MODELING THE DEMAND FOR DURABLE INPUTS: DISTRIBUTED LAGS AND CAUSALITY

H. W. Mui, G. L. Bradford, and M. M. Ali

Abstract

Vector-autoregressive-moving-average (VARMA) modeling was used to identify distributed lag relationships among farm tractor derived demand variables and to provide a basis for formally testing the hypothesis that the price of new tractor horsepower is exogeneous to its quantity demanded. Similar causality tests were used for a number of other explanatory variables, including the interest rate, price of diesel fuel, and price of used tractors. Results indicate that several lagged variables are significant causal factors and that the dynamic nature of the demand structure cannot be ignored when explaining tractor demand.

Key words: distributed lags, causality, farm tractors.

Durable inputs, such as farm tractors, usually play double roles in farming, as current productive resources and as relatively long-lasting capital goods. Since farm tractors are factors of production, changes in their aggregate quantity demanded over time affect aggregate farm output levels, production costs, output prices, and aggregate farm income. Changes in aggregate output, costs, income, and prices feed back to aggregate demand functions. Furthermore, since farm tractors are durable capital goods, the effects may also be in the form of distributed lags. In short, basic economic logic points toward feedback and distributed lag relationships among relevant variables which may be very complex.

Distributed lags and feedback are problems which must inevitably be addressed if appropriate models are to be structured for durable inputs. However, authors of most previous tractor demand studies (Griliches; Heady and Tweeten; Hughes and Penson; Conley and Lambert) have adopted a single equation approach and have either ignored the problem of distributed lags or have arbitrarily introduced only some one-period lag variables.

These studies attempted to justify, or at least rationalize, the *a priori* adoption of a partial equilibrium approach for studying aggregate demand on the basis that data for important supply side variables are unavailable. This is true and serves as a practical convenience. However, it is not necessarily a sound justification. Unless one can show the lines of "causality" run only from the tractor price and other input and output price variables to the aggregate quantity demanded, there is no conclusive basis for adopting a single equation demand model.

The feedback and distributed lag relationships among economic variables are often complex and economic theory is of little help to specify such relationships. A number of recently developed time series modeling techniques seem to provide some promising alternatives. Some of these have been applied to a number of macroeconomic agricultural problems. Studies by Bessler and Schrader, Weaver, Bessler (1980 and 1984), Bessler and Brandt, and Barnett et al. are prominent examples. However, the time series modeling techniques in these research works, with the exception of Bessler's study (1984), are used only to the extent of bivariate models. Such modeling techniques rely heavily on the analysis of cross-correlation of the time series involved. In the presence of a background variable influencing these time series varia-

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H. W. Mui is an Economist, China State Farm Agribusiness Corporation, Xisi, Beijing; G. L. Bradford is a Professor, Agricultural Economics; and M. M. Ali is a Professor, Economics, University of Kentucky.

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bles, the cross-correlations are not interpretable and misleading conclusions can be reached.

This paper examines a method for identifying distributed lag structures among the variables of the derived demand for farm tractors (measured in terms of total tractor horsepower) by applying vector autoregressive-moving-average (VARMA) modeling techniques. Secondly, the paper shows how to formally test the unidirectional causality hypotheses; that is, how to test for the absence of feedback from tractor prices and other input and output prices with respect to the quantity demanded for total tractor horsepower by using a parametric test with the identified VARMA model. In the first section, the methods and methodology of VARMA modeling techniques for identifying the lag structures and a parametric test of unidirectional causality are delineated. Then, these methods and hypothesis tests are applied to a derived input demand model of farm tractors. Finally, implications of the methods and results are discussed.

METHODS AND METHODOLOGY

Sims (1974 and 1980) contended that, under fairly general conditions, economic theory should only be used to the extent of choosing relevant variables in modeling. He suggested estimation of unconstrained vector autoregressive (VAR) models by treating all variables as endogenous at the first stage, in order to avoid infecting the model with spurious or false restrictions, and then formulating and testing hypotheses with economic content at the second stage. It is the first stage VAR model from which the distributed lag structure can be obtained. The hypothesis regarding causality can be tested at the second stage. A recent application of this approach was presented by Bessler (1984). Using a VAR model, he investigated the dynamic relationships between monthly observations on money supply, agricultural prices, and industrial prices in Brazil over the period 1964-81. A methodology to build such models also can be found in this study.

