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A FINITE-LIFE PRIVATE-INFORMATION THEORY OF UNSECURED CONSUMER DEBT

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Abstract

We present a theory of unsecured consumer debt that does not rely on utility costs of default or on enforcement mechanisms that arise in repeated-interaction settings. The theory is based on private information about a person's type and on a person's incentive to signal his type to entities other than creditors. Specifically, debtors signal their low-risk status to insurers by avoiding default in credit markets. The signal is credible because in equilibrium people who repay are more likely to be the low-risk type and so receive better insurance terms. We explore two different mechanisms through which repayment behavior in the credit market can be positively correlated with low-risk status in the insurance market. Our theory is motivated in part by some facts regarding the role of credit scores in consumer credit and auto insurance markets.

Key Words: Unsecured Consumer Debt, Bankruptcy, Default, Adverse Selection, Credit Score, Insurance

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1 Introduction

The question we address in this paper is: how can *unsecured* consumer debt coexist with debtor's right to invoke bankruptcy? We propose a theory of unsecured consumer debt that is based on the existence of private information about a person's *type* and on the fact that some debtors have the incentive to forgo bankruptcy in order to *signal* their type. The theory formalizes the notion that a person's type may be relevant to different trading partners and if a person has some advantage in resisting opportunistic behavior against one trading partner, he may credibly signal information about his type to other trading partners by doing so.

Our theory of unsecured consumer debt is distinct from some other approaches to explaining debt when enforcement is imperfect. Our model has finite-lived people to whom the opportunity to borrow and repay is presented only *once*. So our theory does not rely on enforcement mechanisms that arise in repeated-interaction settings. Also, our theory does not depend on any utility cost of failing to honor debt contracts – there is no stigma attached to default.

There are three periods and two markets in the model. There is an asset market that links periods 1 and 2, and a period 3 insurance market that provides insurance against an observable loss. People in our model differ with respect to the likelihood of this loss – some are more likely to suffer the loss than others. But insurers do not directly observe the risk status of people, and so the low-risk types have an incentive to signal their type. This signalling can take two forms: accepting limited insurance in the insurance market for a lower premium and resisting opportunistic behavior in the credit market, i.e., not invoking the right to default in the credit market. Crucially, repayment of debt is a credible signal of low-risk status because low-risk people have an advantage in resisting opportunistic behavior in the credit market, and therefore, in equilibrium those who repay are more likely to be the low-risk type.

We explore two distinct reasons why low-risk people may have an advantage in resisting opportunistic behavior in credit markets. In the first case, we assume that people who are low-risk are also more patient. Then repayment is a credible signal of low-risk status because the benefit of a lower insurance premium comes in the future – in period 3 – and the low-risk type is also the type who values the future more. Put differently, the high-risk type does not mimic the low-risk type and does not send the same signal (i.e., repay their debt) because they do not care enough about the future and so do not value the benefit of a lower insurance premium to the same extent.

In the second case, we assume that there is a difference in the stochastic environment facing the low-risk and high-risk types. In particular, we assume that the high-risk type is more likely to experience shocks that trigger default. Consequently, repayment of debt can act as a signal of low-risk status because the kinds of shocks that lead to default are less likely to be experienced by the low-risk types.

The two explanations can be differentiated as follows. In the first case repayment is a signal of low-risk status because the low-risk type is more willing to repay; in the second case repayment is a signal of low-risk status because the low-risk type is less likely to be in situations that trigger default. The first explanation relies on differences in behavior, while the second explanation relies on differences in luck.

Our theory of unsecured consumer debt is motivated by some features of the U.S. economy. Most important, reputation appears to play an important role in credit and insurance markets. In particular, both industries make use of credit scores – a summary measure of a person's creditworthiness (the likelihood that a person will repay his or her debt). In the credit market the following facts are well established: (i) people with higher scores obtain credit on cheaper terms, (ii) default lowers a person's credit score, and (iii) holding fixed a person's credit limit, more borrowing tends to lower a person's credit score.[1] In the insurance market (iv) people with higher scores tend to receive insurance at a lower price.[2] In

[1] On these points see `http://www.myfico.com/Downloads/Files/myFICO_CFA`

[2] Consumers with low credit scores pay 20 to 50 percent more in auto insurance premiums than consumers with high credit scores. And two-thirds of policy holders have lower premiums because of their high credit scores. On this point, see `http://www.bankrate.com/brm/news/insurance/credit-scores1.asp`

turn people with high credit scores default less frequently and file fewer insurance claims. The key contribution of this paper is to present a theory of unsecured consumer debt in which something like a credit score emerges endogenously and which is consistent with facts (i)-(iv).

In the theory presented in this paper, the insurance benefit of a good credit-market reputation is the sole reason people repay their debt. In other words, no credit is extended to any person if all types are equally risky with respect to the insurable loss, since there is no reward to curtailing opportunistic behavior in the credit market. Clearly, this is a simplification – people may wish to maintain a good reputation in the credit market in order to maintain access to credit on cheap terms in the future. In a companion paper (Chatterjee *et al.* [4]) we study an infinite horizon environment where this particular benefit of good credit market behavior is analyzed.

The theory of debt presented in this paper is a reputation-based theory that employs an "adverse selection approach" wherein creditors attempt to learn about the borrower's type.[3] In this it bears a resemblance to the reputation-based model of sovereign debt presented in Cole and Kehoe [5]. However, both the motivation and details in Cole and Kehoe are different from ours. Cole and Kehoe's important insight was a response to Bulow and Rogoff's [2] criticism of models of sovereign debt based on exclusion costs of default. Bulow and Rogoff showed that if a country can default and simultaneously purchase consumption insurance out of its resulting savings, exclusion costs of default have no bite and equilibria with positive debt will fail to exist.[4] Cole and Kehoe wrote down a model where there are two types of sovereigns and one type suffers a large disutility of reneging on a debt contract – a disutility

[3]An alternative repeated-interaction approach assumes there is no hidden information about types but agents resist opportunistic behavior in order to avoid a less favorable continuation path in the game. In this approach, the punishment-triggering deviation is sometimes interpreted as "loss of one's reputation." See Chapter 15 of Mailath and Samuelson [9] for a discussion of the two approaches and for arguments in favor of modeling reputation using the adverse selection approach.

[4]Earlier work, such as the seminal study of Eaton and Gersovitz [6], had justified the assumption of exclusion from international borrowing *and* lending following default as resulting from a loss in "reputation" of the sovereign and Bulow and Rogoff's demonstration was, therefore, viewed as a criticism of reputation-based models of sovereign debt.

thought to come from default tarnishing the government's reputation in another market (e.g., the labor market). Even if type is not directly observable to lenders, the utility cost of a bad reputation can support positive amounts of (unsecured) debt if sovereigns are sufficiently patient.

The first point to note is that Bulow and Rogoff's criticism of exclusion-based models of sovereign debt are not germane to correctly specified exclusion-based models of unsecured *consumer* debt. U.S. bankruptcy law does *not* permit those invoking bankruptcy to simultaneously accumulate assets: a bankrupt must relinquish all (non-exempt) assets to creditors at the time that discharge of debt is granted by a bankruptcy court.[5] However, there is no restriction on accumulating assets *after* discharge has been granted. Thus the exclusion costs of U.S. consumer bankruptcy are the costs of not being able to save in the period of default and not being able to borrow for some time following default (these are the exclusion costs of default in Chatterjee *et al.* [3]). The fact that saving is not permitted in the period of default means that positive amounts of unsecured debt can be supported in equilibrium, so existence is not an issue. However, the fact that defaulters can save in the periods following default implies that the costs of default tend to be too low, which makes it challenging to match certain moments of the debt and default data in a plausibly calibrated model like that of Chatterjee *et al.* [3]. Part of our motivation in considering a model of debt with reputation effects is to explore costs of default that go beyond those of credit market exclusion.

A second difference between our paper and Cole and Kehoe's is that we are explicit about how default can lead to a cost in a market other than the credit market. As mentioned above, Cole and Kehoe assume that there is a utility cost to default, but they do not derive this utility cost from fundamentals. We do. Being explicit about the cost is important because alternative signalling devices may be used in this other market to circumvent the problem of hidden information about type. For instance, if insurers can figure out who they are dealing

[5] Some U.S. states (Texas and Florida) exempt home equity from bankruptcy proceedings. However, even in those states, diversion of assets into home equity during bankruptcy proceedings would be considered an abuse of the bankruptcy provision and would not be permitted.

with by using appropriately designed contracts, they need not care about their client's credit market behavior.

The paper is organized as follows. In section 2 we lay out the environment. In section 3 we go through the decision problem of each agent. In section 4 we give the definition of a competitive equilibrium and prove some basic properties. In section 5 we analyze the equilibrium of the period 3 insurance market and explain how the insurance contracts depend on the credit score. At the start of section 6 we point out that differences in loss probability across types and private information about types are essential for debt to be valued in our environment. Then, in section 6.1 we specialize the model to one where there are only 2 asset levels that people can choose from in period 1 and analyze the credit market equilibrium for the case where low- and high-risk types differ with respect to their discount factors. In section 6.2 we permit any finite number of asset level choices in period 1 and analyze the credit market equilibrium for the case where people differ in terms of the probability distributions from which idiosyncratic shocks are drawn. The key result here is that there exist distributions from which these shocks are drawn such that an equilibrium with positive debt characterized by properties (i)-(iv) exists. Section 7 concludes.

2 Environment

There is a single good. There are 3 periods, denoted $t = 1, 2,$ and 3 and a unit measure of people. We will describe people who live in this economy, the legal environment they face, the market arrangement, and the timing of events.

2.1 People

There are two types of people, denoted $i \in \{b, g\}$. The measure of type g is $0 < \gamma < 1$ and the measure of type b is $(1 - \gamma)$.

The endowment of each person is given by $e > 0$ and is constant over time. In period 3 a person of type i faces a probability $\pi^i > 0$ of experiencing a loss in wealth L, where L is a given number in $(0, e)$. For each type, the loss is an independent draw from the corresponding loss distribution. The loss incurred by a person is denoted by $z \in \{0, L\}$.

The preferences of each type is given by

$$E^i \left[\theta_1 u(c_1) + \beta^i \, \theta_2 u(c_2) + (\beta^i)^2 u(c_3)\right] \tag{1}$$

where c_t is consumption of the single good in period t and $\theta_t \in R_+$ is a period-t preference shock drawn independently for each person of type i from the probability space $\{R_+, \mathcal{B}(R_+), F_t^i\}$. The utility function $u(c) : R_+ \to R$ is strictly concave and twice continuously differentiable.

A person's type and preference shocks are not directly observable to others. The loss in period 3, however, is observable to others.

Types differ by the probability of the observable loss, by discount factors, and by the probability distribution of preference shocks. Throughout this study we assume that the probability of the observable loss is lower for type g than for type b, i.e., $0 < \pi^g < \pi^b < 1$. Since the type of a person is not directly observable to others, type g people have an incentive to signal their type to insurers. We are interested in situations where type g people attempt to signal their type to insurers by resisting opportunistic behavior in the credit market. For this signalling to be credible, the cost of any signal must be lower for type g than for type b (otherwise the type b people will always mimic the type g).

We explore two distinct ways in which the cost of signalling could be lower for type g than for type b. The first mechanism relies on differences in discount factors. In particular, type g people are assumed to be more patient than type b people, i.e., $\beta^b < \beta^g$. The second mechanism relies on differences in the distribution of preference shocks across types. In particular, the required differences are consistent with $F_2^b(\theta_2)$ first-order stochastic dominating $F_2^g(\theta_2)$, i.e., with $F_2^b(\theta_2) \leq F_2^g(\theta_2)$ for all $\theta_2 \geq 0$ (inequality holding strictly for some θ_2).

2.2 Legal Environment

A key feature of the environment is the existence of a bankruptcy law. This law gives people the right to disavow their financial obligations. As is generally true for actual bankruptcy law, this “right to bankruptcy” is assumed to be inalienable – meaning that a debtor cannot waive his or her right to bankruptcy at the time of taking out a loan. For simplicity, we assume that invoking the right to bankruptcy does not cost the debtor any fees or expenses.

2.3 Market Arrangements

There are two sets of markets. In one market people borrow from or lend to banks, and in the other market, people purchase insurance against the observable loss L. Since household type is private information, and type will matter for the propensity of a person to declare bankruptcy or suffer a loss, banks and insurance companies must make an assessment of a person's type when selling a loan or an insurance policy.[6] We will study a market structure that permits the terms of financial contracts to depend on such an assessment. In what follows, we will use σ as the generic symbol to denote the probability that a person is type g. Then σ is the assessment of a person's type or, more succinctly, a person's *type score*. A person's type score will evolve over time because a person's actions in the asset and insurance markets can (and will) reveal information about a person's true type. We imagine that there is an information processing agency (resembling real-world credit bureaus) that keeps track of a person's actions in the asset and insurance markets – i.e., keeps track of a person's financial history.

[6] If the preference shocks are correlated over time, banks may also have an incentive to form an assessment of the preference shock hitting a person because this information will be valuable in predicting future default. Since we have assumed that shocks are *i.i.d*, this assessment is not necessary.

The asset market operates only in period 1.[7] The asset market offers one-period bond contracts $y \in A$, where A is a finite subset of R. These bond contracts are offered at prices $q(y, \sigma)$. A person with type score σ who purchases the contract y pays $q(y, \sigma) \cdot y$ in period 1 and receives y in period 2. A positive y signifies a deposit and a negative y signifies a loan. If $y < 0$, the person promises to repay y in period 2 conditional on not declaring bankruptcy. Finally, we assume that banks have access to a world credit market in which they borrow or save at the interest rate r.[8]

The insurance market in period 3 operates as follows. Insurers offer contracts x in I, where I is a compact subset of R. A person with type score σ who purchases the contract x pays $p(x, \sigma) \cdot x$ as a premium and, in the event of loss, collects the indemnity x. This notation emphasizes the symmetry between the loan and insurance markets. Just as the probability of default on a loan will depend on the size of the loan, so too can the probability of loss on an insurance contract depend on the amount of insurance purchased – for the usual moral hazard reasons. However, in this paper we abstract from moral hazard issues – for each type of person, the probability of loss cannot be affected by any action that the person can take. Therefore, in equilibrium, the price of insurance will turn out to depend only on σ and not on x.

