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First-Order Conditions for $C^{0,1}$ Constrained Vector Optimization

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Abstract

For a Fritz John type vector optimization problem with $C^{0,1}$ data we define different type of solutions, give their scalar characterizations applying the so called oriented distance, and give necessary and sufficient first order optimality conditions in terms of the Dini derivative. While establishing the sufficiency, we introduce new type of efficient points referred to as *isolated minimizers of first order*, and show their relation to properly efficient points. More precisely, the obtained necessary conditions are necessary for weakly efficiency, and the sufficient conditions are both sufficient and necessary for a point to be an isolated minimizer of first order.

Key words: Vector optimization, Nonsmooth optimization, $C^{0,1}$ functions, Dini derivatives, First-order optimality conditions, Lagrange multipliers.

Math. Subject Classification: 90C29, 90C30, 49J52.

1 Introduction

In this paper we consider the vector optimization problem

$$f(x) \to \min_C , \quad g(x) \in -K , \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^p$. Here n, m and p are positive integers and $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ are closed convex cones.

Problem (1) generalizes from scalar to vector optimization the Fritz John problem [20]. The latter is the scalar constrained problem obtained by (1) for $n = 1$, $C = \mathbb{R}_+$ and $K = \mathbb{R}^p_+$.

There are different type of solutions of problem (1). Usually the solutions are called points of efficiency. We prefer, like in the scalar optimization, to call them minimizers. In Section 2 we define different type of minimizers and give their scalar characterizations applying the so called oriented distance.

We assume that the functions f and g are $C^{0,1}$, that is f and g are locally Lipschitz. The purpose of the paper is to give necessary and sufficient first-order optimality conditions in terms of Dini directional derivatives. This result is obtained in Section 3. While establishing it we introduce new type of efficient points referred to as *isolated minimizers of first order*, and show their relation to properly efficient points. More precisely, the obtained necessary conditions are necessary for weakly efficiency, and the sufficient conditions are both sufficient and necessary for a point to be an isolated minimizer of first order.

We confine to functions f, g defined on the whole space $\mathbb{R}^n$. Usually in optimization functions on open subsets are considered, but such a more general assumption does not introduce new features in the problem.

The present paper is a part of a project, whose aim is to establish first and higher-order optimality conditions for $C^{k,1}$ vector optimization problems in terms of Dini derivatives. The class of $C^{0,1}$ functions is the natural environment, when looking for first-order conditions, while the class $C^{1,1}$ is the natural environment for second-order conditions. Second-order theory for unconstrained problems is developed in Ginchev, Guerraggio, Rocca [13]. The present paper opens the perspective for second-order theory of Fritz John type constrained problems. The direction for further development is the general constrained vector optimization problem, this means problems containing also equality constraints. Some hints on classical optimization level for the relation of F. John type problem and general constrained problems we find in the textbook of Kenderov, Christov, Dontchev [21]. or in some monographs like Alekseev, Tikhomirov, Fomin [2]. In the framework of these perspectives, recall that a vector function is said to be of class $C^{k,1}$ if it is k-times Fréchet differentiable with locally Lipschitz k-th derivative. The functions from the class $C^{0,1}$ are simply called locally Lipschitz and are traditionally in the limelight of the nonsmooth analysis, see e. g. Clarke [8] and Rockafellar, Wets [34]. The $C^{1,1}$ functions in optimization and second-order optimality conditions have been introduced in Hiriart-Urruty, Strodiot, Hien Nguen [17]. Thereafter an intensive study of various aspects of $C^{1,1}$ functions was undertaken, let us mention the papers Klatte, Tammer [22], Yang , Jeyakumar [35], Yang [36, 37], La Torre, Rocca [24]. For Taylor expansion formula and other aspects of $C^{k,1}$ functions with arbitrary k see Luc [29]. The optimality conditions in vector optimization are studied lately intensively, e.g. in Aghezzaf [1], Bolintenéanu, El Maghri [6], Amahroq, Taa [3], Ciligot-Travain [7], Ginchev, Guerraggio, Rocca [12]. Through scalarization this problem naturally transforms into scalar optimization with nonsmooth data, which gives some relations to Demyanov, Rubinov [10], Ginchev [11], Luc [28], Yang [37]. For optimization problems with $C^{0,1}$ and $C^{1,1}$ data (including vector problems and constrained problems) see Hiriart-Urruty, Strodiot, Hien Nguen [17], Klatte, Tammer [22], Yang, Jeyakumar [35], Yang [36], Liu [25], Liu, Křířek [26], Liu, Neittaanmäki, Křířek [27], Guerraggio, Luc [15, 16], Ginchev, Guerraggio, Rocca [13].

2 Concepts of optimality and scalar characterizations

We denote the unit sphere and the open unit ball in $\mathbb{R}^n$ respectively by $S = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ and $B = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$. For the norm and the scalar product in the considered finite-dimensional spaces we write $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$. From the context it should be clear to exactly which spaces these notations are applied.

We consider problem (1) with $C \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^p$ closed convex cones. The point x is said to be feasible if $g(x) \in -K$ (equivalently $x \in g^{-1}(-K)$). There are different concepts of solutions of this problem. In any case a solution x^0 should be a feasible point, which is assumed in the following definitions. As for the assumption C and K closed convex cones, we consider it as a natural and do not care for possible relaxations to non-closed cones. At the same time often results in vector optimization deal with pointed cones with non-empty interior. Our point of view is to avoid assumptions of this type as far as possible. However, let us underline, that such a more general point of view may meet with obstacles by possible generalization to infinite-dimensional spaces. For instance, in the proof of Proposition 1 we use that $\operatorname{int} C = \emptyset$ implies that C is contained in a hyperplane, which is not true in general for infinite-dimensional spaces.

The feasible point x^0 is said to be weakly efficient (efficient) point, if there is a neighbourhood U of x^0, such that if $x \in U \cap g^{-1}(-K)$ then $f(x) - f(x^0) \notin -\operatorname{int} C$ (respectively $f(x) - f(x^0) \notin$

$-(C \setminus \{0\})$). The feasible point x^0 is said to be properly efficient if there exists a closed convex cone $\tilde{C} \subset \mathbb{R}^n$, such that $C \setminus \{0\} \subset \operatorname{int} \tilde{C}$ and x^0 is weakly efficient point with respect to $\tilde{C}$ (that is x^0 is weakly efficient for the problem $f(x) \to \min_{\tilde{C}}$, $g(x) \in -K$). In this paper the weakly efficient, the efficient and the properly efficient points of problem (1) are called respectively w-minimizers, e-minimizers and p-minimizers.

The unconstrained problem

$$f(x) \to \min_C \tag{2}$$

should be considered as a particular case of problem (1). The concepts of efficiency are obviously valid also for this problem. For instance, the point x^0 is said to be weakly efficient, here called w-minimizer (or efficient, here called e-minimizer), if there is a neighbourhood U of x^0, such that if $x \in U$ then $f(x) - f(x^0) \notin -\operatorname{int} C$ (respectively $f(x) - f(x^0) \notin -(C \setminus \{0\})$).

Each p-minimizer is e-minimizer, which follows from the implication $f(x) - f(x^0) \notin -\operatorname{int} \tilde{C}$ $\Rightarrow f(x) - f(x^0) \notin -(C \setminus \{0\})$, a consequence of $C \setminus \{0\} \subset \operatorname{int} \tilde{C}$. Assuming $C \neq \mathbb{R}^m$, each e-minimizer is w-minimizer, which follows from the implication $f(x) - f(x^0) \notin -(C \setminus \{0\}) \Rightarrow$ $f(x) - f(x^0) \notin -\operatorname{int} C$, a consequence of $\operatorname{int} C \subset C \setminus \{0\}$.