The VARMA models are generalizations of the VAR models. In particular, if the process is invertible, it can be represented by an autoregressive process and the implications of lag structure and causality from such models are easily derived. It has been argued (Tiao and Box, p. 807) that an adequate VARMA model can be parsimonious in that the number of parameters of the model is fewer than that of a similarly effective VAR model. This is a big advantage when the data are scarce. The following presents methodologies to build and draw implications from VARMA models in the context of an input demand model.

Consider a simple derived input demand function obtained from profit maximization subject to a technology constraint:

$$(1)\ X_1 = f(P_1, P_2, P_Y),$$

where X_1 is the quantity demanded for the input and P_1, P_2, P_Y are prices of input 1, input 2, and output, respectively. Thus, economic theory identifies four variables (X_1, P_1, P_2, P_Y) as constituting the demand function. There are other variables which are related to X_1 through the supply function of input 1 and demand and supply functions of input 2 as well as of the output. However, these relationships are indirect and such variables may be ignored at least at the preliminary stage of an analysis. Sim's (1974 and 1980) arguments suggest that the variables X_1, P_1, P_2, and P_Y are jointly determined. Let the vector W_t be expressed as:

$$(2)\ W_t = (P_{1t}, P_{2t}, P_{Yt}, X_{1t})',$$

for $t = 1, 2, ..., n$ (the number of periods in the time series). Suppose W_t is covariance stationary. Then, W_t can be described by a vector autoregressive-moving-average model of order p, q (VARMA (p, q)) in the form of:

$$(3)\ \varphi(B)W_t = \theta_0 + \theta(B)\ a_t,$$

where $a_t = (a_{1t}, a_{2t}, a_{3t}, a_{4t})'$ is a 4x1 column vector of white noise processes (identically, independently distributed random shocks), with a mean of a vector of zeroes and a covariance matrix; $\varphi(B) = I - \varphi_1 B - \varphi_2 B^2 - ... - \varphi_p B^p$ and $\theta(B) = I - \theta_1 B - \varphi_2 B^2 - ... - \theta_q B^q$. Both $\varphi(B)$ and $\theta(B)$ are 4x4 matrices of polynomials in the backshift operator B (the backshift operator B is defined so that $BX_t = X_{t-1}$, $B^2 X_t = X_{t-2}$, etc.); each of I; $\varphi_1, ..., \varphi_p$; and $\theta_1, \theta_2, ..., \theta_q$, is of order 4x4. When $q = 0$, this is a VAR model of order p (VAR(p)); and when $p = 0$, this is a VMA model of order q (VMA(q)).

Suppose the model for W_t is identified to be following a VAR process of the second order, i.e., $p = 2$ and $q = 0$ (see the Appendix for tools of identification). The VARMA (2, 0) model can be expressed as:

$$(4)\ (I - \varphi_1 B - \varphi_2 B^2)\, W_t = a_t,$$

or

$$(5)\ W_t = \varphi_1 W_{t-1} + \varphi_2 W_{t-2} + a_t,$$

which, if expanded, is written as:

$$(6)\ P_{1t} = \varphi_{111} P_{1,t-1} + \varphi_{112} P_{2,t-1} + \ldots + \varphi_{214} X_{1,t-2} + a_{1t},$$

$$(7)\ P_{2t} = \varphi_{121} P_{1,t-1} + \varphi_{122} P_{2,t-1} + \ldots + \varphi_{224} X_{1,t-2} + a_{2t},$$

$$(8)\ P_{yt} = \varphi_{131} P_{1,t-1} + \varphi_{132} P_{2,t-1} + \ldots + \varphi_{234} X_{1,t-2} + a_{3t},$$

$$(9)\ X_{1t} = \varphi_{141} P_{1,t-1} + \varphi_{142} P_{2,t-1} + \ldots + \varphi_{244} X_{1,t-2} + a_{4t}.$$

The lag structure of the input demand model can be obtained from equation (9) of the so-called first stage result.

