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# NOTE OF THE PATH FOLLOWING APPROACH OF EQUILIBRIUM PROGRAMMING

by

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ABSTRACT.

Recently Zangwill and Garcia introduced a general formulation of equilibrium problems. To prove the existence of an equilibrium they discussed a path following procedure. In this note we consider the application to the exchange economy problem. An economic equilibrium may be found by applying a simplicial variable dimension algorithm developed by Van der Laan and Talman.
We will show that when an appropriate triangulation and labelling rule is taken the limiting path of this algorithm coincides with the adjustment process induced by the procedure of Zangwill and Garcia.

KEY WORDS: Equilibrium programming, exchange economy, simplicial algorithms, limiting path.

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## 1. The economic equilibrium problem.

In this section we consider a path following procedure to obtain an economic equilibrium. This procedure has been given by Zangwill and Garcia [5] as an application to a general approach of equilibrium programming. In the next section we show that the procedure is very similar to the variable dimension approach of Van der Laan and Talman (see [1]-[4]).

Consider an exchange economy of m agents with n commodities. Let $w^i=(w^i_1,\dots,w^i_n)>0$ be the endowment of agent i, and let the utility function of agent i be given by $f^i$: $R^n \to R$. Let $w=\Sigma^m_{i=1} w^i$ be the total endowments.

Definition 1.1.

A competitive equilibrium is a pair of vectors $(\bar{x},\bar{p})$ where $\bar{x}=(\bar{x}^1,\dots,\bar{x}^m)\in R^{nm}$ and $\bar{p}\in R^n$ such that

(a) $f^i(\bar{x}^i) = \max f^i(x^i), \quad i=1,\dots,m,$

for $\bar{p}x^i \leq \bar{p}w^i$ and $0\leq x^i \leq w$,

(b) $\Sigma^m_{i=1} \bar{x}^i \leq w$, (1.1)

(c) $\bar{p}\geq 0$ and $\Sigma^n_{j=1}\bar{p}_j=1$.

To show the existence of a competitive equilibrium, Zangwill and Garcia [5] introduced the following equilibrium program.

(a) For i=1,...,m, given p,

$\max f^i(x^i)$

for $px^i \leq pw^i$ and $0\leq x^i \leq w$, (1.2)

(b) Given x,

$$\max p(\Sigma_{i=1}^{m} x^{i} - w)$$
$$\text{for } p \in S^{n-1}(t) = \{p \in S^{n-1} \mid p \geq \hat{p} - (t+\varepsilon)e\},$$

where $S^{n-1} = \{p \in R_{+}^{n} \mid \Sigma_{j=1}^{n} p_j = 1\}$, $\varepsilon > 0$ is very small, e is a vector with all components 1, $0 \leq t \leq 1$, and $\hat{p}$ is some arbitrarily chosen initial price vector. For given p, let z(p) be the excess demand, i.e. $z(p) = \Sigma_{i=1}^{m} x^{i}(p) - w$, where $x^{i}(p)$ solves (1.2a). It is assumed that $z: S^{n-1} \to R^{n}$ is a continuous function.
Without loss of generality we can assume that the excess demand in $\hat{p}$ has a maximum at a unique index, say k. Clearly, for $t = -\varepsilon$, (1.2) has a unique solution $(x(\hat{p}), \hat{p})$, where $x^{i}(\hat{p})$ solves (1.2a) given $p = \hat{p}$.

For $\varepsilon > 0$ small enough we have that $z_k(p)$ is still the unique maximum excess demand for t=0 and p satisfying the conditions of (1.2b). Therefore, given x(p) with $p \in S^{n-1}(t)$, the solution of (1.2b) is

$$\mathring{p}_k = \hat{p}_k + (n-1)\varepsilon$$
$$\mathring{p}_j = \hat{p}_j - \varepsilon \qquad j \neq k \tag{1.3}$$

when t=0. Hence $(x(\mathring{p}), \mathring{p})$ is the unique solution of (1.2) at t=0. In their paper Zangwill and Garcia prove the following theorem.

Theorem 1.2.

Suppose for all i, $f^{i}$ is 3-differentiable and strictly concave. If the equilibrium program (1.2) is regular, then starting from $(x,p,t) = (x(\mathring{p}), \mathring{p}, 0)$ there is a path of solutions to (1.2) that reach a solution to (1.1) at t=1.