For the cone $M \subset \mathbb{R}^k$ its positive polar cone M' is defined by $M' = \{\zeta \in \mathbb{R}^k \mid \langle \zeta, \phi \rangle \geq 0 \text{ for all } \phi \in M\}$. The cone M' is closed and convex. It is well known that $M'' := (M')' = \operatorname{cl} \operatorname{co} M$, see e. g. Rockafellar [33, Chapter III, § 15]. In particular for the closed convex cone M we have $M' = \{\zeta \in \mathbb{R}^k \mid \langle \zeta, \phi \rangle \geq 0 \text{ for all } \phi \in M\}$ and $M = M'' = \{\phi \in \mathbb{R}^k \mid \langle \zeta, \phi \rangle \geq 0 \text{ for all } \zeta \in M'\}$.

The linear span of the cone $M \subset \mathbb{R}^k$, that is the smallest subspace of $\mathbb{R}^k$ containing M, is denoted $\mathbf{L}_M$. The positive polar cone of M related to the linear span of M is

$$M'_{\mathbf{L}_M} = \{\zeta \in \mathbf{L}_M \mid \langle \zeta, \phi \rangle \geq 0 \text{ for all } \phi \in M\} = M' \cap \mathbf{L}_M\,.$$

The relative interior $\operatorname{ri} M$ of M is defined as the interior of M with respect to the relative topology of the linear span $\mathbf{L}_M \subset \mathbb{R}^k$ of M, that is $\operatorname{ri} M = \operatorname{int}_{\mathbf{L}_M} M$.

The closed convex cone M and its relative interior admit the following description in terms of positive polar cones.

$$M = \{\phi \in \mathbf{L}_M \mid \langle \zeta, \phi \rangle \geq 0 \text{ for all } \zeta \in M'_{\mathbf{L}_M}\}\,,$$
$$\operatorname{ri} M = \{\phi \in \mathbf{L}_M \mid \langle \zeta, \phi \rangle > 0 \text{ for all } \zeta \in M'_{\mathbf{L}_M}\}\,.$$

An essential and important for the next considerations property is that $\operatorname{ri} M \neq \emptyset$ for any convex cone M.

Let $\phi \in -\operatorname{cl} \operatorname{co} M$. Then $\langle \zeta, \phi \rangle \leq 0$ for all $\zeta \in M'$. We denote $M'(\phi) = \{\zeta \in M' \mid \langle \zeta, \phi \rangle = 0\}$. Then $M'(\phi)$ is a closed convex cone and $M'(\phi) \subset M'$. Consequently its positive polar cone $M(\phi) = (M'(\phi))'$ is a close convex cone, $M \subset M(\phi)$ and its positive polar cone satisfies $(M(\phi))' = M'(\phi)$. In this paper we apply this notation for $M = K$ and $\phi = g(x^0)$. Then we write for short $K'(x^0)$ instead of $K'(g(x^0))$ (and call this cone the index set of problem (1) at x^0) and $K(x^0)$ instead of $K(g(x^0))$. We find this abbreviation convenient and not ambiguous, since further this is the unique case, in which we make use of the cones $M'(\phi)$ and $M(\phi)$.

For the closed convex cone M' we apply in the sequel the notations $\Gamma_{M'} = \{\zeta \in M' \mid \|\zeta\| = 1\}$ and $\Gamma_{M' \cap \mathbf{L}_M} = \{\zeta \in M' \cap \mathbf{L}_M \mid \|\zeta\| = 1\} = \{\zeta \in M'_{\mathbf{L}_M} \mid \|\zeta\| = 1\}$. The sets $\Gamma_{M'}$ and $\Gamma_{M' \cap \mathbf{L}_M}$ are compact, since they are closed and bounded.

Further we make use of the orthogonal projection. Let $\mathbf{L} \subset \mathbb{R}^k$ be a given subspace of $\mathbb{R}^k$. The orthogonal projection is a linear function $\pi_{\mathbf{L}} : \mathbb{R}^k \to \mathbf{L}$ determined by $\pi_{\mathbf{L}} \phi \in \mathbf{L}$ and $\langle \zeta, \phi - \pi_{\mathbf{L}} \phi \rangle = 0 \Leftrightarrow \langle \zeta, \phi \rangle = \langle \zeta, \pi_{\mathbf{L}} \phi \rangle$ for all $\zeta \in \mathbf{L}$. It follows easily from the Cauchy inequality that $\|\pi_{\mathbf{L}}\| := \max(\|\pi_{\mathbf{L}} \phi / \|\phi\|) = 1$ if $\mathbf{L} \neq \{0\}$ and $\|\pi_{\mathbf{L}}\| = 0$ if $\mathbf{L} = \{0\}$.

If $\mathbf{L} \subset \mathbb{R}^k$ is a subspace fixed from the context and $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ is a given function, then we usually denote by $\bar{\Phi}$ the composition $\bar{\Phi} = \pi_{\mathbf{L}} \circ \Phi$.

A relation of the vector optimization problem (1) to some scalar optimization problem can be obtained in terms of positive polar cones.

Proposition 1 *The feasible point $x^0 \in \mathbb{R}^n$ is w-minimizer of problem (1), with C and K closed convex cones, if and only if x^0 is a minimizer of the scalar problem*

$$\varphi(x) = \max \left\{ \langle \xi, f(x) - f(x^0) \rangle \mid \xi \in C', \ \|\xi\| = 1 \right\} \to \min , \quad g(x) \in -K . \tag{3}$$

Proof 1^0. Let $\operatorname{int} C = \emptyset$. Then each feasible point x^0 is w-minimizer. At the same time C is contained in some hyperplane $H = \{z \in \mathbb{R}^m \mid \langle \xi^0, z \rangle = 0\}$ with $\xi^0 \in \mathbb{R}^m$, $\|\xi^0\| = 1$. Then both $\xi^0 \in C'$ and $-\xi^0 \in C'$, whence

$$\varphi(x) \geq \max \left(\langle \xi^0, f(x) - f(x^0) \rangle, -\langle \xi^0, f(x) - f(x^0) \rangle \right) = |\langle \xi^0, f(x) - f(x^0) \rangle| \geq 0 = \varphi(x^0) ,$$

which shows that each feasible point x^0 is a minimizer of the corresponding scalar problem (3).

$\mathbf{2}^0$. Let $\operatorname{int} C \neq \emptyset$. Suppose x^0 is w-minimizer of problem (1). Let U be the neighbourhood from the definition of a w-minimizer and fix $x \in U \cap g^{-1}(-K)$. Then $f(x) - f(x^0) \notin -\operatorname{int} C \neq \emptyset$. From the well known Separation Theorem there exists $\xi^x \in \mathbb{R}^m$, $\|\xi^x\| = 1$, such that $\langle \xi^x, f(x) - f(x^0) \rangle \geq 0$ and $\langle \xi^x, -y \rangle = -\langle \xi^x, y \rangle \leq 0$ for all $y \in C$. The latter inequality shows that $\xi^x \in C'$ and the former one shows that $\varphi(x) \geq \langle \xi^x, f(x) - f(x^0) \rangle \geq 0 = \varphi(x^0)$. Thus $\varphi(x) \geq \varphi(x^0)$, $x \in U \cap g^{-1}(-K)$, and therefore x^0 is a minimizer of the scalar problem (3).

Let now x^0 be a minimizer of the scalar problem (3). Choose the neighbourhood U of x^0, such that $\varphi(x) \geq \varphi(x^0)$ for all $x \in U \cap g^{-1}(-K)$ and fix one such x. Then there exists $\xi^x \in C'$, $\|\xi^x\| = 1$, such that $\varphi(x) = \langle \xi^x, f(x) - f(x^0) \rangle \geq \varphi(x^0) = 0$ (here we use the compactness of the set $\{\xi \in C' \mid \|\xi\| = 1\}$). From $\xi^x \in C'$ it follows $\langle \xi^x, -y \rangle < 0$ for $y \in \operatorname{int} C$. Therefore $f(x) - f(x^0) \notin -\operatorname{int} C$. Consequently x^0 is w-minimizer of problem (1). □

If $\operatorname{int} C = \emptyset$, then each feasible point x^0 of problem (1) is w-minimizer. For this case the concept of a relatively weakly efficient point (rw-minimizer) turns to be reacher in content. We use in the sequel the concept of rw-minimizer instead of w-minimizer in some of the results for the case if $\operatorname{int} C = \emptyset$ or $\operatorname{int} K = \emptyset$ (and rather $\operatorname{int} K(x^0) = \emptyset$). Let us say in advance that if both $\operatorname{int} C \neq \emptyset$ and $\operatorname{int} K \neq \emptyset$ the concepts of rw-minimizer and w-minimizer coincide.

