# On the relationship between the open-loop Nash equilibrium in LQ-games and the inertia of a matrix

by

Jacob C. Engwerda
and
Arie J.T.M. Weeren*

both
Tilburg University
Department of Econometrics
P.O. Box 90153
5000 LE Tilburg
The Netherlands

* The work of Ir. Weeren is supported by a grant from the common research pool of Tilburg and Eindhoven Universities (SOBU).

**Abstract**

In this paper we consider the location of the eigenvalues of the composite matrix $\begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}$, where the matrices $S_i$ and $Q_i$ are assumed to be semi-positive definite.
Two interesting observations, which are not or only partially mentioned in literature before, challenge this study. The first observation is that this matrix appears naturally in a both necessary and sufficient condition for the existence of a unique open-loop Nash solution in the 2-player linear-quadratic dynamic game and, more in particular, its inertia play an important role in the analysis of the convergence of the associated state in this game. The second observation is that from the eigenvalue and eigenstructure of this matrix all solutions for the algebraic Riccati equations corresponding with the above mentioned dynamic game can be directly calculated and, moreover, also the eigenvalues of the associated closed-loop system.
Simulation experiments suggest that the composite matrix will have at least n eigenvalues (here n is the state dimension of the system) with a positive real part. Unfortunately, it turns out that this property of the inertia of this matrix in general does not hold. Some specific cases for which the property does hold are discussed.

**Keywords**: Linear Quadratic games, open-loop Nash equilibrium, asymptotic analysis, inertia of a matrix

## I. Introduction

A well known problem studied in the literature on dynamic games is the existence of a unique open-loop Nash equilibrium in the two-player linear quadratic differential game defined by (see e.g. Starr and Ho (1969), Simaan and Cruz (1973) or Başar and Olsder (1982)):

$$\dot{x} = Ax + B_1u_1 + B_2u_2, x(0) = x_0 \tag{1}$$

with cost functionals:

$$\begin{aligned} J_1(u_1,u_2) &:= x(t_f)^T K_{1f} x(t_f) + \int_0^{t_f} \{x(t)^T Q_1 x(t) \\ &+ u_1(t)^T R_{11} u_1(t) + u_2(t)^T R_{12} u_2(t)\} dt, \end{aligned}$$

and

$$\begin{aligned} J_2(u_1,u_2) &:= x(t_f)^T K_{2f} x(t_f) + \int_0^{t_f} \{x(t)^T Q_2 x(t) \\ &+ u_1(t)^T R_{21} u_1(t) + u_2(t)^T R_{22} u_2(t)\} dt, \end{aligned}$$

in which all matrices are symmetric and, moreover, $Q_i$ are semi-positive definite and $R_{ii}$ are positive definite.

It is well known (see e.g. Starr and Ho (1969)) that the unique open-loop Nash solution for this game is given by

$$u_1^*(t) = -R_{11}^{-1} B_1^T K_1(t) \Phi(t,0) x_0$$

$$u_2^*(t) = -R_{22}^{-1} B_2^T K_2(t) \Phi(t,0) x_0$$

provided that there exists a unique solution set $K_1(t)$ and $K_2(t)$ satisfying the coupled asymmetric Riccati-type differential equations

$$\dot{K}_1 = -A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2;\ K_1(t_f) = K_{1f}$$

$$\dot{K}_2 = -A^T K_2 - K_2 A - Q_2 + K_2 S_2 K_2 + K_2 S_1 K_1;\ K_2(t_f) = K_{2f}$$

Here $\Phi(t,0)$ satisfies the transition equation

$$\dot{\Phi}(t,0) = (A - S_1 K_1 - S_2 K_2)\Phi(t,0);\ \Phi(t,t) = I$$

and $S_i = B_i R_{ii}^{-1} B_i^T, i = 1,2$.

