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NOTE ON A TWO-PLAYER ALL-PAY AUCTION WITH ASYMMETRICAL BIDDERS AND INCOMPLETE INFORMATION

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Abstract

The present paper analyzes a general class of first-price all-pay auctions where two players have different "bidding technologies" and one bidder has a head start advantage over his/her opponent. Equilibria are characterized for the complete information setting and for the case where there is incomplete asymmetrical information. In particular, the handicapped player is uncertain about the size of the opponent's advantage.
Keywords: All-pay auctions and Auction theory and Games with asymmetrical players and Incomplete information games

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INTRODUCTION

In all-pay auctions, players must pay for their bids whether or not they win. In other words, participation involves a resource commitment which is not conditional on the event of winning, but is certain from the beginning. This description fits a large number of circumstances where competition between economic agents requires making an irreversible investment before its results are observed. After the seminal contributions by Hillman and Riley (1989) and Baye et al. (1993 and 1996), all-pay auctions have been widely studied and applied to several fields, in economics ranging from rent-seeking activities (Tullock (1980); Becker (1983); Hillman and Riley (1989)) to redistributive politics (Sahuguet and Persico (2006)), from competition for patents (Fudenberg et al. (1983)) and for monopoly positions (Ellingsen (1991)), to waiting in line (Clark and Riis (1998)), sales (Varian (1980)) and firm decisions over R&D investments (Dasgupta and Stiglitz (1980))[1]. The present paper contributes to this debate, analyzing an issue which has received growing attention in recent years: the case of asymmetrical players (see Siegel (2009)).

A first-price all-pay auction is analyzed where each player has his/her own valuation of the prize and also his/her own "bidding technology". Identical investments thus define different bids. This is due to asymmetries in money marginal productivity and to a fixed head start advantage awarded to a favored player. Adding this element has some relevant effects. If, indeed, it is possible to account for differences in productivity rescaling players' prize values, as proven by Baye et al. (1996), the head start advantage requires a redefinition of the equilibrium strategies.

Studying a setting with a favored player is of some interest because in many real world circumstances there are incumbency advantages. Moreover, it allows another fairly common situation to be considered, where asymmetries involve the distribution of information. Usually, the size of the advantage of an incumbent is uncertain for potential entrants.

The main result of the paper is the characterization of an equilibrium for an auction where there is asymmetrical incomplete information and the handicapped player does not know the size of the head start advantage of his/her opponent. The present setting is novel to the literature which only considers uncertainty over money marginal productivity (see Moldovanu and Sela (2001)) but not over the

[1] See Konrad (2007) for an exhaustive review of the literature on contests and their applications.

size of the head start advantage.

The equilibrium with complete information is also characterized allowing for different valuation of the auctioned object by the players. In this sense the model is a generalization of Konrad (2002) which defines the equilibrium for the complete information setting when the players have the same prize value. Furthermore it complements Siegel (2009) which defines players' payoffs when there is complete information but does not characterize the equilibrium strategies.

There are several circumstances where contest rules require that players are treated differently and the applications of the present analysis are potentially large. This is the case for instance in criminal law, where the "in dubio pro reo" principle applies. Bernardo et al. (2000) study this problem in a litigation game where costly evidence presented by two agents is weighed unequally by the court. In addition, incumbency is often an issue, as pointed out by Clark and Riis (2000) who consider the case of a government contract awarded to those who succeed in bribing a corrupt official. In this setting firms have private information over the value of winning and bribe marginal productivities are different. Konrad (2002) also considers contests where asymmetries are due to productivity and head start advantage, and analyzes incentives to invest when property rights are absent.

A rather different setting where asymmetries matter are elections. The strategic equivalence between a first-price all-pay auction and the problem of two candidates competing in an election on the basis of redistributive politics was first established by Sahuguet and Persico (2006). Competition with asymmetrical parties and different productivity of redistributive politics is studied by Kovenock and Roberson (2008). The case of a head start advantage available to one candidate is considered by Magnani (2010).

Analyzing the effects of incomplete information provides useful insights for many of the applications listed above where assuming that the handicapped player does not know the size of the advantage of his/her opponent is realistic.

The paper has the following structure. An equilibrium for an all-pay auction with asymmetrical players and complete information is characterized in Section 3.2. In Section 3.3, equilibria for the incomplete information setting are studied. Section 3.4 duly concludes.

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COMPLETE INFORMATION

Consider a first-price all-pay auction with complete information, where two risk-neutral players, 1 and 2, compete, and assume, without loss of generality, that Player 1 has a head start advantage $\alpha > 0$. Every bid x_2 by Player 2 is equivalent to a bid $x_1 = x_2 - \alpha$ by Player 1. Money marginal productivity is the same for both bidders but rescaling the prize values and the size of the advantage easily allows accounting for differences in that[1]. Player i's prize value is $V_i > 0$ $(i = 1, 2)$.

In the case of a tie, each player has the same probability of winning.

Different equilibria emerge depending on the comparison between V_2 and α, which is crucial for Player 2's participation in the auction.

Proposition 2.0.1 *If $V_2 < \alpha$, in the unique equilibrium, Player 1 and Player 2 bid 0 with probability 1. Player 1's payoff amounts to V_1 while Player 2 gets 0.*

Proof. Any strictly positive bid smaller than α has a nil probability of winning and gives Player 2 a strictly negative payoff. However, any $x_2 \geq \alpha$ requires Player 2 to pay a sum that exceeds his/her prize value and is a dominated action. Bidding $x_2 = 0$ is a dominant strategy for Player 2, who obtains a nil payoff.

Since Player 2 does not submit positive bids, Player 1 always wins. Any strictly positive offer does not increase the probability of winning, but decreases his/her expected payoff. Hence bidding 0 with probability 1 is a dominant strategy and gives a payoff equal to V_1. ■

The head start advantage represents a fixed participation cost for Player 2 who decides not to bid if it exceeds his/her prize value.

If $V_2 \geq \alpha$, Player 2 participates in the auction and only mixed strategy equilibria survive; this is a well known result for the standard all-pay auction and is easily extended to the present setting. The comparison between V_2 and $V_1 + \alpha$ defines the unique equilibrium of the game.

Proposition 2.0.2 *If $V_2 \geq \alpha$ and $V_2 \leq V_1 + \alpha$ hold, a unique equilibrium exists where Player 1 randomizes over the support $[0, V_2 - \alpha]$ according to the cumulative distribution function:*

$$F_1(x_1) \left\{ \begin{array}{ll} \frac{\alpha}{V_2} & \text{if } x_1 = 0 \\ \frac{x_1+\alpha}{V_2} & \text{if } x_1 > 0 \end{array} \right\}.$$

The expected payoff amounts to $V_1 + \alpha - V_2 \geq 0$.

[1] This is proven by Baye et al. (1996) in a model with no head start advantage; their results though are straightforwardly extended to the present setting.

Player 2 *randomizes over the support* $(\alpha, V_2] \cup \{0\}$ *according to the cumulative distribution function:*

$$F_2(x_2) \left\{ \begin{array}{cc} 1 - \frac{V_2 - \alpha}{V_1} & \text{if } x_2 = 0 \\ 1 - \frac{V_2}{V_1} + \frac{x_2}{V_1} & \text{if } x_2 > \alpha \end{array} \right\}.$$

He/She gets a nil expected payoff.

Proof. See the Appendix. ■

If prize values are different, each player has a different reach. Using Siegel's definition (2009) a player's reach is the highest bid which gives a positive payoff if he/she wins with certainty. In the previous case Player 1 bids up to V_1 which is also his/her reach. This quantity exceeds Player 2's prize value, net of the participation cost represented by the opponent's head start advantage, $V_2 - \alpha$ which defines his/her reach.

Therefore as in the standard all-pay auction Player 2 submits positive bids with probability less than 1. This allows him/her to compete on equal terms with the opponent when he/she actively participates in the auction. The size of the atom in 0 is increasing in the ratio between the player's reaches $\frac{V_2-\alpha}{V_1}$, and equals 1 if $V_2 = \alpha$.

The atom in Player 2's equilibrium distribution provides Player 1 with the incentive to reduce his/her expenditures by bidding 0 with some positive probability. Indeed, if both submit the same bid, the favored player wins because of the head start advantage. The size of the atom in 0 depends on the relative size of Player 1's advantage compared to Player 2's prize value and it is increasing in α and decreasing in V_2.

Consider now the case where $V_2 > V_1 + \alpha$.

Proposition 2.0.3 *If* $V_2 > V_1 + \alpha$, *there is a unique equilibrium where Player* 1 *randomizes over the support* $[0, V_1]$ *according to:*

$$F_1(x_1) \left\{ \begin{array}{cc} 1 - \frac{V_1}{V_2} & \text{if } x_1 = 0 \\ 1 - \frac{V_1}{V_2} + \frac{x_1}{V_2} & \text{if } x_1 > 0 \end{array} \right\}.$$

He/She gets a nil expected payoff.

Player 2 *randomizes continuously over the support* $(\alpha, \alpha + V_1]$ *according to:*

$$F_2(x_2) = \frac{x_2 - \alpha}{V_1}.$$

The expected payoff is $V_2 - \alpha - V_1 > 0$

Proof. Analogous to the proof of Proposition 3.2. ■

Player 1 bids 0 with positive probability because his/her reach is less than that of the opponent. Since he/she does not pay any participation cost, the size of the atom in 0 is increasing in the ratio $\frac{V_1}{V_2}$. The complete information setting payoffs reproduce the results by Siegel (2009).

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INCOMPLETE ASYMMETRICAL INFORMATION

3.1 Uncertainty over the Existence of a Head-Start Advantage

Consider the situation where Player 2 is uncertain about the auctioneer increasing the opponent's bid while Player 1 observes his/her move. Let λ be the probability that this happens. In this setting there are in facts two types of Player 1: one who has the advantage and one who doesn't. Denote with $F_1^{\alpha}(x_1)$ and $F_1^{0}(x_1)$ Player 1's equilibrium strategies respectively for the case where he/she has the head start advantage and for the case where he/she has not. Players' payoffs are defined as follows:

$$E\left[U_1\left(x_1, F_2\left(\alpha + x_1^{\alpha}\right)\right)\right] = V_1 \cdot F_2\left(\alpha + x_1\right) - x_1^{\alpha} = p_1^{\alpha} \tag{3.1}$$

$$E\left[U_1\left(x_1, F_2\left(x_1^{0}\right)\right)\right] = V_1 \cdot F_2\left(x_1\right) - x_1^{0} = p_1^{0} \tag{3.2}$$

$$\begin{aligned} E\left[U_2\left(x_2, F_1^{\alpha}\left(x_2 - \alpha\right), F_1^{0}\left(x_2\right)\right)\right] &= \\ \lambda \cdot V_2 \cdot F_1^{\alpha}\left(x_2 - \alpha\right) + \left(1 - \lambda\right) V_2 \cdot F_1^{0}\left(x_2\right) - x_2 &= p_2 \end{aligned} \tag{3.3}$$

This implies that $p_1^{\alpha} \geq p_1^{0}$ since $F_2\left(\alpha + x_1^{\alpha}\right) \geq F_2\left(x_1^{0}\right)$.

Different equilibria emerge depending either from the size of the head start advantage or from the probability that the advantage is actually awarded.

In a first class of equilibria $\alpha > V_2$ holds and the size of the advantage exceeds any bid possibly submitted by Player 2. As a consequence he/she never wins when the advantage is awarded. In this case a unique equilibrium exists for the game where Player 1 does not actively participate in the auction if he/she has the head start advantage

A second circumstance entails $\alpha \leq V_2$. When this happens Player 1 always participate in the auction if he/she has the advantage. Multiple equilibria emerge when $\lambda > \frac{V_1 + \alpha}{V_2}$. If the probability that the opponent is favoured by the auctioneer is high, Player 2 adopts a strategy that discourages the active participation of Player 1 when he/she has no advantage. Another possible equilibrium entails instead the active participation of Player 1 in both states of the world.

