x_{N\setminus S})$ to x after y_S is chosen.
The answer is simple. Because all the members of S know x and y_S and all the types of all members of S prefer $(y_S, x_{N\setminus S})$ to x, S can choose to play y_S without a change in the beliefs. In order to be completely precise we supply the following simple model for choice (by the members of S) between x_S and y_S.
Let $G^\star = \langle S, (A_i^\star)_{i\in S}, (T_i)_{i\in N^+}, (p_i)_{i\in S}, (u_i^\star)_{i\in S}\rangle$ be the following EBG: $A_i^\star = \{y_S, x_S\}$ for all $i \in S$ and $u_i^\star : A^\star \times T \to \mathbb{R}$ for all $i \in S$ be given by

$$u_i^\star((a_j^\star)_{j\in S}, t) = \begin{cases} U_i(x_{N\setminus S}, y_S|t_i) & \text{if } a_j^\star = y_S \text{ for all } j \in S \\ U_i(x|t_i) & \text{otherwise.} \end{cases}$$

We shall say that y_S *is chosen by* S if each $i \in S$ plays in $G^\star$ the strategy $y_i^\star$, where $y_i^\star(t_i) := y_S$ for every $t_i \in T_i$. As the reader may easily verify $y_S^\star$ is dominant in $G^\star$ and the posterior probability attributed by a player $i \in S$ to an n-tuple t of types does not change when y_S is chosen by S (in th (ii) The second question is more subtle.
When will S indeed choose y_S over x_S ? The obvious answer is that y_S will be chosen by S because it is an improvement upon x. However, we shall show, by means of two examples, that this is true only if the players are short-sighted. This kind of criticism of the SBE is not directly stemming from the incomplete information environment, it applies already to SNE for games with complete information. Indeed, our two examples are games with complete information.

4.3 Example

We consider the following (numerical) version of the prisoner's dilemma.

	L	R
T	$2,2$	$0,3$
B	$3,0$	$1,1$

table 4.

This game has no strong Nash equilibrium. The unique Nash equilibrium (B, R) can be improved upon by (T, L). However, (T, L) is not an NE and therefore it cannot be implemented. In this sense the rejection of (B, R) in favor of (T, L) is short-sighted.
The reader might think that players who want to implement improvements that are also NE's, are not short-sighted. The following example shows that this is not necessarily true.

4.4 Example

Consider the following 3-person game with complete information.

M_1:

	L	R
T	$0,0,0$	$0,0,0$
B	$0,0,0$	$0,0,0$

M_2:

	L	R
T	$2,2,-1$	$-1,-1,-1$
B	$-1,-1,-1$	$1,1,1$

table 5.

Here $x = (T, L, M_1)$ is not an SNE because $y = (B, R, M_2)$ is an improvement. Now y is a strict NE which is Pareto optimal. However, from the point of view of player 3, playing y is short-sighted behaviour. Indeed, if players 1 and 2 know that 3 will play M_2, then they will play (T, L).

The basic assumption underlying the definitions is the following.
During the interim phase of a Bayesian game (i.e. when the players know their types but have not yet implemented their strategies) transmission of information between the players is not allowed.
Thus, in our model the players may communicate in order to coordinate the choice of strategies, but they are not allowed to reveal any part of private information to each other. Under these conditions SBE's may be the only stable points of G if the players are sufficiently short-sighted. As discussed, the above assumption does not eliminate short-sighted behaviour.

The foregoing assumption is common in applications of BG's to economics. Transmission of information or signaling is usually done by the actual use of strategies (or local strategies in extensive form games).

As an illustration we consider the following simple game.

4.5 Example

Let $N = N^+ = \{1,2\},\ A_1 = \{T, B\},\ A_2 = \{L, R\},\ T_1 = \{\alpha\},\ T_2 = \{\gamma, \delta\},\ p_1(\alpha, \gamma) = 0.1,\ p_1(\alpha, \delta) = 0.9,\ p_2(\alpha, \gamma) = p_2(\alpha, \delta) = 0.5$ and u_1 and u_2 be given by the following matrices.

		$\boxed{\gamma}$			$\boxed{\delta}$	
		L	R		L	R
$\boxed{\alpha}$	T	$-1,2$	$-1,3$	T	$2,5$	$2,-5$
	B	$1,-1$	$1,1$	B	$-1,-1$	$-1,-1$

table 6.