In order to define a rw-minimizer we consider the problem

$$\bar{f}(x) \to \min\nolimits_C , \quad \bar{g}(x) \in -K , \tag{4}$$

where $\bar{f} = \pi_{\mathbf{L}_C} \circ f$ and $\bar{g} = \pi_{\mathbf{L}_C} \circ g$. Then we call the feasible point x^0 of problem (1) its rw-minimizer, if there exists a neighbourhood U of x^0 such that $\bar{f}(x) - \bar{f}(x^0) \notin -\operatorname{ri} C$ for $x \in U \cap \bar{g}^{-1}(-K)$. The following proposition characterizes the rw-minimizers.

Proposition 2 *The feasible point x^0 is rw-minimizer of problem (1), with C and K closed convex cones, if and only if x^0 is a minimizer for the scalar problem*

$$\psi(x) = \max \left\{ \langle \xi, f(x) - f(x^0) \rangle \mid \xi \in C'_{\mathbf{L}_C} = C' \cap \mathbf{L}_C, \ \|\xi\| = 1 \right\} \to \min , \quad \bar{g}(x) \in -K , \tag{5}$$

where $\bar{f} = \pi_{\mathbf{L}_C} \circ f$ and $\bar{g} = \pi_{\mathbf{L}_C} \circ g$.

Proof Due to $\langle \xi,\, f(x)\rangle = \langle \xi,\, \bar f(x)\rangle$ and $\langle \xi,\, f(x^0)\rangle = \langle \xi,\, \bar f(x^0)\rangle$ for $\xi \in \mathbf{L}_C$, the scalar product in (scp-r) can be written into the form $\langle \xi, f(x) - f(x^0)\rangle = \langle \xi, \bar f(x) - \bar f(x^0)\rangle$.

Let x^0 be a minimizer of problem (5). Then there exists a neighbourhood U of x^0, such that $\psi(x) \ge \psi(x^0)$ for $x \in U \cap \bar g^{-1}(-K)$. Fix one such x. From the definition of ψ and the compactness of $\Gamma_{C' \cap \mathbf{L}_C}$, there exists $\xi^0 \in \Gamma_{C' \cap \mathbf{L}_C}$, such that $\psi(x) = \langle \xi^0,\, \bar f(x) - \bar f(x^0)\rangle \ge \psi(x^0) = 0$, whence $\bar f(x) - \bar f(x^0) \notin -\mathrm{ri}\, C$ and consequently x^0 is rw-minimizer.

Conversely, let x^0 be rw-minimizer and let U be the neighbourhood from the definition of the rw-minimizer. Fix $x \in U \cap \bar g^{-1}(-K)$. Since $\bar f(x) - \bar f(x^0) \notin -\mathrm{ri}\, C \ne \emptyset$, there exists $\xi^0 \in \Gamma_{C' \cap \mathbf{L}_C}$, such that $\langle \xi^0,\, \bar f(x) - \bar f(x^0)\rangle \ge 0$. Then $\psi(x) \ge \langle \xi^0,\, \bar f(x) - \bar f(x^0)\rangle \ge 0 = \psi(x^0) = 0$. Therefore x^0 is a minimizer of problem (5). $\square$

We see that the proof of Proposition 2 repeats in some sense the proof of Proposition 1, and is even simpler, since $\mathrm{ri}\, C$ in Proposition 2, being an analogue of $\mathrm{int}\, C$ from Proposition 1, is never empty. While the phase space in Proposition 1 is $\mathbb{R}^m$, in Proposition 2 it is $\mathbf{L}_C$.
After Proposition 2 the following definitions look natural. We call the feasible point x^0 of problem (1) relatively efficient point, for short re-minimizer, (relatively properly efficient point, for short rp-minimizer) if x^0 is efficient (properly efficient) point for problem (4).

We call x^0 a strong e-minimizer (strong re-minimizer), if there is a neighbourhood U of x^0, such that $f(x) - f(x^0) \notin -C$ for $x \in (U \setminus \{x^0\}) \cap g^{-1}(-K)$ ($\bar f(x) - \bar f(x^0) \notin -C$ for $x \in (U \setminus \{x^0\}) \cap \bar g^{-1}(-K)$). Obviously, each strong e-minimizer (strong re-minimizer) is e-minimizer (re-minimizer). The following characterization of the strong e-minimizers (strong re-minimizers) holds. The proof is omitted, since it nearly repeats the one from Proposition 1 (Proposition 2).

Proposition 3 *The feasible point x^0 is a strong e-minimizer (strong re-minimizer) of problem (1) with C and K closed convex cones, if and only if x^0 is a strong minimizer of problem (3) (problem (5)).*

Proposition 1 claims that the statement x^0 *is w-minimizer of problem (1)* is equivalent to the statement x^0 *is a minimizer of the scalar problem (3).* Applying some first or second-order sufficient optimality conditions to check the latter, we usually get more, namely that x^0 *is an isolated minimizer respectively of first and second order of (3).* Recall, that the feasible point x^0 is said to be an isolated minimizer of order κ (κ positive) of problem (3) if there is a constant $A > 0$ such that $\varphi(x) \ge \varphi(x^0) + A\, \|x - x^0\|^\kappa$ for all $x \in U \cap g^{-1}(-K)$. The concept of an isolated minimizer has been popularized by Auslender [4].

It is natural to introduce the following concept of optimality for the vector problem (1):

Definition 1 *We say that the feasible point x^0 is an isolated minimizer of order κ for vector problem (1) if it is an isolated minimizer of order κ for scalar problem (3).*

Obviously, also a “relative” variant of an isolated minimizer, and as well for other type of efficient points, does exist. From here on we skip such definitions.

To interpret geometrically the property that x^0 is a minimizer of problem (1) of certain type we introduce the so called oriented distance. Given a set $A \subset \mathbb{R}^k$, then the distance from $y \in \mathbb{R}^k$ to A is given by $d(y, A) = \inf\{\|a - y\| \mid a \in A\}$. The oriented distance from y to A is defined by $D(y, A) = d(y, A) - d(y, \mathbb{R}^k \setminus A)$. The function D is introduced in Hiriart-Urruty [18, 19] and is used later in Ciligot-Travain [7], Amahroq, Taa [3], Miglierina [31], Miglierina, Molho [32]. Zaffaroni [38] gives different notions of efficiency and uses the function D for their scalarization and comparison. Ginchev, Hoffmann [14] use the oriented distance to study approximation of set-valued functions by single-valued ones and in case of a convex set A show the representation

$D(y, A) = \sup_{\|\xi\|=1} (\inf_{a \in A} \langle \xi,\, a \rangle - \langle \xi,\, y \rangle)$. From this representation, if C is a convex cone and taking into account

$$\inf_{a \in C} \langle \xi,\, a \rangle = \begin{cases} 0 & , \quad \xi \in C', \\ -\infty & , \quad \xi \notin C', \end{cases}$$

we get easily $D(y, -C) = \sup_{\|\xi\|=1,\, \xi \in C'} (\langle \xi,\, y \rangle)$. In particular the function φ in (3) is expressed by $\varphi(x) = D(f(x) - f(x^0),\, -C)$. Propositions 1 and 3 are easily reformulated in terms of the oriented distance, namely:

$$\begin{array}{rcl} x^0 \ w\text{-minimizer} & \Leftrightarrow & D(f(x) - f(x^0), -C) \geq 0 \text{ for } x \in U \cap g^{-1}(-K)\,, \\ x^0 \text{ strong } e\text{-minimizer} & \Leftrightarrow & D(f(x) - f(x^0), -C) > 0 \text{ for } x \in (U \setminus \{x^0\}) \cap g^{-1}(-K)\,. \end{array}$$

The definition of the isolated minimizers gives

$$\begin{gathered} x^0 \text{ isolated minimizer of order } \kappa \quad \Leftrightarrow \\ D(f(x) - f(x^0), -C) \geq O(\|x - x^0\|^\kappa) \text{ as } x \to x^0,\ x \in g^{-1}(-K)\,. \end{gathered}$$

We see, that the isolated minimizers (of a positive order) are strong e-minimizers. The next proposition gives a relation of the p-minimizers and the isolated minimizers of first order. The proof for the unconstrained case can be found in Crespi, Ginchev, Rocca [9].