Consider initially the first circumstance. A preliminary result is worth mentioning.

Lemma 3.1.1 *If $\alpha > V_2$ holds, bidding 0 with probability 1 is a dominant strategy for Player 1 when he/she has the head start advantage.*

Proof. Analogous to that of Proposition 3.1. ■

In equilibrium $p_1^\alpha = V_1$ thus holds and Player 2's expected payoff is:

$$E\left[U_2\left(x_2, F_1^0\left(x_2\right)\right)\right] = \left(1-\lambda\right)V_2 \cdot F_1^0\left(x_2\right) - x_2 = p_2$$

Player 1's expected payoff when he/she has not the head start advantage is defined as in Equation 3.2. The game thus is strategically equivalent to a single all-pay auction with complete information where the prize values are respectively V_1 for Player 1 and$\left(1-\lambda\right)V_2$ for Player 2.

In this setting two circumstances are possible: $\left(1-\lambda\right)V_2 > V_1$ and $\left(1-\lambda\right)V_2 \leq V_1$. The results of Siegel (2009) apply and the equilibrium has the following characteristics.

Proposition 3.1.2 *If $\alpha > V_2$ and $\left(1-\lambda\right)V_2 > V_1$ hold, a unique equilibrium exists characterized by the following elements:*

When he/she does not have the head start advantage, Player 1 randomizes over the support $\left[0, V_1\right]$according to the following cumulative distribution function:

$$F_1^0\left(x_1^0\right)\left\{\begin{array}{ll} 1-\frac{V_1}{(1-\lambda)V_2} & \text{if } x_1^0 = 0 \\ 1-\frac{V_1}{(1-\lambda)V_2}+\frac{x_1^0}{(1-\lambda)V_2} & \text{if } x_1^0 > 0 \end{array}\right\}$$

The expected payoff is $p_1^0 = 0$

When he/she has the head start advantage, Player 1 bids zero with probability 1.

His/her payoff is $p_1^\alpha = V_1$

Player 2 randomizes over the support $\left[0, V_1\right]$according to the following cumulative distribution function:

$F_2\left(x_2\right) = \frac{x_2}{V_1}$

The expected payoff is $p_2 = \left(1-\lambda\right)V_2 - V_1$.

Proof. The proof comes directly from Lemma 3.4 and from the results of Baye et al. (1996). ■

Consider now the case where $\left(1-\lambda\right)V_2 \leq V_1$.

Proposition 3.1.3 *If $\alpha > V_2$ and $\left(1-\lambda\right)V_2 \leq V_1$ hold, a unique equilibrium exists characterized by the following elements:*

When he/she does not have the head start advantage, Player 1 randomizes over the support $\left[0, \left(1-\lambda\right)V_2\right]$according to the following cumulative distribution function:

$F_1^0\left(x_1\right) = \frac{x_1^0}{(1-\lambda)V_2}$

The expected payoff is $p_1^0 = V_1 - \left(1-\lambda\right)V_2$

When he/she has the head start advantage, Player 1 bids zero with probability 1.

His/her payoff is $p_1^\alpha = V_1$

Player 2 randomizes over the support $[0,(1-\lambda)V_2]$*according to the following cumulative distribution function:*

$$F_2(x_2)\left\{\begin{array}{cc} 1-\frac{(1-\lambda)V_2}{V_1} & \text{if } x_2=0 \\ 1-\frac{(1-\lambda)V_2}{V_1}+\frac{x_2}{V_1} & \text{if } x_2>0 \end{array}\right\}$$

The expected payoff is $p_2=0$.

Proof. Analogous to that of Proposition 3.5. ■

If α is large, Player 1 wins with probability 1 and does not submit positive bids when the head start advantage is awarded. Player 2's probability to win thus is scaled down by a factor $1-\lambda$ and so is his/her reach.

Note that when $(1-\lambda)V_2 \leq V_1$ holds, uncertainty guarantees an information rent to Player 1. When he/she has no advantage, the payoff is $p_1^0 = V_1-(1-\lambda)V_2$ which is greater than V_1-V_2 and zero, i.e. the payoffs that he/she would have got in a complete information setting respectively if $V_1 \geq V_2$ and $V_1 < V_2$.

Consider now what happens when Player 1 actively participates in the auction if he/she has the head start advantage. A preliminary result is worth mentioning.

Lemma 3.1.4 *If* $\alpha \leq V_2$ *holds, no equilibrium strategies exist such that* $p_1^0 > 0$ *and* $p_2 > 0$.

Proof. The same argument used in Proposition 3.2 establishes that in equilibrium $\underline{x}_1^0 = 0$ and $\underline{x}_2 = 0$ must hold. Note that $p_1^0 > 0$ requires $F_2(0) > 0$ since $V_1 \cdot F_2(0) > 0$ must hold when $x_1^0 = 0$. The same argument can be used to show that $p_2 > 0$ requires $F_1^0(0) > 0$. A profitable deviation thus exists since players' probability of getting the item increases by a finite amount, if they bid slightly more than 0. ■

In this setting two classes of equilibria emerge; one involves the active participation of Player 1 in both states of the world while the second is characterized by the fact that Player 1 bids zero with probability 1 when he/she has not the head start advantage. Consider the case where both players submit positive bids in all the states of the world.

Proposition 3.1.5 *If* $1 \geq \frac{V_1}{V_2} \geq \lambda$ *and* $\alpha \leq V_2$ *hold, an equilibrium characterized by the following elements exists*

Player 1 randomizes over the support $[0, V_1]$ *according to the cumulative distribution function:*

$$F_1^0(x_1^0)\left\{\begin{array}{cc} \frac{1}{1-\lambda}\left(1-\frac{V_1}{V_2}\right)+\frac{x_1^0}{(1-\lambda)V_2} & \text{if } x_1^0<\alpha \\ x_1^0\cdot\frac{\frac{V_1}{V_2}-\frac{\alpha}{V_2}-\lambda}{(1-\lambda)(V_1-\alpha)}+\frac{\alpha}{V_1-\alpha}+V_1\cdot\frac{1-\frac{V_1-\alpha}{V_2}}{(1-\lambda)(V_1-\alpha)} & \text{if } x_1^0\geq\alpha \end{array}\right\}$$

when he/she does not have the head start advantage.

The expected payoff is $p_1^0 = 0$

When he/she has the head start advantage Player 1 randomizes over the support$[0, V_1-\alpha]$ *according to the following cumulative distribution:*

$F_1^\alpha(x_1^\alpha) = \frac{x_1^\alpha}{V_1 - \alpha}$

The expected payoff is $p_1^\alpha = \alpha$

Player 2 randomizes over the support $[0, V_1]$ *according to the cumulative distribution function:*

$F_2(x_2) = \frac{x_2}{V_1}$

The expected payoff is $p_2 = V_2 - V_1$.

Proof. See the Appendix ■

Note again that uncertainty guarantees an information rent to Player 1 whose payoff is nil if he/she has no advantage as in the corresponding complete information setting but increases to $\alpha \geq V_1 - V_2 + \alpha$ in case he/she has the advantage.

Proposition 3.1.6 *The equilibrium characterized above is unique.*

Proof. Consider the case where $\lambda \leq \frac{V_1}{V_2}$ holds. As proven in Proposition 3.8, $p_1^0 = 0$ must hold and Player 1 is indifferent between bidding or not if he/she has not the head start advantage. Assume now that Player 1 does not actively participates in the auction. Player 2 competes with probability λ with an opponent whose reach is $V_1 + \alpha$. Two circumstances are possible in this setting and namely $V_2 \geq V_1 + \alpha$ or $V_2 < V_1 + \alpha$

Consider the latter case. The same argument used in the complete information setting establishes the following equilibrium condition must hold:

$$V_1 \cdot F_2(x_1^\alpha + \alpha) - x_1^\alpha = V_1 - V_2 + \alpha$$

so that

$$F_2(x_2) - x_1^\alpha = 1 - \frac{V_2}{V_1} + \frac{x_2}{V_1}$$

implying that Player 2's distribution has an atom of probability in zero amounting to $1 - \frac{V_2}{V_1}$. If this is the case though profitable deviation exists and Player 1 submits a bid $x_1^0 = \varepsilon > 0$ with ε arbitrarily small to get a positive expected payoff when he/she has not the head start advantage. This excludes that if $V_2 \leq V_1 + \alpha$ holds, in equilibrium Player 1 does not participate in the auction when no head start advantage is awarded.

Consider now the case $V_2 \geq V_1 + \alpha$ and assume again that Player 1 does not actively participates in the auction if he/she has not the head start advantage. Player 2 can choose to contend the prize to both types of Player 1 or just bid $x_2 = \varepsilon$ with probability 1 where $\varepsilon > 0$ is arbitrarily small. This allows to win with certainty if the head start advantage is not awarded and gives a payoff $p_2 = (1 - \lambda) V_2 - \varepsilon$. Contending the prize in both states of the world gives Player 2 the payoff $p_2 = V_2 - V_1 - \alpha$. This is easily proven using same argument presented in the complete information setting.

Bidding $x_2 = \varepsilon$ is an optimal strategy only if $(1 - \lambda) V_2 \geq V_2 - V_1 - \alpha$ or $\lambda < \frac{V_1 + \alpha}{V_2}$ holds. The previous condition is trivially satisfied since it is $\lambda \leq \frac{V_1}{V_2}$ and thus Player 2 only contest the prize in the state of the world where the advantage is not awarded. Player 1 though has the profitable deviation to submit a bid slightly bigger than ε when he/she has not the head start advantage, to get a positive payoff. This excludes that in equilibrium Player 1 does not participate in the auction when no head start advantage is awarded. As a consequence when it is $1 \geq \frac{V_1}{V_2} \geq \lambda$, the equilibrium characterized in Proposition 3.8 is unique. Uniqueness follows from the argument used in the complete information setting which establishes that player strategies in each state of the world are continuous and that atoms of probability can be placed only in $x_1^0 = 0$. In particular it follows from the fact that $\underline{x}_1^\alpha = 0$ and $\bar{x}_1^\alpha = V_1 - \alpha$ hold if Player 1 has the head start advantage, while $\underline{x}_1^0 = 0$ and $\bar{x}_1^0 = V_1$ hold otherwise.The same argument establishes uniqueness with regard to Player 2 strategy. ■

Corollary 3.1.7 *If* $1 \geq \lambda > \frac{V_1}{V_2}$, $\alpha \leq V_2$ *and* $V_2 < V_1 + \alpha$ *hold, no equilibrium for the game exists*

Proof. Follows straightforwardly from Proposition 3.9. ■

A second class of equilibria emerges when the probability that the head start advantage is awarded is sufficiently high and Player 2's reach exceeds that of his/her opponent.

Corollary 3.1.8 *If* $1 \geq \lambda > \frac{V_1}{V_2}$, $\alpha \leq V_2$ *and* $V_2 \geq V_1 + \alpha$ *hold, a unique equilibrium exists where Player* 1 *bids zero with probability* 1 *if he/she does not have the head start advantage, and randomizes over the support* $[0, V_1]$ *according to the following cumulative distribution function otherwise:*

$$F_1^\alpha\left(x_1^\alpha\right) = 1 - \frac{V_1}{\lambda \cdot V_2} + \frac{x_1}{\lambda \cdot V_2}$$

His/her expected payoff is $p_1^\alpha = 0$.

Player 2 *randomizes over the support* $[\alpha, V_1 + \alpha]$ *according to the following cumulative distribution function:*

$$F_2\left(x_2\right) = \frac{x_2 - \alpha}{V_1}$$

His/her expected payoff is $p_2 = V_2 - V_1 - \alpha$.