More recently, the asymptotic behaviour of these Riccati equations and convergence of the associated closed-loop state of the system has been considered by Abou-Kandil and Bertrand in (1986), Abou-Kandil, Freiling and Jank in (1993) and Weeren, Schumacher and Engwerda in (1994). Abou-Kandil et al used in (1986) already indirectly the matrix $M := \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}$ to study the asymptotic behaviour of the associated closed-loop system for a special subclass of games of the above mentioned type (1). We will show in sections 2 and 3 that their approach can be generalized straightforwardly if one analyses problem (1) from its roots: the corresponding Hamiltonian equations. In section 2 we will show how both necessary and sufficient conditions for the existence of a unique open-loop Nash equilibrium from these Hamiltonian equations can be derived, which are tightly connected with matrix M. Then, in section 3 we will study the asymptotic behaviour of the associated state if the planning horizon $t_f$ tends to infinity and show in particular that if matrix M has at least n eigenvalues (counted with their multiplicities) with a positive real part, this state will converge to zero. In Abou-Kandil et al (1993) it was shown, amongst other things, that under some technical conditions solutions of the with the above set of differential equations corresponding set of algebraic Riccati equations

$$\left.\begin{array}{l} 0 = -A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2; \\ 0 = -A^T K_2 - K_2 A - Q_2 + K_2 S_2 K_2 + K_2 S_1 K_1; \end{array}\right\} \quad (ARE)$$

can be calculated from the eigenstructure of matrix M. In section 4 we will show that also the eigenvalues of the associated closed-loop system, obtained by applying the state feedback control $u_i^*(t) = -R_{ii}^{-1} B_i^T K_i(t) x(t)$, is completely determined by the eigenvalues of matrix M. In particular we have that if matrix M has only real eigenvalues containing n positive ones then from the corresponding eigenvectors of M one can derive a solution for the algebraic Riccati equations which stabilize the associated closed-loop system.

The above observations naturally lead to the question whether matrix M will always have n eigenvalues with a positive real part. Simulation experiments suggest that this conjecture might be true. Therefore, we study the location of the eigenvalues of this matrix in section 5 in more detail. In literature it is usual to summarize the number of eigenvalues with a positive real part, denoted by $\pi(M)$, the number of eigenvalues with a negative real part, denoted by $\nu(M)$, and the number of eigenvalues on the imaginary axis, denoted by $\delta(M)$ of a matrix $M$ in the notion of "inertia" of $M$. So the inertia of $M$, written In $M$, is the triple of integers $(\pi(M), \nu(M), \delta(M))$ (counted with their algebraic multiplicities) (see e.g. Lancaster et al.(1985) pp.186 et seq.). Obviously, $\pi(M) + \nu(M) + \delta(M) = 3n$, and the matrix $M$ is nonsingular if $\delta(M) = 0$. Unfortunately, it turns out that in general In $M \not\geq (n, 0, 0)$ (here we use the convention to write In $M \geq (p, q, r)$ if as well $\pi(M) \geq p$, $\nu(M) \geq q$ and $\delta(M) \geq r$). On the other hand we will see that for a number of special cases the relationship In $M \geq (n, 0, 0)$ does hold. The paper ends with some concluding remarks.

## II. The open-loop Nash equilibrium of the LQ differential game revisited

In this section we consider the existence of a unique open-loop Nash equilibrium of the differential game (1) in some more detail. This is done for two reasons. On the one hand we like to stress that there is a simple necessary and sufficient condition for the existence of a unique open-loop Nash equilibrium of the game, and that the existence of a solution to the Riccati-type differential equations is (generally speaking) just a sufficient condition. On the other hand we like to show how the matrix M we introduced in the introduction plays a crucial role in the analysis of the properties of this Nash equilibrium.

So, reconsider the existence of a unique open-loop feedback Nash equilibrium for problem (1). Due to the stated assumptions both cost functionals $J_i, i = 1, 2$, are strictly convex functions of $u_i$ for all admissible control functions $u_j, j \neq i$ and for all $x_0$. This implies that the necessary conditions following from the minimum principle are also sufficient (see e.g. Başar and Olsder (1982, section 6.5)).

