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n-Person Nonconvex Bargaining: Efficient Proportional Solutions

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Abstract

For n-person bargaining problems the family of proportional solutions (introduced and characterized by Kalai) is generalized to bargaining problems with non-convex payoff sets. The so-called "efficient proportional solutions" are characterized axiomatically using natural extensions of the original axioms provided by Kalai.

Keywords: n-person non-convex bargaining, proportional solutions, the egalitarian solution.

JEL-classification: C78.

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1 Introduction

In axiomatic bargaining theory it has traditionally been assumed that the set of payoffs is convex since if two different payoffs (considered as von Neumann-Morgenstern utilities) are feasible so is any lottery between them. However, there are bargaining problems where randomization seems unreasonable, for example, in moral hazard problems where random contracts may not be allowed. Moreover, even though randomization may be reasonable, agents may still violate the von Neumann-Morgenstern axioms.

Consequently a series of recent papers have examined well known bargaining solutions when the set of payoffs is non-convex: The Nash solution ([12]) has been considered in e.g. [8], [5], [14] and [13]. The Kalai-Smorodinsky solution ([9]) has been considered in e.g. [1], [4] and [7].

The family of *proportional solutions* (by Kalai [8]), comprising the egalitarian solution, has received less attention in this respect. [4] consider a characterization of the egalitarian solution when the set of payoffs is non-convex by relaxing Pareto optimality but it seems difficult to justify bargaining solutions which are not Pareto optimal.

In the present paper we define a generalization of the family of proportional solutions to bargaining problems when the set of payoffs is non-convex insisting on Pareto optimality - hence called *efficient proportional solutions*. We demonstrate that a natural extension of Kalai's original axioms (Pareto optimality, Scale invariance and Monotonicity) for bargaining problems where the set of payoffs is convex may be used to characterize efficient proportional solutions for bargaining problems where the set of payoffs is non-convex.

By insisting on Pareto-optimality, the efficient proportional solution need not be unique for some bargaining problems but it can be shown that typically the efficient proportional solution will in fact be unique by adapting the proof of Theorem 2 in [7] to the present set-up.

2 The model

An n-person bargaining problem is described by a threat point $a \in \mathbb{R}^n$, that is the result in case of disagreement, and a set of feasible payoffs $S \subset \mathbb{R}^n$. Let $\mathcal{U}$ be the set of n-person bargaining problems (a, S) where $S \cap (\{a\} + \mathbb{R}^n_+)$ is compact and $S \cap (\{a\} + \mathbb{R}^n_{++})$ is non-empty. A *solution* is map f from the set of bargaining problems $\mathcal{U}$ to the set of payoffs $\mathbb{R}^n$ such that $f(a, S) \in S$ for all $(a, S) \in \mathcal{U}$.

Let $E(S) \subset \mathbb{R}^n$ be the set of *Pareto optimal payoffs* in S and let $D(a, S) \subset \mathbb{R}^n$ be the set of *individually rational payoffs* so $D(a, S) = \{x \in S | x \geq a\}$. Let the *reference point* $s_v : \mathcal{U} \to \mathbb{R}^n$ be the efficient point in the intersection of the comprehensive hull of the set of feasible payoff and the line through a in direction v so

$$s_v(a, S) \;=\; E((S - \mathbb{R}^n_+) \cap \{x \in \mathbb{R}^n | x = a + tv \text{ for some } t \in \mathbb{R}\}).$$

Let $\mathcal{U}^c \subset \mathcal{U}$ be the set of bargaining problems where the set of feasible payoffs is comprehensive and convex, then for all $v \in \mathbb{R}^n_+ \setminus \{0\}$ the reference point $s_v : \mathcal{U}^c \to \mathbb{R}^n$ is a *proportional solution*. The family of proportional solutions is analyzed in [8] where it is shown that in $\mathcal{U}^c$ the proportional solution is characterized by the following properties:

- Pareto optimality so $f(a, S) \in E(S)$.
- Scale Invariance so $\alpha(f(a, S)) = f(\alpha(a), \alpha(S))$ for all strictly increasing maps, $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ with $\alpha_j(x) = \gamma x_j + \delta_j$.
- Monotonicity so $f(a, S) \leq f(a, T)$ for $S \subset T$.

Note, that the 'proportional solution' in [3] differs from the above definition and is closer related to the Kalai-Smorodinsky solution ([9]).

2.1 Efficient proportional solutions

For the set of n-person bargaining problems where the threat point is zero and the set of feasible payoffs is compact and comprehensive, [4] consider an extension of the egalitarian solution s_v for $v = (1, \ldots, 1)$. However in their characterization of the egalitarian solution they use weak Pareto optimality and we find it difficult to justify bargaining solutions that are not Pareto optimal.