Proof. Note that Player 2 can always get a positive payoff by bidding slightly more than $V_1 + \alpha$ and this implies $p_2 > 0$. The same argument used in Proposition 3.2 establishes that in equilibrium $\underline{x}_1^\alpha = 0$ and $\underline{x}_2 = \alpha$ must hold; this implies further that in order to have $p_2 > 0$, it must be $F_1^\alpha(0) > 0$. It is possible then to exclude that an equilibrium exist where $p_1^\alpha > 0$. If that were the case indeed $F_2(\alpha) > 0$ must hold, implying that both players have the profitable deviation to bid slightly more respectively than $\underline{x}_1^\alpha = 0$ and $\underline{x}_2 = \alpha$ since their probability of winning increases discretely at these bids.

In equilibrium thus it is $p_1^\alpha = 0$ implying that Player 1 is indifferent between actively participating in the auction or bid 0 with probability 1. The latter circumstance is never verified in equilibrium. If that were the case indeed, Player 2 optimal response entails bidding with probability 1, $x_2 = \alpha + \varepsilon$ with $\varepsilon > 0$ arbitrarily small. A profitable deviation thus exists for Player 1 when he/she has the advantage because he/she can bid slightly more than $x_1^\alpha = \varepsilon$ to get a positive expected payoff.

As a consequence Player 1 participates in the auction and the following condition must hold:

$$V_1 \cdot F_2\left(x_1^\alpha + \alpha\right) - x_1^\alpha = 0$$

so that

$$F_2\left(x_1^\alpha + \alpha\right) = \frac{x_1^\alpha}{V_1}$$

or

$$F_2\left(x_2\right) = \frac{x_2 - \alpha}{V_1}$$

implying that Player 2 randomizes continuously over the support $[\alpha, V_1 + \alpha]$.

Given this distribution for Player 2's bids, it is easy to verify that when the head start advantage is not awarded, every bid in the set of Player 1's undominated actions gives a negative payoff. Hence Player 1 never actively participates in the auction.

Since the upper bound of Player 2' distribution is $V_1 - \alpha$, it must be the case that in equilibrium:

$$(1 - \lambda) V_2 + \lambda \cdot V_2 \cdot F_1^\alpha\left(x_2 - \alpha\right) - x_2 = V_2 - V_1 - \alpha$$

so that

$$F_1^\alpha\left(x_2 - \alpha\right) = 1 - \frac{V_1}{\lambda \cdot V_2} + \frac{x_2 - \alpha}{\lambda \cdot V_2}$$

holds or

$$F_1^\alpha\left(x_1^\alpha\right) = 1 - \frac{V_1}{\lambda \cdot V_2} + \frac{x_1^\alpha}{\lambda \cdot V_2}$$

implying that Player 1 randomizes over the support $[0, V_1]$ with an atom in zero amounting to $1 - \frac{V_1}{\lambda \cdot V_2} > 0$. Uniqueness is proven using the standard arguments.
■

Consider now the complementary case where $V_1 > V_2$ holds and both players participate in the auction in all the states of the world.

Proposition 3.1.9 *If $\frac{V_1}{V_2} > 1 \geq \lambda$ and $\alpha \leq V_2$ hold, a unique equilibrium characterized by the following elements exists.*

Player 1 *randomizes over the support* $[0, V_2]$ *according to the following cumulative distribution function:*

$$F_1^0\left(x_1^0\right)\left\{\begin{array}{ll}\frac{x_1^0}{(1-\lambda)V_2} & if\ x_1^0<\alpha \\ \frac{x_1^0\left(1-\lambda-\frac{\alpha}{V_2}\right)}{(1-\lambda)(V_2-\alpha)}+\frac{\lambda\cdot\alpha}{(1-\lambda)(V_2-\alpha)} & if\ x_1^0\geq\alpha\end{array}\right\}$$

when he/she does not have the head start advantage.

The expected payoff is $p_1^0 = V_1 - V_2$

When he/she has the head start advantage Player 1 *randomizes over the support*$[0, V_2 - \alpha]$ *according to the following cumulative distribution:*

$F_1^\alpha\left(x_1^\alpha\right) = \frac{x_1^\alpha}{V_2-\alpha}$

The expected payoff is $p_1^\alpha = V_1 - V_2 + \alpha$

Player 2 *randomizes over the support* $[0, V_2]$ *according to the following cumulative distribution function:*

$F_2\left(x_2\right) = 1 - \frac{V_2}{V_1} + \frac{x_2}{V_1}$

The expected payoff is $p_2 = 0$

Proof. See the Appendix. ■

In this case the upper bound for players' distributions remains unchanged and coincides with the valuation of the prize by Player 2. As a consequence there are no variations in player payoffs with respect to the complete information setting. Player 2's payoff is equal to that obtained with complete information when competing with an opponent with the head start advantage. If he/she has no advantage Player 1's payoff is the same as in the corresponding complete information setting.

3.2 Uncertainty over the Size of the Head-Start Advantage

Consider a setting where the size of the head start advantage is private information of Player 1. Suppose for instance that an incumbent and a new entrant are racing for the same patent and the latter is uncertain about the amount of the incumbent's previous R&D expenditures.

Player 2 faces $n \geq 2$ types of Player 1, each characterized by a specific advantage α^i $(i = 1, 2, ..., n)$ with $0 \leq \alpha^1 < \alpha^2 ... < \alpha^n$. Let π_i be the probability that Player 1's type is i. Assume initially that the head start advantage does not exceed Player 2's prize value and $\alpha^n \leq V_2$. Hence there are no states of the world where Player 2's reach is negative and he/she never wins.

Denote with $F_1^i\left(x_1^i\right)$ Player 1's equilibrium strategy when his/her type is i. Players' expected payoffs are:

$$E\left[U_1^i\left(x_1^i, F_2\left(\alpha_i + x_1^i\right)\right)\right] = V_1 \cdot F_2\left(x_1^i + \alpha_i\right) - x_1^i = p_1^i$$

$$E\left[U_2\left(x_2, F_1^i\left(x_2 - \alpha_i\right)\right)\right] = \Sigma_{i=1}^n \pi_i \cdot V_2 \cdot F_1^i\left(x_2 - \alpha_i\right) - x_2 = p_2.$$

The characterization of the equilibrium requires that Player 2's reach be compared with that of Player 1, type 1.

Proposition 3.2.1 *If $V_1 > V_2 - \alpha^1$, a unique equilibrium exists.*

Player 1 bids 0 with probability 1 when his/her advantage is $\alpha^j < \alpha^t$ where $t \leq n$ is the lowest type such that $\frac{\alpha^t}{\pi_t \cdot V_2} - \sum_{j=1}^{t-1} \frac{\pi_j}{\pi_t} < 1$ holds. The expected payoff is $p_1^j = V_1 - V_2 + \alpha^t$.

Consider a type $i \geq t$. If $i = t$, Player 1 randomizes over the support $[0, V_2 - \alpha^i]$. The equilibrium cumulative distribution function in the interval $[0, \alpha^{t+1} - \alpha^t]$ is:

$$F_1^t\left(x_1^t\right) \left\{ \begin{array}{ll} \frac{\alpha^t}{\pi_t \cdot V_2} - \sum_{j=1}^{t-1} \frac{\pi_j}{\pi_t} & \text{if } x_1^t = 0 \\ \frac{x_1^t + \alpha^t}{\pi_t \cdot V_2} - \sum_{j=1}^{t-1} \frac{\pi_j}{\pi_t} & \text{if } 0 < x_1^t < \alpha^{t+1} - \alpha^t \end{array} \right\}.$$

If $i > t$ he/she randomizes continuously over the support $(0, V_2 - \alpha^i]$; his /her cumulative distribution function in the interval $[0, \alpha^{i+1} - \alpha^i]$ is:

$$F_1^i\left(x_1^i\right) = \frac{x_1^i}{V_2 \cdot \sum_{p=1}^{i} \pi_p}.$$

In the interval $\alpha^k - \alpha^i \leq x_1^i < \alpha^{k+1} - \alpha^i$ with $k \geq i + 1$, Player 1, type $i \geq j$, randomizes according to:

$$F_1^i\left(x_1^i\right) = \sum_{j=0}^{k-i-1} \frac{\alpha^{k-j} - \alpha^{k-j-1}}{V_2 \cdot \sum_{p=1}^{k-j-1} \pi_p} + \frac{x_1^i + \alpha^i - \alpha^k}{V_2 \cdot \sum_{p=1}^{k} \pi_p}.$$

If $\alpha^n - \alpha^i \leq x_1^i$ the cumulative distribution function is:

$$F_1^i\left(x_1^i\right) = \frac{x_1^i + \alpha^i - \alpha^n}{V_2 - \alpha^n} \left(1 - \sum_{j=0}^{n-i-1} \frac{\alpha^{n-j} - \alpha^{n-j-1}}{V_2 \cdot \sum_{p=1}^{n-j-1} \pi_p}\right) + \sum_{j=0}^{n-i-1} \frac{\alpha^{n-j} - \alpha^{n-j-1}}{V_2 \cdot \sum_{p=1}^{n-j-1} \pi_p}.$$

The expected payoff is $p_1^i = V_1 - V_2 + \alpha^i$.

Player 2 randomizes over the support $(\alpha^t, V_2] \cup \{0\}$ according to:

$$F_2\left(x_2\right) \left\{ \begin{array}{ll} 1 - \frac{V_2}{V_1} + \frac{\alpha^t}{V_1} & \text{if } x_2 = 0 \\ 1 - \frac{V_2}{V_1} + \frac{x_2}{V_1} & \text{if } x_2 > \alpha^t \end{array} \right\}.$$

The expected payoff is $p_2 = 0$

Proof. See the Appendix. ■

An interesting feature of the equilibrium is that players' payoffs and expenditures are as in the complete information setting, no matter which type of Player 1 is considered. This is the case because the reach of Player 1, type 1, exceeds that of Player 2. Introducing uncertainty over the size of the advantage thus does not change the upper bound for players' distributions which depends on Player 2's prize value. Indeed, the ranking over players' reaches is unchanged.

A different situation emerges if the reach of Player 1, type 1, is lower than that of the opponent. In this case Player 1 obtains an information rent which follows from a redefinition in the ranking of the bidders' reaches. A preliminary result must be mentioned before characterizing the equilibria for the game.

Lemma 3.2.2 *When $V_2 - \alpha^t < V_1 \leq V_2 - \alpha^1$ with $t \leq n$, no equilibria exist where all the types $j < t$ do not actively participate in the auction, if there is a $\hat{\jmath} < t$ such that $\left(\sum_{l=1}^{\hat{\jmath}} \pi_l\right) V_2 - \alpha^{\hat{\jmath}} \geq 0$ holds.*

Proof. No equilibria exist where Player 1, type 1, participates in the auction and gets a positive payoff. Suppose indeed that $p_1^1 > 0$. The same argument presented in Proposition 3.13 establishes that $p_2 = 0$ and requires that Player 2's distribution has an upper bound $\bar{x}_2 < V_1 + \alpha^1$. Since $\bar{x}_1^1 = \bar{x}_2 - \alpha^1$ holds[1], the condition $V_1 \cdot F_2\left(\bar{x}_1^1 + \alpha^1\right) - \bar{x}_1^1 > 0$ also holds. Every bid greater than $\bar{x}_2 - \alpha^i$ is thus a dominated action for Player 1, type $i \geq 1$ and Player 2 has the profitable deviation to submit a bid in the interval $(\bar{x}_2, V_2)$ and get a positive payoff given that $\bar{x}_2 < V_1 + \alpha^1 < V_2$ holds.

Hence $p_1^1 = 0$ and Player 1, type 1, is indifferent between participating in the auction or not. Assume that he/she bids 0 with probability 1. The same argument presented above establishes that $p_1^2 = 0$ and $p_1^j = 0$ with $j < t$ hold if Player 1 does not actively participate in the auction when his/her advantage is strictly smaller than α^j.