Minimization of the Hamiltonian

$$H_i = (x^T Q_i x + u_1^T R_{i1} u_1 + u_2^T R_{i2} u_2) + \psi_i^T (Ax + B_1 u_1 + B_2 u_2),\ i = 1, 2$$

with respect to $u_i$ yields the optimality conditions:

$$u_1^*(t) = -R_{11}^{-1} B_1^T \psi_1(t) \tag{2}$$

$$u_2^*(t) = -R_{22}^{-1} B_2^T \psi_2(t), \tag{3}$$

where the n-dimensional vectors $\psi_1(t)$ and $\psi_2(t)$ satisfy

$$\dot{\psi}_1(t) = -Q_1 x(t) - A^T \psi_1(t),\ \text{with } \psi_1(t_f) = K_{1f} x(t_f)$$

$$\dot{\psi}_2(t) = -Q_2 x(t) - A^T \psi_2(t),\ \text{with } \psi_2(t_f) = K_{2f} x(t_f)$$

and

$$\dot{x}(t) = Ax(t) - S_1 \psi_1 - S_2 \psi_2;\ x(0) = x_0.$$

In other words, the problem has a unique open-loop Nash equilibrium if and only if the differential equation

$$\frac{d}{dt} \begin{pmatrix} x(t) \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix} = - \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix} \begin{pmatrix} x(t) \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix}$$

with boundary conditions $x(0) = x_0,\ \psi_1(t_f) - K_{1f} x(t_f) = 0$ and $\psi_2(t_f) - K_{2f} x(t_f) = 0$, has a unique solution. Denoting the state variable $(x^T(t)\ \psi_1^T(t)\ \psi_2^T(t))^T$ by $y(t)$, we can rewrite this two-point boundary value problem in the standard form

$$\dot{y}(t) = -My(t),\ \text{with } Py(0) + Qy(t_f) = (x_0^T\ 0\ 0)^T, \tag{4}$$

where $P = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 & 0 \\ -K_{1f} & I & 0 \\ -K_{2f} & 0 & I \end{pmatrix}$

Elementary matrix analysis shows then that

Theorem 1:

For every initial state the two-player linear quadratic differential game (1) has a unique open-loop Nash equilibrium if and only if, with $W(-t_f) = (W_{ij}(-t_f))$ $\{i,j = 1,2,3;\ W_{ij} \in R^{n\times n}\} := \exp(-Mt_f)$, the following matrix is invertible

$$\begin{pmatrix} W_{22}(-t_f) - K_{1f}W_{12}(-t_f) & W_{23}(-t_f) - K_{1f}W_{13}(-t_f) \\ W_{32}(-t_f) - K_{2f}W_{12}(-t_f) & W_{33}(-t_f) - K_{2f}W_{13}(-t_f) \end{pmatrix},$$

or, equivalently (since matrix $\exp(Mt_f)$ is invertible, $P + Q\exp(-Mt_f)$ is invertible if and only if $P\exp(Mt_f) + Q$ is invertible) the next matrix is invertible

$$W_{11}(t_f) + W_{12}(t_f)K_{1f} + W_{13}(t_f)K_{2f}.$$

Moreover, this solution is given by (2,3). The optimal control trajectories together with the associated state trajectory can be calculated from the linear two-point boundary value problem (4). □

It is easily verified that if the Riccati-type differential equations mentioned in the introduction have a solution, then $\psi_i(t) = K_i(t)x^*(t)$ satisfies the above two-point boundary differential equation which yields the result as stated by Starr and Ho.

## III. Asymptotic analysis of the Nash equilibrium

In this section we study the asymptotic behaviour of the equilibrium state trajectory, that is $x^*(t_f)$, if the planning horizon $t_f$ approaches infinity. Since $u_i^*(t_f) = -R_{ii}^{-1}B_i^T\psi_i(t_f)$, with $\psi_i(t_f) = K_{1f}x(t_f)$, this also completely determines the limiting behaviour of $u_i^*$, $i = 1,2$.