There are various definitions of causality and none is free from pitfalls (Zellner). The primary one used in practice is by Granger. Granger's notion has some attractive implications, such as (a) it is consistent with the notion of econometric exogeneity (Sims, 1972) and (b) it is closely related to an accepted notion of lead-lag indicators and rational expectations (Pierce). However, one should realize that this is a statistics-oriented notion rather than an economic one; it is based solely on an incremental predictability criterion.

There are alternative ways to test Granger-type causality (e.g., Sims, 1972, and Geweke). However, the one which seems best suited for VAR models is the parametric test of Granger which was generalized to VMA models by Sims and by Pierce and Haugh. This can be adapted to VARMA models as shown by Kang. Specifics of this parametric test are as follows: let $\varphi_{ij}(B)$ and $\theta_{ij}(B)$ be the (i,j)th polynomial element of the matrix $\varphi(B)$ and $\theta(B)$ of equation (3), respectively. Then, a sufficient condition that the variable j does not cause variable i is that $\varphi_{ij}(B) = \theta_{ij}(B) = 0$. In the VAR or VMA models, this condition is also necessary. Thus, in equation (4), P_1 does not cause X_1 if and only if $\varphi_{41}(B) = 0$; i.e., $\varphi_{141} = \varphi_{241} = 0$. The hypothesis test of whether $\varphi_{41}(B)$ equals zero can be performed by a likelihood ratio test. In this test, it is assumed that the a_t's are normally distributed. Let the null hypothesis and the alternative hypothesis be:

$$H_0 : \varphi_{41}(B) = 0$$

and

$$H_A : \varphi_{41}(B) \neq 0.$$

In essence, one takes the VAR model, equation (4), as the unconstrained model, obtains the maximum of the unconstrained log likelihood function, L_u, sets $\varphi_{41}(B)$ equal to zero, then considers the resulting model as the constrained model and obtains the maximum of the constrained log likelihood function, L_c. The test statistic is $2(L_u - L_c)$. The null hypothesis is rejected if this statistic is larger than a critical value fixed by the choice of a level of significance. The distribution of $2(L_u - L_c)$ is approximately a χ^2, with degrees of freedom equal to the number of constraints (Silvey, pp. 113-4).

To test whether the own price (P_1) is exogenous to the quantity demanded (X_1), i.e., whether P_1 is unidirectionally causing X_1 and has no feedback, the hypothesis test of whether $\varphi_{14}(B)$ equals zero is performed in addition to the above hypothesis test of $\varphi_{41}(B) = 0$. If it is found that $\varphi_{41}(B) \neq 0$ and $\varphi_{14}(B) = 0$, then one may conclude that P_1 is unidirectionally causing X_1.

APPLICATION EXAMPLE

The United States demand for new farm tractors can be conceptualized using comparative static theory of the firm. Since farm tractors are made primarily to provide mechanical power in agricultural production, the derived demand for tractors can be approximated by the derived demand for tractor horsepower.

The aggregate input demand function for tractor horsepower can be derived from a neoclassical optimization approach by hypothesizing that: (a) American farmers are profit maximizers; (b) the quantity of tractor horsepower is a function of not only the purchase price of tractors but also of the rate of interest (cost of capital) and the price of diesel fuel; (c) input prices are not normalized by the output price (crop prices), i.e., it is the absolute price that matters; and (d) that used tractor horsepower is distinguishable from new tractor horsepower, or it is separable from "other production items." The resultant input demand function is:

$$(10)\ X1 = f(P1, IR, PD, P2, P3, P4, PY),$$

where X1 is the quantity of new tractor horsepower, P1 is the price per unit of new tractor horsepower, IR is the rate of interest (measured by 6-month commercial paper), PD is the price index of diesel fuel, P2 is the farm wage rate for field workers, P3 is the price index of other inputs, P4 is the price per unit of used tractor horsepower, and PY is the price index of crops.

Monthly time series data for X1 for 1973-82 were obtained through the Farm and Industrial Equipment Institute (FIEI). These data are not published.[1] Data on IR were obtained from the Board of Governors, Federal Reserve System (rates for 6-month commercial paper were used). Data on P4 were obtained from various issues of *Implement and Tractor (Intertec Publishing Corporation)*. Data on P1, PD, P2, P3, and PY were obtained from appropriate U.S. Bureau of the Census and USDA periodicals.