Suppose now that every type $j < t$ bids 0 with probability 1. If $i \geq t$, $\underline{x}_1^i = 0$ and $\underline{x}_2 = \alpha^t$ hold and players' expected payoffs are $p_1^i > 0$ and $p_2 = 0$[2]. This requires that Player 2's distribution has an atom of probability at some bid in the interval $[0, \alpha^t]$, so that $F_2\left(\alpha^t\right) > 0$ holds. Player 2 places the atom in 0 unless a type $\hat{\jmath}$ exists such that $\left(\sum_{l=1}^{\hat{\jmath}} \pi_l\right) V_2 - \alpha^{\hat{\jmath}} \geq 0$. If the latter inequality holds, he/she can indeed get a positive expected payoff by bidding slightly more than $\alpha^{\hat{\jmath}}$. This though provides Player 1, type $\hat{\jmath}$, with the profitable deviation to bid $x_1^{\hat{\jmath}} = \varepsilon > 0$ with ε arbitrarily small and get a positive payoff. As a consequence, no equilibria exist where all the types $j < t$ do not participate in the auction while $\left(\sum_{l=1}^{\hat{\jmath}} \pi_l\right) V_2 - \alpha^{\hat{\jmath}} \geq 0$. ■

Proposition 3.2.3 *When $V_2 - \alpha^t < V_1 \leq V_2 - \alpha^1$ and $\left(\sum_{l=1}^{\hat{\jmath}} \pi_l\right) V_2 - \alpha^{\hat{\jmath}} \geq 0$ holds for some $\hat{\jmath} < t$, multiple equilibria exist. In particular there are $t - \hat{t}$ equilibria where $\hat{t} < t$ denotes the lowest type such that $1 - \sum_{p=1}^{\hat{t}} \pi_p < \frac{V_1}{V_2}$ holds. Each of them is characterized as follows.*

Player 1 bids 0 with probability 1 and gets a nil payoff if his/her advantage is strictly smaller than α^j with $t > j > \hat{t}$.

[1] This is proven as in Proposition 3.2.

[2] This again is proven as in Proposition 3.2.

Consider a type $i \geq j$. If $i = j$, Player 1 randomizes over the support $[0, V_1 - \alpha^i + \alpha^j]$. The equilibrium cumulative distribution function on the interval $[0, \alpha^{j+1} - \alpha^j]$ is:

$$F_1^j\left(x_1^j\right)\left\{\begin{array}{ll}\frac{1}{\pi_j}\left(1-\sum_{p=1}^{j-1}\pi_p-\frac{V_1}{V_2}\right) & \textit{if } x_1^j=0 \\ \frac{1}{\pi_j}\left(1-\sum_{p=1}^{j-1}\pi_p-\frac{V_1}{V_2}\right)+\frac{x_1^j}{\pi_j\cdot V_2} & \textit{if } 0<x_1^j<\alpha^{t+1}-\alpha^t\end{array}\right\}.$$

If $i > j$, he/she randomizes continuously over the support $(0, V_1 - \alpha^i + \alpha^j]$. His/her cumulative distribution function on the interval $[0, \alpha^{i+1} - \alpha^i]$ is:

$$F_1^i\left(x_1^i\right)=\frac{x_1^i}{V_2\cdot\sum_{q=j}^{i}\pi_q}.$$

In the interval $\alpha^{k-1} - \alpha^i \leq x_1^i < \alpha^k - \alpha^i$ with $k \geq i+2$, Player 1, type $i \geq j$, randomizes according to:

$$F_1^i\left(x_1^i\right)=\sum_{l=0}^{k-i-1}\frac{\alpha^{k-l}-\alpha^{k-l-1}}{V_2\cdot\sum_{r=1}^{k-l-1}\pi_r}+\frac{x_1^i-\alpha^i-\alpha^k}{V_2\cdot\sum_{q=j}^{k}\pi_q}.$$

If $\alpha^n - \alpha^i \leq x_1^i$, the cumulative distribution function is:

$$F_1^i\left(x_1^i\right)=\frac{x_1^i-\alpha^n+\alpha^i}{V_1-\alpha^n+\alpha^j}\left(1-\sum_{l=0}^{n-i-1}\frac{\alpha^{n-l}-\alpha^{n-l-1}}{V_2\cdot\sum_{r=1}^{n-l-1}\pi_r}\right)+\sum_{l=0}^{n-i-1}\frac{\alpha^{n-l}-\alpha^{n-l-1}}{V_2\cdot\sum_{r=1}^{n-l-1}\pi_r}.$$

The expected payoff is $p_1^i = \alpha^i - \alpha^j$.

Player 2 randomizes continuously over the support $(\alpha^j, V_1 + \alpha^j]$ according to:

$$F_2\left(x_2\right)=\frac{x_2-\alpha^j}{V_1}.$$

The expected payoff is $p_2 = V_2 - V_1 - \alpha^j$.

Proof. Lemma 3.14 establishes that in equilibrium there is at least one type $j < t$ of Player 1 who actively participates in the auction obtaining an expected payoff $p_1^j = 0$. Therefore $F_2(V_1 + \alpha^j) = 1$ must hold and every bid greater than $V_1 - \alpha^i + \alpha^j$ is a dominated action for a type $i > j$. As a consequence, no equilibrium exists such that $p_2 = 0$ since Player 2 can get a positive payoff by bidding slightly more than $V_1 + \alpha^j$.

The same argument used in Proposition 3.2 establishes that $\bar{x}_2 = V_1 + \alpha^j$, $\bar{x}_1^i = V_1 + \alpha^j - \alpha^i$, $\underline{x}_2 = \alpha^j$ and $\underline{x}_1^i = 0$. Hence $p_2 = V_2 - V_1 - \alpha^j$ and the following equality must hold for every $x_2 \in (\alpha^j, \alpha^{j+1}]$:

$$V_2\cdot\sum_{p=1}^{j-1}\pi_p+V_2\cdot\pi_j\cdot F_1^j\left(x_2-\alpha^j\right)-x_2=V_2-V_1-\alpha^j$$

such that:

$$F_1^j\left(x_2-\alpha^j\right)=\frac{V_2\left(1-\sum_{p=1}^{j-1}\pi_p\right)-V_1}{\pi_j\cdot V_2}+\frac{x_2-\alpha^j}{\pi_j\cdot V_2}$$

or exploiting the equivalence $x_2-\alpha^j=x_1^j$:

$$F_1^j\left(x_1^j\right)=\frac{1}{\pi_j}\left(1-\sum_{p=1}^{j-1}\pi_p-\frac{V_1}{V_2}\right)+\frac{x_1^j}{\pi_j\cdot V_2}.$$

The distribution of Player 1, type j, has an atom of probability whose size is $\frac{1}{\pi_j}\left(1-\sum_{p=1}^{j-1}\pi_p-\frac{V_1}{V_2}\right)$ which can be placed only in 0[3]. His/her active participation in the auction thus requires:

$$1-\sum_{p=1}^{j}\pi_p<\frac{V_1}{V_2}.$$

When this happens, for every bid $x_2\in\left(\alpha^k,\alpha^{k+1}\right]$ with $k\geq j+1$, the following equilibrium condition must hold:

$$\begin{aligned}
& V_2\cdot\sum_{p=1}^{j-1}\pi_p+V_2\cdot\sum_{i=j}^{k}\pi_i\cdot F_1^i\left(\alpha^k-\alpha^i\right)+\\
& +\left[\sum_{i=t}^{j}\pi_i\cdot F_1^i\left(\alpha^{k+1}-\alpha^i\right)-\sum_{i=j}^{k}\pi_i\cdot F_1^i\left(\alpha^k-\alpha^i\right)\right]\cdot\\
& \cdot V_2\cdot\sum_{i=j}^{k}\frac{\pi_i}{\sum_{q=j}^{k}\pi_q}\cdot\Pr\left[x_2-\alpha^i>x_1^i|\alpha^{k+1}\geq x_2\geq\alpha^k\right]-x_2\\
= \; & V_2-V_1-\alpha^j
\end{aligned}$$

Through simple algebra it is easy to get

$$\begin{aligned}
& \left[\sum_{i=j}^{k}\pi_i\cdot F_1^i\left(\alpha^{k+1}-\alpha^j\right)-\sum_{i=j}^{k}\pi_i\cdot F_1^i\left(\alpha^k-\alpha^i\right)\right]\cdot\\
& \cdot V_2\cdot\sum_{i=j}^{k}\frac{\pi_i}{\sum_{q=j}^{k}\pi_q}\left[\frac{F_1^i\left(x_2-\alpha^i\right)-F_1^i\left(\alpha^k-\alpha^i\right)}{F_1^i\left(\alpha^{k+1}-\alpha^i\right)-F_1^i\left(\alpha^k-\alpha^i\right)}\right]\\
= \; & 1-\sum_{p=1}^{j-1}\pi_p-\sum_{i=j}^{k}\pi_i\cdot F_1^i\left(\alpha^k-\alpha^i\right)+\frac{x_2-V_1-\alpha^j}{V_2}.
\end{aligned}$$

[3]This is proven as in Proposition 3.2.

Following the same steps presented in Proposition 3.13 it is possible to characterize the equilibrium cumulative distribution functions.

Consider now Player 1 and note that if his/her advantage is α^j, he/she gets a nil expected payoff since $\bar{x}_1^j = V_1$. Therefore the following condition holds:

$$V_1 \cdot F_2\left(x_1^j + \alpha^j\right) - x_1^j = 0$$

or exploiting the equivalence $x_1^j + \alpha^j = x_2$ and using simple algebra:

$$F_2\left(x_2\right) = \frac{x_2 - \alpha^j}{V_1}.$$

Since $\bar{x}_1^i = V_1 + \alpha^j - \alpha^i$, Player 1's expected payoff if his/her advantage is $\alpha^i > \alpha^t$, is $p_1^i = \alpha^i - \alpha^j$ and the condition

$$V_1 \cdot F_2\left(x_1^i + \alpha^i\right) - x_1^i = \alpha^i - \alpha^j$$

holds such that once again $F_2\left(x_2\right) = \frac{x_2 - \alpha^j}{V_1}$.

Given Player 2's equilibrium strategy, Player 1 gets a negative payoff submitting a strictly positive bid when his/her advantage is strictly smaller than α^j. This excludes that a profitable deviation exists. ■

Corollary 3.2.4 *If* $V_2 - \alpha^t < V_1 \leq V_2 - \alpha^1$ *and* $\left(\sum_{l=1}^{\hat{j}} \pi_l\right) V_2 - \alpha^{\hat{j}} \geq 0$ *holds for some* $\hat{j} < t$, *no equilibrium exists when* $1 - \sum_{p=1}^{t-1} \pi_p < \frac{V_1}{V_2}$.

Proof. This follows straightforwardly from Proposition 3.15. ■

Uncertainty reduces players expenditures. In particular, auctioneer's expected revenues are as if there were perfect information and Player 1's head start advantage were α^j. This circumstance guarantees an information rent to Player 1. Indeed, his/her payoff is nil when the advantage is α^j as in the complete information setting but increases to $\alpha^i - \alpha^j \geq V_1 - V_2 + \alpha^i$ if $i > j$. Player 1 "mimics" the behavior of the type with the lowest reach which participates in the auction to reduce expected expenditures and increase his/her expected payoff. In fact, the equilibrium strategies defined above guarantee that the following condition is verified:

$$V_2 \cdot \sum_{k=j}^{n} \pi_k \cdot F_1^k\left(x_2 - \alpha^k\right) - x_2 = V_2 - V_1 - \alpha^j.$$

This result could be interpreted as a reverse of the "exclusion principle"[4] which states that a selection among the players increases the auctioneer's revenues by increasing competition. Uncertainty widening the set of (potential) players reduces competition. Compare indeed the case of complete information where

[4]See Baye et al. (1993)

Player 2 faces an opponent with a head start advantage α^i with the incomplete information setting. Now the potential competitors are n and their reaches are distributed over a range where at least one element is smaller than or equal to $V_1 + \alpha^i$. This reduces competition because it allows for the circumstance where Player 2 faces an opponent who is "weaker" than type i.