A financial firm takes the set of contracts and prices $\{q(y, \sigma), y \in A\}$ and $\{p(x, \sigma), x \in I\}$ as given. Any given contract is viewed as a distinct financial product. There is free-entry in the provision of each of these financial products. In equilibrium each of these financial products will fetch zero profits in expectation.

An important feature of the environment is the possibility that a person's actions in the asset and insurance markets may reveal information about a person's type. The possibility

[7] There is obviously no role for an asset market in period 3, since no one lives beyond period 3. In period 2 it is impossible to support borrowing if agents can default after their insurance choices. Hence the assumption that there is no loan market in period 2 is without loss of generality. For simplicity we also rule out savings in period 2.

[8] Alternatively, we could imagine that banks have access to a storage technology that allows them to transform 1 unit of output in period 1 into $(1+r)$ units of output in period 2. This would require incorporating the restriction that aggregate consumer assets cannot be negative.

of information transmission is captured by the following three *belief-updating* functions.

- The function $s = \Psi_1(\ell)$ gives the person's type score at the end of period 1 if the person chooses asset level ℓ.
- The function $s' = \Psi_2(d, \ell, \Psi_1(\ell))$ gives a person's type score at the end of period 2 if he starts period 2 with asset ℓ, type score $\Psi_1(\ell)$, and chooses a bankruptcy decision d ($d = 1$ means file for bankruptcy and $d = 0$ means no filing).
- The function $s'' = \Psi_3(x, \Psi_2(d, \ell, \Psi_1(\ell)))$ gives a person's type score if he starts period 3 with type score $\Psi_2(d, \ell, \Psi_1(\ell))$ and chooses an insurance level x.

All market participants take these functions and the sets A and I as given. Therefore, the prices faced by participants in the loan and insurance markets are $q(\ell, \Psi_1(\ell)), \ell \in A$ and $p(x, \Psi_3(x, \Psi_2(d, \ell, \Psi_1(\ell))), x \in I$.

2.4 The Timing of Events

The timing of events in each period is as follows.

People begin period 1 with type score γ and learn their true type i and the realization of the preference shock θ_1. They then choose how much to borrow or save from the set A. This choice is used to update a person's type score from γ to s. Then they consume and period 1 ends.

People begin period 2 with their type score s and learn the realization of their preference shock θ_2. If a person borrowed in period 1, the person then chooses whether or not to default. This choice is used to update a person's type score from s to s'. After the default decision is made they consume and period 2 comes to an end.

People begin the period with type score s' and purchase insurance from the set I. Their choice of insurance is used to update the type score to s''. Then the shock z is realized and the person receives the insurance payment if $z = L$. Then people consume and die.[9]

3 Decision Problems

In this section we describe the decision problem of people, insurers, and banks.

3.1 People

It's convenient to start with the final period and work backward.

3.1.1 Period 3

At the start of period 3 each person will have a type score s' that is determined by the person's choices in the asset market via the function $s' = \Psi_2(d, \ell, \Psi_1(\ell))$. Then, a type i person's insurance decision problem is as follows.

$$V_3^i(\Psi_2(d, \ell, \Psi_1(\ell))) = \max{}_x \, \pi^i u\,(e - p(x, s'') \cdot x - L + x) + (1 - \pi^i) u\,(e - p(x, s'') \cdot x)$$

s.t.

$$x \in I$$

$$s'' = \Psi_3(x, \Psi_2(d, \ell, \Psi_1(\ell)))$$

Observe that the set of insurance choices available to a person may be constrained by a person's beginning-of-period 3 type score $\Psi_2(d, \ell, \Psi_1(\ell))$. These constraints will be elaborated upon in the next section. We will denote a type i person's decision rule regarding insurance

[9] An alternative model where periods 2 and 3 are lumped together and the timing is a default decision followed by an insurance choice cannot support the type of equilibrium we describe in section 6.1 of the paper because differences in discount factors cannot then play the role they do.

purchase as $x^i(\Psi_2(d, \ell, \Psi_1(\ell)))$.

3.1.2 Period 2

People begin period 2 with some asset holding ℓ and a corresponding type score $\Psi_1(\ell)$. At the start of the period they learn their preference shock θ_2. If a person of type i is a debtor and chooses to default, i.e., $d = 1$, the person's utility is given by

$$V_2^{i,\, d=1}(\theta_2, \ell, \Psi_1(\ell)) = \theta_2 u(e) + \beta^i V_3^i(\Psi_2(1, \ell, \Psi_1(\ell))).$$

If the person chooses not to default (or the person is not a debtor), then the person's utility is given by

$$V_2^{i,\, d=0}(\theta_2, \ell, \Psi_1(\ell)) = \theta_2 u(e + \ell) + \beta^i V_3^i(\Psi_2(0, \ell, \Psi_1(\ell)))$$

Therefore,

$$V_2^i(\theta_2, \ell, \Psi_1(\ell)) = \max \{V_2^{i,\, 1}(\theta_2, \ell, \Psi_1(\ell)),\ V_2^{i,\, 0}(\theta_2, \ell, \Psi_1(\ell))\}. \tag{2}$$

If both options fetch the same utility, we assume that the person chooses $d = 0$.

If the person is not a debtor, then he has no choice to make – the person simply consumes $e + \ell$. However, it will be be convenient to assume that in this case his "choice" of d is also 0. With this convention, the choice of d is defined for all ℓ. We will denote a type i's period 2 decision rules as $d^i(\theta_2, \ell, \Psi_1(\ell))$.

3.1.3 Period 1

At the start of period 1 people learn their type i and their preference shock θ_1. The decision problem of a person of type i is then

$$V_1^i(\theta_1) = \max_\ell \theta_1 u(e - q(\ell, \Psi_1(\ell)) \cdot \ell) + \beta^i \int_{R_+} V_2^i(\theta_2, \ell, \Psi_1(\ell)) dF_2(\theta_2)$$

s.t.

$e - q(\ell, \Psi_1(\ell)) \cdot \ell \geq 0$ and $\ell \in A$

We will denote a type i's period 1 decision rule as $\ell^i(\theta_1)$.

3.2 Insurers

Insurers face a set of insurance contracts $x \in I$ and prices $\{p(x, \Psi_2(d, \ell, \Psi_1(\ell)))\}$. The decision problem of insurers is to choose how many of these different types of contracts to sell. Clearly insurers will participate in selling any contract $x \in I$ that makes non-negative profits in expectation. That is if

$$p(x, \Psi_3(x, \Psi_2(d, \ell, \Psi_1(\ell)))) \cdot x \geq \Psi_3(x, \Psi_2(d, \ell, \Psi_1(\ell))) \cdot \pi^g \cdot x + (1 - \Psi_3(x, \Psi_2(d, \ell, \Psi_1(\ell)))) \cdot \pi^b \cdot x.$$

Eliminating x, this condition reduces to

$$p(x, \Psi_3(x, \Psi_2(d, \ell, \Psi_1(\ell)))) \geq \pi^b - \Psi_3(x, \Psi_2(d, \ell, \Psi_1(\ell)))[\pi^b - \pi^g].$$

3.3 Banks

In period 1, banks face a set of loan contracts $y \in A$ and prices $\{q(y, \Psi_1(y)), y \in A\}$. As in the case of insurers, the decision problem of banks is to choose how many of these different types of contracts to sell. And, as in the case of insurers, banks will participate in selling only those contracts that make non-negative profits in expectation.

For $y < 0$, non-negative profits require

$$q(y, \Psi_1(y)) \cdot y \geq \Psi_1(y) \left[1 - \mu^g(y)\right] \frac{y}{(1+r)} + (1 - \Psi_1(y)) \left[1 - \mu^b(y)\right] \frac{y}{(1+r)},$$

where r is the risk-free rate available to banks and $\mu^i(y)$ is the probability that a person of type i will default on a loan of size y. Eliminating y yields the condition

$$q(y, \Psi_1(y)) \leq \left(\Psi_1(y) \left[1 - \mu^g(y)\right] + (1 - \Psi_1(y)) \left[1 - \mu^b(y)\right]\right) (1+r)^{-1}.$$

For $y > 0$, non-negative profits require that

$$q(y, \Psi_1(y)) \geq (1+r)^{-1}.$$

4 Equilibrium

We can now state the definition of a competitive equilibrium.

Definition 1 A competitive equilibrium is: (i) a set of belief-updating functions $\Psi_1^*(\ell)$, $\Psi_2^*(d, \ell, \Psi_1^*(\ell))$, and $\Psi_3^*(x, \Psi_2^*(d, \ell, \Psi_1^*(\ell))$, (ii) a set of loan prices $q^*(\ell, \Psi_1^*(\ell))$; (iii) a set of insurance prices $p^*(x, \Psi_3^*(x, \Psi_2^*(d, \ell, \Psi_1^*(\ell)))$; (iv) a set of default probabilities $\mu^{i*}(\ell)$; (iv) a set of decision rules $\ell^{i*}(\theta_1)$, $d^{i*}(\theta_2, \ell, \Psi_1^*(\ell))$, and $x^{i*}(\Psi_2^*(d, \ell, \Psi_1^*(\ell)))$ such that

1. Each loan earns zero profits.

   (a) For $\ell \geq 0$ this requires

$$q^*(\ell, \Psi_1^*(\ell)) = \frac{1}{1+r}. \tag{3}$$

(b) For $\ell < 0$, this requires

$$q^*(\ell, \Psi_1^*(\ell)) = \frac{\Psi_1^*(\ell)\left[1 - \mu^{g*}(\ell)\right] + (1 - \Psi_1^*(\ell))\left[1 - \mu^{b*}(\ell)\right]}{1 + r}. \tag{4}$$

2. Each insurance contract earns zero profits. For $x \in I(\Psi_2^*(\ell, d, \Psi_1^*(\ell)))$

$$p^*(x, \Psi_3^*(x, \Psi_2^*(d, \ell, \Psi_1^*(\ell))) = \pi^b - \Psi_3^*(x, \Psi_2^*(d, \ell, \Psi_1^*(\ell)))[\pi^b - \pi^g]. \tag{5}$$

3. Default probabilities are consistent with decision rules. This requires

$$\mu^{i*}(\ell) = \int_{R_+} d^{i*}(\theta_2, \ell, \Psi_1^*(\ell)) dF_2(\theta_2) \text{ for } \ell < 0. \tag{6}$$

4. The decision rules $\ell^{i*}(\theta_1)$, $d^{i*}(\theta_2, \ell, \Psi_1^*(\ell))$ and $x^{*i}(\Psi_2^*(d, \ell, \Psi_1^*(\ell)))$ solve each household type's optimization problem given (i) the loan pricing function $q^*(\ell, \Psi_1^*(\ell)))$, $\ell \in A$, (ii) the insurance pricing function $p^*(x, \Psi_3^*(x, \Psi_2^*(d, \ell, \Psi_1^*(\ell)))$, $x \in I$, and (iii) belief-updating functions $\Psi_1^*(\ell)$, $\Psi_2^*(d, \ell, \Psi_1^*(\ell))$ and $\Psi_3^*(x, \Psi_2^*(d, \ell, \Psi_1^*(\ell)))$.

5. Updating functions are consistent with decision rules and satisfy Bayes' Rule whenever possible. To state these conditions, define $H_1^{i*}(\ell) = \{\theta_1 : \ell^{i*}(\theta_1) = \ell\}$ and $H_2^{i*}(d; \ell) = \{\theta_2 : d^{i*}(\theta_2, \ell, \Psi_1^*(\ell)) = d\}$. Then,

(a) For $\Psi_1^*(\ell)$ this requires

$$\Psi_1^*(\ell) = \frac{\gamma \int 1_{\{\theta_1 \in H_1^{g*}(\ell)\}} dF_1(\theta_1)}{\gamma \int 1_{\{\theta_1 \in H_1^{g*}(\ell)\}} dF_1(\theta_1) + (1 - \gamma) \int 1_{\{\theta_1 \in H_1^{b*}(\ell)\}} dF_1(\theta_1)} \tag{7}$$

provided the denominator is positive – that is, provided a positive measure of people choose ℓ in period 1.

(b) For $\Psi_2^*(d, \ell, \Psi_1^*(\ell))$ this requires

$$\Psi_2^*(d, \ell, \Psi_1^*(\ell)) = \frac{\Psi_1^*(\ell) \int 1_{\{\theta_2 \in H_2^{g*}(d;\ell)\}} dF_2(\theta_2)}{\Psi_1^*(\ell) \int 1_{\{\theta_2 \in H_2^{g*}(d;\ell)\}} dF_2(\theta_2) + (1 - \Psi_1^*(\ell)) \int 1_{\{\theta_2 \in H_2^{b*}(d;\ell)\}} dF_2(\theta_2)} \tag{8}$$

provided, again, the denominator is positive.

(c) For $\Psi_3^*(x, \Psi_2^*(d, \ell, \Psi_1^*(\ell)))$ this requires

$$\Psi_3^*(x, \Psi_2^*(d, \ell, \Psi_1^*(\ell))) = \frac{\Psi_2^*(d, \ell, \Psi_1^*(\ell)) \cdot 1_{\{x^{g*}(\Psi_2^*(d,\ell,\Psi_1^*(\ell))=x\}}}{\Psi_2^*(d, \ell, \Psi_1^*(\ell)) \cdot 1_{\{x^{g*}(\Psi_2^*(d,\ell,\Psi_1^*(\ell)))=x\}} + (1 - \Psi_2^*(d, \ell, \Psi_1^*(\ell))) \cdot 1_{\{x^{b*}(\Psi_2^*(d,\ell,\Psi_1^*(\ell)))=x\}}} \tag{9}$$

provided, again, the denominator is positive.

We can now state a basic result, namely, that if there exists an equilibrium in which borrowing is not permitted, then there always exists a competitive equilibrium in which borrowing is permitted but no borrowing actually occurs in equilibrium. In such an equilibrium, insurers ignore a person's behavior in the credit market and therefore all borrowers have an incentive to default. In that case default behavior is, in fact, uninformative about a borrower's type and no borrowing can be supported.

Proposition 1: Let $B \subset R$ be a finite set with positive and negative elements. Suppose there exists a competitive equilibrium for an economy for which $A = B \cap R_+$ (i.e., no borrowing is permitted). Then there always exists another competitive equilibrium in which $A = B$ but $q^*(\ell, s) = 0$ for all $\ell < 0$ (i.e., no loans are made).