With this input demand function, the identified VARMA model is:

$$(11)\ (I - \varphi_1 B - \varphi_2 B^2 - \varphi_6 B^6)(I - \varphi_{12} B^{12} - \varphi_{13} B^{13} - \varphi_{14} B^{14})\overline{W}_t = a_t,$$

where $\overline{W}_t = (\overline{P1}_t, \overline{IR}_t, \overline{PD}_t, \overline{P2}_t, \overline{P3}_t, \overline{P4}_t, \overline{PY}_t, \overline{X1}_t)'$, the φ's are matrices of order 8x8, and $a_t = (a_{1t}, a_{2t}, \ldots, a_{8t})'$. Component variables of $\overline{W}_t$ are the original variables in W_t after transformations, so that $\overline{W}_t$ is stationary. Thus, $\overline{P1}_t$, $\overline{IR}_t$, $\overline{PD}_t$, $\overline{P2}_t$, $\overline{P3}_t$, $\overline{P4}_t$, $\overline{PY}_t$, and $\overline{X1}_t$ stand for $(1-B)\mathrm{LnP1}_t$, $(1-B)\mathrm{LnIR}_t$, $(1-B)^2 PD_t$, $(1-B)^2(1-B^{12})P2_t$, $(1-B)(1-B^{12})P3_t$, $(1-B)\mathrm{LnP4}_t$, $(1-B)^2(1-B^{12})PY_t$ and $(1-B)(1-B^{12})X1_t$, respectively. Note, that $\varphi_3, \varphi_4, \varphi_5, \varphi_7, \ldots, \varphi_{11}$ are not included in equation (11) because they were not found to be significant. Equation (11) was estimated by the likelihood method (Tiao and Box). Cross-correlations for the residuals from the fitted model were found to be insignificant. The white noise property of the residuals justifies the adequacy of the model.

There are eight equations in the system corresponding to eight component variables in $\overline{W}_t$. Each equation describes the lag structure determining a particular variable. The estimated equation for $\overline{XI}_t$, which is the last equation in the system is (t-values shown in parentheses):

$$
\begin{aligned}
(12)\ \overline{X1}_t = &- \underset{(-3.95)}{.888}\ \overline{IR}_{t-1} + \underset{(1.97)}{1.099}\ \overline{P2}_{t-1} \\
&- \underset{(-6.12)}{.487}\ \overline{X1}_{t-1} + \underset{(3.07)}{1.674}\overline{P2}_{t-2} \\
&- \underset{(-2.79)}{.468}\ \overline{P4}_{t-6} - \underset{(-2.76)}{1.067}\ \overline{P1}_{t-12} \\
&- \underset{(-1.80)}{9.21}\ \overline{P2}_{t-12} + \underset{(3.21)}{.675}\ \overline{IR}_{t-13} \\
&+ \underset{(4.06)}{2.126}\ \overline{P2}_{t-13} + \underset{(2.83)}{1.436}\ \overline{P2}_{t-14}.
\end{aligned}
$$

This equation is in transformed variables. When these variables are expressed in terms of their original forms, one obtains the variables with their lag structures determining X1. In this case, it can be achieved in two steps. First, replace each transformed variable by its relation with the original form. Thus, replace $\overline{X1}_t$ by $(1 - B)(1 - B^{12})\ X1_t$, $\overline{P1}_t$ by $(1 - B)\ \mathrm{LnP1}_t$, etc. Second, divide both sides of the equation by $(1 - B)(1 - B^{12})$. When this is done, it can be seen that only variable P2 has lags of finite lengths and all other variables (IR, P1, and P4) have lags of infinite lengths in determining X1. The longest lag of P2 is fifteen (15) and the significant lags are 1, 2, 3, 12, 13, 14, and 15.

The next step is to determine, through appropriate causality tests, whether or not lagged P1, IR, P2, P4, and XI are causal factors for X1. The lagged P3, PD, and PY do not appear in equation (12) and hence cannot be causal factors to X1.