To that end, we first rewrite the boundary condition as $(P + Q\exp(-Mt_f))y(0) = (x_0^T\ 0\ 0)^T$. Consequently,

$$\begin{aligned} y(t_f) &= \exp(-Mt_f)y(0) \\ &= \exp(-Mt_f)(P + Q\exp(-Mt_f))^{-1}(x_0^T\ 0\ 0)^T \\ &= (P\exp(Mt_f) + Q)^{-1}(x_0^T\ 0\ 0)^T. \end{aligned}$$

So that

$$x(t_f) = (I\ 0\ 0)(P\exp(Mt_f) + Q)^{-1}(x_0^T\ 0\ 0)^T.$$

Now, using the previously introduced notation $W(-t_f) = W_{ij}(-t_f),\ i,j = 1,2,3$ for matrix $\exp(-Mt_f)$, we have that

$$P\exp(Mt_f) + Q = \begin{pmatrix} W_{11}(t_f) & W_{12}(t_f) & W_{13}(t_f) \\ -K_{1f} & I & 0 \\ -K_{2f} & 0 & I \end{pmatrix}.$$

Elementary calculation shows that the left-upper $n \times n$-block of the inverse of this matrix is given by $(W_{11}(t_f) + W_{12}(t_f)K_{1f} + W_{13}(t_f)K_{2f})^{-1}$. So, we find the following expression for $x(t_f)$:

$$x(t_f) = (W_{11}(t_f) + W_{12}(t_f)K_{1f} + W_{13}(t_f)K_{2f})^{-1}x_0 \tag{5}$$

This yields then the following result

<u>Theorem 2:</u>

For every initial state the final state corresponding with the open-loop Nash equilibrium in the two-player linear quadratic differential game (1) converges if and only if the matrix $(W_{11}(t_f) + W_{12}(t_f)K_{1f} + W_{13}(t_f)K_{2f})^{-1}$ converges. □

Note that this result differs from the result obtained by Abou-Kandil et al. in (1993) in the sense that their "dichotomic separability" condition is just a sufficient condition to conclude that the Riccati-type differential equations converge. They do not say anything on the convergence of the associated state of the open-loop Nash equilibrium of the game.

Next we discuss a sufficient condition under which one can conclude that the open-loop Nash-equilibrium and its associated state will converge to zero. Roughly spoken this condition says that if matrix $-M$ has at least $n$ eigenvalues (counted with multiplicities) with a negative real part then the associated equilibrium state of the game will converge to zero.

<u>Theorem 3:</u>

To avoid some technicalities assume that the eigenvalues of $-M$ are ordered as follows: $\mathrm{Re}\,\lambda_1 \leq \mathrm{Re}\,\lambda_2 \leq \ldots \leq \mathrm{Re}\,\lambda_n < \mathrm{Re}\,\lambda_{n+1} \leq \ldots \leq \mathrm{Re}\,\lambda_{3n}$.

Then, under the assumption that the first $n$ eigenvalues $\lambda_1, \ldots, \lambda_n$ have a negative real part and the matrices $T_{11}$ and $H$ (see proof below) are invertible, the state $x^*(t_f)$ corresponding with the Nash equilibrium for the differential game (1) converges to zero.

<u>Proof:</u>

To prove this theorem, we first recall from linear algebra that it is always possible to make a Jordan decomposition of matrix $M$, that is, there exists an invertible matrix $S$ such that $-M = S^{-1}JS$, where $J$ is in Jordan canonical form (see e.g. Lancaster et al. (1985)). Then, $J$ can be partitioned as

$$J = \begin{pmatrix} J_1 & \\ & J_2 \end{pmatrix}.$$

Partition $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ and $S^{-1} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ accordingly. Then it is easy to verify that

$$\begin{aligned} W_{11}(t_f) + W_{12}(t_f)K_{1f} + W_{13}(t_f)K_{2f} &= T_{11}\exp(-J_1t_f)(S_{11} + S_{12}(K_{1f}\ K_{2f})^T) \\ &\quad + T_{12}\exp(-J_2t_f)(S_{21} + S_{22}(K_{1f}\ K_{2f})^T) \end{aligned}$$