Let $x = (B, LR)$. Then $y = (T, RL)$ is an improvement upon x by N. However, if player 2 is of type γ and his type is somehow revealed to player 1 (before y is implemented), then y is no longer an improvement upon x for N. By the foregoing assumption player 2 is not allowed to tell his type to player 1 and vice versa. By the foregoing discussion y can be implemented without revelation of information.

4.6 Definition

A solution ϕ on a set Γ of EBG's satisfies *weak Pareto-optimality (WPO)* if, for all $G \in \Gamma$, no strategy combination in $\phi(G)$ can be impr

The next lemma shows that SBE satisfies OPR ,WPO and (on closed sets) $CONS$.

4.7 Lemma

Let Γ be a closed set of EBG's.

Then SBE satisfies OPR , WPO and $CONS$ on Γ.

Proof.

(i) Let $G = \langle \{i\}, A_i, (T_j)_{j \in \{i\}^+}, p_i, u_i \rangle$ be a one-person game in Γ. To prove OPR it suffices to show that $SBE(G) = BE(G)$, because BE satisfies OPR.

Let $x_i \in BE(G)$. Then for every $t_i \in T_i, y_i \in X_i$:

$$U_i(x_i|t_i) \geq U_i(y_i|t_i)$$

so $\{i\}$ does not have an improvement upon x_i, hence $x_i \in SBE(G)$.

(ii) Let $G \in \Gamma, x \in SBE(G)$. Then in particular, the coalition N has no improvement upon x. This means that SBE satisfies WPO .

(iii) Let $G \in \Gamma, x \in SBE(G)$ and $S \subseteq N, S \neq \emptyset$. We prove that $x_S \in SBE(G^{S,x})$.

We know that $x_S \in BE(G^{S,x})$ because $x \in BE(G)$ and BE satisfies $CONS$.

Suppose $R \subseteq S$ is a coalition which has an improvement y_R upon x_S in $G^{S,x}$. Then y_R is also an improvement upon x in G, which contradicts the fact that $x \in SBE(G)$. Hence there is no coalition which has an improvement upon x. So SBE satisfies $CONS$. $\square$

Since, in general, the sets of Bayesian equilibria and strong Bayesian equilibria do not coincide, we know that SBE will not satisfy $COCONS$. However, we can formulate a weaker version of converse consistency that is satisfied by SBE.

4.8 Definition

Let ϕ be a solution on a closed set of EBG's.

Then ϕ satisfies $COCONS\text{-}S$ if for every $G \in \Gamma$ with $|N| \geq 2$ and every $x \in X$:

$$\text{if } x \in \tilde{\phi}(G) \text{ and } x \text{ cannot be improved upon by } N, \text{ then } x \in \phi(G).$$

Recall that $\tilde{\phi}(G) = \{x \in X \mid \text{for every } S \subsetneqq N, S \neq \emptyset \ : \ x_S \in \phi(G^{S,x})\}$.

4.9 Lemma

Let Γ be a closed set of EBG's.

Then SBE satisfies $COCONS\text{-}S$ on Γ.

Proof.

Let $G \in \Gamma$, $|N| \geq 2$ and $x \in \widetilde{SBE}(G)$ such that x cannot be improved upon by N. Then in particular $x \in \widetilde{BE}(G)$ and consequently $x \in BE(G)$ since BE satisfies $COCONS$.

Suppose $S \subsetneqq N$ is a coalition which has an improvement y_S upon x. Then y_S is also an improvement upon x_S in $G^{S,x}$, which contradicts the fact that $x \in \widetilde{SBE}(G)$. Because x cannot be improved by N either, there $\square$

4.10 Theorem

Let Γ be a closed set of EBG's.

There is a unique solution on Γ that satisfies OPR , WPO , $CONS$ and $COCONS\text{-}S$, and it is SBE.

Proof.

From the previous lemmas we know that SBE satisfies OPR , WPO , $CONS$ and $COCONS\text{-}S$. Now let ϕ be a solution on Γ that satisfies the foregoing four axioms. We prove by induction on the number of players that $\phi(G) = SBE(G)$ for every $G \in \Gamma$.