Proposition 4 *Let in problem (1) f be Lipschitz in a neighbourhood of the feasible point x^0 and let x^0 be isolated minimizer of first order. Then x^0 is p-minimizer of (1).*

Proof Assume in the contrary, that x^0 is isolated minimizer of first order, but not p-minimizer. Let f be Lipschitz with constant L in $x^0 + r\,\mathrm{cl}\, B$. Take sequences $\delta_k \to +0$ and $\varepsilon_k \to +0$ and define the cones $\tilde{C}_k = \mathrm{cone}\,\{y \in \mathbb{R}^m \mid D(y,\, C) \leq \varepsilon_k,\, \|y\| = 1\}$. It holds $\mathrm{int}\,\tilde{C}_k \supset C \setminus \{0\}$. From our assumption, there exists a sequence of feasible points $x^k \in (x^0 + \delta_k B) \cap g^{-1}(-K)$, such that $f(x^k) - f(x^0) \in -\mathrm{int}\,\tilde{C}_k$, and in particular $f(x^k) - f(x^0) \neq 0$. From the definition of $\tilde{C}_k$ we get

$$D(f(x^k) - f(x^0), -C) \leq \varepsilon_k\, \|f(x^k) - f(x^0)\| \leq \varepsilon_k\, L\, \|x^k - x^0\|\,,$$

which contradicts to x^0 isolated minimizer of first order. □

We introduce now two other concepts of efficiency.

Definition 2 *We say that the feasible point x^0 for problem (1) is linearly scalarized weakly efficient, for short lw-minimizer (linearly scalarized properly efficient, for short lp-minimizer), if there exists a pair $(\xi^0,\, \eta^0) \in C' \times K'(x^0) \setminus \{(0,\, 0)\}$, such that x^0 is a minimizer (isolated minimizer of first order) for the scalar function*

$$\varphi^0(x) = \langle \xi^0,\, f(x) - f(x^0) \rangle + \langle \eta^0,\, g(x) \rangle\,. \tag{6}$$

Proposition 5 *If x^0 is lw-minimizer for problem (1) with $\xi^0 \neq 0$, then x^0 is w-minimizer. By the way, let*

$$\begin{gathered} \text{for each neighbourhood } U \text{ of } x^0 \text{ there exists } x \in U \cap g^{-1}(-K) \\ \text{such that } \langle \eta,\, g(x) \rangle < 0 \text{ for all } \eta \in K'(x^0) \setminus \{0\}\,. \end{gathered} \tag{7}$$

Then if x^0 is a minimizer of some function (6) with $(\xi^0,\, \eta^0) \in C' \times K'(x^0) \setminus \{(0,\, 0)\}$, that is if x^0 is lw-minimizer, we have $\xi^0 \neq 0$.

Proof We show, that the made assumptions imply that x^0 is a minimizer of the scalar problem (3), whence according to Proposition 1 x^0 is w-minimizer. Let U be the neighbourhood of x^0, for which $\varphi^0(x) \geq \varphi^0(x^0)$ for $x \in U \cap g^{-1}(-K)$. Without loss of generality, we may assume that $\|\xi^0\| = 1$, otherwise we replace in (6) ξ^0 by $\xi^0/\|\xi^0\|$. Fix $x \in U \cap g^{-1}(-K)$. Then for the function φ in (3) we have

$$\varphi(x) \geq \langle \xi^0,\, f(x) - f(x^0)\rangle \geq \langle \xi^0,\, f(x) - f(x^0)\rangle + \langle \eta^0,\, g(x)\rangle = \varphi^0(x) \geq \varphi^0(x^0) = 0 = \varphi(x^0)\,,$$

which had to be demonstrated. Here we have applied that $\langle \eta^0,\, g(x)\rangle \leq 0$ coming from $g(x) \in -K$, and $\langle \eta^0,\, g(x^0)\rangle = 0$ coming from $\eta^0 \in K'(x^0)$. □

Preassigned properties of the constraints are called constraint qualifications. The given constraint qualification (7) is referred usually to as a qualification of Slater type.

In Section 3 we show that each lp-minimizers is p-minimizers, see Proposition 10.

Developing second-order optimality conditions for $C^{1,1}$ functions, we meet with isolated minimizers of second order, compare with Ginchev, Guerraggio, Rocca [13]. The property x^0 isolated minimizer of second order can be considered as some refinement of the property x^0 is p-minimizer. The isolated minimizers of second order are related to strictly efficient points, referred to as s-minimizers of problem (1), and defined as follows.

Definition 3 (Bednarczuk, Song [5]) *A feasible point x^0 is said to be strictly efficient point of problem (1) (or s-minimizer), if there exists a neighborhood U of x^0 such that for every $\varepsilon > 0$ there exists $\delta > 0$ with*

$$(f(x) - f(x^0)) \cap (\delta B - C) \subseteq \varepsilon B \text{ for all } x \in U \cap g^{-1}(-K)\,.$$

It is known, see Zălinescu [39], that if x^0 is p-minimizer of the unconstrained problem (2), then it is also s-minimizer. Hence, strictly efficient points form an intermediate class between efficient and properly efficient points. The following proposition gives a relation to isolated minimizers of second order and is proved in Crespi, Ginchev, Rocca [9].

Proposition 6 *Let f and g be a continuous function. If x^0 is an isolated minimizer of second-order of the unconstrained problem (2), then x^0 is s-minimizer of (2).*

Let C be a closed convex cone with $\operatorname{int} C \neq \emptyset$. Then its positive polar C' is a pointed closed convex cone. Recall that the set Ξ is a base for C', if Ξ is convex with $0 \notin \Xi$ and $C' = \operatorname{cone} \Xi := \{y \mid y = \lambda\, \xi,\ \lambda \geq 0,\ \xi \in \Xi\}$. The property C' pointed closed convex cone in $\mathbb{R}^m$ implies that C' possesses a compact base Ξ and

$$0 < \alpha = \min\{\|\xi\| \mid \xi \in \Xi\} \leq \max\{\|\xi\| \mid \xi \in \Xi\} = \beta < +\infty\,. \tag{8}$$

Further we assume that Ξ_0 is compact and $\Xi = \operatorname{conv} \Xi_0$. With the help of Ξ_0 we define the problem

$$\varphi_0(x) = \max\left\{\langle \xi, f(x) - f(x^0)\rangle \mid \xi \in \Xi_0\right\} \to \min\,, \quad g(x) \in -K\,. \tag{9}$$

Proposition 7 *Let Ξ be a base of C' satisfying (8), φ be the function in (3) and*

$$\varphi_\Xi(x) = \max\{\langle \xi,\, f(x) - f(x^0)\rangle \mid \xi \in \Xi\}\,.$$

Then $\alpha\, \varphi(x) \leq \varphi_\Xi(x) \leq \beta\, \varphi(x)$.

Proof If $\xi \in \Gamma_{C'} = \{\xi \in \mathbb{R}^m \mid \xi \in C',\ \|\xi\| = 1\}$, then there exists $\lambda_\xi > 0$, such that $\lambda_\xi\, \xi \in \Xi$. In fact, $\lambda_\xi = \|\lambda_\xi\, \xi\|$, whence from inequality (8) we have $0 < \alpha \le \lambda_\xi = \|\lambda_\xi\, \xi\| \le \beta$.