Proof We will establish the Proposition by extending the "no-borrowing" equilibrium belief-updating function $\Psi_1^*(\ell)$ to negative values of ℓ and the "no-borrowing" equilibrium belief-updating function $\Psi_2^*(d, \ell, \Psi_1^*(\ell))$ to allow for debt and default in such a way as to leave budget sets and signalling opportunities unchanged. In this new extended equilibrium $q^*(\ell, s)$ will be 0 for all $\ell < 0$.

For any given $\Psi_1^*(\ell)$, $\ell < 0$, set $\Psi_2^*(0, \ell, \Psi_1^*(\ell)) = \Psi_2^*(1, \ell, \Psi_1^*(\ell)) = \Psi_1^*(\ell)$. Then we have that $V_3^i(\Psi_2^*(0, \ell, \Psi_1^*(\ell))) = V_3^i(\Psi_2^*(1, \ell, \Psi_1^*(\ell))) = V_3^i(\Psi_1^*(\ell))$ and so $d^i(\theta_2, \ell, \Psi_1^*(\ell)) = 1$ is optimal for all $\theta_2 \geq 0$ and $\ell < 0$. Given this decision rule it follows from (6) that for $\ell < 0$, $\mu^i(\ell) = 1$, from (4) that $q^*(\ell, \Psi_1^*(\ell)) = 0$ and from (8) that $\Psi_2^*(1, \ell, \Psi_1^*(\ell)) = \Psi_1^*(\ell)$ satisfies the Bayesian updating conditions. Since for $\Psi_2^*(0, \ell, \Psi_1^*(\ell))$ the Bayesian updating formula is not applicable, we are free to set $\Psi_2^*(0, \ell, \Psi_1^*(\ell)) = \Psi_1^*(\ell)$.

Next, set $\Psi_1^*(\ell) = \Psi_1^*(0)$ for all $\ell < 0$ so that any consumption stream a person can get by choosing $\ell < 0$ in period 1 and defaulting in period 2 can be obtained by choosing $\ell = 0$. Given this it follows that $\ell^*(\theta_1) \geq 0$ – the optimal period-1 decision rule for the "no-borrowing" equilibrium – continues to be the optimal period-1 decision rule for the economy in which borrowing is permitted but all prices on loans are zero. Since no one chooses $\ell < 0$ in equilibrium the Bayesian updating formula is not applicable and we are free to set $\Psi_1^*(\ell) = \Psi_1^*(0)$.

$\square$

In the rest of the paper, we explore the possibility of equilibria in which behavior in the credit market conveys useful information to insurers. It is a useful fact that the equilibrium of this model can be analyzed in three blocks – corresponding to the three periods. In period 3, the equilibrium decision rule $x^{*i}(\Psi_2^*(d, \ell, \Psi_1^*(\ell)))$ must be the optimal decision rule given $I(\Psi_2^*(d, \ell, \Psi_1^*(\ell)))$, $p^*(x, \Psi_3^*(x, \Psi_2^*(d, \ell, \Psi_1^*(\ell)))$ and, in turn, $p^*(x, \Psi_3^*(x, \Psi_2^*(d, \ell, \Psi_1^*(\ell)))$ must satisfy zero profits and $\Psi_3^*(x, \Psi_2^*(d, \ell, \Psi_1^*(\ell)))$ must satisfy Bayes' Law whenever possible. In period 2, $d^{i*}(\theta_2, \ell, \Psi_1^*(\ell))$ must be the optimal decision rule given the period-3 *equilibrium* value function and the belief-updating function $\Psi_2^*(d, \ell, \Psi_1^*(\ell))$ and, in turn, $\Psi_2^*(d, \ell, \Psi_1^*(\ell))$ must satisfy Bayes' Law whenever possible. Finally, in period 1, the equilibrium decision rule $\ell^{i*}(\theta_1)$ must be the optimal decision rule given the period-2 *equilibrium* value function, $q^*(\ell, \Psi_1^*(\ell))$ and the belief-updating function $\Psi_1^*(\ell)$ and, in turn, $q^*(\ell, \Psi_1^*(\ell))$ must satisfy zero profits and $\Psi_1^*(\ell)$ satisfies Bayes' Law whenever possible.

Since type g people are better insurance risks than type b people, the equilibria of interest are those in which people resist opportunistic behavior in the credit market in an effort to signal that they are more likely to be type g. For this logic to work, it must be true that people with a high score are treated better in the insurance market. But insurers may offer contracts that separate people into good and bad insurance risks, so it's not obvious that a person's type score will matter in the insurance market. Therefore, the nature of the equilibrium in the period-3 insurance market is key and it is to this issue we now turn.

5 Equilibrium in the Insurance Market: Pooling, Separation and Type Score

The microeconomic literature on the provision of insurance indicates that for a population with two hidden types, competition among insurers could result in one of two kinds of equilibrium - pooling or separating. In a pooling equilibrium insurers offer one full-insurance contract at a price that reflects the composition of low- and high-risk types in the population. In a separating equilibrium insurers offer two contracts, one with a low price but enough limitations on insurance so as to attract only the low-risk types and another with full insurance at a high price for the remaining high-risk types.

An important insight of the microeconomic insurance literature is that competitive insurers have an incentive to break away from pooling contracts. In a pooling contract where insurers make zero profits, the low-risk types must subsidize the high-risk types, since both types pay the same price but the high-risk types have a higher probability of loss. Because of this subsidy, insurers have an incentive to entice away the low-risk types by offering them slightly less-than-full-insurance at a price that is significantly below the pooling contract price but above the price that would be actuarially fair for low-risk types (a so-called "cream skimming" strategy). Given this incentive toward "cream skimming," Rothschild and Stiglitz [12] claimed that pooling contracts cannot exist in equilibrium. But Wilson [13] (see also Miyazaki [11]) pointed out that if firms anticipate that contracts rendered unprofitable by cream-skimming will be withdrawn from the market, they will also anticipate that the cream-skimming contract will end up attracting both high- and low-risk types and therefore turn out to be unprofitable. Given these anticipations, Wilson showed that firms cannot offer separating contracts on which they anticipate earning non-negative profits and so equilibrium with pooling contracts can exist.[10]

We will proceed in the spirit of Wilson's notion of equilibrium in the insurance market. In particular, we will assume that people are offered the full insurance pooling contract unless there exists a separating contract that the low-risk types will strictly prefer over the pooling contract *and* that the high-risk types will not strictly prefer over an actuarially fair full insurance contract. This latter requirement takes Wilson's point into account, namely, that if a limited insurance contract is successful in attracting only the low-risk types then the insurance alternatives facing the high-risk types no longer include the original pooling contract. Therefore, for the limited insurance separating contract to be feasible, we require that the high-risk types prefer obtaining full insurance at a high but actuarially fair price over obtaining limited insurance at a lower price that is actuarially fair for the low-risk types.

In what follows, we will establish that there is a threshold value for the type score, σ^*, such that separating contracts are not feasible for pools with type score equal to or greater than σ^* but are feasible for pools with type score less than σ^*. Furthermore, if a separating contract is feasible then there is a unique best separating contract that can be offered to the low-risk types regardless of their type score. These results are used to define the insurance choice

[10] A similar point was made by Hellwig [8], who analyzed a 3-period extensive form game in which in the first stage firms offer insurance contracts, in the second stage customers apply for insurance under one of these contracts and, in the third stage, firms accept or reject customers' applications for insurance. Hellwig argued that the possibility that a firm may reject customers' applications in the third stage implies that customers cannot rationally believe that an application for a low-priced limited insurance contract will be accepted by the offering firm in the final stage. Given this belief, no customer applies for insurance under a low-priced limited insurance contract and the pooling contract survives. Also, see Elul [7] for a recent application of the Wilson-Miyazaki concept of equilibrium in a related context.

correspondence $I(s')$ used later in the paper.

First, some terminology and definitions. Denote an *insurance contract* as a pair (X, m), where X is the indemnity and m is the price per unit of insurance (so that the premium on the contract is $m \cdot X$). Denote the utility of a type i household from purchasing a contract (X, m) by $W^i(X, m) = \pi^i u(e - mX - L + X) + (1 - \pi^i)u(e - mX)$. Denote the price of an *actuarially fair*, or zero profit, insurance contract offered to people with type score σ by $m(\sigma) = \sigma\pi^g + (1 - \sigma)\pi^b$. Then we have the following definition of a feasible full insurance pooling contract.

Definition: The full insurance pooling contract $(L, m(\sigma))$ is *feasible* if there does not exist a contract (X, π^g), $X < L$, such that (i) $W^g(X, \pi^g) > W(L, m(\sigma))$ and (ii) $W^b(L, \pi^b) \geq W^b(X, \pi^g)$.

This definition of feasibility captures the idea behind Wilson's notion of equilibrium in the insurance market. Basically, for a pooling contract for people with type score σ to be infeasible there must exist a limited insurance actuarially fair contract that type g people strictly prefer over the pooling contract and which is *not* strictly preferred by the type b people over an actuarially fair full insurance contract.

We make the following assumption on the utility function.

Assumption 1: $u(e - \pi^b L) > \pi^g u(e - L) + (1 - \pi^g)u(e)$.

The inequality effectively asserts that a type g person prefers to purchase full insurance at a price that is appropriate for the type b households to not purchase insurance at all. Clearly this assumption is a restriction on curvature of the u function and the loss L – the function must be sufficiently concave or the loss sufficiently large.

Lemma 1 There exists a unique $X^* \in (0, L)$ such that $W^b(X^*, \pi^g)= W^b(L, \pi^b)$.

Proof Since $\pi^g < \pi^b$ it's clear that $W^b(X = L, \pi^g) > W^b(L, \pi^b)$. And, by virtue of the strict concavity of u, no-insurance is worse than full-insurance at an actuarially fair price so $W^b(X = 0, \pi^g) < W^b(L, \pi^b)$. Clearly $W^b(X, \pi^g)$ is a continuous function of $X \in [0, L]$. Therefore, the existence of $X^* \in (0, L)$ follows from the Intermediate Value Theorem. Uniqueness of X^* follows from the fact that $W^b(X, \pi^g)$ is strictly increasing in X, since insurance is being offered at a price that is lower than the probability of loss.

□

Since $W^b(X, \pi^g)$ is strictly increasing in X, X^* has the interpretation of being the *most generous* actuarially fair (or, zero-profits) insurance that can be offered to type g people who are in a pool with type score $\sigma \in [0, 1]$ without necessarily attracting the type b people in the pool.

Lemma 2 There exists a unique $x(\sigma) \in [0, L]$ such that $[W^g(x(\sigma), \pi^g) - W^g(L, m(\sigma))] = 0$. Furthermore, $x(\sigma)$ is continuous and strictly increasing in σ.

Proof First, observe that $W^g(X, \pi^g)$ is clearly continuous in X and, because the price of the insurance is actuarially fair, it is strictly increasing in X. Consider first the case where $\sigma = 1$. Clearly a unique $x(1)$ exists and is equal to L. Next consider any $\sigma \in [0, 1)$. Then $W^g(X = L, \pi^g) > W^g(L, m(\sigma))$ because $\pi^g < m(\sigma)$. Furthermore, $W^g(X = 0, \pi^g) < W^g(L, m(\sigma)$ because by Assumption 1 $W^g(X = 0, \pi^g) < W^g(L, \pi^b)$ and because $m(\sigma)$ is increasing in σ and $W^g(L, m)$ is increasing in m. Therefore, by the continuity of $W^g(X, \pi^g)$ and the Intermediate Value Theorem, $x(\sigma) \in (0, L)$ exists. And by the strict monotonicity of $W^g(X, \pi^g)$ in X, $x(\sigma) \in (0, L)$ is unique. From the continuity of $W^g(X, \pi^g)$ with respect to X, the continuity of $m(\sigma)$ with respect to σ, and the continuity of $W^g(L, m)$ with respect to m, it follows that $x(\sigma)$ is a continuous function of $\sigma \in [0, 1]$. Furthermore, since $W^g(L, m(\sigma))$ is strictly increasing in σ and $W^g(X, \pi^g)$ is strictly increasing in X, it follows that $x(\sigma)$ is strictly increasing in σ.

□

Since $W^g(X, \pi^g)$ is strictly increasing in X, $x(\sigma)$ has the interpretation of being the *least* *generous* actuarially fair insurance that can be offered to type g people in a pool with type score σ that weakly dominates a full-insurance contract offered at the price $m(\sigma)$.

Corollary to Lemmas 1 and 2: A full-insurance pooling contract $(L, m(\sigma))$ offered to people with type score σ is not feasible if $x(\sigma) < X^*$ and is feasible if $x(\sigma) \geq X^*$.

Proof: Suppose $x(\sigma) < X^*$. Consider a contract $(\tilde{X}, \pi^g)$, where $X^* > \tilde{X} > x(\sigma)$. Type g people will strictly prefer such a contract to the pooling contract, but type b people will not prefer such a contract over the contract (L, π^b). Therefore the contract $(L, m(\sigma))$ is not feasible. Suppose $x(\sigma) \geq X^*$. Suppose, to get a contradiction, that the pooling contract $(L, m(\sigma))$ is not feasible. Then there must exist a contract (X, π^g) that is strictly better than $(L, m(\sigma))$ for type g people and that the type b people do not strictly prefer over (L, π^b). The first implies that $X > x(\sigma)$ and the second implies that $X \leq X^*$. Together they imply that $x(\sigma) < X^*$, which contradicts the assertion that $x(\sigma) \geq X^*$.

□

Proposition 2: (i) There exists a cut-off type score $\sigma^* \in (0, 1)$ below which a full-insurance pooling contract is not feasible and at or above which it is feasible. (ii) Furthermore when a pooling contract is not feasible the best separating contract that can be offered to type g is the contract (X^*, π^g).