- If G is a one-person game, then by OPR, $\phi(G) = SBE(G)$.
- Now assume $k \in \{2, 3, ..\}$ and that $\phi(G) = SBE(G)$ for every $G \in \Gamma$ with less than k players. Let $G \in \Gamma$ be a k-person game and let $x \in \phi(G)$. By $CONS$ of ϕ,

$x \in \tilde{\phi}(G)$ and by induction, $x \in \widetilde{SBE}(G)$. Hence, by WPO of ϕ and $COCONS$-S of SBE, $x \in SBE(G)$. Thus $\phi(G) \subseteq SBE(G)$. Similarly, we can prove that $SBE(G) \subseteq \phi(G)$.

□

Now we provide a characterization of SBE using non-emptiness in the spirit of the ancestor property of definition 3.5. Let Γ^{SBE} denote the set of EBG's which have at least one SBE.

4.11 Theorem

Let ϕ be a solution on Γ^{SBE}.

Then ϕ satisfies NEM, OPR, $CONS$ and WPO iff $\phi = SBE$.

Proof.

We know that a solution which satisfies OPR, $CONS$ and WPO, is a refinement of SBE (see the proof of theorem 4.10). We are left to prove that $Graph(\Gamma^{SBE}, SBE)$ has the ancestor property.

If $G \in \Gamma^{SBE}$ and $x \in SBE(G)$, we can consider the same H as in the proof of theorem 3.8. It immediately follows that the unique Bayesian equilibrium (x, α) is also a SBE. □

Now we shall define coalition-proof Bayesian equilibria.

4.12 Definition

Let G be an EBG.

(i) Let $x \in BE(G)$ and $S \subseteq N, S \neq \emptyset$.

We define an *internally consistent improvement (ICI) of* S *upon* x *in* G by induction on $|S|$.

- If $S = \{i\}$, then $y_i \in X_i$ is an ICI of S upon x if it is an improvement (see definition 4.1).
- If $|S| > 1$, then $y_S \in X_S$ is an ICI of S upon x if y_S is an improvement upon x and no coalition $T \subsetneqq S, T \neq \emptyset$ has an ICI upon $(y_S, x_{N \setminus S})$.

(ii) Let $x \in BE(G)$.

x is a *coalition-proof Bayesian equilibrium (CPBE)* if no coalition has an ICI upon x.

For 2-person games we have the following characterization.

4.13 Lemma

Let G be a 2-person EBG.

Then, for every $x \in X$: $x \in CPBE(G)$ iff $x \in BE(G)$ and the grand coalition N has no improvement y upon x, such that $y \in BE(G)$.

Proof.

Let $x \in CPBE(G)$. By definition $x \in BE(G)$. Therefore $\{1\}$ and $\{2\}$ have no improvements upon x. Suppose N has an improvement $y \in BE(G)$ upon x. Th For the converse case, it suffices to note that if N has an ICI y upon x, then $y \in BE(G)$. □

Using the following modified forms of Pareto optimality and converse consistency, we are able to provide an axiomatic characterization for the coalition-proof Bayesi

4.14 Definition

Let ϕ be a solution on a closed set Γ of EBG's.

(i) ϕ satisfies *relative Pareto-optimality (RPO)* if for every $G \in \Gamma$:
if $x \in \phi(G)$ then there is no $y \in \tilde{\phi}(G)$ which is an improvement of N upon x.

(ii) ϕ satisfies *COCONS-CP* if for every $G \in \Gamma$:
if $x \in \tilde{\phi}(G)$ and there is no $y \in \tilde{\phi}(G)$ which is an improvement of N upon x, then $x \in \phi(G)$.

It is not difficult to prove that $CPBE$ satisfies OPR , RPO and $CONS$. In the next lemma we prove that $CPBE$ also satisfies $COCONS\text{-}CP$.

4.15 Lemma

Let Γ be a closed set of EBG's.

Then $CPBE$ satisfies $COCONS\text{-}CP$ on Γ.

Proof.

Let $G \in \Gamma$ and $x \in X$.

Suppose $x \in \widetilde{CPBE}(G)$ and suppose $x \notin CPBE(G)$. We show that there is an $y \in \widetilde{CPBE}(G)$ which is an improvement of N upon x.

Choose a coalition $S \subseteq N$ which has an ICI y_S upon x. Then y_S is also an ICI of S upon x_S in $G^{S,x}$.