A last circumstance must be considered.

Corollary 3.2.5 *If $V_2 - \alpha^t < V_1 \leq V_2 - \alpha^1$ and $\left(\sum_{l=1}^{j} \pi_l\right) V_2 - \alpha^j < 0$ for every type $j < t$, two classes of equilibria emerge.*

A first class includes equilibria where every type $j < t$ does not actively participate in the auction. The equilibrium strategies for Player 2 and for all the types $i \geq t$ are the same as for the case where Player 1's advantage is always greater than or equal to α^t.

A second class includes equilibria where some type $j < t$ actively participates in the auction while lower types bid 0 with probability 1. The equilibrium strategies for Player 2 and for all the types $i \geq j$ are the same as for the case where Player 1's advantage is always greater than or equal to α^j.

Proof. Consider the first type of equilibrium and assume that no type $j < t$ actively participates in the auction. Then $p_2 = 0$ and $p_1^t > 0$ must hold implying that $F_2(\alpha^t) > 0$. Player 2's equilibrium distribution has an atom of probability in the interval $[0, \alpha^t]$. Since the condition $\left(\sum_{l=1}^{j} \pi_l\right) V_2 - \alpha^j < 0$ holds for every type $j < t$, an optimal strategy requires the atom to be placed in 0.

Player 2 in fact competes only with types $i > t$; hence, the equilibrium for the game is qualitatively analogous to that arising when Player 1' advantage is always greater or equal to α^t which is characterized in Proposition 3.13.

Consider now what happens when a type $j < t$ actively participates in the auction. Following the argument presented in Proposition 3.15 it is possible to show that $t - 1$ equilibria emerge where Player 1 submits positive bids only if his/her advantage is bigger than α^j. ∎

Consider now what happens if the hypothesis $\alpha^n \leq V_2$ is dropped. A first preliminary result is worth noting.

Proposition 3.2.6 *When $\alpha^n \geq V_2$ holds, Player 2 actively participates in the auction only if a type of Player 1, $t < n$, exists such that $\alpha^t \geq V_2 \left(1 - \sum_{l=t}^{n} \pi_t\right) \geq \alpha^{t-1}$ holds.*

Proof. If $\alpha^n \geq V_2$, there must be at least a type j of Player 1 such that for every $i > j$ the inequality $\alpha^i \geq V_2$ holds. Since every bid $x_2 > V_2$ is a dominated action, Player 2 with probability $\sum_{l=j}^{n} \pi_l$ submits a bid which never wins. Player 2's reach is thus reduced to $V_2 \left(1 - \sum_{l=j+1}^{n} \pi_l\right)$. As a consequence, it might also happen that $V_2 \left(1 - \sum_{l=j+1}^{n} \pi_l\right) < \alpha^j$ holds. If this is the case Player 2's reach is further scaled down by a factor $\pi_j \cdot V_2$. The set of undominated actions must again be revised, comparing $V_2\left(1 - \sum_{l=j+1}^{n} \pi_l\right) - \pi_j \cdot V_2$ and α^{j-1}. Player 2 thus submits strictly positive bids only if there is at least a type $t \leq j$ of Player 1 whose head start advantage does not exceed his/her reach. This requires that $\alpha^t - \alpha^{t-1} \geq V_2 \cdot \pi_t$ holds or equivalently $\alpha^t \geq V_2 (1 - \sum_{i=t}^{n} \pi_i) \geq \alpha^{t-1}$. ■

Corollary 3.2.7 *If there is a type $t < n$ such that $\alpha^t \geq V_2 (1 - \sum_{l=t}^{n} \pi_t) \geq \alpha^{t-1}$, the game is strategically equivalent to an auction where prize values are V_1 and $V_2 (1 - \sum_{i=t}^{n} \pi_i)$ and Player 1's head start advantage is smaller than or equal to α^{t-1}.*

Proof. The proof follows straightforwardly from Proposition 3.18. ■

Corollary 3.2.8 *If no type t exists such that $\alpha^t \geq V_2 (1 - \sum_{i=t}^{n} \pi_i) \geq \alpha^{t-1}$ holds, in the unique equilibrium of the game Player 1 and Player 2 bid 0 with probability 1 and get a payoff respectively of $p_1^i = V_1$ and $p_2 = 0$.*

Proof. The proof descends straightforwardly from Proposition 3.18. ■

4

FINAL REMARKS

The paper provides a brief analysis of the game where two players with different "bidding technologies" compete in a first-price all-pay auction. Indeed there is a favored player who has a head start advantage whose size is exogenously given. This setting fits many contests where one player has an incumbency advantage which depends on the auctioneer's preferences. The head start advantage captures, for instance, the bias toward a briber of a corrupt official motivated by trust deriving from previous involvement in the same business. In elections, it defines the initial preference which some or all voters may have for a specific candidate due to ideology or personal charisma, regardless of electoral platforms. Finally, an asymmetry between players may stem from the decision of the legislator to promote specific principles as in the case of criminal law where the counsel for the defense starts ahead of the prosecution. The circumstances listed above are fairly common in the economic literature and at least for the first two examples, it is plausible that the size of the incumbent's head start advantage is uncertain.

The main effect of uncertainty is to reduce bidders' expected expenditures and hence the revenues collected by the auctioneer. In particular, this happens if there is at least one type of Player 1 whose reach is smaller than that of Player 2. This circumstance also defines an information rent for the favored player. If revenue maximization is the aim of the contest, as is the case in rent-seeking activities, then the auctioneer should try to reduce uncertainty. Analogously in patent races a reduction in R&D investments of both the incumbent firm and the new entrant occurs, slowing down the pace of innovation.

In the present setting uncertainty is limited to the size of the head start advantage. However, in some real world circumstances the prize value is also private information of the players. Studying uncertainty on both these dimensions could represent a valuable extension to the present paper. Finally many contests involve more than one player. The introduction of multiple bidders could be a fruitful avenue for future research.

5

APPENDIX: MAIN PROOFS

5.1 Proof of Proposition 2.0.2

Successive rounds of elimination of strictly dominated strategies restrict players' bids to the interval $[0, V_2 - \alpha]$ for Player 1 and to $[\alpha, V_2] \cup \{0\}$ for Player 2. Every strictly positive bid smaller than α is strictly dominated by $x_2 = 0$; any such offer has a nil probability of winning and gives a negative payoff. Moreover, Player 2 never bids more than his/her own prize value because again this gives a negative payoff; $x_2 = 0$ strictly dominates $x_2 > V_2$. Since Player 2 never bids more than V_2, Player 1 wins with certainty by offering $x_1 \geq V_2 - \alpha$. However, $x_1 = V_2 - \alpha$ strictly dominates every greater bid; Player 1 wins in both cases but in the latter his/her payment is smaller and the payoff higher.

Following Hillman and Riley (1989) (see also Baye et al. (1993), Che and Gale (1998) and Ellingsen (1991)) it is possible to exclude that Player 1 submits a bid, $\kappa \in (0, V_2 - \alpha]$, with strictly positive probability. If this happens, Player 2's probability of winning rises discontinuously at $x_2 = \kappa$ and there is some $\varepsilon > 0$ such that Player 2 bids on the interval $[\kappa - \varepsilon, \kappa]$ with nil probability. Player 1 thus is better off shifting the mass of probability down from κ to $\kappa - \varepsilon$ to reduce his/her spending level without affecting the probability of winning. A symmetrical argument applies to Player 2's strategies on the interval $(\alpha, V_2]$.

A strictly positive probability can only be attached to $x_1 = 0$, $x_2 = \alpha$ or to $x_2 = 0$. However, it cannot be the case that Player 1 places an atom of probability in 0 and that, at the same time, Player 2's distribution has an atom of probability in α. If this happens, Player 1 increases the probability of winning by a finite amount, by bidding slightly more than 0. Player 2 can do the same by bidding slightly more than α, and a profitable deviation exists for both players. Analogously in equilibrium it cannot happen that $F_1(0) > 0$ and Player 2 randomizes over a support which includes $x_2 = \alpha$. Since ties are broken randomly, his/her probability of winning decreases discontinuously at α and a profitable deviation exists, i.e. to shift probability to $x_2 > \alpha$. The same argument excludes that $x_1 = 0$ is submitted with positive probability if Player 2's distribution has an atom of probability in α. Given that players' strategies are continuous over $(0, V_2 - \alpha]$ and $(\alpha, V_2]$, the probability of a tie is nil.

Let $\bar{x}_i$ be the upper bound for the support of Player i's distribution. In equilibrium the equality $\bar{x}_1 = \bar{x}_2 - \alpha$ holds. Suppose instead that $\bar{x}_1 > \bar{x}_2 - \alpha$. When $x_1 = \bar{x}_1$, Player 1 gets the auctioned item with probability 1; every bid such that $\bar{x}_2 - \alpha < x_1 < \bar{x}_1$ has the same probability of winning but involves a smaller payment and strictly dominates $\bar{x}_1$. A symmetrical argument applied to Player 2 excludes that $\bar{x}_1 < \bar{x}_2 - \alpha$ holds.

Let $\underline{x}_i$ denote the lower bound of Player i's support. The equality $\underline{x}_1 = \underline{x}_2 - \alpha$ holds in equilibrium. Suppose that, instead, $\underline{x}_1 < \underline{x}_2 - \alpha$. Any bid such that $\underline{x}_1 \leq x_1 < \underline{x}_2 - \alpha$ gives Player 1 a nil probability of winning and a negative payoff. This means that it is strictly dominated by $x_1 = 0$. Assume now that $\underline{x}_1 > \underline{x}_2 - \alpha$; a symmetrical argument excludes that this inequality is verified.

Note now that $\underline{x}_1 = 0$ and $\underline{x}_2 = \alpha$ must hold. Indeed, if $\underline{x}_1 > 0$, Player 1 can reduce his/her spending level by shifting down the lower bound of the distribution. This does not affect the probability of winning since $\underline{x}_1 = \underline{x}_2 - \alpha$ and $F_2(\underline{x}_2) = 0$ hold.

Player 1's expected payoff, p_1, is:

$$\begin{aligned} E\left[U_1\left(x_1, F_2\left(\alpha + x_1\right)\right)\right] &= \\ \left(V_1 - x_1\right) F_2\left(\alpha + x_1\right) - x_1\left[1 - F_2\left(\alpha + x_1\right)\right] &= p_1. \end{aligned}$$

Player 2's expected payoff, p_2, is:

$$\begin{aligned} E\left[U_2\left(x_2, F_1\left(x_2 - \alpha\right)\right)\right] &= \\ \left(V_2 - x_2\right) F_1\left(x_2 - \alpha\right) - x_2\left[1 - F_1\left(x_2 - \alpha\right)\right] &= p_2. \end{aligned}$$

In equilibrium, the players obtain the same expected payoff from each pure strategy over which they randomize. Since Player 2 never bids more than V_2 and $V_2 \leq V_1 + \alpha$ holds, Player 1 can always secure a positive payoff offering $x_1 \geq V_2$. If in equilibrium $p_1 > 0$, then $F_2(\alpha) > 0$ when $\underline{x}_1 = 0$ holds. Moreover, Player 2's expected payoff must be nil. Suppose that this is not the case; $p_2 > 0$ requires that when $x_2 = \alpha$, $F_1(0) > 0$ holds. However, the inequality $F_2(\alpha) > 0$ must be verified as well, implying that the atom in Player 2's distribution is placed at 0 as proven above. Since $x_2 = 0$ always gives a nil payoff, $p_2 > 0$ cannot hold.