Results of these hypothesis tests are summarized in Table 1. Test statistics of lagged P1 (price of new tractor horsepower), lagged P2 (farm wage rate), lagged P4 (price of used tractor horsepower), lagged IR (interest rate), and lagged X1 (quantity of new tractor horsepower) are significant at the 1 percent level. This means the lagged variables are significant causal factors. However, lagged Xl does not appear in the Pl equation, which means lagged Pl is exogenous to Xl, or it is unidirectionally "causing" Xl. Consequently, it seems doubtful that the direction of causality of Pl and Xl is other than unidirectional.

CONCLUDING REMARKS

As indicated by the estimated distributed lag system of the derived input demand for tractor horsepower, the rate of interest (IR), farm wage rate (P2), new tractor price (P1), used tractor price (P4), and lagged quantity demanded (X1) are shown to have distributed-lag effects on the current demand for tractor horsepower. The range of effects goes from a lag of finite length (15 periods) to a lag of infinite length. Such results demonstrate that there is no way to hypothesize the lag lengths in a definitive manner. The in-

[1] Obviously, more recent monthly data could alter parameter estimates, lag lengths, and test results.

TABLE 1. LIKELIHOOD RATIO TESTS TO DETECT CAUSALITY FROM LAGGED P1, P2, P4, IR, AND X1[a]

	X1 caused by				
Item[a]	Lagged P1	Lagged P2	Lagged P4	Lagged IR	Lagged X1
L_u	−63.4472	−63.4472	−63.4472	−63.4472	−63.4472
L_c	−66.9924[b]	−75.4450[b]	−67.0083[b]	−75.6466[b]	−76.9932[b]
$2(L_u - L_c)$	7.0304	23.9556	7.0633	24.3388	27.0320
d.f.	1	5	1	2	1
$\chi^2_{.05}$	3.84	11.07	3.84	5.99	3.84
$\chi^2_{.01}$	6.63	15.09	6.63	9.21	6.63

[a] As given in equation (10), P1 is the price per unit of new tractor horsepower, P2 is the farm wage rate, P4 is the price per unit of used tractor horsepower, IR is the rate of interest, and X1 is the quantity of new tractor horsepower. All results are based on monthly time series data for the U.S. for 1973-1982. L_u is the maximum value of the log likelihood function of the unconstrained model and L_c is the maximum value of the log likelihood function when the parameters are constrained by H_0.
[b] Significant at the 1 percent level.

formation of the lag structure can be used to build an appropriate structural econometric model for input demand (for more detail, see Mui). However, care must be exercised in interpreting these findings. All the relevant variables may not have been included in the analysis. If there is an omitted variable which has a significant influence on all or a subset of the variables included, the lag structure can be distorted.

The causality test of exogeneity of tractor price with respect to quantity demand for tractor horsepower has provided a positive answer, at least for the lagged variables, about the unidirectional causality of the new tractor price to the quantity demanded, and not conversely. If the contemporaneous tractor price can be shown further to be exogenous, the input demand model may be built independent of its price determining equation. Otherwise, a simultaneous system would have to be structured.

These modeling techniques can be applied to a variety of durable inputs. When there is suspicion of any kind of distributed-lag effects, VARMA modeling techniques should be applied before actually building a structured econometric model. Similarly, when a partial equilibrium approach is attempted to model a demand phenomenon of durable inputs, a causality test of exogeneity of the input price (and other relevant variables) with the identified VARMA model should be performed before a single equation demand model is developed and estimated.

Comparative static theory of the firm obviously is limited when applied to the study of demand for durable inputs. However, certain drawbacks can be overcome when the theory is applied in combination with VARMA modeling techniques. Static theory is used, in this study, only to the extent of selecting variables. VARMA modeling techniques are used to identify specific features of lag structures. There is no reason why an investment modeling approach cannot be used instead of neoclassical static theory; for example, use the dynamic optimization approach commonly associated with Jorgenson. Even so, if the researcher is trying to realistically estimate the structure (and parameters) of an aggregate durable input demand function, causality and lag distribution specification problems are not surmounted simply by relying upon capital investment theory.

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APPENDIX

For a stationary series W_t which follows VARMA (p,q), the following tools can be used to determine appropriate values for p and q (Tiao and Box).