If $S \neq N$ then we have a contradiction because $x \in \widetilde{CPBE}(G)$, so $S = N$. Clearly $y = y_N$ is an improvement upon x, moreover $y \in \widetilde{CPBE}(G)$ since $y_R \in CPBE(G^{R,y})$ for all $R \subsetneqq N$. For, if $T \subsetneqq R$ has an ICI z_T upon y_R in $G^{R,y}$ then $zICIof$T$uponyin$G$,whichcontradictsthefactthat$

For the proof of the following theorem we refer to the analogue in Peleg & Tijs [1992] for coalition-proof Nash equilibria.

4.16 Theorem

Let Γ be a closed set of EBG's.
Then there is a unique solution on Γ that satisfies OPR , RPO, $CONS$ and $COCONS$-CP, and it is the $CPBE$.

We conclude this section with a modification of the definition of strong Bayesian equilibrium.

4.17 Definition

Let G be an EBG and $x \in X$.
x is a *strictly strong Bayesian equilibrium (SSBE)* if there is no coalition $S \subseteq N, S \neq \emptyset$ which has a *weak improvement upon* x, which means that there is no $y_S \in X_S$ such that, for all $i \in S, t_i \in T_i$: $U_i(y_S, x_{N \setminus S}|t_i) \geq U_i(x|t_i)$ and there is a least one $i \in S$ such that for all $t_i \in T_i$: $U_i(y_S, x_{N \setminus S}|t_i) > U_i(x|t_i)$.

In the definition of $SSBE$ we look at a coalition in which at least one player gains *in every type*. One can also imagine a concept in which every player in a certain coalition gains *in at least one type*. It will not be difficult to characterize this concept by OPR , $CONS$ and slightly modified versions of Pareto optimality and converse consistency.

Finally we want to mention that some of the theorems of section 3 and 4 can be strenghtened by replacing 'closedness' by 'ϕ-closedness', where ϕ is the solution in question. We call a set Γ of EBG's *ϕ-closed* if, for

5 Bayesian potential games

In this section we introduce a specific closed class of EBG's namely the class of Bayesian potential games. Bayesian potential games are generalizations of (strategic) potential games, introduced by Monderer & Shapley [1991]. It turns out that, under a special condition on the priors, each Bayesian potential game has a pure Bayesian equilibrium.

5.1 Definition

Let G be an EBG.

G is a *Bayesian potential game (BPG)* if there exists a function $q : A \times T \to \mathbb{R}$ such that, for every $i \in N, a \in A, b_i \in A_i$ and $t \in T$

$$u_i(a,t) - u_i((a^{-i}, b_i), t) = q(a,t) - q((a^{-i}, b_i), t).$$

Such a function q is called a *potential* for G.

One can easily verify that the class of BPG's is closed. More precisely:
if G is a BPG with potential q, $S \subseteq N$ and $x \in X$, then $q^x : A_S \times T \to \mathbb{R}$ defined by

$$q^x(a_S, t) := q(((x_i(t_i))_{i \in N \setminus S}, a_S), t) \qquad (a_S \in A_S, t \in T)$$

is a potential for $G^{S,x}$.
Moreover, if G is a BPG with potential q then, in order to determine the set of Bayesian equilibria of G, we can replace each player's utility function by q.

5.2 Definition

Let G be an EBG.

(i) We say that G has *consistent priors* if each player has the same prior p on T.
If G has consistent priors, we write $G = \langle N, (A_i)_{i \in N}, (T_i)_{i \in N^+}, p, (u_i)_{i \in N} \rangle$.

(ii) We define the *ex ante game* $\hat{G}$ as the strategic game $\langle X_1, ..., X_n, \hat{u}_1, ..., \hat{u}_n \rangle$ where, for every $i \in N, X_i$ is the set of pure strategies of player i, and for every $x \in X, i \in N$:

$$\hat{u}_i(x) := \sum_{t \in T} p_i(t) u_i((x_j(t_j))_{j \in N}, t).$$

Note that the ex ante game $\hat{G}$ is a potential game if G is a Bayesian potential game with consistent priors.

We mention the following important relation between an EBG and the corresponding ex ante game. For the proof we refer to Harsanyi [1967], part II, theorem I.

5.3 Theorem

Let G be an EBG with consistent priors.