Proof (i) Given the fact that $x(\sigma)$ is strictly increasing in σ (Lemma 2) and the corollary to Lemmas 1 and 2, it is sufficient to establish that there is a unique $\sigma^* \in (0, 1)$ that satisfies the equation $x(\sigma^*) = X^*$. From Lemmas 1 and 2 we know $x(\sigma = 1) = L > X^*$. Now consider $x(0)$, which solves

$$\pi^g u(e - \pi^g x(0) - L + x(0)) + (1 - \pi^g)u(e - \pi^g x(0)) = u(e - \pi^b L). \tag{10}$$

We know that X^* solves

$$\pi^b u(e - \pi^g X - L + X) + (1 - \pi^b)u(e - \pi^g X) = u(e - \pi^b L). \tag{11}$$

Since $X^* < L$ (by Lemma 1), we know that $u(e - \pi^b L) < u(e - \pi^g X))$. Therefore, $u(e - \pi^b L)$ being the average of the two terms in (11), it follows that

$$u(e - \pi^g X^* - L + X^*) < u(e - \pi^b L) < u(e - \pi^g X^*).$$

Therefore, since $(1 - \pi^g) > (1 - \pi^b)$,

$$\pi^g u(e - \pi^g X^* - L + X^*) + (1 - \pi^g) u(e - \pi^g X^*) > u(e - \pi^b L). \tag{12}$$

Hence (10) and (12) imply $x(0) < X^*$. By Lemma 2, $x(\sigma)$ is continuous and strictly increasing in σ. Therefore there must exist a unique $\sigma^* \in (0, 1)$ such that $x(\sigma^*) = X^*$.

(ii) When $\sigma < \sigma^*$ then the least generous limited-insurance contract that a type g would take over a full-insurance contract at price $m(\sigma)$ is less than X^*. Therefore, any contract $(\widetilde{X}, \pi^g)$, $\widetilde{X} \in (x(\sigma), X^*]$ will induce type g to migrate to the limited-insurance contract without giving type b a strict incentive to choose the limited-insurance contract over a full-insurance contract offered at the price π^b. However, among the set of separating contracts, the contract (X^*, π^g) gives the highest utility to type g because u is strictly concave and the insurance is offered at a price that is actuarially fair for type g people.

□

The reason there is a cut-off value of σ^* at or above which a pooling contract is feasible is that we require that when a separating contract is offered to the low-risk types, the high-risk types must not find it in their interest to re-pool with the low-risk types (and accept limited insurance) over a contract that offers them full insurance at a high but actuarially fair (i.e., *non-subsidized*) price. Importantly, the incentive of high-risk types to "re-pool" varies inversely with the proportion of high-risk types in the pool. When the pool contains relatively few high-risk types, the price of insurance in the limited-insurance contract cannot be too much below the price of insurance in the original pooling contract, and therefore, the insurance offered at the lower price cannot be too much below full insurance. Such a contract will appear quite attractive to a high-risk type whose alternative is to purchase somewhat more insurance at a potentially *much* higher price. Therefore, a pooling contract cannot be broken by a separating contract when the fraction of high-risk types in the pool is sufficiently low.

To summarize, Proposition 2 establishes that the kind of insurance opportunities a person will face depends on the person's type score s' – specifically, people with relatively high s' will be offered full insurance pooling contracts while people with relatively low s' will be offered separating contracts. This result is incorporated into our environment *via* the insurance choice correspondence $I(s')$. Specifically, we assume that

$$I(s') = \begin{cases} \{0, X^*, L\} & \text{if } s' < \sigma^* \\ \{0, L\} & \text{if } s' \geq \sigma^* \end{cases} \tag{13}$$

With this definition of $I(s')$, we can characterize the period-3 insurance market equilibrium.[11]

Proposition 3: Given the insurance contract correspondence (13), the following functions constitute a period-3 insurance market equilibrium: (i) $p^*(x, \Psi_3^*(x, s')) = \pi^b - \Psi_3^*(x, s')(\pi^b - \pi^g)$, (ii) for $s' \geq \sigma^*$, $x^{*i}(s') = L$ for all i, for $s' < \sigma^*$, $x^{*g}(s') = X^*$ and $x^{*b}(s') = L$; and (iii)

$$\Psi_3^*(x, s') = \begin{cases} s' & \text{if } s' \geq \sigma^* \text{ and } x \in \{0, L\} \\ 1 & \text{if } s' < \sigma^* \text{ and } x = X^* \\ s' & \text{if } s' < \sigma^* \text{ and } x = 0 \\ 0 & \text{if } s' < \sigma^* \text{ and } x = L \end{cases}$$

Proof Obviously, $p^*(x, \Psi_3^*(x, s'))$ satisfies the zero profit condition (5) in the definition of

[11] In the interest of keeping the analysis simple, we do not explore the robustness of the credit market equilibrium to sets of insurance contracts other than the one that provides the best possible separating contract to the low-risk type. Furthermore, we do not consider the possible benefits of long-term contracts that can presumably be offered by a banking-insurance conglomerate.

equilibrium. So we need only establish that $x^{*i}(s')$ are optimal and $\Psi_3^*(x, s')$ satisfy condition (9) in the definition of competitive equilibrium. Observe that a type-i person's insurance choice problem reduces to:

$$V_3^i(s') =$$
$$\max_x \pi^i u(e - [\pi^b - s''(\pi^b - \pi^g)]x - L + x) + (1 - \pi^i)u(e - [\pi^b - s''(\pi^b - \pi^g)]x)$$
s.t
$$x \in \{0, L\} \text{ if } s' \geq \sigma^* \text{ and } x \in \{0, X^*, L\} \text{ otherwise}$$
$$s'' = \Psi_3^*(x, s').$$

Consider first households with $s' \geq \sigma^*$. For these households $\Psi_3^*(x, s') = s'$ for $x \in \{0, L\}$. By Assumption 1 it follows that the optimal insurance choice for these households, regardless of type, is $x = L$. Consider next a household with $s' < \sigma^*$. If this household chooses $x = X^*$ then Ψ_3^* implies that the household's $s'' = 1$. By the pricing function p^*, the household will face the price π^g. Now suppose the household is of type g. Since the price π^g is actuarially fair for him, *conditional* on choosing $x \leq X^*$, it is optimal for him to choose $x = X^*$ by Assumption 1. If the household is of type b, then the price π^g is *better than* actuarially fair. So, conditional on choosing $x \leq X$, it also optimal for a type b household to choose $x = X^*$ by concavity of $u(.)$. On the other hand, for any household that chooses $x = L$, Ψ_3^* implies that the household's $s'' = 0$. By the pricing function p^*, it follows that the household will face the price π^b. Consequently, the choice of insurance reduces to a choice between the contract (X^*, π^g) and the contract (L, π^b) regardless of type. Now we know from Lemma 1 that a type b person is indifferent between these two contracts. And by Lemma 2 we know that a type g household with $s' < \sigma^*$ strictly prefers the contract (X^*, π^g) to (L, π^b). Therefore $x^{i*}(s')$ is the optimal decision rule.

We now verify that $\Psi_3^*(x, s')$ satisfies Bayes' Rule whenever possible. Consider first $s' \geq \sigma^*$ in which case $I(s') = \{0, L\}$. The decision rules imply that both types choose $x = L$.

Therefore,

$$s' \cdot 1_{\{x^{g*}(s')=L\}} = s' \text{ and } s' \cdot 1_{\{x^{g*}(s')=L\}} + (1-s') \cdot 1_{\{x^{b*}(s')=L\}} = 1.$$

These expressions imply that for $s' \geq \sigma^*$, $\Psi_3^*(x = L, s')$ satisfies the requirements in (9). For $x = 0$, however, the requirements lead to an indeterminacy (0/0). In this case, we assume that the off-equilibrium choice is uninformative about a person's type. That is $\Psi_3^*(x = 0, s') = s'$.

Next, consider $s' < \sigma^*$ in which case $I(s') = \{0, X^*, L\}$. The decision rules imply that type g people choose $x = X^*$ and type b choose $x = L$. Therefore,

$$s' \cdot 1_{\{x^{g*}(s')=L\}} = 0 \text{ and } s' \cdot 1_{\{x^{g*}(s')=L\}} + (1-s') \cdot 1_{\{x^{b*}(s')=L\}} = 1 - s',$$

and

$$s' \cdot 1_{\{x^{g*}(s')=X^*\}} = s' \text{ and } s' \cdot 1_{\{x^{g*}(s')=X^*\}} + (1-s') \cdot 1_{\{x^{b*}(s')=X^*)\}} = s'.$$

These expressions imply that the requirements in (9) are met for $x = L$ and $x = X^*$. For $x = 0$, however, the requirements lead to the indeterminacy (0/0). In this case, we assume that the off-equilibrium choice is uninformative about a person's type so that $\Psi_3^*(x = 0, s') = s'$.

$\square$

Proposition 3 allows us to fully characterize the period-3 equilibrium value function for each type of individual. We have:

Proposition 4

$$V_3^{g*}(s') = \begin{cases} \pi^g u(e - \pi^g \cdot X^* + X^* - L) + (1-\pi^g)u(e - \pi^g \cdot X) & \text{if } s' < \sigma^* \\ u(e - [\pi^b - s'(\pi^b - \pi^g)] \cdot L) & \text{if } s' \geq \sigma^* \end{cases}$$

$$V_3^{b*}(s') = \begin{cases} u(e - \pi^b \cdot L) & \text{if } s' < \sigma^* \\ u(e - [\pi^b - s'(\pi^b - \pi^g)] \cdot L) & \text{if } s' \geq \sigma^* \end{cases}$$

Furthermore, $V_3^{g*}(s')$ is continuous in s' and concave over $[\sigma^*, 1]$, $V_3^{b*}(s') < V_3^{g*}(s')$ for $s' < \sigma^*$, and $V_3^{b*}(s) = V_3^{g*}(s)$ for $s' \geq \sigma^*$.

Proof σ^* solves the equation $x(\sigma^*) = X^*$. By definition $x(\sigma^*)$ is such that $\pi^g u(e - \pi^g \cdot x(\sigma^*) + x(\sigma^*) - L) + (1-\pi^g)u(e - \pi^g \cdot x(\sigma^*)) = u(e - [\pi^b - \sigma^*(\pi^b - \pi^g)] \cdot L)$. Therefore it follows that $V_3^{g*}(s') = V_3^{g*}(\sigma^*)$ for all $s' \leq \sigma^*$. Since $V_3^{g*}(s')$ is clearly continuous over $[\sigma^*, 1]$, continuity of $V_3^{g*}(s')$ follows. Concavity of $V_3^{g*}(s')$ over $[\sigma^*, 1]$ can be established by differentiating with respect to s' twice. Finally, observe that $u(e - \pi^b \cdot L) < u(e - [\pi^b - \sigma^*(\pi^b - \pi^g)] \cdot L)$. Therefore $V_3^{b*}(s') < V_3^{g*}(s')$ for all $s' < \sigma^*$.

□

Figure 1 illustrates the properties of V_3^{i*} functions. For $s' < \sigma^*$, the value functions are constant for both types but it is constant at a lower level for type b than for type g. For $s' \geq \sigma^*$, the value functions for both types coincide and is strictly increasing in the score. The existence of the increasing segment in these value functions is the reason why it is desirable for people to emerge from the credit market with a high type score.

6 Credit Market Equilibrium: Debt, Default, and Signalling

We are interested in the possibility of equilibria in which people resist opportunistic behavior in the credit market with a view to obtaining better terms in the insurance market. The fact that $V_3^{i*}(s')$ is strictly increasing in s' for $s' > \sigma^*$ provides people with the incentive to signal that they are more likely to be of type g. However it is an interesting (if somewhat inconvenient) fact that from the perspective of period 3 it is the type b people who lose more from default than type g people. To see this consider Figure 1. Suppose that in equilibrium repayment of debt leads to a type score $s' = s^R$ and default leads to a type score $s' = s^D < s^R$. By Proposition 4 (or simply from inspection of Figure 1) it is evident that the loss from default for type g $\left[V_3^{g*}(s^R) - V_3^{*g}(s^D)\right]$ cannot be any greater than the loss from default for type b $\left[V_3^{b*}(s^R) - V_3^{b*}(s^D)\right]$ and can, in fact, be strictly less. On this count we would expect type b to have *less* of an incentive to default than type g. However, as noted earlier, it is possible to make assumptions on either discount factors or the distribution of preference shocks that ensure that default is less likely for type g than type b.

In what follows we seek conditions under which the following three basic properties hold in equilibrium.

- **P1**: $\Psi_1^*(\ell) > 0$ is strictly increasing in ℓ, with $\Psi^*(\ell_{\min}) < \gamma < \Psi^*(\ell_{\max})$. That is, borrowing less or saving more improves the likelihood that a person is of type g and this likelihood is less than γ for the largest loan and greater than γ for the largest deposit.

- **P2**: $0 < \Psi_2^*(1, \ell, \Psi_1^*(\ell)) < \Psi_1^*(\ell) < \Psi_2^*(0, \ell, \Psi_1^*(\ell))$ for $\ell < 0$. That is, repaying a loan improves the likelihood that a person is of type g while default worsens it.

- **P3**: $\Psi_2^*(0, \ell, \Psi_1^*(\ell)) > 0$ is non-decreasing in ℓ. This assumption rules out the possibility that a person who borrowed more can end period 2 with a better score (upon repayment) than a person who borrowed less (and repaid) or who saved.

In addition, we make the following consistency assumption.

- **P4:** $\Psi_2^*(0, \ell, \Psi_1^*(\ell)) = \Psi_1^*(\ell)$ for $\ell > 0$. That is, for people without debt there is no change in their score in period 2. This assumption simply recognizes that since these people do not make any choices in period 2, nothing further is revealed about them.

Our strategy is to assume that there are functions Ψ_t^* that satisfy properties P1-P4 (and some additional properties specific to each of the following subsections) and then show that there exist distribution functions that make Ψ_t^* consistent with the equilibrium requirements on belief-updating functions. In the first approach we require that $F_t^g(\theta) = F_t^b(\theta)$ (no difference in distribution of preference shocks across types) and in the second approach we require that $\beta^b = \beta^g$ (no difference in discount factors across types).

We begin by establishing a simple property of every competitive equilibrium, namely, that the default decision has a threshold property with respect to θ_2.

Lemma 3 Let $\ell < 0$. There exists $\theta_2^{i*} \geq 0$ such that $d^{i*}(\theta_2, \ell, \Psi_1^*(\ell)) = 1$ for $\theta_2 > \theta_2^{i*}$ and 0 otherwise.