Now let $r := \max\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(J_1)\}$, and $\Lambda := rI$.
Note that $r < 0$ and for all $\lambda \in \sigma(J_2)$ we have $r - \mathrm{Re}(\lambda) < 0$. Hence $\exp((\Lambda - J_2)t_f) \to 0$ for $t_f \to \infty$. Consequently, under the assumption that both $T_{11}$ and $H := S_{11} + S_{12}(K_{1f}\ K_{2f})^T$ are invertible we have:

$$\begin{aligned} x(t_f) &= e^{rt_f} \cdot \Big(T_{11}\exp((\Lambda - J_1)t_f))(S_{11} + S_{12}(K_{1f}\ K_{2f})^T) \\ &\quad + T_{12}\exp((\Lambda - J_2)t_f)(S_{21} + S_{22}(K_{1f}\ K_{2f})^T)\Big)^{-1} x_0 \\ &= e^{rt_f}H^{-1}\Big(\exp((\Lambda - J_1)t_f) \\ &\quad + T_{11}^{-1}T_{12}\exp((\Lambda - J_2)t_f)(S_{21} + S_{22}(K_{1f}\ K_{2f})^T)H^{-1}\Big)^{-1} T_{11}^{-1}x_0 \\ &= e^{rt_f}H^{-1}\exp(-(\Lambda - J_1)t_f)T_{11}^{-1}x_0 + O(e^{rt_f}) \\ &= H^{-1}\exp(J_1t_f)T_{11}^{-1}x_0 + O(e^{rt_f}). \end{aligned}$$

Hence $x(t_f) \to 0$ for $t_f \to \infty$.

□

## IV. The relationship between ARE and M

In this section we consider the relationship between the eigenstructure of matrix M and the solutions of the algebraic Riccati equations (ARE) in some more detail. We have the following relationship between the spectra of M and the associated closed-loop system matrix that results by using the state feedback control, $u_i^*(t) = -R_{ii}^{-1}B_i^TK_i(t)x(t)$, as mentioned in the introduction:

Lemma 4:
Assume that (ARE) has real solutions $K_1$ and $K_2$.

Then, $\sigma(A - S_1K_1 - S_2K_2) \subset \sigma(-M)$ (Here $\sigma(A)$ denotes the spectrum of matrix $A$).

Proof:

Let $\{K_1,\ K_2\}$ be a set of solutions satisfying (ARE). Consider the matrix $T = \begin{pmatrix} I & 0 & 0 \\ -K_1 & I & 0 \\ -K_2 & 0 & I \end{pmatrix}$. Then simple calculations show that

$$-TMT^{-1} = \begin{pmatrix} A - S_1K_1 - S_2K_2 & -S_1 & -S_2 \\ 0 & K_1S_1 - A^T & K_1S_2 \\ 0 & K_2S_1 & K_2S_2 - A^T \end{pmatrix},$$

which yields the advertised result. □

The next theorem sharpens this result considerably in two ways. First it gives an exact relationship between solutions of (ARE) and eigenvalues of M and second it gives us precise information on the spectrum of the closed-loop system matrix. The price we pay is that we make some technical assumptions on the eigenstructure of matrix M. How far these assumptions can be relaxed remains a matter of future research. Results obtained by Abou-Kandil et al in (1993) give hope that the assumptions may be considerably relaxed.

Theorem 5:

Assume that matrix $M$ has only real eigenvalues and that their corresponding algebraic multiplicities equal their geometric multiplicities. Denote the eigenvector corresponding to the eigenvalue $\lambda_i$ of $-M$ by $(x_i^T y_i^T z_i^T)^T$, $i = 1, .., 3n$. Let $K^{pos}$ be the set of all 3nxn matrices $(X^TY^TZ^T)^T := ((x_i^T y_i^T z_i^T)^T)\ i = 1, .., n$, where all matrices $X, Y$ and $Z$ are square nxn matrices, which can be formed this way.
Then, (ARE) has a real solution $K_1, K_2$ if and only if $K_1 = YX^{-1}$ and $K_2 = ZX^{-1}$ for some $(X^TY^TZ^T)^T \in K^{pos}$. Moreover, assuming that the matrix $(X^TY^TZ^T)^T$ is determined by the n eigenvalues $\lambda_i,\ i = 1, .., n$ of $-M$, we have that $\sigma(A - S_1K_1 - S_2K_2) = \{\lambda_i,\ i = 1, .., n\}$.