Therefore insisting on Pareto optimal solutions we must accept that proportions (as given by the direction v) may only remain fixed up to a certain point simply because the boundary of S may not be Pareto optimal for all directions. Intuitively, a straightforward efficient generalization of the family of proportinal solutions is to 'move' from the threat point a in direction v until the reference point $s_v(a, S)$ is reached and if it is not Pareto optimal, then jump to an Pareto optimal point $x \in E(S)$ dominating the reference point, that is to some $x \geq s_v(a, S)$. Formally: a bargaining solution f is called *efficient proportional* if and only if there exists a vector $v \in \mathbb{R}^n_+ \setminus \{0\}$ such that $f(a, S) \in \{x \in E(S) | x \geq s_v(a, S)\}$ for all $(a, S) \in \mathcal{U}$. Let f_v be an efficient proportional solution with direction v.

Clearly, on the set of bargaining problems where the set of feasible payoffs is comprehensive and convex $\mathcal{U}^c$, the family of efficient proportional solutions and family of the proportional solutions of [8] coincide. Moreover for a bargaining problem (a, S) and a fixed direction v the efficient proportional solution need not be unique. Indeed the set of Pareto optimal payoffs, that dominate $s_v(a, S)$, $\{x \in E(S) | x \geq s_v(a, S)\}$ may contain many points. However the approach and proof of Theorem 2 in [7] may be adapted to the present framework to demonstrate that typically the efficient proportional solution is in fact unique.

3 Characterization of efficient proportional solutions

Let $\mathcal{U}^h \subset \mathcal{U}^c$ be the set of bargaining problems where the sets of feasible payoffs is the intersection of a finite number of half spaces containing the threat point. Therefore, $(a, S) \in \mathcal{U}^h$ if and only if there exist a finite number of strictly positive vectors and numbers $(p_k, b_k)_k$ where $p_k \in \mathbb{R}^m_{++}$ and $b_k > 0$ for all k such that

$$S = \bigcap_k \{x \in \mathbb{R}^n | p_i \cdot x \leq p_i \cdot a + b_i \text{ and } x \geq a\}.$$

For all bargaining problems $(a, S) \in \mathcal{U}^h$ and directions $v \in \mathbb{R}_{++}$, the reference point is Pareto optimal because $\mathcal{U}^h \subset \mathcal{U}^c$ so $s_v(a, S) \in E(S)$.

Efficient proportional solutions can be characterized by four axioms: Pareto Optimality, Restricted Scale Invariance (that is scale invariance restricted to problems in $\mathcal{U}^h$), Positive Directions (that is the solution must improve upon the threat point in all directions) and Restricted Monotonicity (that is monotonicity restricted to comparing pairs of problems where one problem is in $\mathcal{U}^h$ and the other problem is in $\mathcal{U}$).

Axiom 1 *(Pareto Optimality) $f(a, S) \in E(S)$ for all $(a, S) \in \mathcal{U}$.*

Axiom 2 *(Restricted Scale Invariance) $\alpha(f(a, S)) = f(\alpha(a), \alpha(S))$ for all strictly increasing maps $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ where $\alpha_j(x) = \gamma x_j + \delta_j$ for all $(a, S) \in \mathcal{U}^h$.*

Axiom 3 *(Positive Directions) $f(a, S) - a \in \mathbb{R}^n_{++}$ for all $(a, S) \in \mathcal{U}$.*

Axiom 4 *(Restricted Monotonicity) $f(a, S) \leq f(a, S')$ for all $(a, S) \in \mathcal{U}^h$ and $(a, S') \in \mathcal{U}$ where $S \subset S' - \mathbb{R}^n_+$.*

The main result of the present paper is the following characterization of the family of efficient proportional solutions.

Theorem 1 *A bargaining solution is an efficient proportional bargaining solution if and only if it satisfies Pareto Optimality, Restricted Scale Invariance, Positive Directions and Restricted Monotonicity.*

Proof: Clearly, for all $v \in \mathbb{R}_{++}$ the solution $f_v : \mathcal{U} \to \mathbb{R}^n$ satisfies Pareto Optimality, Restricted Scale Invariance, Positive Directions and Restricted Monotonicity. Next, we prove the converse.

Firstly, we show that if a solution satisfies Pareto Optimality, Restricted Scale Invariance, Positive Directions, and Restricted Monotonicity, then it is efficient proportional on the class $\mathcal{U}^h$. Secondly, we extend the result to the class $\mathcal{U}$.

Suppose that there exist two problems (a, S) and (a', S') (both in $\mathcal{U}^h$) and two directions v and v', where $v, v' \in \mathbb{R}^n_{++}$ and $v' \neq v$, such that $f(a, S) = s_v(a, S)$ and $f(a', S') = s_{v'}(a', S')$.