Fix $x \in \mathbb{R}^n$. From the compactness of $\Gamma_{C'}$ there exists $\xi^x \in \Gamma_{C'}$, such that

$$\varphi(x) = \langle \xi^x,\ f(x) - f(x^0)\rangle = \frac{1}{\lambda_{\xi^x}} \langle \lambda_{\xi^x}\, \xi^x,\ f(x) - f(x^0)\rangle \le \frac{1}{\lambda_{\xi^x}} \varphi_\Xi(x) \le \frac{1}{\alpha}\, \varphi_\Xi(x)\,,$$

whence $\alpha\, \varphi(x) \le \varphi_\Xi(x)$. For the other inequality, from the compactness of Ξ there exists $\eta^x \in \Xi$, such that $\varphi_\Xi(x) = \langle \eta^x,\ f(x) - f(x^0)\rangle$. Put $\lambda = \lambda_{\eta^x/\|\eta^x\|}$. Then

$$\varphi_\Xi(x) = \langle \eta^x,\ f(x) - f(x^0)\rangle = \lambda\, \langle \frac{\eta^x}{\lambda}\,,\ f(x) - f(x^0)\rangle \le \lambda\, \varphi(x) \le \beta\, \varphi(x)\,.$$

$\square$

Proposition 8 *Propositions 1 and 3, and Definition 1 remain true, if in their formulation* problem (3) *is replaced by* problem (9)*.*

Proof We show first, that $\varphi_0(x) = \varphi_\Xi(x)$, where $\varphi_\Xi(x)$ is the function from Proposition 7.

The inequality $\varphi_0(x) \le \varphi_\Xi(x)$ follows directly from $\Xi_0 \subset \Xi$. To prove the converse inequality, fix x and let $\varphi_\Xi(x) = \langle \xi^x,\ f(x) - f(x^0)\rangle$, $\xi^x \in \Xi$. Let ξ^x be the convex combination $\xi^x = \sum_j \lambda_j \xi^j$, where $\xi^j \in \Xi_0$, $\sum_j \lambda_j = 1$, $\lambda_j \ge 0$. Then

$$\varphi_\Xi(x) = \langle \xi^x,\ f(x) - f(x^0)\rangle = \sum_j \lambda_j \langle \xi^j,\ f(x) - f(x^0)\rangle \le \sum_j \lambda_j \varphi_0(x) = \varphi_0(x)\,.$$

A consequence of the proved equality and Proposition 7 is the inequality $\alpha\, \varphi(x) \le \varphi_0(x) \le \beta\, \varphi(x)$. In order to prove the proposition, we have to show that x^0 is a minimizer of problem (3) if and only if it is minimizer of (9). Assume x^0 is a minimizer of (3) and $\varphi(x) \ge \varphi(x^0)$ for $x \in U \cap g^{-1}(-K)$. Then $\varphi_0(x) \ge \alpha\, \varphi(x) \ge \alpha\, \varphi(x^0) = 0 = \varphi_0(x)$, whence x^0 is a minimizer of (9). Conversely, if x^0 is a minimizer of (9), then $\varphi(x) \ge \frac{1}{\beta}\, \varphi_0(x) \ge \frac{1}{\beta}\, \varphi_0(x^0) = 0 = \varphi(x^0)$. $\square$

Corollary 1 *In the important case $C = \mathbb{R}^n_+$ (and suitable choice of Ξ) the function φ_0 in (9) transforms into*

$$\varphi_0(x) = \max_{1 \le i \le n} \left(f_i(x) - f_i(x^0) \right)\,. \tag{10}$$

Proof Clearly, $C' = \mathbb{R}^n_+$ has a base $\Xi = \operatorname{conv} \Xi_0$, where $\Xi_0 = \{e^1, \dots, e^n\}$ are the unit vectors on the coordinate axes. With this set we get immediately that the function φ_0 in (9) transforms into that in (10). $\square$

More generally, the cone C is said to be polyhedral, if $C' = \operatorname{cone} \Xi_0$ with some finite set of nonzero vectors $\Xi_0 = \{\xi^1, \dots, \xi^k\}$. In this case, similarly to Corollary 1 the function φ_0 in (9) transforms into the maximum of the finite number of functions

$$\varphi_0(x) = \max_{1 \le i \le k} \langle \xi^i,\ f_i(x) - f_i(x^0)\rangle\,.$$

3 First-order conditions for $C^{0,1}$ problems

In this section we investigate problem (1) under the assumption that f and g are $C^{0,1}$ functions. We obtain optimality conditions in terms of the first-order Dini directional derivative.

Given a $C^{0,1}$ function $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ we define the Dini directional derivative (we use to say just Dini derivative) $\Phi'_u(x^0)$ of Φ at x^0 in direction $u \in \mathbb{R}^n$ as the set of the cluster points of $(1/t)(\Phi(x^0+tu)-\Phi(x^0))$ as $t \to +0$, that is as the Kuratowski limit

$$\Phi'_u(x^0) = \operatorname*{Limsup}_{t \to +0} \frac{1}{t}\left(\Phi(x^0+tu)-\Phi(x^0)\right) .$$

If Φ is Fréchet differentiable at x^0 then the Dini derivative is a singleton, coincides with the usual directional derivative and can be expressed in terms of the Fréchet derivative $\Phi'(x^0)$ (called sometimes the Jacobian of Φ at x^0) by

$$\Phi'_u(x^0) = \lim_{t\to+0} \frac{1}{t}\left(\Phi(x^0+tu)-\Phi(x^0)\right) = \Phi'(x^0)u\,.$$

In connection with problem (1) we deal with the Dini directional derivative of the function $\Phi : \mathbb{R}^n \to \mathbb{R}^{m+p}$, $\Phi(x) = (f(x),\, g(x))$, and then we use to write $\Phi'_u(x^0) = (f(x^0),\, g(x^0))'_u$. If at least one of the derivatives $f'_u(x^0)$ and $g'_u(x^0)$ is a singleton, then $(f(x^0), g(x^0))'_u = (f'_u(x^0), g'_u(x^0))$. Let us turn attention that always $(f(x^0),\, g(x^0))'_u \subset f'_u(x^0) \times g'_u(x^0)$, but in general these two sets do not coincide. Indeed, for f any $C^{0,1}$ function, $(f(x^0),\, f(x^0))'_u$ is the diagonal of $f'_u(x^0) \times f'_u(x^0)$. If $f'_u(x^0)$ is not a singleton, then the two sets are different.

Lemma 1 *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ be Lipschitz with constant L in $x^0 + r\,\mathrm{cl}\,B$, where $x^0 \in \mathbb{R}^n$ and $r > 0$. Then for $u,\, v \in \mathbb{R}^n$ and $0 < t < r/\max(\|u\|,\, \|v\|)$ it holds*

$$\left\| \frac{1}{t}\left(\Phi(x^0+tv)-\Phi(x^0)\right) - \frac{1}{t}\left(\Phi(x^0+tu)-\Phi(x^0)\right) \right\| \le L\,\|v-u\|\,, \tag{11}$$

In particular for $v = 0$ and $0 < t < r/\|u\|$ we get

$$\left\| \frac{1}{t}\left(\Phi(x^0+tu)-\Phi(x^0)\right) \right\| \le L\,\|u\|\,. \tag{12}$$

Proof The left hand side of (11) is obviously transformed and estimated by

$$\left\| \frac{1}{t}\left(\Phi(x^0+tv)-\Phi(x^0+tu)\right) \right\| \le L\,\|v-u\|\,.$$

□

Lemma 2 *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ be Lipschitz with constant L in $x^0 + r\,\mathrm{cl}\,B$, where $x^0 \in \mathbb{R}^n$ and $r > 0$. Then $\Phi'_u(x^0)$, $u \in \mathbb{R}^n$, is non-empty compact set, bounded by $\sup\{\|\phi\| \mid \phi \in \Phi'_u(x^0)\} \le L\,\|u\|$. For each $u,\, v \in \mathbb{R}^n$ and $\phi_u \in \Phi'_u(x^0)$, there exists a point $\phi_v \in \Phi'_v(x^0)$, such that $\|\phi_v - \phi_v\| \le L\,\|v-u\|$. Consequently, the set-valued function $u \to \Phi'_u(x^0)$ is Lipschitz with constant L (and hence continuous) with respect to the Hausdorff distance in $\mathbb{R}^k$.*