Proof: For $\ell < 0$, the benefit from default is $\theta_2^0 \left[u(e) - u(e + \ell)\right]$ and the cost of default is $\beta^i \left[V_3^i(\Psi_2^*(0, \ell, \Psi_1^*(\ell))) - V_3^i(\Psi_2^*(1, \ell, \Psi_1^*(\ell)))\right]$. The benefit of default is strictly positive. By Proposition 4 and $P2$, the cost of default is non-negative. If the cost is 0 then the benefit of default exceeds the cost for all $\theta_2 > 0$ and is equal to the cost for $\theta_2 = 0$. Therefore $\theta_2^{i*} = 0$. If the cost is strictly positive it is bounded above by $\beta^i \left[V_3^{i*}(1) - V_3^{i*}(0)\right]$. Therefore, there must exist $\theta_2^0 > 0$ for which the cost and the benefit are exactly equal. Then $\theta_2^{i*} = \theta_2^0$.

□

Next we point out that equilibrium loan prices are zero if there is no private information about types or if both types have the same probability of loss in period 3. Thus in our environment, the desire to signal private information about type is the fundamental reason why active trade can occur in the credit market.

Corollary to Lemma 3: If type is observable or if $\pi^g = \pi^b$, then $q^*(\ell, \Psi_1^*(\ell)) = 0$, provided the probability of $\theta_2 = 0$ is 0.

Proof: If type is observable, then competition implies that full insurance is available to type g and b at prices π^g and π^b, respectively. Since both types are risk averse it is optimal for them to buy full insurance. Then, in period 2, it is strictly optimal for both types to default for any $\theta_2 > 0$ since defaulting yields utility $\theta_2 u(e) + \beta^i u(e - \pi^i L)$ and not defaulting yields utility $\theta_2 u(e+\ell) + \beta^i u(e - \pi^i L)$, where $\ell < 0$. If the probability of $\theta_2 = 0$ is zero, both types will default with certainty, i.e., $\mu^*(\ell) = 1$. Then $q^*(\ell, \Psi^*(\ell))$ must be 0.

If $\pi^g = \pi^b = \pi$ then insurers do not care about a person's type and, under competition, both types will be offered full insurance at the price π. Once again, it is strictly optimal for both types to default on any $\ell < 0$ for any $\theta_2 > 0$.

$\square$

6.1 Credit Market Equilibrium with Differences in Discount Factors

In this subsection we will assume that type g is more patient than type b but the distribution functions for the two types are the same. That is, $\beta^b < \beta^g$ and $F_t^b(\theta) = F_t^g(\theta)$. Furthermore, we take $A = \{-a, 0\}$. That is, people in period 1 have the option of choosing to borrow a or not. This is a strong assumption. At the end of this section we discuss what happens when more asset choices are permitted.

We make the following assumption on discount factors.

Assumption 2: $\beta^b \left[V_3^{b*}(1) - V_3^{*b}(0)\right] < \beta^g \left[V_3^{g*}(1) - V_3^{g*}(0)\right]$.

The assumption requires that even if the period-3 cost of default for type b exceeds the period-3 cost of default for type g by as much as is possible, the *discounted* cost of default is less for type b than type g. With this assumption we have the following useful result.

Lemma 4: Given Assumption 2, there exists $\tilde{\sigma} \in (\sigma^*, 1)$ such that $\beta^g \left[V_3^{g*}(s') - V_3^{g*}(0)\right] > \beta^b \left[V_3^{b*}(s') - V_3^{b*}(0)\right]$ if and only if $s' \in (\tilde{\sigma}, 1]$,

Proof: Consider the function $w(s') = \beta^g \left[V_3^{g*}(s') - V_3^{g*}(0)\right] - \beta^b \left[V_3^{b*}(s') - V_3^{b*}(0)\right]$ defined for $s' \in [\sigma^*, 1]$. Since $V_3^{i*}(1) = u(e - \pi^g L)$, it follows from Assumption 2 that $w(1) > 0$. By Proposition 4, $V_3^{g*}(s') = V_3^{b*}(s')$ for all $s' \in [\sigma^*, 1]$ so that $w(s')$ is continuous and strictly increasing in s' for $s' \in [\sigma^*, 1]$. Furthermore by Proposition 4, $V_3^{g*}(\sigma^*) = V_3^{*g}(0) > V_3^{b*}(0)$. Hence $w(\sigma^*) < 0$. Therefore, there must exist $\tilde{\sigma} \in (\sigma^*, 1)$ such that $w(\tilde{\sigma}) = 0$. The Proposition follows.

$\square$

In what follows we assume that we have given Ψ_t^* functions satisfying P1-P4. In addition, in this subsection we will make the following additional assumption on the Ψ_2^* function.

Assumption 3: $\Psi_2^*(0, \ell, \Psi_1^*(\ell)) > \tilde{\sigma}$ for $\ell = -a$.

This assumption requires that repaying the debt a puts a person's score in the $\tilde{\sigma}$ to 1 range. This assumption will ensure that type b people have a greater incentive to default than type g people in period 2.

Proposition 5 Given Assumptions 2 and 3, the optimal default thresholds θ_2^{i*} are (i) strictly positive and (ii) satisfy $\theta_2^{b*} < \theta_2^{g*}$.

Proof: By Lemma 3 there exist $\theta_2^{i*} \geq 0$ such that $d^{i*}(\theta_2, -a, \Psi_1^*(-a)) = 1$ for $\theta_2 > \theta_2^{i*}$ and zero otherwise. (i) To establish that the default thresholds θ_2^{i*} are strictly positive, consider a type i with debt a. The benefit from default is given by $\theta_2[u(e) - u(e-a)]$ and the cost of default is given by $\beta^i[V_3^{i*}(\Psi_2^*(0, -a, \Psi_1^*(-a))) - V_3^{i*}(\Psi_2^*(1, -a, \Psi_1^*(-a)))]$. By Assumption 3 and Lemma 4 we know that $\Psi_2^*(0, -a, \Psi_1^*(-a)) > \sigma^*$. By P2 we know that

$\Psi_2^*(1,-a,\Psi_1^*(-a)) < \Psi_2^*(0,-a,\Psi_1^*(-a))$. By Proposition 4 it follows that the cost of default is strictly positive for both types. Since the benefit of default is clearly positive, the default thresholds given by $\theta_2^{i*}[u(e)-u(e-a)] = \beta^i[V_3^{i*}(\Psi_2^*(0,-a,\Psi_1^*(-a)))-V_3^{i*}(\Psi_2^*(1,-a,\Psi_1^*(-a)))]$ are strictly positive.

(ii) To establish that $\theta_2^{b*} < \theta_2^{g*}$ we need to consider two cases. First, consider the case where $\Psi_2^*(1,-a,\Psi_1^*(-a))$ is also at least as large as σ^*. Then it follows from Proposition 4 that the undiscounted cost of default $[V_3^{i*}(\Psi_2^*(0,-a,\Psi_1^*(-a))) - V_3^{i*}(\Psi_2^*(1,-a,\Psi_1^*(-a)))]$ is the *same* for both types. But $\beta^b < \beta^g$ so the *discounted* cost of default is strictly lower for type b compared to type g. Therefore $\theta_2^{b*} < \theta_2^{g*}$. Second, consider the case where $\Psi_2^*(1,-a,\Psi_1^*(-a))$ is strictly less than σ^*. Then by Proposition 4 the discounted cost of default for type i is given by $\beta^i[V_3^{i*}(\Psi_2^*(0,-a,\Psi_1^*(-a)))-V_3^{i*}(0)]$. By Assumption 3 and Lemma 4, the discounted cost of default for type b is strictly lower than the discounted cost for type g and, therefore, $\theta_2^{b*} < \theta_2^{g*}$.

□

The following proposition establishes that there exists a distribution of preference shocks that will make the default thresholds obtained in the previous Proposition consistent with the belief-updating functions. The belief-updating function implicitly fixes the probability of repayment (and therefore of default) of the two types and property P2 implies that the probability of repayment by type g is greater than the probability of repayment by type b. Since the default threshold for type g is higher than the default threshold for type b as established in the previous Proposition, the ordering of repayment probabilities and the ordering of default thresholds coincide. Hence a distribution function can be found for which the probability mass below the default thresholds of type b and type g equals the repayment probabilities implied by the belief-updating functions.

Proposition 6: For $\Psi_2^*(d,\ell,\Psi_1^*(\ell))$ satisfying Assumption 3 there exist distribution functions $F_2(\theta)$ (same for both types) for which $\Psi_2^*(d,\ell,\Psi_1^*(\ell))$ constitutes a period-2 equilibrium and for which both types default or repay with strictly positive probability.

Proof: Recall that a period-2 equilibrium is a pair of decision rules $d^{i*}(\theta_2, \ell, \Psi_1^*(\ell))$, $i \in \{b, g\}$, and a belief-updating function $\Psi_2^*(d, \ell, \Psi_1^*(\ell))$ such that the decision rules are optimal given the belief updating function and the belief-updating function satisfies Bayes' Law (wherever applicable) given the optimal decision rules.

Consider the decision rules $d^{i*}(\theta_2, -a, \Psi_1^*(-a)) = 1$ for $\theta_2 > \theta_2^{i*}$ and $d^{i*}(\theta_2, -a, \Psi_1^*(-a)) = 0$ for $\theta_2 \leq \theta_2^{i*}$. By Proposition 5 these decision rules are optimal given the belief-updating function $\Psi_2^*(d, \ell, \Psi_1(\ell))$.

We will now establish that there exists a probability distribution function $F_2(\theta)$ for which $\Psi_2^*(d, \ell, \Psi_1(\ell))$ satisfies Bayes' Law wherever applicable given these decision rules. Let Δ_2^{i*} denote the probability that a type i person repays the loan $-a$. Then

$$\Psi_2^*(0, -a, \Psi_1^*(-a)) = \frac{\Delta_2^{g*} \cdot \Psi_1^*(-a))}{\Delta_2^{g*} \cdot \Psi_1^*(-a)) + \Delta_2^{b*} \cdot [(1 - \Psi_1^*(-a))]} \tag{14}$$

and

$$\Psi_2^*(1, -a, \Psi_1^*(-a)) = \frac{(1 - \Delta_2^{g*}) \cdot \Psi_1^*(-a)}{(1 - \Delta_2^{g*}) \cdot \Psi_1^*(-a)) + (1 - \Delta_2^{b*}) \cdot [(1 - \Psi_1^*(-a))]}. \tag{15}$$

Given $\Psi_2^*(d, -a, \Psi_1^*(-a))$, $d \in \{0, 1\}$, these two equations pin down the values of Δ_2^{i*}, $i \in \{b, g\}$. To see this assume (this will be verified to be true) that $\Delta_2^{*g} \in (0, 1)$ and let $\rho_2^* = \Delta_2^{b*}/\Delta_2^{g*}$, where ρ_2^* is the repayment likelihood ratio. Since $\Psi_1^*(-a) > 0$ by P1, (14) can be written as

$$\Psi_2^*(0, -a, \Psi_1^*(-a)) = \frac{1}{1 + \rho_2^* \cdot [(1 - \Psi_1^*(-a))/\Psi_1^*(-a)]} \tag{16}$$

Viewed as a function of ρ_2^*, the r.h.s. of (16) is strictly decreasing in ρ_2^*. Therefore $\Psi_1^*(0, -a, \Psi_1^*(-a))$ determines a unique strictly positive value of ρ_2^*. Furthermore, the r.h.s of (16) is equal to $\Psi_1^*(-a)$ for $\rho_2^* = 1$. Therefore it follows from the last inequality in P2 that $\rho_2^* < 1$, namely, the probability that type b repays is less than the probability that type g

repays. Similarly, letting $\delta_2^* = (1-\Delta_2^{g*})/(1-\Delta_2^{g*})$, where δ_2^* is the default likelihood ratio, (15) can be written as

$$\Psi_2^*(1,-a,\Psi_1^*(-a)) = \frac{1}{1+\delta_2^*\cdot[(1-\Psi_1^*(-a))/\Psi_1^*(-a)]}. \tag{17}$$

The r.h.s of (17) is strictly decreasing in δ_2^* and, given $\Psi_2^*(1,-a,\Psi_1^*)$, determines a unique strictly positive value of δ_2^*. And by the second inequality in P2, it follows that $\delta_2^* > 1$, i.e., the probability of a type b person defaulting is greater than the probability of a type g person defaulting. Now note that from the definitions of ρ_2^* and δ_2^* we have that $\Delta_2^{*g} = [1-\delta_2^*]/[\rho_2^*-\delta_2^*]$. Since $0 < \rho_2^* < 1 < \delta_2^*$ it follows that Δ_2^{g*} is in $(0,1)$ and $\Delta_2^{b*} = \rho_2^*{\cdot}\Delta_2^{g*}$ is also in $(0,1)$. Now consider any distribution function for which $F_2(\theta_2^{g*}) = \Delta_2^{g*}$ and $F_2(\theta_2^{b*}) = \Delta_2^{b*}$. Since $\theta_2^{b*} < \theta_2^{g*}$ and $\Delta_2^{b*} < \Delta_2^{g*}$ such a distribution can always be found. Denote the distribution function by $F_2^*(\theta)$. Given the optimal decision rules and the distribution function $F_2^*(\theta)$, the belief-updating function will, by construction, satisfy Bayes' Law. Therefore there exists a distribution $F_2^*(\theta)$ that delivers the Ψ_2^* consistent with Assumption 3.

$\square$

We turn now to period 1.

Lemma 5: Given $\Psi_1^*(\ell)$ and the default thresholds θ_2^{i*} the period-1 equilibrium price function $q^*(-a,\Psi_1^*(-a)) > 0$.

Proof: The probability of repayment on debt a is $\Psi_1^*(-a)\cdot F_2(\theta_2^{g*}) + (1-\Psi_1^*(-a))\cdot F_2(\theta_2^{b*})$. By Proposition 6 and P1 the probability of repayment is strictly positive. By the zero profit condition it follows that $q^*(-a,\Psi_1^*(-a))$ is also strictly positive.

$\square$

The next proposition establishes that the period-1 decision rules also display a threshold property and the thresholds for the two types can be ordered.