Player 2 is indifferent between actively participating in the auction and bidding 0 with probability 1. The latter circumstance though never occurs in equilibrium. If this were the case, Player 1's optimal response would be to bid 0 with certainty implying that Player 2 has the profitable deviation to bid $x_2 = \alpha + \varepsilon$ with $\varepsilon > 0$ arbitrarily small.

Since $p_2 = 0$, in equilibrium $\bar{x}_1 = V_2 - \alpha$ and $\bar{x}_2 = \bar{x}_1 + \alpha = V_2$ must hold. If this were not the case and $\bar{x}_2 < V_2$, one would also obtain $\bar{x}_1 < V_2 - \alpha$ and Player 2 would have the profitable deviation to bid slightly more than $\bar{x}_2$ to get a positive payoff.

The equilibrium conditions may now be defined. The following equality must hold for Player 2:

$$\left(V_2 - x_2\right) F_1\left(x_2 - \alpha\right) - x_2\left[1 - F_1\left(x_2 - \alpha\right)\right] = 0$$

so that $F_1(x_2 - \alpha) = \frac{x_2 - \alpha}{V_2} + \frac{\alpha}{V_2}$ holds or, exploiting the equality $x_1 = x_2 - \alpha$,

$$F_1(x_1) = \frac{x_1 + \alpha}{V_2}.$$

Since no negative bids are allowed, Player 1's equilibrium distribution has an atom in 0 amounting to $\frac{\alpha}{V_2}$; hence he/she bids 0 with probability $\frac{\alpha}{V_2}$ and with complementary probability randomizes according to a uniform distribution over the interval $(0, V_2 - \alpha]$.

Bidding $x_1 = V_2 - \alpha$ gives Player 1 an expected payoff amounting to $V_1 + \alpha - V_2 > 0$; in equilibrium then the condition:

$$(V_1 - x_1) F_2(x_1 + \alpha) - x_1 [1 - F_2(x_1 + \alpha)] = V_1 + \alpha - V_2$$

holds, or $F_2(x_1 + \alpha) = 1 - \frac{V_2}{V_1} + \frac{x_1 + \alpha}{V_1}$ such that also

$$F_2(x_2) = 1 - \frac{V_2}{V_1} + \frac{x_2}{V_1}.$$

Given that $\underline{x}_2 = \alpha$, the previous distribution has an atom of probability. Since $F_1(0) > 0$ holds, the atom must be placed in 0. Player 2 thus bids 0 with probability $1 - \frac{V_2}{V_1} + \frac{\alpha}{V_1} \geq 0$ and randomizes according to the cumulative distribution function $\frac{x_2}{V_1}$ over the support $(\alpha, V_2]$ with complementary probability. Uniqueness of the equilibrium follows from the continuity of players' strategies and from the definition of the upper and lower bounds of the supports. Since no atoms of probability can be placed at any bid $x_1 > 0$, only the proposed strategies satisfy the equilibrium conditions.

5.2 Proof of Proposition 3.1.5

Note initially that $p_1^0 = 0$ must hold. Suppose this is not the case. If it were $p_1^0 > 0$, by Lemma 3.7 it should be also $p_2 = 0$ and $F_2(0) > 0$. Player 2's distribution would have an upper bound $\bar{x}_2 < V_1$ and the following inequality would hold for $x_1^0 = \bar{x}_2$:

$$V_1 \cdot F_2\left(x_1^0\right) - x_1^0 > 0$$

Every bid greater than x_1^0 and greater than $x_1^\alpha = \bar{x}_2 - \alpha$ is a dominated action for Player 1 respectively if he/she has not and if he/she has the head start advantage. Hence Player 2 has the profitable deviation to submit a bid x_2 such that $\bar{x}_2 < x_2 < V_2$ and get a positive payoff since $\bar{x}_2 < V_1 < V_2$. This excludes that an equilibrium where $p_1^0 > 0$ and $p_2 = 0$ exists.

If $p_1^0 = 0$, Player 1 is indifferent between actively participating in the auction or bidding zero with probability 1 when he/she does not have the head start advantage. Assume that he/she actively participates in the auction. This requires that in equilibrium, for $x_1^0 = V_1$ the equality $V_1 \cdot F_2(V_1) = V_1$ holds, implying that $F_2(V_1) = 1$. Every bid greater than $V_1 - \alpha$ thus is a dominated action for Player 1 if he/she has the advantage. As a consequence it is possible to exclude that an equilibrium exists where $p_2 = 0$. Player 2 indeed can obtain a certain positive payoff by bidding slightly more than $x_2 = V_1$. Since it is $\bar{x}_2 = V_1$ the same argument used in Proposition 3.2 establishes that $\bar{x}_1^0 = \bar{x}_2 = V_1$ and $\bar{x}_1^\alpha = \bar{x}_2 - \alpha = V_1 - \alpha$ hold and that $\underline{x}_2 = \underline{x}_1^0 = \underline{x}_1^\alpha = 0$; as a consequence it is $p_2 = V_2 - V_1$ and the following equality must be verified for every $x_2 \in [0, \alpha)$:

$$(1-\lambda) V_2 \cdot F_1^0(x_2) - x_2 = V_2 - V_1$$

so that:

$$F_1^0(x_2) = \frac{V_2 - V_1}{(1-\lambda) V_2} + \frac{x_2}{(1-\lambda) V_2}$$

or

$$F_1^0(x_1^0) = \frac{1}{1-\lambda}\left(1 - \frac{V_1}{V_2}\right) + \frac{x_1^0}{(1-\lambda) V_2}$$

Player 1's distribution thus has an atom in 0 amounting to $\frac{1}{1-\lambda}\left(1 - \frac{V_1}{V_2}\right)$. This means that he/she actively participates in the auction if

$$\frac{1}{1-\lambda}\left(1 - \frac{V_1}{V_2}\right) \leq 1$$

holds or

$$\lambda \leq \frac{V_1}{V_2}$$

Consider now the equilibrium conditions for $x_2 \in [\alpha, V_1]$. It must be the case that the following equality holds:

$$\begin{aligned} & V_2(1-\lambda) F_1^0(\alpha) + V_2\left[1 - (1-\lambda) F_1^0(\alpha)\right] \lambda \cdot \Pr\left[x_2 - \alpha > x_1^\alpha | x_2 \geq \alpha\right] + \\ & + V_2\left[1 - (1-\lambda) F_1^0(\alpha)\right] (1-\lambda) \Pr\left[x_2 > x_1^0 | x_2 \geq \alpha\right] - x_2 \\ = \; & V_2 - V_1 \end{aligned}$$

From the previous equilibrium condition it is $F_1^0(\alpha) = \frac{1}{1-\lambda}\left(1 - \frac{V_1}{V_2}\right) + \frac{\alpha}{(1-\lambda)V_2}$. Substituting into the equation above gives:

$$\begin{aligned} & V_2\left(1 - \frac{V_1}{V_2} + \frac{\alpha}{V_2}\right) + V_2\left(\frac{V_1}{V_2} - \frac{\alpha}{V_2}\right) \lambda \cdot \Pr\left[x_2 - \alpha > x_1^\alpha | x_2 \geq \alpha\right] + \\ & + V_2\left(\frac{V_1}{V_2} - \frac{\alpha}{V_2}\right) (1-\lambda) \Pr\left[x_2 > x_1^0 | x_2 \geq \alpha\right] - x_2 \\ = \; & V_2 - V_1 \end{aligned}$$

Simplifying and reordering the terms gives:

$$\begin{aligned} & \lambda \cdot \Pr\left[x_2 - \alpha > x_1^\alpha | x_2 \geq \alpha\right] + (1-\lambda) \Pr\left[x_2 > x_1^0 | x_2 \geq \alpha\right] \\ = \; & \frac{x_2 - \alpha}{V_1 - \alpha} \end{aligned}$$

implying that:

$$\Pr\left[x_2-\alpha>x_1^\alpha|x_2\geq\alpha\right]=\frac{x_2-\alpha}{V_1-\alpha}$$

and

$$\Pr\left[x_2>x_1^0|x_2\geq\alpha\right]=\frac{x_2-\alpha}{V_1-\alpha}$$

Using the same argument presented in Proposition 3.2 it is possible to show that in equilibrium $\underline{x}_1^0=\underline{x}_2=\underline{x}_1^\alpha=0$ holds, implying that $F_1^\alpha(0)=0$. As a consequence since

$$\Pr\left[x_2-\alpha>x_1^\alpha|x_2>\alpha\right]=\frac{F_1^\alpha(x_2-\alpha)-F_1^\alpha(0)}{1-F_1^\alpha(0)}=F_1^\alpha(x_2-\alpha)$$

hold, it is also

$$F_1^\alpha(x_2-\alpha)=\frac{x_2-\alpha}{V_1-\alpha}$$

or

$$F_1^\alpha(x_1^\alpha)=\frac{x_1^\alpha}{V_1-\alpha}$$

Moreover it is the case that:

$$\Pr\left[x_2>x_1^0|x_2>\alpha\right]=\frac{F_1^0(x_2)-F_1^0(\alpha)}{1-F_1^0(\alpha)}$$

or

$$\Pr\left[x_2>x_1^0|x_2>\alpha\right]=\frac{F_1^0(x_2)-\frac{1}{1-\lambda}\left(1-\frac{V_1}{V_2}\right)-\frac{\alpha}{(1-\lambda)V_2}}{1-\frac{1}{1-\lambda}\left(1-\frac{V_1}{V_2}\right)-\frac{\alpha}{(1-\lambda)V_2}}$$

Substituting into the previous equation, simplifying and reordering the terms, gives:

$$F_1^0\left(x_1^0\right)=\frac{x_1^0\left(\frac{V_1}{V_2}-\frac{\alpha}{V_2}-\lambda\right)}{(1-\lambda)(V_1-\alpha)}-\frac{\alpha}{V_1-\alpha}+V_1\cdot\frac{1-\frac{V_1-\alpha}{V_2}}{(1-\lambda)(V_1-\alpha)}$$

Consider now Player 1. When he/she does not have the head start advantage the following condition must hold for every $x_1^0\in[0,V_1]$

$$V_1\cdot F_2\left(x_1^0\right)-x_1^0=0$$

and this implies:

$$F_2(x_2)=\frac{x_2}{V_1}$$

If he/she has the advantage the equality $\bar{x}_1^\alpha=\bar{x}_2-\alpha$ implies that $p_1^\alpha=\alpha$. The following condition thus must be verified for every $x_1^\alpha\in[0,V_1-\alpha]$

$$V_1\cdot F_2\left(x_1^\alpha+\alpha\right)-x_1^\alpha=\alpha$$

which is equivalent to:

$$F_2\left(x_1^{\alpha}+\alpha\right)=\frac{x_1^{\alpha}+\alpha}{V_1}$$

implying that again is $F_2\left(x_2\right)=\frac{x_2}{V_1}$.