As discussed by Zangwill and Garcia, the economic interpretation of this path is as follows. For small $\varepsilon > 0$, when t is increased from zero, the price of good k having the largest excess demand is increased, whereas the other prices are decreased, each with the same amount. In general the price vector p and the variable t are adapted in such a way that the usage of all goods with highest demand is tried to decrease. So, at any p, the process (1.2) works on the worst cases, i.e. on the markets with highest excess demand. As soon as all goods have the same excess demand, by Walras' law we must have $z(p) \leq 0$ and an equilibrium is reached.

Looking at the process in more detail, we first note that obviously t is not necessarily increasing monotonically during the process. The possibility of decreasing t has not made clear by Zangwill and Garcia, who state "the specialist slowly adjusts the prices to bigger and bigger price sets...".

In fact, considering the projection of the path on the set of prices $S^{n-1}$ the adjustment proces behaves as follows. Define, for $T \subset I_n = \{1,\dots,n\}$,

$$C(T) = \{p \in S^{n-1} \mid z_k(p) = \max_j z_j(p),\ k \in T\}$$

and, for $0 \le t \le 1$,

$$P(T,t) = \{p \in S^{n-1}(t) \mid p = \sum_{j \in T} \alpha_j p^j(t),\ \alpha_j \ge 0,\ \sum_{j \in T} \alpha_j = 1\},$$

where $p^j(t)$ is the vertex of $S^{n-1}(t)$ such that the j-th component is maximal. Then, a point (x,p,t) on the path of solutions to (1.2) has the property that for some $T \subset I_n$, $p \in C(T) \cap P(T,t)$. The projection of this path on the price space $S^{n-1}$ is illustrated in figure 1. In figure 1a the procedure starts in $C(\{2\})$ and hence $p_2$ is increased until $C(\{3\})$ is reached. Then the 1-manifold $C(\{2,3\}) = C(\{2\}) \cap C(\{3\})$ is followed until the equilibrium price $\bar{p}$ is obtained. Observe that t increases monotonically. In figure 1b again we have that the procedure starts in $C(\{2\})$, however in a subset of $C(\{2\})$ which is surrounded by $C(\{3\})$. Now t increases on the path from $\hat{p}$ ($\overset{\circ}{p}$) to $p^1$, decreases from $p^1$ to $p^2$ and increases again on the path from $p^2$ to the equilibrium price $\bar{p}$.

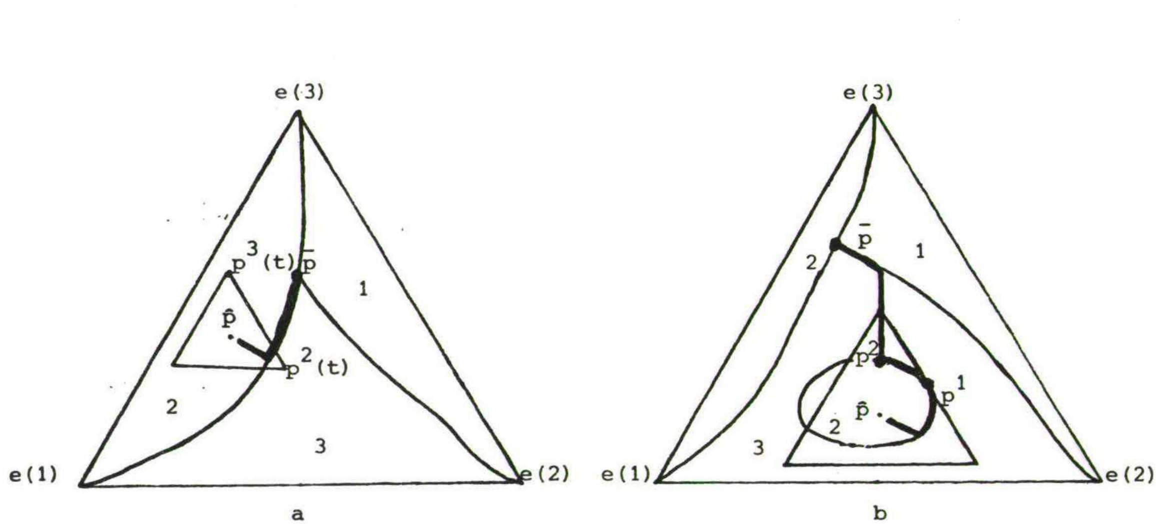


Figure 1. The paths of solution points projected on $S^{n-1}$ have been drawn heavily, e(i) is the i-th unit column.