Proposition 7 (i) For $i \in \{b, g\}$, there exists $\theta_1^{i*} > 0$ such that $\ell^{i*}(\theta_1) = 0$ for $\theta_1 \leq \theta_1^{i*}$ and $\ell^{i*}(\theta_1) = -a$ for $\theta_1 > \theta_1^{i*}$. (ii) Furthermore, for β^b sufficiently close to zero $\theta_1^{b*} < \theta_1^{g*}$

Proof: (i) The "gain" from borrowing a is $\theta_1[u(e + q^*(-a, \Psi_1^*(-a)) \cdot a) - u(e)]$ and the "cost" is $\beta^i E_{\theta_2}\left[V_2^{i*}(\theta_2, 0, \Psi_1^*(0)) - V_2^{i*}(\theta_2, -a, \Psi_1^*(-a))\right]$. By Lemma 5 the "gain" is indeed positive. We will now establish that the "cost" is positive as well. Consider first the set of $\theta_2 \leq \theta_2^{i*}$. For these θ_2 the (discounted) cost is given by

$$\beta^i \cdot \theta_2[u(e) - u(e-a)] + (\beta^i)^2 \cdot [V_3^{i*}(\Psi_2^*(0, 0, \Psi_1^*(0))) - V_3^{i*}(\Psi_2^*(0, -a, \Psi_1^*(-a)))].$$

Clearly the first term above is positive. By property P3 $\Psi_2^*(0, 0, \Psi_1^*(0)) \geq \Psi_2^*(0, -a, \Psi_1^*(-a))$. Therefore by Proposition 4 the second term is non-negative. Next consider the $\theta_2 > \theta_2^{i*}$. For these θ_2 the cost is given by

$$(\beta^i)^2 \cdot [V_3^{i*}(\Psi_2^*(0, 0, \Psi_1^*(0))) - V_3^{i*}(\Psi_2^*(1, -a, \Psi_1^*(-a)))] > 0$$

by properties P2-P3 and Proposition 4. Since the cost is positive for all realizations of θ_2, $\beta^i E_{\theta_2}\left[V_2^{i*}(\theta_2, 0, \Psi_1^*(0)) - V_2^{i*}(\theta_2, -a, \Psi_1^*(-a))\right]$ is positive. Then for each type there exists $\theta_1^{i*} > 0$ such that $\ell^{i*}(\theta_1) = 0$ for $\theta_1 \leq \theta_1^{i*}$ and $\ell^{i*}(\theta_1) = -a$ for $\theta_1 > \theta_1^{i*}$.

(ii) To establish that $\theta_1^{b*} < \theta_1^{g*}$ we need to show that the cost of default is strictly lower for type b than for type g. From the expressions given above, it is evident that for any θ_2 the cost of default converges to 0 as β^i converges to 0. Therefore there must exist some β^b sufficiently small for which $\beta^b E_{\theta_2}\left[V_2^{b*}(\theta_2, 0, \Psi_1^*(0)) - V_2^{b*}(\theta_2, -a, \Psi_1^*(-a))\right]$ is less than $\beta^g E_{\theta_2}\left[V_2^{g*}(\theta_2, 0, \Psi_1^*(0)) - V_2^{g*}(\theta_2, -a, \Psi_1^*(-a))\right]$. Given this it follows that if a type g person is indifferent between borrowing and not borrowing for $\theta_1 = \theta_1^{g*}$, then type b must strictly prefer to borrow at $\theta_1 = \theta_1^{g*}$. Hence, $\theta_1^{b*} < \theta_1^{g*}$.

□

We turn now to establishing that there exist distribution functions $F_1(\theta)$ that can support given belief-updating functions Ψ_1^* and Ψ_2^* satisfying P1-P4 as an equilibrium outcome.

Proposition 8 There exists a distribution function $F_1(\theta)$ for which $\Psi_1^*(\ell)$ constitutes a period-1 equilibrium. Furthermore, for this distribution function both types choose 0 and $-a$ with positive probability.

Proof: Recall that a period-1 equilibrium is a pair of decision rules $\ell^{i*}(\theta_1)$, a belief-updating function $\Psi_1^*(\ell)$, and a pricing function $q^*(\ell, \Psi_1^*(\ell))$ such that the decision rules are optimal given the belief-updating function, the pricing function, and the period-2 equilibrium value functions. And the pricing function satisfies zero profits and the belief-updating function satisfies Bayes' Law (wherever applicable) given the optimal decision rules.

The decision rules $\ell^{i*}(\theta_1)$ were derived for the given $\Psi_1^*(\ell)$ function, the equilibrium period-2 value functions, and pricing function satisfying zero profits. Therefore, all we need to show is that there exists a distribution function $F_1(\theta)$ for which $\Psi_1^*(\ell)$ satisfies Bayes' Rule. The proof closely follows the logic of the proof of Proposition 6 and therefore will be sketched.

Denote the probability that type i chooses $\ell = 0$ with Δ_1^{i*}. Then the probability that type i chooses $\ell = -a$ is $(1 - \Delta_1^{i*})$. Assume (and this will be verified to be true) that Δ_1^{g*} is in $(0, 1)$. Then satisfaction of Bayes' Rule requires:

$$\Psi_1^*(0) = \frac{\Delta_1^{g*} \cdot \gamma}{\Delta_1^{g*} \cdot \gamma + \Delta_1^{b*} \cdot (1 - \gamma)} \text{ and } \Psi_1^*(-a) = \frac{(1 - \Delta_1^{g*}) \cdot \gamma}{(1 - \Delta_1^{g*}) \cdot \gamma + (1 - \Delta_1^{b*}) \cdot (1 - \gamma)}.$$

Given $\Psi_1^*(\ell)$, these two equations determine the values of Δ_1^{b*} and Δ_1^{g*}. It can be shown that P1 implies (i) $\Delta_1^{i*} \in (0, 1)$ and (ii) $\Delta_1^{b*} < \Delta_1^{g*}$ (the proof parallels the one given in Proposition 6 for Δ_2^{i*} and is omitted).

Now consider any distribution function for which $F_1(\theta_1^{g*}) = \Delta_1^{g*}$ and $F_1(\theta_2^{b*}) = \Delta_1^{b*}$. Since $\theta_1^{b*} < \theta_1^{g*}$ (by Proposition 7) and $\Delta_1^{b*} < \Delta_1^{g*}$ such a distribution always exists. Denote the distribution function by $F_1^*(\theta)$. Given the optimal decision rules and the distribution function $F_1^*(\theta)$, the belief-updating function $\Psi_1^*(\ell)$ will, by construction, satisfy Bayes' Law.

□

What have we learned? It is possible to construct equilibria in which what we have called the type score behaves like a credit score. In the constructed equilibrium (i) a person's type score declines as the person borrows more (as in P1), (ii) default on debt lowers a person's type score (as in P2), and (iii) people with low type scores get worse insurance rates and are, on average, more likely to file a claim (suffer the loss L). However, our type score goes beyond a credit score in that it takes into account a person's asset information (as in P1 and P4), which real-world credit scores do not take into account but which credit granters might.[12]

The construction relied on the assumption that people differ with respect to their discount factors and those who are less patient are also more likely to suffer the loss. Although not modeled, the idea underlying the link between patience and loss probability is that the loss probability can be reduced if a person undertakes some costly investment. Since less patient people are less likely to invest, there is likely to be a positive association between the degree of impatience and loss probability.

The construction leaned rather heavily on the fact that there was a single level of debt that people could choose. Can the results be generalized to many asset/loan levels? There is a close connection between properties P1 and P2 and the monotone comparative statics results of Milgrom and Shannon [10] and Athey [1]. These authors study conditions under which a decisionmaker's optimal choice (say, to default or not) is increasing in some parameter (say, β). They show that there is a monotonic relationship between the optimal choice and the parameter, provided the difference in payoff from choosing one action over the other is increasing in the parameter (the so-called *property of increasing difference*). The proofs of Propositions 5 and 7 relied on the property of increasing difference. In Proposition 5 we established that the (expected) benefit from choosing repayment over default was increasing

[12] To see what kinds of information credit scores are based on and what kinds of information offers of credit are based on, see `http://www.myfico.com/Downloads/Files/myFICO_UYFS_Booklet.pdf`

in β. Similarly in Proposition 7 we established that the expected gain from choosing $\ell = 0$ over choosing $\ell = -a$ was also increasing in β. These results were instrumental in establishing the ordering of default and debt thresholds that allowed us to establish the existence of distribution functions that support belief-updating functions with the desired properties. Generalizing the results of this section to many asset choices requires that we generalize the increasing difference property to the many-asset case. This appears to be a challenging exercise.[13] However, the second approach to modeling differences between people obviates the need to rely on the property of increasing differences and can handle multiple asset choices with less difficulty. We turn to this alternative approach next.

6.2 Credit Market Equilibrium with Differences in the Distribution of Preference Shocks

In this subsection we assume that discount factors are the same for both types but permit the distribution of preference shocks to be different across types. We will assume that A is a finite set with potentially many elements. It will be convenient to label the elements of A by ℓ_k, $k = 1, 2, \ldots, K$, where $\ell_{k+1} < \ell_k$. We will assume that A has both negative and positive elements with $\ell_J = 0$, $1 < J < K$. Hence $\ell_k \geq 0$ for $k \leq J$ and $\ell_k < 0$ for $k > J$.

We will continue to suppose that we are given belief-updating functions $\Psi_1^*(\ell)$ and $\Psi_2^*(d, \ell, \Psi_1^*(\ell))$ that satisfy properties P1-P4. However, in this section we will find it necessary to restrict these functions further. The following two assumptions ensure that the cost of default is strictly positive for both types.

Assumption 4: For all ℓ_k, $\Psi_2^*(0, \ell_k, \Psi_1^*(\ell_k)) > \sigma^*$. This assumption requires that repayment on debt lead to a score greater than σ^*. Note that by property P4, it also requires that saving lead to a score greater than σ^*.

[13] Multiple asset choices can be handled if the asset levels are chosen with care, but for arbitrary choice of asset levels, however, the increasing difference property need not hold.

Assumption 5: For $\ell_k < 0$, $\Psi_2^*(1, \ell_k, \Psi_1^*(\ell_k)) < \sigma^*$. This assumption requires that default lead to a score less than σ^*.

By Lemma 3 we know that period-2 default decision rules satisfy the threshold property. What we show next is that, for each type, the default thresholds are strictly positive and that lower the thresholds, the higher the debt. This result has no analog in the previous subsection, since there was only one debt level.

Proposition 9 Given Assumptions 4 and 5, there exist unique and strictly positive default thresholds $\theta_{2,k}^{i*}$ for each i and $\ell_k < 0$. Furthermore, $\theta_{2,k+1}^{i*} < \theta_{2,k}^{i*}$.

Proof: The benefit from default is given by $\theta_2[u(e) - u(e + \ell_k)] > 0$. The cost of default is given by $\beta[V_3^{i*}(\Psi_2^*(0, \ell_k, \Psi_1^*(\ell_k))) - V_3^{i*}(\Psi_2^*(1, \ell_k, \Psi_1^*(\ell_k)))]$. By Assumption 5 and Proposition 4, we can write the cost as $\beta[V_3^{i*}(\Psi_2^*(0, \ell_k, \Psi_1^*(\ell_k))) - V_3^{i*}(0)]$. By Assumption 4 and Proposition 4 $V_3^{i*}(\Psi_2^*(0, \ell_k, \Psi_1^*(\ell_k))) > V_3^{i*}(0)$. Therefore, the cost of default is strictly positive for both types. Hence there must exist $\theta_{2,k}^{i*} > 0$ such that $\theta_{2,k}^{i*}[u(e) - u(e + \ell_k)] = \beta^i[V_3^{i*}(\Psi_2^*(0, \ell_k, \Psi_1^*(\ell_k))) - V_3^{i*}(0)]$. To prove that the thresholds are increasing in ℓ_k, observe that by P3 and Proposition 4 $V_3^{i*}(\Psi_2^*(0, \ell_k, \Psi_1^*(\ell_k)))$ is nondecreasing in ℓ_k. Hence the cost of default is non-decreasing in ℓ_k. Since the benefit of default is strictly decreasing in ℓ_k it follows that default thresholds $\theta_{2,k}^{i*}$ are strictly increasing in ℓ_k, or equivalently, $\theta_{2,k+1}^{i*} < \theta_{2,k}^{i*}$ (recall that higher k means more debt).

$\square$

We now turn to establishing that there exist distribution functions $F_2^i(\theta)$ that can support the given belief-updating function $\Psi_2^*(d, \ell, \Psi_1^*(\ell))$ in a period-2 equilibrium. To establish this we need to restrict the belief-updating function further. Consider debt $\ell_k < 0$ and let $\Delta_{2,k}^{*i}$ be the equilibrium probability of repayment ℓ_k by a type i person. Let $\rho_{2,k}^* = \Delta_{2,k}^{*b}/\Delta_{2,k}^{*g}$ denote the repayment likelihood ratio on debt ℓ_k and $\delta_{2,k}^* = (1 - \Delta_{2,k}^{b*})/(1 - \Delta_{2,k}^{*g})$ denote the default likelihood ratio on debt ℓ_k. We make the following assumption.

Assumption 6: For $k = J + 1, \ldots, K$, (i) $\Delta_{2,k}^{*g}$ is in $(0, 1)$, (ii) $\rho_{2,k}^*$ is strictly decreasing in

k and (iii) $\Delta^{g*}_{2,k}$ is also strictly decreasing in k.