Then, for every $x \in X$, x is a Bayesian equilibrium of G if and only if x is a Nash equilibrium of the ex ante game $\hat{G}$.

5.4 Corollary

Let G be a BPG with consistent priors, such that every type has positive probability. Then $BE(G) \neq \emptyset$.

Proof.

If G is a BPG, then $\hat{G}$ is a potential game, so $\hat{G}$ has an NE (see Monderer & Shapley [1991], corollary 2.3). Hence G has a BE. □

5.5 Example

(A congestion situation : cf. Rosenthal [1973], Monderer & Shapley [1993].)

We consider a situation, corresponding to the network in the following figure.

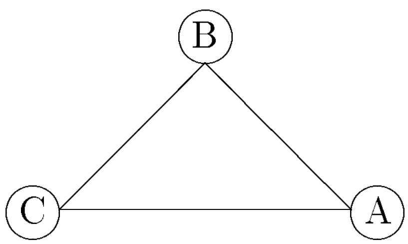

figure 1.

This network gives rise to a 2-person Bayesian potential game with consistent priors.

Suppose player 2 lives in A and has to go to C, either directly using road AC or via the detour ABC. Suppose player 1 lives in C and has to go with probability $\frac{1}{2}$ to A (using the road CA or CBA) and with probability $\frac{1}{2}$ to B (using CB or CAB).

These probabilities are common knowledge to both players. Suppose that if one player uses a road AC, BC or AB he has to pay 2 units and if two players use the same road, then both of them have to pay 8 units. Suppose further that the reward for player 2 going to C is 200 and that the reward for player 1 is 100 (or 50) if he goes to A (or B). This situation corresponds to the following Bayesian potential game:

$N = \{1, 2\}, A_1 = \{CA, CBA, CB, CAB\}, A_2 = \{AC, ABC\}, T_1 = \{A, B\}, T_2 = \{C\},$

where e.g. A is interpreted as 'player 1 has to go to A'. Further $p_i(A,C) = p_i(B,C) = \frac{1}{2}$ for $i \in \{1,2\}$ and u_1, u_2 and a potential are given in table 7.

So e.g. $u_1((CBA, ABC),(A,C)) = 84, \quad u_2((CBA, ABC),(A,C)) = 184$ which we obtain as follows. Player 1 and 2 obtain a reward of 100 and 200, respectively, but both have costs 16 because both use the roads CB and BA.
The Bayesian game in table 7 gives rise to a 16×2- ex ante bimatrix game. In table 8 we only give the relevant 4×2-bimatrix, leaving out 12 dominated rows, and also a 'knotted' 4×2-potential game.

Note that the unique pure Nash equilibrium $((CBA, CB), AC)$ corresponds to the following behaviour in the network:
player 2 goes straight to his goal C using AC and player 1 goes straight to his goal B, using CB, if he is of type B; otherwise player 1 goes to A making the detour CBA. This Nash equilibrium corresponds in the original game to the Bayesian equilibrium (x_1, x_2) where $x_1(A) = CBA, x_1(B) = CB$ and $x_2(C) = AC$.
For a systematic study of congestion situations we refer to Tijs [1994].

the Bayesian game

		type C: AC	type C: ABC
type A	CA	$92, 192$	$98, 196$
	CBA	$96, 198$	$84, 184$
	CB	$-2, 198$	$-8, 190$
	CAB	$-10, 192$	$-10, 190$
type B	CA	$-8, 192$	$-2, 196$
	CBA	$-4, 198$	$-16, 184$
	CB	$48, 198$	$42, 190$
	CAB	$40, 192$	$40, 190$

a potential

		type C: AC	type C: ABC
type A	CA	290	294
	CBA	294	280
	CB	196	188
	CAB	188	186
type B	CA	190	194
	CBA	194	180
	CB	246	238
	CAB	238	236

table 7.

	AC	ABC
(CA, CB)	$70, 195$	$70, 193$
(CA, CAB)	$66, 192$	$69, 193$
(CBA, CB)	$72, 198^{\star}$	$63, 187$
(CBA, CAB)	$68, 195$	$62, 187$

the 'knotted' ex ante game

	AC	ABC
(CA, CB)	268	266
(CA, CAB)	264	265
(CBA, CB)	$270^{\star}$	259
(CBA, CAB)	266	258

a 'knotted' potential

table 8.