## 2. The variable dimension approach.

In this section we show that the projection on the price space $S^{n-1}$ of the adjustment process of Zangwill and Garcia coincides with the limiting path of the simplicial variable dimension algorithm developed by Van der Laan and Talman, provided an appropriate labelling rule and triangulation underly the algorithm.

To do so, for $T \subset I_n$, let the sets A(T) be defined by

$$A(T) = \{p \in S^{n-1} | p = \hat{p} + \sum_{j \in T} \lambda_j u(j),\ \lambda_j \geq 0\} \tag{2.1}$$

where u(j), j=1,...,n, is the j-th column of the n×n matrix

$$U = \begin{bmatrix} (n-1) & 1 & . & . & . & 1 \\ 1 & (n-1) & & & & . \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & 1 \\ 1 & . & . & . & 1 & (n-1) \end{bmatrix}$$

(see Van der Laan and Talman [3, page 282]). Since, for $T \neq 0, n$, C(T) is an (n-|T|)-dimensional subset of $S^{n-1}$ and, for $\hat{p} \in \text{int } S^{n-1}$, A(T) is a |T|-dimensional subset of $S^{n-1}$, the set of points p with

$$p \in \bigcup_{T \subset I_n} (A(T) \cap C(T))$$

is a collection of paths and loops. The set of endpoints of the paths is the set of points $p \in \bigcup_{|T|=0,n} (A(T) \cap C(T))$. Clearly, the point $\hat{p}$ is the unique endpoint for $T=\emptyset$. For all other endpoints p we have $T=I_n$ and hence, $p \in C(I_n)$, i.e. p is an equilibrium price. So, under some regularity conditions, a path from $\hat{p}$ can be followed, leading to an economic equilibrium. Clearly this path yields the projection of the adjustment process described by Zangwill and Garcia on the set $S^{n-1}$ when $\varepsilon > 0$ small.

On the other hand, the path in $\cup(A(T) \cap C(T))$ originated in $\hat{p}$ is the limiting path of the simplicial variable dimension algorithm on $S^{n-1}$ developed by Van der Laan and Talman [2], when the U triangulation of $S^{n-1}$ proposed by the same authors in [3] underlies the algorithm and the following labelling rule is used. In case of vector labelling each point $p \in S^{n-1}$ is labelled according to z(p) and in case of integer labelling p is labelled with the index k of the commodity with the

highest demand. In both cases the same limiting path in $\cup(A(T)\cap C(T))$ is obtained, as discussed in Van der Laan [1, pp. 72, 73 and 83, 84]. Observe that the computational results in [1] were obtained for these labelling rules and in both [1] and [4] also for the U triangulation.
Therefore the adjustment process of Zangwill and Garcia can be interpreted in the following way. Starting in an arbitrarily chosen price vector $p\in P$ with $T=\emptyset$ the process generates for varying T a path of prices p in A(T) such that all goods j have highest excess demand $z_j(p)$, for $j\in T$. As soon as good i, $i\notin T$, has an excess demand $z_i(p)$ equal to the highest one, the process continues in $A(T\cup\{i\})$ with prices such that $z_h(p)=\max_j z_j(p)$ for all $h\in T\cup\{i\}$. If, however, the process generates a price p in $A(T\setminus\{k\})$ for some $k\in T$, the process continues in $A(T\setminus\{k\})$ with prices p such that $z_h(p)=\max_j z_j(p)$ for all $h\in T\setminus\{k\}$. The latter step happens when $\lambda_k$ in (2.1) becomes equal to zero. So, let (x,p,t) be a solution generated by the adjustment process and let T, $|T|<n$, be the unique index set such that

$$p = \hat{p} + \sum_{j\in T} \lambda_j\, u(j), \qquad \lambda_j>0.$$

Then, $p\in A(T)\cap C(T)$, i.e. we have the following complementarity

$$\lambda_j=0 \text{ and } z_j(p)<\max_i z_i(p) \qquad j\notin T$$

$$\lambda_j>0 \text{ and } z_j(p)=\max_i z_i(p) \qquad j\in T.$$

As a final remark we note that it can easily be shown that t is equal to $\sum_{j\in T}\lambda_j$, i.e. roughly speaking t denotes how far the process is from the starting point $\hat{p}$.

Concluding, the variable dimension algorithm can be utilized as a simplicial path following scheme for the projection on $S^{n-1}$ of the path of solutions to (1.2). Moreover, the limiting path of the algorithm generates the path of points of the adjustment process proposed by Zangwill and Garcia.

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