We know from the arguments given in the proof of Proposition 6 that the probabilities of repayment and default by type i are fixed by $\Psi^*_2(d, \ell_k, \Psi^*_1(\ell_k))$, $d \in \{0, 1\}$ and that P2 implies that $\rho^*_{2,k} < 1 < \delta^*_{2,k}$. Before we can proceed we need to verify that Assumption 6 does not conflict with Assumptions 4 and 5 and properties P1-P4. To recap the dependence between likelihood ratios and the probability of repayment note that the definitions of these ratios imply that $\Delta^{g*}_{2,k} = [\delta^*_{2,k} - 1]/[\delta^*_{2,k} - \rho^*_{2,k}]$. Furthermore $\Psi^*_2(0, \ell_k, \Psi^*_1(\ell_k)) = 1/[1 + \rho^*_{2,k} \cdot [(1 - \Psi^*_1(\ell_k))/\Psi^*_1(\ell_k)]]$ and $\Psi^*_1(1, \ell_k, \Psi^*_1(\ell_k)) = 1/[1 + \delta^*_{2,k} \cdot [(1 - \Psi^*_1(\ell_k))/\Psi^*_1(\ell_k)]]$. Hence the likelihood ratios are fixed by the period-2 belief-updating function. Turning to Assumption 6(i) the requirement that $\Delta^{*g}_{2,k}$ be in $(0, 1)$ follows from P2 (namely, from the requirement that $\Psi^*_2(1, \ell, \Psi^*_1) < \Psi^*_1(\ell_k) < \Psi^*_2(0, \ell, \Psi^*_1)$ which implies that $\rho^*_{2,k} < 1 < \delta^*_{2,k}$). Turning to Assumption 6(ii), note that Property P3 requires that the product $\rho^*_{2,k} \cdot [(1 - \Psi^*_1(\ell_k))/\Psi^*_1(\ell_k)]$ be non-decreasing in ℓ_k, or equivalently, non-increasing in k. But by property P1, the term $[(1 - \Psi^*_1(\ell_k))/\Psi^*_1(\ell_k)]$ is decreasing in ℓ_k, or equivalently, increasing in k. Therefore, it is possible for $\rho^*_{2,k}$ to be strictly decreasing in k without violating P3. Finally, by choosing the sequence $\delta^*_{2,k}$ appropriately we can ensure that Assumption 6(iii) is also satisfied, namely $\Delta^{*g}_{2,k} = [\delta^*_{2,k} - 1]/[\delta^*_{2,k} - \rho^*_{2,k}]$ is decreasing in k. It is worth noting that the variation in $\delta^*_{2,k}$ can freely be chosen because Assumption 5 (which restricts $\delta^*_{2,k}$) requires only that $1/[1 + \delta^*_{2,k} \cdot [(1 - \Psi^*_1(\ell_k))/\Psi^*_1(\ell_k)]]$ be less than σ^*.[14]

We can now state the following result.

Proposition 10: Given Assumptions 4-6, there exist distribution functions $F^i_2(\theta_2)$ for which the belief-updating functions $\Psi^*_2(d, \ell_k, \Psi^*_1(\ell_k))$ constitute a period-2 equilibrium.

Proof: The fact that the thresholds $\theta^{i*}_{2,k}$ are optimal given the belief-updating function $\Psi^*_2(d, \ell_k, \Psi^*_1(\ell_k))$ follows from Proposition 9. Let $F^{g*}_2(\theta_2)$ be such that $F^{g*}_2(\theta^{g*}_{2,k}) = \Delta^{g*}_{2,k}$ for $k = J + 1, ..., K$. Since $\Delta^{g*}_{2,k}$ is declining in k by Assumption 6 and $\theta^{g*}_{2,k}$ is declining

[14] This degree of freedom comes from the fact that if $1/[1 + \delta^*_{2_k} \cdot [(1 - \Psi^*_1(\ell_k))/\Psi^*_1(\ell_k)]]$ is less than σ^* then by Proposition 4 $V^{i*}_3(s')$ is *constant* for any $s' < \sigma^*$. Thus the precise value of $\delta^*_{2,k}$ is not important for determining the payoff from default.

in k by Proposition 9 such a distribution exists. Let $F_2^{b*}(\theta_2)$ be such that $F_2^{b*}(\theta_{2,k}^{b*}) = \rho_{2,k}^* \cdot \Delta_{2,k}^{g*}$. Observe that since $\rho_{2,k}^*$ and $\Delta_{2,k}^{*g}$ are both in $(0,1)$ the product $\rho_{2,k}^* \cdot \Delta_{2,k}^{g*}$ is in $(0,1)$. Since the product is declining by Assumption 6 and $\theta_{2,k}^{b*}$ is declining by Proposition 9, such a distribution also exists. Given the optimal decision rules and the distribution function $F_2^{i*}(\theta)$, the belief-updating function $\Psi_2^*(d, \ell_k, \Psi_1^*(\ell))$ will, by construction, satisfy Bayes' Law. Therefore there exist distribution functions $F_2^{i*}(\theta_2)$ that deliver a belief-updating function Ψ_2^* consistent with Assumptions 4, 5 and 6.

□

In the previous subsection we provided an explanation of why a person's type score might decline with default that was based on differences in discount factors between the low-risk and the high-risk types. These differences implied that the default threshold for type b (the high-risk type from the insurer's perspective) was lower than the default threshold for type g. Since the two types faced the same distribution of preference shocks, the ordering of default thresholds implied that default should lower the likelihood that the person is type g. In contrast, Proposition 10 provides a very different explanation of why default leads to a decline in the score. The difference can be encapsulated in the following corollary to Proposition 9.

Corollary to Proposition 9: For $\ell_k < 0$, the default threshold $\theta_{2,k}^{g*}$ is strictly less than the default threshold $\theta_{2,k}^{b*}$.

Proof: The cost of default for type i is given by $\beta[V_3^{i*}(\Psi_2^*(0, \ell_k, \Psi_1^*)) - V_3^{i*}(0)]$. By Proposition 4, $V_3^{b*}(0) < V_3^{g*}(0)$. Therefore, the cost of default is *larger* for type b than for type g. Given a θ_2, the benefit of default is the same for both types, namely, $\theta_2[u(e) - u(e + \ell_k)]$. Therefore, $\theta_{2,k}^{g*} < \theta_{2,k}^{b*}$.

□

From the perspective of decision rules, type b people have *less* of a predilection toward default than type g people, which is exactly opposite to what was true in the case of differences in discount factors. Nevertheless, default leads to a lower type score because type b people are more likely to get high preference shocks. Indeed, combining the corollary with Assumption 6 (ii) and (iii) we have that for each $\ell_k < 0$, $F_2^{b*}(\theta_{2,k}^{b*}) < F_2^{g*}(\theta_{2,k}^{g*})$ even though $\theta_{2,k}^{b*} > \theta_{2,k}^{g*}$. These inequalities imply that $F_2^{b*}(\theta_{2,k}^{i*}) < F_2^{g*}(\theta_{2,k}^{i*})$ for all $\theta_{2,k}^{i*}$ – which is consistent with $F_2^{b*}(\theta_2)$ first-order stochastic dominating $F_2^{g*}(\theta_2)$, i.e., with type b distribution putting relatively more probability mass on high values of θ and the type g distribution putting relatively more probability mass on low values of θ.

We turn now to period 1 and to establishing that the period-1 belief-updating function $\Psi_1^*(\ell)$ can also be supported as an equilibrium for some pair of distribution functions $F_1^i(\theta)$. To do so we establish the following monotonicity result with regard to the expected value function in period 2.

Lemma 6: $\int_{\theta_2} V_2^{i*}(\theta_2, \ell_k, \Psi_1^*(\ell_k))dF_2^i(\theta_2)$ is strictly increasing in ℓ_k or, equivalently, strictly decreasing in k.

Proof: Let $\ell_{k'} > \ell_k$. We will consider 3 mutually exclusive and exhaustive cases.

Case 1: $\ell_{k'} > \ell_k \geq 0$. Then $V_2^{i*}(\theta_2, \ell_{k'}, \Psi_1^*(\ell_{k'})) = \theta_2 u(e + \ell_{k'}) + \beta V_3^{i*}(\Psi_2^*(0, \ell_{k'}, \Psi_1^*(\ell_k)))$. But for $\ell_k' > 0$ we have $\Psi_2^*(0, \ell_k, \Psi_1^*(\ell_k)) = \Psi_1^*(\ell_k)$. Therefore $V_2^{i*}(\theta_2, \ell_{k'}, \Psi_1^*(\ell_{k'})) = \theta_2 u(e + \ell_{k'}) + \beta V_3^{i*}(\Psi_1^*(\ell_{k'}))$. Similarly, $V_2^{i*}(\theta_2, \ell_k, \Psi_1^*(\ell_k)) = \theta_2 u(e + \ell_k) + \beta V_3^{i*}(\Psi_1^*(\ell_k))$. By P1 and Assumption 4, $\Psi_1^*(\ell_{k'}) > \Psi_1^*(\ell_k) > \sigma^*$. Then by Proposition 4 $V_3^{i*}(\Psi_1^*(\ell_{k'})) > V_3^{i*}(\Psi_1^*(\ell_k))$. Since $\ell_{k'} > \ell_k$, $V_2^{i*}(\theta_2, \ell_{k'}, \Psi_1^*(\ell_{k'})) > V_2^{i*}(\theta_2, \ell_k, \Psi_1^*(\ell_k))$ for each θ_2. The result then follows.

Case 2: Let $\ell_{k'} \geq 0 > \ell_k$. Within this case, consider first $\theta_2 \leq \theta_{2,k}^{i*}$. By Proposition 9. these are θ_2 values for which there is repayment of the debt ℓ_k. Hence $V_2^{i*}(\theta_2, \ell_k, \Psi_1^*(\ell_k)) = \theta_2 u(e + \ell_k) + \beta V_3^{i*}(\Psi_2(0, \ell_k, \Psi_1^*))$. The expression for $V_2^{i*}(\theta_2, \ell_{k'}, \Psi_1^*(\ell_{k'}))$ is exactly the same as in case (1). By P4 $\Psi_1^*(\ell_{k'}) = \Psi_2(0, \ell_k', \Psi_1^*(\ell_k'))$ and by P3 and Assumption 4 $\Psi_2(0, \ell_k', \Psi_1^*(\ell_k')) > \Psi_2(0, \ell_k, \Psi_1^*(\ell_k)) > \sigma^*$. By Proposition 4 $V_2^{i*}(\theta_2, \ell_{k'}, \Psi_1^*(\ell_{k'})) > V_2^{i*}(\theta_2, \ell_k, \Psi_1^*(\ell_k))$. Next consider $\theta_2 > \theta_{2,k}^{i*}$. For these θ_2 there is default on the debt ℓ_k. Therefore $V_2^{i*}(\theta_2, \ell_k, \Psi_1^*(\ell_k)) = \theta_2 u(e) + \beta V_3^{i*}(\Psi_2(1, \ell_k, \Psi_1^*)$. The expression for $V_2^{i*}(\theta_2, \ell_{k'}, \Psi_1^*(\ell_{k'}))$ is exactly the same as in case (1) again. By P2, Assumption 5 and Proposition 4 $V_3^{i*}(\Psi_1^*(\ell_{k'})) > V_3^{i*}(\Psi_2^*(1, \ell_k, \Psi_1^*(\ell_k)))$. Since $\ell_{k'} > 0$, $V_2^{i*}(\theta_2, \ell_{k'}, \Psi_1^*(\ell_{k'})) > V_2^{i*}(\theta_2, \ell_k, \Psi_1^*(\ell_k))$ for each θ_2. The result follows.

Case 3: Finally consider the case where $0 > \ell_{k'} > \ell_k$. Within this case there are 3 subcases to consider. Consider first $\theta_2 < \theta_{2,k}^{i*}$. By Proposition 9, this is the set of θ_2 for which there is no default on $\ell_{k'}$ or ℓ_k. Thus $V_2^{i*}(\theta_2, \ell_{k'}, \Psi_1^*(\ell_{k'})) = \theta_2 u(e + \ell_{k'}) + \beta V_3^{i*}(\Psi_2^*(0, \ell_{k'}, \Psi_1^*(\ell_{k'})))$ and $V_2^{i*}(\theta_2, \ell_k, \Psi_1^*(\ell_k)) = \theta_2 u(e + \ell_k) + \beta V_3^{i*}(\Psi_2^*(0, \ell_k, \Psi_1^*(\ell_k)))$. Since $\ell_{k'} > \ell_k$ and by P3 and Assumption 4,$\Psi_2^*(0, \ell_{k'}, \Psi_1^*(\ell_{k'})) > \Psi(0, \ell_k, \Psi_1^*(\ell_k)) > \sigma^*$ it follows by Proposition 4 that $V_2^{i*}(\theta_2, \ell_{k'}, \Psi_1^*(\ell_{k'})) > V_2^{i*}(\theta_2, \ell_k, \Psi_1^*(\ell_k))$ Next consider $\theta_2 \in [\theta_{2,k}^{i*}, \theta_{2,k'}^{i*}]$. For these values there is repayment on ℓ_k' but default on ℓ_k. Therefore $V_2^{i*}(\theta_2, \ell_{k'}, \Psi_1^*(\ell_{k'}))) = \theta_2 u(e + \ell_{k'}) + \beta V_3^{i*}(\Psi(0, \ell_{k'}, \Psi_1^*(\ell_{k'})))$ and $V_2^{i*}(\theta_2, \ell_k, \Psi_1^*(\ell_k))) = \theta_2 u(e) + \beta V_3^{i*}(\Psi_2^*(1, \ell_k, \Psi_1^*(\ell_k)))$. Since default is an option on $\ell_{k'}$, $\theta_2 u(e+\ell_{k'})+\beta V_3^{i*}(\Psi_2^*(0, \ell_{k'}, \Psi_1^*(\ell_{k'}))) \geq \theta_2 u(e)+\beta V_3^{i*}(\Psi_2^*(1, \ell_{k'}, \Psi_1^*(\ell_{k'})))$. By Assumption 5 and Proposition 4, $\theta_2 u(e) + \beta V_3^{i*}(\Psi_2^*(1, \ell_{k'}, \Psi_1^*(\ell_{k'})))$ is equal to $\theta_2 u(e) + \beta V_3^{i*}(\Psi_2^*(1, \ell_k, \Psi_1^*(\ell_k)))$. Therefore $V_2^{i*}(\theta_2, \ell_{k'}, \Psi_1^*(\ell_{k'})) \geq V_2^{i*}(\theta_2, \ell_k, \Psi_1^*(\ell_k))$. Finally, consider $\theta_2 > \theta_{2,k'}^{*i}$. For these θ_2 there is default on both $\ell_{k'}$ and ℓ_k. Consequently, by the argument in the preceding case, the utility obtained is the same for $\ell_{k'}$ and ℓ_k. Since the utility obtained for $\ell_{k'}$ is strictly larger than utility obtained for ℓ_k for $\theta_2 < \theta_{2,k'}^{*i}$ and at least as large for all other θ_2 we may conclude $\int_{\theta_2} V_2^{i*}(\theta_2, \ell_{k'}, \Psi_1^*(\ell_{k'}))dF_2(\theta_2) > \int_{\theta_2} V_2^{i*}(\theta_2, \ell_k, \Psi_1^*(\ell_k))dF_2(\theta_2)$.

□

We can now prove that in period 1 the decision rules $\ell_k^{i*}(\theta_1)$ are decreasing in θ_1, but need the following preliminary Lemma.