5.3 Proof of Proposition 3.1.9

If $V_1>V_2$ holds, it is possible to exclude that $p_1^0=0$, since Player 1 can always obtain a certain positive payoff by bidding slightly more than $x_1^0=V_2$. By Lemma 3.7, $p_2=0$ must hold. Player 2 thus is indifferent between actively participate in the auction and bid zero with probability 1. Note though that no equilibria exist such that $F_2\left(0\right)=1$ holds. If this happens indeed, a best response for Player 1 is to bid $x_1^1=\varepsilon$ with $\varepsilon>0$ arbitrarily small and $x_1^{\alpha}=0$ to win the prize with certainty in both the states of the world. This implies that Player 2 has the profitable deviation to bid slightly more than α to get a certain positive payoff. As a consequence it must be the case that for $x_2=V_2$ the equality $V_2\cdot F_1\left(V_2\right)=V_2$ holds implying $F_1\left(V_2\right)=1$. Every bid greater than V_2 and than $V_2-\alpha$ thus is a dominated action for Player 1 respectively if he/she has not and if he/she has the head start advantage. Since it is $\bar{x}_2=V_2$ the same argument used in Proposition 3.2 establishes that $\bar{x}_1^0=\bar{x}_2=V_2$ and $\bar{x}_1^{\alpha}=\bar{x}_2-\alpha=V_2-\alpha$ hold and that $\underline{x}_2=\underline{x}_1^0=\underline{x}_1^{\alpha}=0$; as a consequence it is $p_2=0$ and the following equality must be verified for every $x_2\in[0,\alpha)$:

$$\left(1-\lambda\right)V_2\cdot F_1^0\left(x_2\right)-x_2=0$$

so that:

$$F_1^0\left(x_2\right)=\frac{x_2}{\left(1-\lambda\right)V_2}$$

or

$$F_1^0\left(x_1^0\right)=\frac{x_1^0}{\left(1-\lambda\right)V_2}$$

For every bid $x_2\in[\alpha,V_2]$ it must be the case that the following equality holds:

$$\begin{aligned}&V_2\left(1-\lambda\right)F_1^0\left(\alpha\right)+\left[1-\left(1-\lambda\right)F_1^0\left(\alpha\right)\right]V_2\cdot\lambda\cdot\Pr\left[x_2-\alpha>x_1^{\alpha}|x_2\geq\alpha\right]+\\&+V_2\left[1-\left(1-\lambda\right)F_1^0\left(\alpha\right)\right]\left(1-\lambda\right)\Pr\left[x_2>x_1^0|x_2>\alpha\right]-x_2=0\end{aligned}$$

From the previous equilibrium condition it is $F_1^0\left(\alpha\right)=\frac{\alpha}{(1-\lambda)V_2}$. Substituting into the equation above gives:

$$\begin{aligned}&\alpha+V_2\left(1-\frac{\alpha}{V_2}\right)\lambda\cdot\Pr\left[x_2-\alpha>x_1^{\alpha}|x_2\geq\alpha\right]+\\&+V_2\left(1-\frac{\alpha}{V_2}\right)\left(1-\lambda\right)\Pr\left[x_2>x_1^0|x_2>\alpha\right]-x_2=0\end{aligned}$$

Note now that is:

$$\Pr\left[x_2-\alpha > x_1^\alpha | x_2 \geq \alpha\right] = \frac{F_1^\alpha\left(x_2-\alpha\right)-F_1^\alpha\left(0\right)}{1-F_1^\alpha\left(0\right)} = F_1^\alpha\left(x_2-\alpha\right)$$

and

$$\Pr\left[x_2 > x_1^0 | x_2 > \alpha\right] = \frac{F_1^0\left(x_2\right)-F_1^0\left(\alpha\right)}{1-F_1^0\left(\alpha\right)}$$

or

$$\Pr\left[x_2 > x_1^0 | x_2 > \alpha\right] = \frac{F_1^0\left(x_2\right)-\frac{\alpha}{(1-\lambda)V_2}}{1-\frac{\alpha}{(1-\lambda)V_2}}$$

Substituting into the previous equation gives:

$$\alpha + V_2\left(1-\frac{\alpha}{V_2}\right)\lambda\cdot F_1^\alpha\left(x_2-\alpha\right)+$$
$$+V_2\left(1-\frac{\alpha}{V_2}\right)(1-\lambda)\frac{F_1^0\left(x_2\right)-\frac{\alpha}{(1-\lambda)V_2}}{1-\frac{\alpha}{(1-\lambda)V_2}} - x_2 = 0$$

Simplifying and reordering the terms gives:

$$\lambda\cdot F_1^\alpha\left(x_2-\alpha\right)+(1-\lambda)\frac{F_1^0\left(x_2\right)-\frac{\alpha}{(1-\lambda)V_2}}{1-\frac{\alpha}{(1-\lambda)V_2}}$$
$$= \frac{x_2-\alpha}{V_2-\alpha}$$

implying that

$$F_1^\alpha\left(x_1^\alpha\right) = \frac{x_1^\alpha}{V_2-\alpha}$$

and also

$$\frac{F_1^0\left(x_2\right)-\frac{\alpha}{(1-\lambda)V_2}}{1-\frac{\alpha}{(1-\lambda)V_2}} = \frac{x_2-\alpha}{V_2-\alpha}$$

so that

$$F_1^0\left(x_1^0\right) = \frac{x_1^0\left(1-\lambda-\frac{\alpha}{V_2}\right)}{(1-\lambda)\left(V_2-\alpha\right)} + \frac{\lambda\cdot\alpha}{(1-\lambda)\left(V_2-\alpha\right)}$$

Uniqueness is proven as in Corollary 3.10.

Consider now Player 1. Since it is $\bar{x}_1^0 = \bar{x}_2$, $p_1^0 = V_1 - V_2$ holds and the following condition must be verified for every $x_1^0 \in [0, V_2]$:

$$V_1\cdot F_2\left(x_1^0\right) - x_1^0 = V_1 - V_2$$

or

$$F_2\left(x_1^0\right) = 1 - \frac{V_2}{V_1} + \frac{x_1^0}{V_1}$$

and this implies:

$$F_2(x_2) = 1 - \frac{V_2}{V_1} + \frac{x_2}{V_1}$$

If he/she has the advantage instead the equality $\bar{x}_1^{\alpha} = \bar{x}_2 - \alpha$ implies that $p_1^{\alpha} = V_1 - V_2 + \alpha$. The following condition thus must hold for every $x_1^{\alpha} \in [0, V_2 - \alpha]$

$$V_1 \cdot F_2(x_1^{\alpha} + \alpha) - x_1^{\alpha} = V_1 - V_2 + \alpha$$

which is equivalent to:

$$F_2(x_1^{\alpha} + \alpha) = 1 - \frac{V_2}{V_1} + \frac{x_1^{\alpha} + \alpha}{V_1}$$

implying that again is $F_2(x_2) = 1 - \frac{V_2}{V_1} + \frac{x_2}{V_1}$.

Uniqueness is proven using the standard argument.

5.4 Proof of Proposition 3.2.1

If $V_1 + \alpha^1 > V_2$, it may be excluded that in equilibrium $p_1^1 = 0$ holds, since Player 1 always obtains a positive payoff by bidding slightly more than $V_2 - \alpha^1$. As a consequence, Player 2 must get a nil expected payoff. Suppose instead that $p_1^1 > 0$ and $p_2 > 0$ hold. The same argument used in Proposition 3.2 establishes that, in equilibrium, we have $\underline{x}_1^1 = 0$ and $\underline{x}_2 = \alpha^1$. But $p_1^1 > 0$ requires $F_2(\alpha^1) > 0$ since $V_1 \cdot F_2(\alpha^1) > 0$ must hold when $x_1^1 = 0$. Analogously $p_2 > 0$ requires $F_1^1(0) > 0$ and the atom of probability in Player 2's equilibrium distribution must be placed at 0. This gives a nil expected payoff and excludes that $p_2 > 0$ holds.

Therefore Player 2 is indifferent between actively participating in the auction or bidding 0 with probability 1. But no equilibria exist such that $F_2(0) = 1$. If this happens, a best response for Player 1 is to bid $x_1^i = 0$ and get the prize with certainty. Player 2 then has the profitable deviation to bid slightly more than α^n and get a positive payoff.

Since $p_2 = 0$, when $x_2 = V_2$, the equality $\Sigma_{i=1}^n \pi_i \cdot F_1^i(V_2 - \alpha^i) = 1$ must hold and every bid greater than $V_2 - \alpha^i$ is a dominated action for Player 1, type i. The same argument used in Proposition 3.2 establishes further that $\bar{x}_1^i = V_2 - \alpha^i$ and $\bar{x}_2 = V_2$ hold.

In equilibrium the following condition must be verified for every $x_2 \in [\alpha^1, \alpha^2]$:

$$V_2 \cdot \pi_1 \cdot F_1^1\left(x_2 - \alpha^1\right) - x_2 = 0.$$

Through simple algebra, exploiting the equivalence $x_1^1 = x_2 - \alpha^1$ it is possible to get:

$$F_1^1\left(x_1^1\right) = \frac{x_1^1 + \alpha^1}{\pi_1 \cdot V_2}.$$

This defines the equilibrium cumulative distribution function of Player 1, type 1, when $0 \leq x_1^1 < \alpha^2 - \alpha^1$. It has an atom of probability which by the argument presented in Proposition 3.2 can only be placed in 0; as a consequence, in order to have Player 1 actually participating in the auction $\frac{\alpha^1}{\pi_1 \cdot V_2} \leq 1$ must hold. If this is not the case, Player 2 never bids in the interval $(\alpha^1, \alpha^2]$ because this gives a negative expected payoff.

Consider the case where Player 1 does not actively participate in the auction when his/her advantage is strictly smaller than α^t. The same argument presented above establishes that the following condition must be verified for every $x_2 \in (\alpha^t, \alpha^{t+1}]$:

$$V_2 \cdot \sum_{j=1}^{t-1} \pi_j + V_2 \cdot \pi_t \cdot F_1^t \left(x_2 - \alpha^t\right) - x_2 = 0$$

so that Player 1, type t, randomizes according to the cumulative distribution function:

$$F_1^t \left(x_1^t\right) = \frac{x_1^t + \alpha^t}{\pi_t \cdot V_2} - \sum_{j=1}^{t-1} \frac{\pi_j}{\pi_t}$$

when $0 \leq x_1^t < \alpha^{t+1} - \alpha^t$. Also in this case Player 1's distribution has an atom of probability in 0 implying that he/she submits positive bids only if

$$\frac{\alpha^t}{\pi_t \cdot V_2} - \sum_{j=1}^{t-1} \frac{\pi_j}{\pi_t} < 1$$

holds.

For every bid $x_2 \in \left(\alpha^k, \alpha^{k+1}\right]$ with $k \geq t+1$, the equilibrium condition:

$$V_2 \cdot \sum_{i=1}^{k} \pi_i \cdot F_1^i \left(\alpha^k - \alpha^i\right) + \left[\sum_{i=1}^{k} \pi_i \cdot F_1^i \left(\alpha^{k+1} - \alpha^i\right) - \sum_{i=\hat{t}}^{k} \pi_i \cdot F_1^i \left(\alpha^k - \alpha^i\right) \right] \cdot$$
$$\cdot V_2 \cdot \sum_{i=1}^{k} \frac{\pi_i}{\sum_{p=1}^{k} \pi_p} \cdot \Pr\left[x_2 - \alpha^i > x_1^i | \alpha^{k+1} \geq x_2 \geq \alpha^k\right] - x_2 = 0$$

must hold. Substituting

$$\Pr\left[x_2 - \alpha^i > x_1^i | \alpha^{k+1} \geq x_2 \geq \alpha^k\right] = \frac{F_1^i \left(x_2 - \alpha^i\right) - F_1^i \left(\alpha^k - \alpha^i\right)}{F_1^i \left(\alpha^{k+1} - \alpha^i\right) - F_1^i \left(\alpha^k - \alpha^i\right)}$$

in the previous equation, it is easy to get through simple algebra:

$$\left[\sum_{i=1}^{k} \pi_i \cdot F_1^i \left(\alpha^{k+1} - \alpha^i\right) - \sum_{i=1}^{k} \pi_i \cdot F_1^i \left(\alpha^k - \alpha^i\right) \right] \cdot$$
$$\cdot \sum_{i=1}^{k} \frac{\pi_i}{\sum_{p=1}^{k} \pi_p} \left[\frac{F_1^i \left(x_2 - \alpha^i\right) - F_1^i \left(\alpha^k - \alpha^i\right)}{F_1^i \left(\alpha^{k+1} - \alpha^i\right) - F_1^i \left(\alpha^k - \alpha^i\right)} \right] = \frac{x_2}{V_2} - \sum_{i=1}^{k} \pi_i \cdot F_1^i \left(\alpha^k - \alpha^i\right).$$