Proof The closedness of $\Phi'_u(x^0)$ follows from the definition of the Dini derivative. Estimation (12) shows that $\Phi'_u(x^0)$ is not empty and $\|\phi_u\| \le L\,\|u\|$ for each $\phi_u \in \Phi'_u(x^0)$. Let $\phi_u = \lim_k (1/t_k)\left(\Phi(x^0+t_k u)-\Phi(x^0)\right)$. Passing to a subsequence we may assume that $\phi_v = \lim_k (1/t_k)\left(\Phi(x^0+t_k v)-\Phi(x^0)\right)$ (to make this conclusion we use also the boundedness expressed in (12)). A passing to a limit in (11) gives $\|\phi_v - \phi_v\| \le L\,\|v-u\|$. Now the Lipschitz property of the set-valued function $u \to \Phi'_u(x^0)$ becomes obvious. $\square$

Recall the definition of the index set. Let x^0 be feasible point for problem (1). Then $g(x^0) \in -K$, which gives $\langle \eta,\, g(x^0)\rangle \le 0$ for all $\eta \in K'$. The index set is defined by $K'(x^0) = \{\eta \in K' \mid \langle \eta,\, g(x^0)\rangle = 0\}$. We put $K(x^0) = (K'(x^0))'$. Then $K'(x^0)$ is the positive polar cone of the cone $K(x^0)$, and $K \subset K(x^0)$, the latter follows from $K'(x^0) \subset K'$.

Lemma 3 *Consider problem (1) with f, g being $C^{0,1}$ functions and C and K closed convex cones. If x^0 is w-minimizer and $(y^0, z^0) \in (f(x^0), g(x^0))'_u$, then $(y^0, z^0) \notin -(\operatorname{int} C \times \operatorname{int} K(x^0))$.*

Proof Suppose that $(y^0, z^0) \in (f(x^0), g(x^0))'_u$ and $(y^0, z^0) \in -\operatorname{int}(C \times K(x^0)) = -(\operatorname{int} C \times \operatorname{int} K(x^0))$. Let

$$y^0 = \lim_k \frac{1}{t_k}\left(f(x^0 + t_k u) - f(x^0)\right), \quad z^0 = \lim_k \frac{1}{t_k}\left(g(x^0 + t_k u) - g(x^0)\right). \tag{13}$$

Without loss of generality, we may assume that $0 < t_k < r/\|u\|$ for all k and that f and g are Lipschitz with constant L in $x^0 + r\,\operatorname{cl} B$.

We show now that there exists k_0, such that $g(x^0 + t_k u) \in -\operatorname{int} K \subset -K$ for $k > k_0$, that is, $x^0 + t_k u$ is feasible for $k > k_0$. Recall the notation $\Gamma_{K'} = \{\eta \in K' \mid \|\eta\| = 1\}$ and $\Gamma_{K'(x^0)} = \{\eta \in K'(x^0) \mid \|\eta\| = 1\}$. The sets $\Gamma_{K'}$ and $\Gamma_{K'(x^0)}$ are compact as being closed and bounded sets in an Euclidean space.

Let $\bar\eta \in \Gamma_{K'}$. We show that there exists a positive integer $k(\bar\eta)$ and a neighbourhood $V(\bar\eta)$ of $\bar\eta$ in $\Gamma_{K'}$, such that $\langle \eta,\, g(x^0 + t_k u)\rangle < 0$ for $k > k(\bar\eta)$ and $\eta \in V(\bar\eta)$.

1^0. Let $\bar\eta \in \Gamma_{K'(x^0)}$. From our assumption, we have $\langle \bar\eta,\, z^0\rangle < -\varepsilon < 0$ for some $\varepsilon = \varepsilon(\bar\eta) > 0$. Then

$$\lim_k \frac{1}{t_k}\langle \bar\eta,\, g(x^0 + t_k u) - g(x^0)\rangle = \langle \bar\eta,\, z^0\rangle < 0\,,$$

whence there exists $k(\bar\eta)$, such that for all $k > k(\bar\eta)$ it holds

$$\langle \bar\eta,\, g(x^0 + t_k u)\rangle < \langle \bar\eta,\, g(x^0)\rangle = 0\,.$$

Let $\langle \bar\eta,\, g(x^0 + t_k u)\rangle < -\varepsilon < 0$ for some $\varepsilon = \varepsilon(\bar\eta) > 0$. Then

$$\begin{aligned}\langle \eta,\, g(x^0 + t_k u)\rangle &= \langle \bar\eta,\, g(x^0 + t_k u)\rangle + \langle \eta - \bar\eta,\, g(x^0 + t_k u)\rangle \\ &< -\varepsilon + \|\eta - \bar\eta\|\left(\|g(x^0 + t_k u) - g(x^0)\| + \|g(x^0)\|\right) \\ &\le -\varepsilon + \|\eta - \bar\eta\|\left(Lr + \|g(x^0)\|\right) < -\varepsilon + \tfrac{1}{2}\varepsilon = -\tfrac{1}{2}\varepsilon < 0\end{aligned}$$

as far as $\|\eta - \bar\eta\| < \varepsilon/(2(Lr + \|g(x^0)\|))$ (which determines $V(\bar\eta)$).

2^0. Let $\bar\eta \in \Gamma_{K'} \setminus \Gamma_{K'(x^0)}$. We have $\langle \bar\eta,\, g(x^0)\rangle < -\varepsilon < 0$ for some $\varepsilon = \varepsilon(\bar\eta) > 0$. Then

$$\begin{aligned}\langle \eta,\, g(x^0 + t_k u)\rangle &= \langle \bar\eta,\, g(x^0)\rangle + \langle \eta,\, g(x^0 + t_k u) - g(x^0)\rangle + \langle \eta - \bar\eta,\, g(x^0)\rangle \\ &< -\varepsilon + \|g(x^0 + t_k u) - g(x^0)\| + \|\eta - \bar\eta\|\|g(x^0)\| \\ &< -\varepsilon + L\,t_k + \|\eta - \bar\eta\|\|g(x^0)\| < -\varepsilon - \tfrac{1}{3}\varepsilon - \tfrac{1}{3}\varepsilon = -\tfrac{1}{3}\varepsilon < 0\end{aligned}$$

as far as $t_k < \varepsilon/(3\,L)$ (we choose $k(\bar\eta)$ in a way that this inequality holds for $k > k(\bar\eta)$) and $\|\eta - \bar\eta\| < \varepsilon/(3\|g(x^0)\|)$ (which determines $V(\bar\eta)$).

Since $\Gamma_{K'}$ is compact, $\Gamma_{K'} \subset V(\bar\eta_1) \cup \ldots \cup V(\bar\eta_s)$. Let $k_0 = \max(k(\bar\eta_1) \cup \ldots \cup k(\bar\eta_s))$. For $k > k_0$ we have $\langle \eta,\, g(x^0 + t_k u)\rangle < 0$ for all $\eta \in \Gamma_{K'}$ (and hence for all $\eta \in K'$). This shows that $g(x^0 + t_k u) \in -\operatorname{int} K \subset -K$, in other words the points $x^0 + t_k u$ for $k > k_0$ are feasible.

According to the made assumption $y^0 \in -\mathrm{int}\, C$. Since $y^0 = \lim_k (1/t_k)(f(x^0+t_k u) - f(x^0))$, we see that $f(x^0+t_k u) - f(x^0) \in \mathrm{int}\, C$ for all sufficiently large k. This fact, together with $x^0 + t_k u$ feasible, contradicts the assumption that x^0 is w-minimizer. $\square$

The following constraint qualification appears in the Sufficient Conditions part of Theorem 1.

$$\mathbb{Q}_{0,1}(x^0)\ :\qquad \begin{array}{c} \text{If } g(x^0) \in -K \text{ and } \dfrac{1}{t_k}\left(g(x^0+t_k u^0) - g(x^0)\right) \to z^0 \in -K(x^0) \\ \text{then } \exists\, u^k \to u^0 : \exists\, k_0 \in \mathbb{N} : \forall\, k > k_0 : g(x^0 + t_k u^k) \in -K\,. \end{array}$$

The next theorem is our main result.