Cross-Correlation: Lag k cross-covariance matrix is defined by $E(W_t W'_{t+k}) = [\gamma_{ij}(k)]$, from which the lag k cross-correlation matrix, $R(k) = [\rho_{ij}(k)]$ where $\rho_{ij}(k) = \gamma_{ij}(k)/[\gamma_{ii}(0)\gamma_{jj}(0)]^{1/2}$, is obtained. If W_t follows VMA(q) = VARMA(0, q), then $R(k) \neq 0$ for $k = q$ and possibly some $k < q$ and $= 0$ for all $k > q$. The sample analog of lag k cross-correlation $r_{ij}(k)$ is a consistent estimate of $\rho_{ij}(k)$ and it is asymptotically normally distributed with appropriate variance $= 1/n$, where n is the sample size. Thus, to test $\rho_{ij} = 0$ for all i, j, one computes $(n)^{1/2}r_{ij}$ which is a standard normal variable, and declares $\rho_{ij} \neq 0$ if $|(n)^{1/2}r_{ij}|$ exceeds the chosen critical value. If $\rho_{ij}(k) = 0$ for all ij, then $R(k)$ is declared to be 0. By testing $R(k) = 0$ for $k = 1, 2, \ldots$ successively, one can determine q.

Partial Autoregression: Consider a regression of W_t on $W_{t-1}, \ldots, W_{t-k}$ with the regression coefficient matrices P(1), P(2), ..., P(k), respectively. If W_t follows VAR (p) = VARMA (p, 0), then $P(k) \neq 0$ for k = p and possibly some k < p and = 0 for all k > p. Thus, one may test P(k) = 0 successively for k = 1, 2, ... to determine the order p. A test of this hypothesis is explained in the following paragraph.

Let e_t (k) be the residual vector when W_t is regressed on $W_{t-1}, \ldots, W_{t-k}$. Let the matrix of residual sum of squares and cross products be:

$$(13)\quad S(k) = \sum_{t=k+1}^{n} e_t(k)e'_t(k).$$

Define $M(k) = -(n - 1/2 - km) \operatorname{Ln} (|S(k)|/|S(k-1)|)$; m is the number of component series. The likelihood ratio statistic M(k) can be used to test for P(k) = 0. Under the null hypothesis, M(k) is asymptotically distributed as χ^2 with m^2 degrees of freedom. This is the test suggested by Tiao and Box.

To find the order p, one may also examine the diagonal elements of S(k)/n which are estimates of the error variances of the component time series. These estimates are expected to decline as k increases from 0 to p and become stable as k exceeds p.

In short, when W_t follows a pure process (VAR or VMA), one examines R(1), R(2), ..., and P(1), P(2), ... to check when they become zero. The cut-off points of R's and P's determine q and p, respectively. Both the significance (or the insignificance) of the M(k) statistics and the stability of the residual variances (diagonal elements of S(k/n)) help to find the cut-off point of P's.

When W_t follows the mixed process, VARMA (p,q), the cut-off property of R's and P's is lost. However, when one regresses W_t on $W_{t-1}, \ldots, W_{t-k}$, the R's for the residual series are expected to have the cut-off property, cutting off at lag q, if $k \geq p$. This helps to determine both p and q.

For illustrative purposes, let W_t follow VARMA (2, 1); i.e., $W_t = \varphi_1 W_{t-1} + \varphi_2 W_{t-2} + a_t - \theta_1 a_{t-1}$. Then, in a regression of W_t on W_{t-1}, the residual is:

$$u_t = \varphi_2 W_{t-2} + a_t - \theta_1 a_{t-1}.$$

As W_t follows VARMA (2,1), it can show that W_{t-2} is a linear function of $a_{t-2}, a_{t-3}, \ldots$ Thus, u_t follows an infinite order moving average process. Hence, the cut-off property of the cross-correlation R's for u_t is lost. However, in a regression of W_t on W_{t-1} and W_{t-2} or, in general, of W_t on $W_{t-1}, \ldots, W_{t-k}$, $k \geq 2$, (here p = 2), the residual $u_t = a_t - \theta_1 a_{t-1}$. This u_t follows a VMA (1) and its cross-correlation R's will cut-off at lag 1. Note that in a regression of W_t on $W_{t-1}, \ldots, W_{t-k}$, the coefficient of W_{t-j}, $j \geq 2$, is zero. In practice, $u_{t's}$ are estimated and hence the above arguments have to be qualified.