The following example shows that the ex ante game of a BPG which does not have consistent priors need not to be a potential game. This shows that one cannot follow the same line of reasoning as in Corollary 5.4 to prove that BPG's with inconsistent priors have a BE.

5.6 Example

Let $G = \langle \{1,2\}, A_1, A_2, T_1, T_2, p_1, p_2, u \rangle$ be the BPG defined by $A_1 := \{T, B\}$, $A_2 := \{L, R\}$, $T_1 := \{\alpha, \beta\}$, $T_2 := \{\gamma, \delta\}$ and p_1, p_2 and a potential q given by the matrices in tables 9 and 10.

p_1 :

	γ	δ
α	$\frac{1}{2}$	0
β	0	$\frac{1}{2}$

p_2 :

	γ	δ
α	0	$\frac{1}{2}$
β	$\frac{1}{2}$	0

q :

		$\boxed{\gamma}$ L	$\boxed{\gamma}$ R	$\boxed{\delta}$ L	$\boxed{\delta}$ R
$\boxed{\alpha}$	T	1	0	0	0
	B	0	0	1	0
$\boxed{\beta}$	T	0	1	0	0
	B	0	0	0	1

table 9.

The corresponding ex ante game is given by

	LL	RR	LR	RL
TT	$\frac{1}{2},0$	$0,\frac{1}{2}$	$\frac{1}{2},0$	$0,\frac{1}{2}$
BB	$0,\frac{1}{2}$	$\frac{1}{2},0$	$\frac{1}{2},0$	$0,\frac{1}{2}$
TB	$\frac{1}{2},0$	$\frac{1}{2},0$	$1,0$	$0,0$
BT	$0,\frac{1}{2}$	$0,\frac{1}{2}$	$0,0$	$0,1$

table 10.

This game is not a potential game, because we have the following cycle of improvements:

$$(TT, LL) \rightarrow (TT, RR) \rightarrow (BB, RR) \rightarrow (BB, LL) \rightarrow (TT, LL).$$

So if q would be a potential for this game, we would have that

$$q(TT, LL) < q(TT, RR) < q(BB, RR) < q(BB, LL) < q(TT, LL)$$

which is a contradiction.

Note that the BPG in example 5.6 does have a BE, for example (TB, LL). The following example shows that a *3-person* BPG with inconsistent priors need not have Bayesian equilibria.

5.7 Example

Let $N = \{1, 2, 3\}$, $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, $A_3 = \{M\}$ and $T_1 = \{\gamma\}$, $T_2 = \{\delta\}$, $T_3 = \{\alpha, \beta\}$. Also $p_1(\gamma, \delta, \alpha) = p_2(\delta, \gamma, \beta) = 1$, $p_1(\gamma, \delta, \beta) = p_2(\gamma, \delta, \alpha) = 0$, $p_3(\gamma, \delta, \alpha) = p_3(\gamma, \delta, \beta,) = \frac{1}{2}$.
A potential is defined by:

$\boxed{\alpha}$

	L	R
T	1	0
B	0	1

$\boxed{\beta}$

	L	R
T	0	1
B	1	0

table 11.

One can easily verify that this game has no Bayesian equilibria in pure strategies.

We do not know whether *2-person* BPG's with inconsistent priors always have Bayesian equilibria. We found the following partial solution.

5.8 Theorem

Every 2-player BPG with complete information on one side and such that every type has positive probability, has a pure BE.

Proof.

Let G be a 2-player BPG with $T_1 = \{\delta\}$. So player 2 has complete information. We prove that player 2's prior p_2 can be replaced by p_1, without changing the set of BE's. Then the priors are consistent, so we know by corollary 5.4 that $BE(G) \neq \emptyset$.

For every $a \in A, t \in T_2$:

$$\begin{aligned} U_2(a|t) &= \sum_{s \in T_1} p_2(s|t) u_2(a,(s,t)) \\ &= p_2(\delta|t) u_2(a,(\delta,t)) \\ &= \frac{p_2(\delta,t)}{\sum_{s \in T_1} p_2(s,t)} u_2(a,(\delta,t)) = u_2(a,(\delta,t)). \end{aligned}$$

So U_2 does not depend on p_2, which means that, to calculate $BE(G)$, we can replace p_2 by p_1. $\square$

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