Theorem 1 (First-order conditions) *Consider problem (1) with f, g being $C^{0,1}$ functions and C and K closed convex cones.*

(Necessary Conditions) *Let x^0 be w-minimizer of problem (1). Then for each $u \in S$ the following condition is satisfied:*

$$\mathbb{N}'_{0,1}\ :\qquad \begin{array}{c} \forall (y^0,\, z^0) \in (f(x^0),\, g(x^0))'_u : \exists (\xi^0,\, \eta^0) \in C' \times K' : \\ (\xi^0,\, \eta^0) \neq (0,\, 0), \quad \langle \eta^0,\, g(x^0)\rangle = 0 \quad \text{and} \quad \langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0 \rangle \geq 0\,. \end{array}$$

(Sufficient Conditions) *Let $x^0 \in \mathbb{R}^n$ and suppose that for each $u \in S$ the following condition is satisfied:*

$$\mathbb{S}'_{0,1}\ :\qquad \begin{array}{c} \forall (y^0,\, z^0) \in (f(x^0),\, g(x^0))'_u : \exists (\xi^0,\, \eta^0) \in C' \times K' : \\ (\xi^0,\, \eta^0) \neq (0,\, 0), \quad \langle \eta^0,\, g(x^0)\rangle = 0 \quad \text{and} \quad \langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0 \rangle > 0\,. \end{array}$$

Then x^0 is an isolated minimizer of first order for problem (1).

Conversely, if x^0 is an isolated minimizer of first order for problem (1) and the constraint qualification $\mathbb{Q}_{0,1}(x^0)$ holds, then condition $\mathbb{S}'_{0,1}$ is satisfied.

Proof of the Necessary Conditions Let $u \in S$ and $(y^0, z^0) \in (f(x^0), g(x^0))'_u$. According to Lemma 3 we have $(y^0, z^0) \notin -\mathrm{int}\,(C \times K(x^0)) = -(\mathrm{int}\,(C) \times \mathrm{int}\,(K(x^0)))$, whence there exists

$$(\xi^0,\, \eta^0) \in (C \times K(x^0))' \setminus \{(0,\, 0)\} = C' \times K'(x^0) \setminus \{(0,\, 0)\}\,,$$

such that $(\xi^0,\, \eta^0)\,(y^0,\, z^0) = (\xi^0,\, y^0) + (\eta^0,\, z^0) \geq 0$, which proves $\mathbb{N}'_{0,1}$ (let us underline that $\eta^0 \in K'(x^0)$ is equivalent to $\eta^0 \in K'$ and $\langle \eta^0,\, g(x^0)\rangle = 0$). $\square$

Proof of the Sufficient Conditions Assume in the contrary, that x^0 is not an isolated minimizer of first order and choose a monotone decreasing sequence $\varepsilon_k \to +0$. From the assumption, there exist sequences $t_k \to +0$ and $u^k \in S$, such that $g(x^0 + t_k u^k) \in -K$ and

$$D(f(x^0+t_k u^k) - f(x^0), -C) = \max_{\xi \in \Gamma_{C'}} \langle \xi,\, f(x^0 + t_k u^k) - f(x^0)\rangle < \varepsilon_k t_k\,.$$

Here, according to the accepted notation, $\Gamma_{C'} = \{\xi \in C' \mid \|\xi\| = 1\}$. We may assume that $0 < t_k < r$ and both f and g are Lipschitz with constant L in $x^0 + r\,\mathrm{cl}\, B$. Passing to a subsequence, we may assume also that $u^k \to u^0$ and that equalities (13) hold with $u = u^0$. From them we have $(y^0,\, z^0) \in (f(x^0), g(x^0))'_{u^0}$.

Denote $z^k = (1/t_k)(g(x^0+t_k u^k) - g(x^0))$ and $z^{0,k} = (1/t_k)(g(x^0+t_k u^0) - g(x^0))$. We show that $z^k \to z^0$. This follows from the estimation

$$\|z^k - z^0\| \leq \frac{1}{t_k}\|g(x^0+t_k u^k) - g(x^0 + t_k u^0)\| + \|z^{0,k} - z^0\| \leq L\,\|u^k - u^0\| + \|z^{0,k} - z^0\|\,.$$

We show that $z^0 \in -K(x^0)$. For this purpose we must check that $\langle \eta,\, z^0\rangle \le 0$ for $\eta \in K'(x^0)$. We observe that $x^0 + t_k u^k$ feasible and $\eta \in K'(x^0)$ gives $\langle \eta,\, g(x^0+t_k u^k)\rangle \le 0$, whence

$$\langle \eta,\, \frac{1}{t_k}(g(x^0+t_k u^k) - g(x^0))\rangle = \frac{1}{t_k}\langle \eta,\, g(x^0+t_k u^k)\rangle \le 0\,.$$

A passing to a limit gives $\langle \eta,\, z^0\rangle \le 0$.

In order to obtain contradiction, we show that $\mathbb{S}'_{0,1}$ is not satisfied at x^0 for $u = u^0$ and $(y^0,\, z^0)$ as above. Denote $y^k = (1/t_k)(f(x^0+t_k u^k) - f(x^0))$ and $y^{0,k} = (1/t_k)(f(x^0+t_k u^0) - f(x^0))$. We have $y^k \to y^0$, which follows from the estimation

$$\|y^k - y^0\| \le \frac{1}{t_k}\|f(x^0+t_k u^k) - f(x^0+t_k u^0)\| + \|y^{0,k} - y^0\| \le L\,\|u^k - u^0\| + \|y^{0,k} - y^0\|\,. \quad (14)$$

Let $\bar\xi \in \Gamma_{C'}$. Then

$$\begin{aligned}\langle \bar\xi,\, y^k\rangle &= \frac{1}{t_k}\,\langle \bar\xi,\, f(x^0+t_k u^k) - f(x^0)\rangle \le \frac{1}{t_k}\,\max_{\xi\in\Gamma_{C'}}\langle \xi,\, f(x^0+t_k u^k) - f(x^0)\rangle\\ &= \frac{1}{t_k}\,D(f(x^0+t_k u^k) - f(x^0), -C) < \frac{1}{t_k}\,\varepsilon_k\, t_k = \varepsilon_k\,.\end{aligned}$$

Passing to a limit with $k \to \infty$ we get $\langle \bar\xi,\, y^0\rangle \le 0$ for arbitrary $\bar\xi \in \Gamma_{C'}$. Therefore $\langle \xi,\, y^0\rangle \le 0$ for arbitrary $\xi \in C'$. The latter for $\xi \ne 0$ follows from $\langle \xi,\, y^0\rangle = \|\xi\|\,\langle (\xi/\|\xi\|,\, y^0\rangle \le 0$. At the same time $\langle \eta,\, z^0\rangle \le 0$ for all $\eta \in K'(x^0)$. Therefore for all $\xi \in C'$ and $\eta \in K'(x^0)$ we have $\langle \xi,\, y^0\rangle + \langle \eta,\, ,z^0\rangle \le 0$, whence the opposite strong inequality from $\mathbb{S}'_{0,1}$ cannot have place. $\square$

Reversal of the Sufficient Conditions Let x^0 be an isolated minimizer of first order for problem (1), which means that $g(x^0) \in -K$ and there exists $r > 0$ and $A > 0$ such that $g(x) \in -K$ and $\|x - x^0\| \le r$ implies

$$D(f(x) - f(x^0), -C) = \max_{\xi\in\Gamma_{C'}}\langle \xi,\, f(x) - f(x^0)\rangle \ge A\,\|x - x^0\|\,. \quad (15)$$

Let $u^0 \in S$ and $(y^0, z^0) \in (f(x^0), g(x^0))'_u$ is determined by (13) with $u = u^0$. We may assume that $0 < t_k < r$ and that f and g are Lipschitz with constant L on $x^0 + r\,\mathrm{cl}\,B$.