Lemma 7: Given Assumption 6 and the period-1 belief-updating function $\Psi_1^*(\ell_k)$, the period-1 equilibrium pricing function $q^*(\ell_k, \Psi_1^*(\ell_k))$ is strictly positive for all ℓ_k.

Proof: Consider $\ell_k < 0$. By Assumption 6, the repayment probability on the debt ℓ_k, $\Delta_{2,k}^{i*}$, is in $(0, 1)$ and by P1 $\Psi_1^*(\ell_k) > 0$. Therefore the equilibrium probability of repayment $1 - \mu^*(\ell_k) = \Psi_1^*(\ell_k)\Delta_{2,k}^{g*} + [1 - \Psi_1^*(\ell_k)]\Delta_{2,k}^{b*}$ is in $(0, 1)$. By the zero profit condition on the pricing of loans it follows that $q(\ell_k, \Psi_1^*) = [1 - \mu^*(\ell_k)]/(1 + r) > 0$. Next, consider $\ell_k > 0$.

By the zero profit condition on pricing of loans $q^*(\ell, \Psi_1^*(\ell_k)) = 1/(1+r) > 0$.

$\square$

Proposition 11: $\hat{\theta}_1 > \tilde{\theta}_1$ implies $\ell^{i*}(\hat{\theta}_1) \leq \ell^{i*}(\tilde{\theta}_1)$.

Proof: Pick a shock $\tilde{\theta}_1$ and let $\tilde{\ell} = \ell^{i*}(\tilde{\theta}_1)$. Consider $\bar{\ell} > \tilde{\ell}$. Then we claim that $q^*(\tilde{\ell}, \Psi_1^*(\tilde{\ell})) \cdot \tilde{\ell} < q^*(\bar{\ell}, \Psi_1^*(\bar{\ell})) \cdot \bar{\ell}$. If not, the person can select $\bar{\ell}$ and guarantee at least as much utility in period 1 and strictly more utility starting period 2 because by Lemma 6 $\int_{\theta_2} V_2^{i*}(\theta_2, \ell, \Psi_1^*(\ell)) dF_2^i(\theta_2)$ is strictly increasing in ℓ. This would contradict the optimality of $\tilde{\ell}$. Next, by optimality, $\tilde{\theta}_1 \left[u(e - q^*(\tilde{\ell}, \Psi_1^*(\tilde{\ell})) \cdot \tilde{\ell}) - u(e - q^*(\bar{\ell}, \Psi_1^*(\bar{\ell})) \cdot \bar{\ell}) \right]$ is at least as large as $\beta E_{\theta_2} \left[V_2^{i*}(\theta_2, \bar{\ell}, \Psi_1^*(\bar{\ell})) - V_2^{i*}(\theta_2, \tilde{\ell}, \Psi_1^*(\tilde{\ell})) \right]$. But $q^*(\tilde{\ell}, \Psi_1^*(\tilde{\ell})) \cdot \tilde{\ell} < q^*(\bar{\ell}, \Psi_1^*(\bar{\ell})) \cdot \bar{\ell}$ implies that $\tilde{\theta}_1 \left[u(e - q^*(\tilde{\ell}, \Psi_1^*(\tilde{\ell})) \cdot \tilde{\ell}) - u(e - q^*(\bar{\ell}, \Psi_1^*(\bar{\ell})) \cdot \bar{\ell}) \right] > 0$. Therefore for $\hat{\theta}_1 > \tilde{\theta}_1$, $\hat{\theta}_1 \left[u(e - q^*(\tilde{\ell}, \Psi_1^*(\tilde{\ell}) \cdot \tilde{\ell})) - u(e - q^*(\bar{\ell}, \Psi_1^*(\bar{\ell}) \cdot \bar{\ell})) \right]$ is greater than $\beta E_{\theta_2}[V_2(\theta_2, \bar{\ell}, \Psi_1^*(\bar{\ell})) - V_2(\theta_2, \tilde{\ell}, \Psi_1^*(\tilde{\ell}))]$. Since $\bar{\ell}$ was arbitrary, it follows that $\ell^{i*}(\hat{\theta}_1) \leq \tilde{\ell} = \ell^{i*}(\tilde{\theta}_1)$.

$\square$

We are now in a position to establish that there exist period-1 distribution functions that support the given period-1 belief-updating function. For ease of exposition, however, we will assume that for the (given) equilibrium $q^*(\ell_k, \Psi_1^*(\ell_k))$ every element of A is chosen by each type for some θ_1. We do this because if there are actions that are not chosen by any type, then those actions do not have to satisfy a Bayes' Rule restriction. But it is important that if a particular loan level $\hat{\ell}_k \in A$ is chosen by a type b person then $\hat{\ell}_k$ must also be chosen by a type g person for some θ_1.

Proposition 12: Let $\ell^{i*}(\theta_1)$ be such that for each k and i, $\ell^{i*}(\theta_1) = \ell_k$ for some θ_1. Then there exist distribution functions $F_1^i(\theta_1)$ for which the belief-updating functions $\Psi_1^*(\ell_k)$ constitute a period-1 equilibrium and for which each ℓ_k is chosen with positive probability.

Proof: By Proposition 11, $\ell^{i*}(\theta_1)$ is decreasing in θ_1. Since each element of A is chosen for some θ_1, it follows that each decision rule delivers a set of strictly increasing threshold values $\{\theta^{*i}_{1,k}\}_{k=1}^{K}$ such that $\ell^{i*}(\theta_1) = \ell_k$ iff $\theta_1 \in [\theta^{*i}_{1,k}, \theta^{*i}_{1,k+1})$, where $\theta^{*i}_{1,1} = 0$ and $\theta^{*i}_{1,K+1} = \infty$. Satisfaction of Bayes' Rule requires

$$\Psi_1^*(\ell_k) = \frac{\gamma[F^g(\theta^{g*}_{1,k+1}) - F^g(\theta^{g*}_{1,k})]}{\gamma[F^g(\theta^{g*}_{1,k+1}) - F^g(\theta^{g*}_{1,k})] + (1-\gamma)[F^b(\theta^{b*}_{1,k+1}) - F^b(\theta^{b*}_{1,k})]} \text{ for all } k.$$

Denote $\Delta^{i*}_{1,k} = F^i(\theta^{i*}_{1,k+1}) - F^i(\theta^{i*}_{1,k})$ and assume that $\Delta^{i*}_{1,k} > 0$. Let $\lambda^*_{1,k} = \Delta^{b*}_k/\Delta^{g*}_k$. Then $\Psi_1^*(\ell_k) = \left[1 + \left(\frac{1-\gamma}{\gamma}\right)\lambda^*_{1,k}\right]^{-1}$. Clearly, $\Psi_1^*(\ell_k)$ fixes a unique value $\lambda^*_{1,k}$ for each k. Since $\Psi_1^*(\ell_k)$ is strictly decreasing in k by P1, $\lambda^*_{1,k}$ is strictly increasing in k. Then, satisfaction of Bayes' Rule requires that there exist strictly positive probability weights $\{\Delta^{i*}_{1,k}\}$ that satisfy the following two conditions:

$$\Delta^{b*}_{1,k} = \lambda^*_{1,k}\Delta^{g*}_{1,k}, \text{ for all } k \text{ and } i \tag{18}$$

$$\sum_{k=1}^{K} \Delta^{i*}_{1,k} = 1 \text{ for } \text{ for all } i \tag{19}$$

If such weights can be found, then any two distributions that satisfy $F_1^g(\theta^{g*}_{1,k+1}) = \sum_{j=1}^{k} \Delta^{g*}_{1,k}$ and $F_1^b(\theta^{b*}_{1,k+1}) = \sum_{j=1}^{k} \lambda^*_{1,k}\Delta^{g*}_{1,k}$ will satisfy the Bayes' Rule requirements.

Observe that the condition $\Psi_1^*(\ell_1) > \gamma > \Psi_1^*(\ell_K)$ in P1 is equivalent to the condition $\lambda^*_{1,1} < 1 < \lambda^*_{1,K}$. Then there must exist strictly positive $\overline{\Delta}^g_{1,1}$ and $\overline{\Delta}^g_{1,K}$ that sum to 1 and for which $\lambda^*_{1,1}\overline{\Delta}^g_{1,1} + \lambda^*_{1,K}\overline{\Delta}^g_{1,K} = 1$. Let $\overline{\lambda_1} = \sum_{k=2}^{K-1} \lambda^*_{1,k}/(K-2)$ and let $\epsilon > 0$. Now set (i) $\Delta^{g*}_{1,2} = \cdots = \Delta^{g*}_{1,k} = \cdots \Delta^{g*}_{1,K-1}$ to $\epsilon/K-2$, (ii) $\Delta^{g*}_{1,1} = \overline{\Delta}^g_{1,1} - [(\lambda^*_{1,K} - \overline{\lambda_1})\epsilon/(\lambda^*_{1,K} - \lambda^*_{1,1})]$, (iii) $\Delta^{g*}_{1,K} = \overline{\Delta}^g_{1,K} - [(\overline{\lambda_1} - \lambda^*_{1,1})\epsilon/(\lambda^*_{1,K} - \lambda^*_{1,1})]$, and (iv) $\Delta^{b*}_{1,k} = \lambda^*_{1,k}\Delta^{g*}_{1,k}$. For these settings one can verify that both $\{\Delta^{g*}_{1,k}\}$ and $\{\Delta^{b*}_{1,k}\}$ sum to 1. Therefore both (18) and (19) are satisfied.

□

What have we learned? We showed that it is possible to construct equilibria with multiple asset levels in which the type score again behaves like a credit score. We were able to do so because there was no need to prove the analogs of Propositions 5(ii) and 7(ii) in this subsection. Instead the construction relied on differences in the probability distribution of shocks to deliver the requisite properties of the belief-updating functions. For period 2, these differences were consistent with the type g shock distribution being dominated by the type b shock distribution in the first-order stochastic dominance sense.

7 Conclusion

In this paper we present a theory of unsecured consumer debt that does not rely on utility costs of default or on enforcement mechanisms that arise in repeated-interaction settings. The theory is based on private information about a person's type and on a person's incentive to signal his type to entities other than creditors. Specifically, debtors signal their low-risk status to insurers by avoiding default in credit markets. The signal is credible because in equilibrium people who repay are more likely to be the low-risk type and so receive better insurance terms.

Our theory was motivated, in part, by facts regarding the role of credit scores in consumer credit and auto insurance markets. Facts indicate that people with high scores receive credit on cheaper terms, that scores decline with default, and that (given credit limits) greater borrowing leads to lower scores. Also, drivers with high scores pay lower premiums for auto insurance. In the theory presented in this paper, the likelihood that a person is the low-risk type (what we labeled type score) displays these four properties of a credit score. In that sense, we have provided a theory of credit scores as well.

Interestingly, we presented two rather different mechanisms via which a credit score with the four properties could arise. In the first mechanism, the low-risk types were also assumed to be the more patient type. In this case, the low-risk people avoided default because they cared more about the future insurance benefit of a higher score. In the second mechanism, the low-risk people were equally as patient as the high-risk people but they were less likely to suffer shocks that trigger default. Once again, repayment became a signal of low-risk status. Loosely speaking, the first mechanism relies on differences in behavior while the second mechanism relies on differences in luck.

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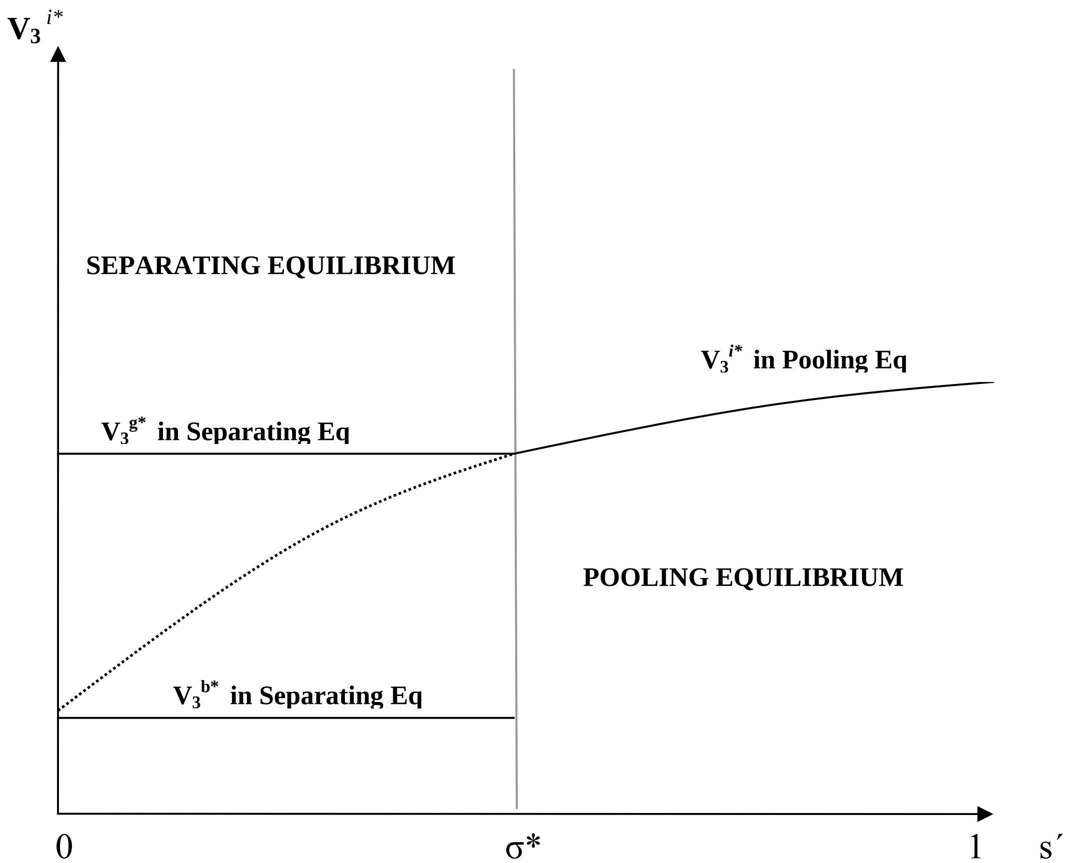
Figure 1



Properties of Equilibrium in the Insurance Market