Proof:

"$\Rightarrow$" According to (the proof) of lemma 4, we have that matrix $A - S_1K_1 - S_2K_2$ contains n real eigenvalues which have the additional property that their algebraic multiplicities coincide with their geometric multiplicity. Let $\lambda_i$ be an arbitrary eigenvalue of $A - S_1K_1 - S_2K_2$ and $x_i$ its corresponding eigenvalue. Then it is easily verified that $(I\ K_1^T\ K_2^T)^T x_i$ is an eigenvector of M corresponding with the eigenvalue $\lambda_i$. Denote $K_1^T x_i$, $K_2^T x_i$ by $y_i$ and $z_i$, respectively. Next form the matrix $(X^T Y^T Z^T)^T :=$ $((x_i^T y_i^T z_i^T)^T)$ $i = 1, .., n$, where all matrices $X, Y$ and $Z$ are square nxn matrices. Then, obviously, $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ equals $\begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} X$. Since $X$ consists of the n different eigenvectors of matrix $A - S_1K_1 - S_2K_2$, it is invertible. So the above matrix equation immediately yields that $K_1 = YX^{-1}$ and $K_2 = ZX^{-1}$.

"$\Leftarrow$" Note that -M$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \Lambda$, where $\Lambda = \text{diag}(\lambda_i)$, $i = 1, .., n$. Denote $X\Lambda X^{-1}$ by $R$. Then this matrix equation can be rewritten as -M$\begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} R$, where $K_1 := YX^{-1}$ and $K_2 := ZX^{-1}$. Simply writing out this equation (see also Abou-Kandil et al (1993)) shows then that this pair of matrices $K_1, K_2$ satisfy (ARE). $\square$

The previous theorem states in particular that if matrix M has n positive real eigenvalues (with the appropriate multiplicities) then the algebraic Riccati equations will have a solution which will stabilize the closed-loop system. Moreover, if M has more than n positive eigenvalues one may expect that there is more than one stabilizing solution for (ARE).

## V. On the inertia of matrix $M$

In the previous sections we saw that if we can show that the eigenvalues of matrix $M$ always satisfy the condition $\nu(-M) \geq n$ then, almost always, the associated state of the open-loop Nash equilibrium of the differential game will converge to zero if either the open-loop control, in which the planning horizon is extended to infinity, or the deterministic state feedback, using the appropriate solutions of the algebraic Riccati equations, is used. The next example shows that this inertia property is, unfortunately, not always satisfied by matrix M.

Example 4:

Let $A = \begin{pmatrix} -1 & 0 \\ 0 & -0.9 \end{pmatrix}$; $S_1 = \begin{pmatrix} 500 & -200 \\ -200 & 100 \end{pmatrix}$; $Q_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$;

$S_2 = \begin{pmatrix} 1000 & 200 \\ 200 & 50 \end{pmatrix}$; $Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; Then, the eigenvalues of $-M$ are (according to Matlab) $\{-42.1096, 0.3168, 0.3441 \pm 4.6285i, 0.8866, 42.1181\}$. So, In $-M = (5,1,0)$. □

It is interesting to note that with e.g. the choice of $K_{1f} := Q_1$ and $K_{2f} := Q_2$ elementary calculations show that the condition of theorem 2 is not satisfied whereas $M$ is dichotomic seperable. In other words, this is also a non-trivial example of a game in which the final state does not converge whereas the Riccati-type differential equations do converge (see Abou-Kandil (1993)). We like to stress here that this result differs from the results obtained in LQ theory. There, a sufficient condition to conclude that the solution of the final horizon optimization problem converges to the solution of the infinite horizon problem is that the system is both controllable and observable. Furthermore, under these conditions the state of the system converges to zero. Obviously, the system is in this example both controllable and observable, but the convergence properties do not hold.
Next, we consider three special cases in which the inertia property does hold.

Case 1: $A$ is symmetric and commutes with either $Q_i$ or $S_i$, $i = 1, 2$.
We prove the case that $A$ commutes with $Q_i$. The other case is proved similarly. Consider the characteristic polynomial of $M$. We have that

$$\det(M - \lambda I) =$$
$$\det{}^2(A^T - \lambda I)\det\left(-A - \lambda I - (S_1\ S_2)\begin{pmatrix} A^T - \lambda I & 0 \\ 0 & A^T - \lambda I \end{pmatrix}^{-1}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}\right) =$$
$$\det{}^2(A^T - \lambda I)\det(-A - \lambda I - S_1(A^T - \lambda I)^{-1}Q_1 - S_2(A^T - \lambda I)^{-1}Q_2).$$