Note now that if $x_2 = \alpha^{k+1}$ then

$$\sum_{i=1}^{k} \pi_i \cdot F_1^i \left(\alpha^{k+1} - \alpha^i\right) = \frac{\alpha^{k+1}}{V_2}$$

implying further

$$\sum_{i=1}^{k} \pi_i \cdot F_1^i \left(\alpha^{k+1} - \alpha^i\right) - \sum_{i=1}^{k-1} \pi_i \cdot F_1^i \left(\alpha^k - \alpha^i\right) = \frac{\alpha^{k+1} - \alpha^k}{V_2}. \tag{5.1}$$

Substitute into the previous expression and exploit the equivalence $x_1^i + \alpha^i = x_2$ to get after simplifying and reordering the terms:

$$\sum_{i=1}^{k} \frac{\pi_i}{\sum_{p=1}^{k} \pi_p} \left[\frac{F_1^i\left(x_1^i\right) - F_1^i\left(\alpha^k - \alpha^i\right)}{F_1^i\left(\alpha^{k+1} - \alpha^i\right) - F_1^i\left(\alpha^k - \alpha^i\right)} \right] = \frac{x_2 - \alpha^k}{\alpha^{k+1} - \alpha^k} \tag{5.2}$$

implying that:

$$\frac{F_1^i\left(x_2 - \alpha^i\right) - F_1^i\left(\alpha^k - \alpha^i\right)}{F_1^i\left(\alpha^{k+1} - \alpha^i\right) - F_1^i\left(\alpha^k - \alpha^i\right)} = \frac{x_2 - \alpha^k}{\alpha^{k+1} - \alpha^k}. \tag{5.3}$$

Suppose that this were not the case and assume without loss of generality that a type i exists such that $\frac{F_1^i\left(x_2 - \alpha^i\right) - F_1^i\left(\alpha^k - \alpha^i\right)}{F_1^i\left(\alpha^{k+1} - \alpha^i\right) - F_1^i\left(\alpha^k - \alpha^i\right)} > \frac{x_2 - \alpha^k}{\alpha^{k+1} - \alpha^k}$. Then in order for Equation 3.5 to be verified it must be the case that there is at least one type j such that $\frac{F_1^j\left(x_2 - \alpha^j\right) - F_1^j\left(\alpha^k - \alpha^j\right)}{F_1^j\left(\alpha^{k+1} - \alpha^i\right) - F_1^j\left(\alpha^k - \alpha^i\right)} < \frac{x_2 - \alpha^k}{\alpha^{k+1} - \alpha^k}$. Note now that if $x_2 = \alpha^{k+1}$, $\frac{F_1^j\left(x_2 - \alpha^j\right) - F_1^j\left(\alpha^k - \alpha^j\right)}{F_1^j\left(\alpha^{k+1} - \alpha^i\right) - F_1^j\left(\alpha^k - \alpha^i\right)} < 1$ holds, implying that $F_1^j\left(x_2 - \alpha^j\right)$ must have an atom of probability at $\alpha^{k+1} - \alpha^j$. This can never be an equilibrium though, because a profitable deviation for Player 2 exists. Indeed, if Player 1 places an atom at $\alpha^{k+1} - \alpha^j$ the probability of a tie is strictly positive. Since ties are broken randomly, Player 2's probability of winning decreases discontinuously at $x_2 = \alpha^{k+1}$ and he/she has the profitable deviation to shift probability to bids $x_2 > \alpha^{k+1}$.

From Equation 3.6, since atoms of probability can be placed only at $x_1^t = 0$[1] and $F_1^k(0) = 0$ holds, it follows that:

$$\frac{F_1^k\left(x_1^k\right)}{F_1^k\left(\alpha^{k+1} - \alpha^k\right)} = \frac{x_2 - \alpha^k}{\alpha^{k+1} - \alpha^k}. \tag{5.4}$$

Divide now Equation 3.4 by $\sum_{p=1}^{k} \pi_p$, and reorder the terms to get:

$$\frac{\pi_k}{\sum_{p=1}^{k} \pi_p} \cdot F_1^k\left(\alpha^{k+1} - \alpha^k\right) + \sum_{i=1}^{k-1} \frac{\pi_i}{\sum_{p=1}^{k} \pi_p} \left[F_1^i\left(\alpha^{k+1} - \alpha^i\right) - F_1^i\left(\alpha^k - \alpha^i\right)\right]$$
$$= \frac{\alpha^{k+1} - \alpha^k}{V_2 \cdot \sum_{p=1}^{k} \pi_p}.$$

[1] This is proven as in Proposition 3.2.

Note that if the probability that $x_2 - \alpha^i$ is bigger than x_1^i, conditional on $x_2 \in \left(\alpha^k, \alpha^{k+1}\right]$, is the same for all types of Player 1, then also $F_1^i\left(x_2 - \alpha^i\right) - F_1^i\left(\alpha^k - \alpha^i\right)$ must be the same for every $i \leq k$. Hence:

$$F_1^k\left(\alpha^{k+1} - \alpha^k\right) = \frac{\alpha^{k+1} - \alpha^k}{V_2 \cdot \sum_{p=1}^k \pi_p}$$

and

$$F_1^i\left(\alpha^{k+1} - \alpha^i\right) - F_1^i\left(\alpha^k - \alpha^i\right) = \frac{\alpha^{k+1} - \alpha^k}{V_2 \cdot \sum_{p=1}^k \pi_p}.$$

Substituting $F_1^k\left(\alpha^{k+1} - \alpha^k\right)$ in Equations 3.7 gives:

$$F_1^k\left(x_1^k\right) = \frac{x_1^k}{V_2 \cdot \sum_{p=1}^k \pi_p}.$$

Substitute further $F_1^i\left(\alpha^{k+1} - \alpha^i\right) - F_1^i\left(\alpha^k - \alpha^i\right)$ in Equation 3.6 to get:

$$F_1^i\left(x_1^i\right) - F_1^i\left(\alpha^k - \alpha^i\right) = \frac{x_2 - \alpha^k}{V_2 \cdot \sum_{p=1}^k \pi_p}. \tag{5.5}$$

Therefore if $x_2 = \alpha^{k+1}$, then

$$F_1^i\left(\alpha^{k+1} - \alpha^i\right) = F_1^i\left(\alpha^k - \alpha^i\right) + \frac{\alpha^{k+1} - \alpha^k}{V_2 \cdot \sum_{p=1}^k \pi_p}$$

such that

$$F_1^i\left(\alpha^k - \alpha^i\right) = \sum_{j=0}^{k-i-1} \frac{\alpha^{k-j} - \alpha^{k-j-1}}{V_2 \cdot \sum_{p=1}^{k-j-1} \pi_p}$$

Substituting in Equation 3.8 allows us to get through simple algebra

$$F_1^i\left(x_1^i\right) = \sum_{j=0}^{k-i-1} \frac{\alpha^{k-j} - \alpha^{k-j-1}}{V_2 \cdot \sum_{p=1}^{k-j-1} \pi_p} + \frac{x_1^i + \alpha^i - \alpha^k}{V_2 \cdot \sum_{p=1}^k \pi_p}.$$

If $x_2 \geq \alpha^n$ the following equality holds:

$$V_2 \cdot \sum_{i=1}^n \pi_i \cdot F_1^i\left(\alpha^n - \alpha^i\right) + \left[1 - \sum_{i=1}^n \pi_i \cdot F_1^i\left(\alpha^n - \alpha^i\right)\right] \cdot$$
$$\cdot V_2 \cdot \sum_{i=1}^n \pi_i \left[\frac{F_1^i\left(x_1^i\right) - F_1^i\left(\alpha^n - \alpha^i\right)}{1 - F_1^i\left(\alpha^n - \alpha^i\right)}\right] - x_2 = 0.$$

Substituting $\sum_{i=1}^{n} \pi_i \cdot F_1^i(\alpha^n - \alpha^i) = \frac{\alpha^n}{V_2}$, simplifying and reordering the terms gives

$$\sum_{i=1}^{n} \pi_i \left[\frac{F_1^i(x_1^i) - F_1^i(\alpha^n - \alpha^i)}{1 - F_1^i(\alpha^n - \alpha^i)} \right] = \frac{x_2 - \alpha^n}{V_2 - \alpha^n}.$$

Hence:

$$F_1^n(x_1^n) = \frac{x_1^n}{V_2 - \alpha^n}$$

and

$$\frac{F_1^i(x_1^i) - F_1^i(\alpha^n - \alpha^i)}{1 - F_1^i(\alpha^n - \alpha^i)} = \frac{x_2 - \alpha^n}{V_2 - \alpha^n}.$$

Substitute $F_1^i(\alpha^n - \alpha^i) = \sum_{j=0}^{n-i-1} \frac{\alpha^{n-j} - \alpha^{n-j-1}}{V_2 \cdot \sum_{p=1}^{n-j-1} \pi_p}$ to get

$$\frac{F_1^i(x_2 - \alpha^i) - \sum_{j=0}^{n-i-1} \frac{\alpha^{n-j} - \alpha^{n-j-1}}{V_2 \cdot \sum_{p=1}^{n-j-1} \pi_p}}{1 - \sum_{j=0}^{n-i-1} \frac{\alpha^{n-j} - \alpha^{n-j-1}}{V_2 \cdot \sum_{p=1}^{n-j-1} \pi_p}} = \frac{x_2 - \alpha^n}{V_2 - \alpha^n}.$$

Reordering the terms, it is possible to get through simple algebra

$$F_1^i(x_1^i) = \frac{x_1^i + \alpha^i - \alpha^n}{V_2 - \alpha^n} \left(1 - \sum_{j=0}^{n-i-1} \frac{\alpha^{n-j} - \alpha^{n-j-1}}{V_2 \cdot \sum_{p=1}^{n-j-1} \pi_p} \right) + \sum_{j=0}^{n-i-1} \frac{\alpha^{n-j} - \alpha^{n-j-1}}{V_2 \cdot \sum_{p=1}^{n-j-1} \pi_p}.$$

Consider now Player 1. Since $\bar{x}_1^i = V_2 - \alpha^i$ holds, it must be the case that $p_1^i = V_1 - V_2 + \alpha^i$ implying that

$$V_1 \cdot F_2\left(x_1^i + \alpha^i\right) - x_1^i = V_1 - V_2 + \alpha^i$$

holds in equilibrium, or

$$F_2\left(x_1^i + \alpha^i\right) = 1 - \frac{V_2}{V_1} + \frac{x_1^i + \alpha^i}{V_1}.$$

Given that $x_1^i + \alpha^i = x_2$, then also

$$F_2(x_2) = 1 - \frac{V_2}{V_1} + \frac{x_2}{V_1}.$$

The equilibrium condition $\underline{x}_1^t + \alpha^t = \underline{x}_2 = \alpha^t$ requires that Player 2's distribution has an atom in 0 amounting to $1 - \frac{V_2}{V_1} + \frac{\alpha^t}{V_1}$ implying that not bidding in the auction is in fact an optimal strategy for Player 1 when his/her type is $j < t$. Indeed, the expected payoff $V_1 - V_2 + \alpha^t$ is greater than $V_1 - V_2 + \alpha^j$ which is obtained submitting a bid $x_1^j \in (\alpha^t, V_2 - \alpha^j]$.

The argument presented in Proposition 3.2 excludes discontinuity in the expected joint distribution function deriving from Player 1's equilibrium strategy and establishes that atoms of probability are placed only at $x_1^t = 0$. It also proves that $\underline{x}_1^i = 0$ and $\bar{x}_1^i = V_2 - \alpha^i$ hold for Player 1 while $\underline{x}_2 = \alpha^t$ and $\bar{x}_1 = V_2$ hold for Player 2. Analogously also Player 2's distribution must be continuous for every bid $x_2 > 0$. Uniqueness of the equilibrium follows from the continuity of players' strategies and from the definition of the upper and lower bounds of the supports. Hence only the proposed strategies satisfy the equilibrium conditions.

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