Let an increasing affine map $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ where $\alpha_j(x) = \gamma x_j + \delta_j$ for $\gamma > 0$ be defined by $\alpha(a) = a'$ and $\alpha(s_v(a, S)) \in E(S')$. Then $s_v(a', S') = \alpha(s_v(a, S))$ and $f(\alpha(a), \alpha(S)) = \alpha(f(a, S))$ according to Restricted Scale Invariance so $f(\alpha(a), \alpha(S)) = s_v(a', S')$. Next, let $S'' = S' \cap \alpha(S)$, then $(a', S'') \in \mathcal{U}^h$ so $f(a', S'') = s_v(a', S')$ according to Pareto Optimality and Restricted Monotonicity applied to the problems $(a', \alpha(S))$ and (a', S''). However $f(a', S') \geq f(a', S'')$ according to Restricted Monotonicity applied to the problems (a', S') and (a', S''), but this contradicts that $f(a', S') = s_{v'}(a', S')$ and $f(a', S'') = s_v(a', S')$, because $s_{v'}(a', S'), s_v(a', S') \in E(S')$. Therefore, we conclude that there exists a unique direction $v \in \mathbb{R}^n_{++}$.

Suppose that there exists $(a, S) \in \mathcal{U} \setminus \mathcal{U}^h$ such that $f(a, S) \notin \{x \in E(S) | x \geq s_v(a, S)\}$. Then there exist $t \in]0, 1[$ and $j \in \{1, \ldots, n\}$ such that $(1-t)a^j + t s_v^j(a, S) > f^j(a, S)$. Let $e_j \in \mathbb{R}^n$ be the vector, where the j'th coordinate is one and all other coordinates are zero, and let $\mathbf{1} \in \mathbb{R}^n$ be the vector, where all coordinates are one. For $\varepsilon > 0$ such that

$$\max_k \{s_v^k(a, S) - a^k\} \ \geq \varepsilon \frac{t}{1-t} \sum_j (s_v^j(a, S) - a^j)$$

let S_j defined by

$$S_j \ = \ \{x | (e_j + \varepsilon \mathbf{1}) \cdot (x - (1-t)a - t s_v(a, S)) \leq 0 \text{ and } x \geq a\}.$$

Then $x^j \leq s_v^j(a, S)$ for all $x \in S_j$. Let $S' = \cap_j S_j$, then $(a, S') \in \mathcal{U}^h$ so $f(a, S') = (1-t)a + t s_v(a, S)$. Moreover $S' \subset S$ because $x \leq s_v(a, S)$ for all $x \in S'$. Therefore Restricted Monotonicity is violated for the problems (a, S) and (a, S').

Q.E.D

The following four examples will demonstrate logical independence of the above axioms.

Example 1 For $t \in]0,1[$ solution $f(a,S) = (1-t)a + ts_v(a,S)$ satisfies Restricted Scale Invariance, Positive Directions and Restricted Monotonicity, but not Pareto Optimality.

Example 2 Let $\theta(t)$ be a path in t starting in 0 where $\theta^j(t') > \theta^j(t)$ for all $t' > t$ such that $\|\theta(\infty)\| = \infty$. Moreover, let $t(a,S) = \sup\{t | a + \theta(t) \in S\}$. Then for $(a,S) \in \mathcal{U}^h$ the solution $f(a,S) = \theta(t(a,S))$ and otherwise $f(a,S) \in \{y \in E(S) | y \geq \theta(t(a,S))\}$. This solution satisfies Pareto Optimality, Positive Directions and Restricted Monotonicity, but not Restricted Scale Invariance.

Example 3 The solution $f(a,S) \in \arg\max\{\prod_{j=1}^{n}(x_j - a_j) | x \in D(a,S)\}$ satisfies Pareto Optimality, Restricted Scale Invariance, Positive Directions but not Restricted Monotonicity.

Example 4 The solution $f_v(a,S)$ where $v = e_j$ satisfies Pareto Optimality, Restricted Scale Invariance and Restricted Monotonicity. but not Positive Directions.

Remark An appropriate generalization of the egalitarian solution is singled out by replacing the axiom of positive directions with an axiom of restricted symmetry: *(Restricted Symmetry)* if $a = 0$ and $S = \{x | \mathbf{1} \cdot x \leq 1\}$, then $f_i(a,S) = 1/n$ for all i.

4 Final remarks

Axiomatic characterization of bargaining solutions may alternatively be interpreted as characterizations of benchmark selections within a production economic framework, see e.g. [6]. The family of Kalai solutions characterized in the present paper resembles selections by the so-called directional distance functions introduced in [11], see e.g. also [2]. Our axioms all have natural interpretations in this respect. Pareto Optimality is equivalent to a requirement of technical efficiency. Scale Invariance is obviously also relevant in production space. Positive Direction implies that the benchmark should be strictly better in all production factors and Restricted Monotonicity may refer to monotonicity in production sets. The present characterization therefore also characterizes benchmark selection by the directional distance function on non-convex production sets with the obvious changes in the modeling framework.

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