Using the facts that $A$ is symmetric and commutes with $Q_i$, $i = 1, 2$ we can rewrite this last expression as

$$(-1)^n \det(A^T - \lambda I)\det((A + \lambda I)(A^T - \lambda I) + S_1Q_1 + S_2Q_2) =$$
$$(-1)^n \det(A^T - \lambda I)\det(-\lambda^2 I + S_1Q_1 + S_2Q_2 + A^2).$$

From this last formula we deduce that In $-M \geq (0, n, 0)$ if e.g. i) $\nu(A) = n$ or ii) $\delta(S_1Q_1 + S_2Q_2 + A^2) = 0$. Note that the zeros of $\det(-\lambda^2 I + S_1Q_1 + S_2Q_2 + A^2)$ are symmetrically distributed w.r.t. to the imaginary axis (i.e. if $\lambda_0$ is a zero, then $-\lambda_0$ is a zero too). □

Case 2: $Q_2 = \alpha Q_1$ or (symmetrically) $S_2 = \alpha S_1$.

Assume that $S_2 = \alpha S_1$ then, with $T := \begin{pmatrix} I & 0 & 0 \\ 0 & I & \alpha I \\ 0 & 0 & I \end{pmatrix}$, $TMT^{-1} = \begin{pmatrix} -A & S_1 & 0 \\ Q_1 + \alpha Q_2 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}$. Since matrix $\begin{pmatrix} -A & S_1 \\ Q_1 + \alpha Q_2 & A^T \end{pmatrix}$ is a Hamiltonian matrix (see e.g. Lancaster (1985)) we conclude that In $-M \geq (0, n, 0)$ if $\delta(A) \neq 0$. □

<u>Case 3:</u> $A$ is a real matrix in "Jordan complex canonical" form and $S_i, Q_i,\ i = 1,2$ are diagonal.

This case deals with the situation in which $A$ has a (modified-diagonal) Jordan form in which the entries are restricted to be real.

In case matrix $A$ is a diagonal matrix $D$ the result follows immediately from case 1. So, consider the case that $A = D + J$, where $D = dI$ and $J$ is a nilpotent matrix with zeros everywhere except for the elements (equal to 1) along the diagonal just above the principal diagonal. Then, from case 1, we have that

$$\det(M - \lambda I) =$$
$$\det{}^2(A^T - \lambda I)\det(-A - \lambda I - S_1(A^T - \lambda I)^{-1}Q_1 - S_2(A^T - \lambda I)^{-1}Q_2). \quad (i)$$

Since $A = D + J$, we have that

$$(A^T - \lambda I)^{-1} = (D - \lambda I + J^T)^{-1} =$$
$$(I + (D - \lambda I)^{-1}J^T)^{-1}(D - \lambda I)^{-1} = \sum_{k=0}^{n} \left((\lambda I - D)^{-1}J^T\right)^k (\lambda I - D)^{-1}.$$

Substitution of this expression into (i) yields:

$$\det{}^2(A^T - \lambda I)\det(-A - \lambda I + S_1 \sum_{k=0}^{n} \left((\lambda I - D)^{-1}J^T\right)^k (\lambda I - D)^{-1}Q_1 +$$
$$S_2 \sum_{k=0}^{n} \left((\lambda I - D)^{-1}J^T\right)^k (\lambda I - D)^{-1}Q_2).$$

Now, $\det(A^T - \lambda I) = \det(D - \lambda I)$. Therefore, we can rewrite this determinant as

$$\det(A^T - \lambda I)\det((-D - \lambda I)(\lambda I - D) - J(\lambda I - D) + S_1 \sum_{k=0}^{n} \left((\lambda I - D)^{-1}J^T\right)^k Q_1 +$$
$$S_2 \sum_{k=0}^{n} \left((\lambda I - D)^{-1}J^T\right)^k Q_2) =:$$
$$\det(A^T - \lambda I)\det(G(\lambda)).$$

Next, consider the $n \times n$ matrix $S(\mu) = \text{diag}(1, \mu, \mu^2, .., \mu^{(n-1)})$. Obviously, $S(\mu)S(\frac{1}{\mu}) = I$. So, $\det(G(\lambda)) = \det(S(\lambda - d))G(\lambda)S(\frac{1}{\lambda - d}))$. By induction one can show that the product of the lastmentioned three matrices equals

$$\tilde{G}(\lambda) := (-D - \lambda I)(\lambda I - D) - J + S_1 \sum_{k=0}^{n} \left(J^T\right)^k Q_1 + S_2 \sum_{k=0}^{n} \left(J^T\right)^k Q_2.$$

So, we see that if $\tilde{G}(\lambda_0)$ is singular then also $\tilde{G}(-\lambda_0)$ will be singular. Furthermore, simple calculations show that if $d \neq 0$ for every number $ix$ on the imaginary axis $\det(\tilde{G}(ix)) \neq 0$ (note that all entries on or below the principal diagonal are semi-positive and use elementary row operations to eliminate subsequently all entries of the second column, then those of the third column etc.). The conclusion is thus that the eigenvalues of $M$ are $d$ (with multiplicity $n$) and that the other eigenvalues are distributed again symmetrically w.r.t. the imaginary axis (and are not located on it) if $d \neq 0$.
Note that by combining both cases we get the result as advertised for a matrix $M$ in general "complex Jordan canonical" form. □

## VI. Concluding remarks

In this note we reconsidered the existence and asymptotic behaviour of a unique open-loop Nash equilibrium in the two-player Linear Quadratic game. We analyzed the problem starting from its basics: the Hamiltonian equations. We derived necessary and sufficients conditions for the existence of a unique open-loop Nash equilibrium in terms of a full rank condition on a modified fundamental matrix. Furthermore, this direct approach made it possible to analyze the asymptotic behaviour of the final state of the game. We showed that this state converges if and only if a certain matrix converges in time. A more detailed analysis of this matrix shows on the one hand that an almost sufficient condition for this matrix to converge is that the matrix $M = \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}$ has n eigenvalues with a positive real part. On the other hand, it is clear that this condition can be relaxed. Probably, by considering the problem from a geometric point of view, more detailed results can be achieved which also give more insight into the basics of the problem. But this remains a topic for future research.
Since simulation experiments suggested that the above inertia condition is almost always satisfied, we tried to prove this conjecture. Unfortunately, it turned out that our supposition was wrong. We gave a counterexample, which has the additional interesting property that it satisfies the dichotomic seperability conditions mentioned by Abou-Kandil. As a consequence the corresponding set of Riccati-type differential equations in this example will converge. As we pointed out, this raises a number of interesting questions in comparison with the theory developed for LQ systems.

Though we were not able to prove our conjecture on the inertia of matrix $M$ for the general case, we succeeded in showing its correctness for a number of special cases. These results may be helpfull in deriving more general properties on the inertia of this matrix. We conclude this paper by noting that the results obtained in the lemma and theorems can be straightforwardly generalized to the N-player linear quadratic differential game.

## References

Abou-Kandil H. and Bertrand P., 1986, Analytic solution for a class of linear quadratic open-loop Nash games, International Journal of Control, vol.43 no.3, pp.997-1002.

Abou-Kandil H., Freiling G. and Jank G., 1993, Necessary and sufficient conditions for constant solutions of coupled Riccati equations in Nash games; Systems&Control Letters 21, pp.295-306.

Başar T. and Olsder G.J., 1982, Dynamic Noncooperative Game Theory, Academic Press London.

Lancaster P. and Tismenetsky M., 1985, The Theory of Matrices, Academic Press London.

Simaan M. and Cruz J.B., Jr., 1973, On the solution of the open-loop Nash Riccati equations in linear quadratic differential games, International Journal of Control 18, no.1, pp.57-63.

Starr A.W. and Ho Y.C., 1969, Nonzero-sum differential games, Journal of Optimization Theory and its Applications 3, pp.184-206.

Weeren A.J.T.M., Schumacher J.M. and Engwerda J.C., 1994, Asymptotic analysis of Nash equilibria in nonzero-sum Linear-Quadratic differential games. The two-player case, Research Memorandum FEW 634, Tilburg University The Netherlands.