One of the following two cases has place:

1^0. $z^0 \notin -K(x^0)$. Then there exists $\eta^0 \in K'(x^0)$, such that $\langle \eta^0,\, z^0\rangle > 0$ (obviously, the strong inequality gives $\eta^0 \ne 0$). Putting $\xi^0 = 0$, we get the pair $(\xi^0,\, \eta^0)$ satisfying condition $\mathbb{S}'_{0,1}$.

2^0. $z^0 \in -K(x^0)$. Then from the constraint qualification $\mathbb{Q}_{0,1}(x^0)$ it follows $g(x^0+t_k u^k) \in -K$ for some sequence $u^k \to u^0$ and all sufficiently large k. Taking a subsequence, we may assume that this holds for all k. From inequality (15) we get that there exists $\xi^0 \in \Gamma_{C'}$ (and hence $\xi^0 \in C'$, $\xi^0 \ne 0$), such that

$$\langle \xi^0,\, \frac{1}{t_k}\Big(f(x^0+t_k u^k) - f(x^0)\Big)\rangle \ge A\|u^k\|\,.$$

Putting $y^k = (1/t_k)(f(x^0+t_k u^k) - f(x^0))$ and $y^{0,k} = (1/t_k)(f(x^0+t_k u^0) - f(x^0))$, we have $y^k \to y^0$, which follows from (14). A passing to a limit gives $\langle \xi^0,\, y^0\rangle \ge A > 0$. Putting $\eta^0 = 0$, we get the pair $(\xi^0,\, \eta^0)$ satisfying condition $\mathbb{S}'_{0,1}$. $\square$

Obviously, the proved theorem is valid also for the unconstrained problem (2). We give this case, since then some of the conditions simplify.

Theorem 2 *Consider problem (2) with f being $C^{0,1}$ function and C closed convex cones.*

(Necessary Conditions) *Let x^0 be w-minimizer of problem (2). Then for each $u \in S$ and $y^0 \in f'_u(x^0)$ there exists $\xi^0 \in C' \setminus \{0\}$ such that $\langle \xi^0, y^0 \rangle \geq 0$.*

(Sufficient Conditions) *Let $x^0 \in \mathbb{R}^n$. Suppose that for each $u \in S$ and $y^0 \in f'_u(x^0)$ there exists $\xi^0 \in C' \setminus \{0\}$ such that $\langle \xi^0, y^0 \rangle > 0$. Then x^0 is an isolated minimizer of first order for problem (2).*

Conversely, the given condition is not only sufficient, but also necessary the point x^0 to be an isolated minimizer of first order.

While the Sufficient Conditions in Theorem 1 admit a reversal, already from the scalar optimization we know, that this is not the case for the Necessary Conditions.

Example 1 *Consider the unconstrained problem (2) with $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ and $C = \mathbb{R}_+$. Then Condition $\mathbb{N}'_{0,1}$ is satisfied at $x^0 = 0$, but x^0 is not w-minimizer.*

In this example $f'_u(x) = 3x^2 u$, which for $x^0 = 0$ gives $y^0 = f'_u(x^0) = 0$. The positive polar cone is $C' = \mathbb{R}_+$ and for any $\xi^0 > 0$ we have $\xi^0 y^0 = 0$. Hence Condition $\mathbb{N}'_{0,1}$ is satisfied, while obviously x^0 is not a minimizer.

The following simple example on one hand illustrates Theorem 1 in practice and on the other hand is applied in the forthcoming discussion.

Example 2 *Consider the unconstrained problem (2) with*

$$f : \mathbb{R} \to \mathbb{R}^2, \quad f(x) = \begin{cases} (x,\, -2x) & , \quad x \geq 0, \\ (2x,\, -x) & , \quad x < 0, \end{cases}$$

optimized with respect to $C = \mathbb{R}^2_+$. The function f is $C^{0,1}$ but not C^1. Then the point $x^0 = 0$ is both p-minimizer and isolated minimizer of first order, the latter can be established on the base of the Sufficient Conditions of Theorem 1.

Here the positive polar cone is $C' = \mathbb{R}^2_+$. For $u = 1$ we have $y^0 = f'_u(x^0) = (1, -2)$ and $\langle \xi^0, y^0 \rangle = \xi^0_1 - 2\xi^0_2 > 0$ if we choose $\xi^0 = (1, 0) \in \mathbb{R}^2_+ \setminus \{(0, 0)\}$. For $u = -1$ we have $y^0 = f'_u(x^0) = (-2, 1)$ and $\langle \xi^0, y^0 \rangle = -2\xi^0_1 + \xi^0_2 > 0$ if we choose $\xi^0 = (0, 1) \in \mathbb{R}^2_+ \setminus \{(0, 0)\}$.

The constraint qualification $\mathbb{Q}_{0,1}(x^0)$ is of Kuhn-Tucker type. In 1951 Kuhn, Tucker [23] published the classical variant for differentiable functions and since then it is the best known constraint qualification. One may be astonished, that in the hypothesis of $\mathbb{Q}_{0,1}(x^0)$ we have $z^0 \in -K(x^0)$, while in the conclusion $g(x^0 + t_k u^0) \in -K$ it stands K instead of $K(x^0)$. If the cone K is polyhedral, we may take in the conclusion $g(x^0 + t_k u^0) \in -K(x^0)$, but in general with such a weaker conclusion the reversal of the Sufficient Conditions of Theorem 1 is not true. This is shown in the next example.

Example 3 *Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}^3$ with $C = \mathbb{R}_+$, $K = \{z \in \mathbb{R}^3 \mid z_3^2 \geq z_1^2 + z_2^2\}$ and $f(x) = x^2$, $g(x) = (x\,|x|, -1, -1)$. Then f and g are C^1 functions, $x^0 = 0$ is an isolated minimizer of first order, $\mathbb{Q}_{0,1}(x^0)$ does not hold, but we have similar condition with $g(x^0 + t_k u^0) \in -K(x^0)$ in the conclusion, instead of $g(x^0 + t_k u^0) \in -K$. At the same time, whatever $u \in \mathbb{R}$ be, there is no pair $(\xi^0, \eta^0) \in C' \times K'(x^0)$ for which $\langle \xi^0, f'(x^0)u \rangle + \langle \eta^0, g'(x^0)u \rangle > 0$.*

Here x^0 is the only feasible point, and according to the definition x^0 is an isolated minimizer of first order. (This means $D(f(x) - f(x^0), -C) \geq A\|x - x^0\|$ for $x \in U \cap g^{-1}(-K)$, which is true, since $U \cap g^{-1}(-K) = \{x^0\}$). The index sets $K(x^0)$ is a half-space determined by the unique tangent plane to the cone $-K$ at $g(x^0)$, whence the modified constraint qualification is checked immediately. More precisely, $-K(x^0) = \{z \in \mathbb{R}^3 \mid -z_2 + z_3 \geq 0\}$. For any $u \in \mathbb{R}$ we have $\lim_k (1/t_k)(g(x^0 + t_k u) - g(x^0)) = (0,\, 0,\, 0) \in -K(x^0)$. At the same time $g(x^0 + t_k u) = (t_k^2 u|u|, -1, -1) \notin -K$, but $g(x^0 + t_k u) \in -K(x^0)$. Now, for any $u \in \mathbb{R}$ we have $f'(x^0)u = 0$, $g'(x^0)u = (0,\, 0,\, 0)$ and therefore $\langle \xi^0,\, f'(x^0)u\rangle + \langle \eta^0,\, g'(x^0)u\rangle = 0$ for all pairs $(\xi^0,\, \eta^0)$.

If g is Fréchet differentiable at x^0, then instead of constraint qualification $\mathbb{Q}_{0,1}(x^0)$ we may consider the constraint qualification $\mathbb{Q}_1(x^0)$ given below.

$$\mathbb{Q}_1(x^0): \quad \begin{array}{c} \text{If } g(x^0) \in -K \text{ and } g'(x^0)u^0 = z^0 \in -K(x^0) \text{ then} \\ \text{there exists } \delta > 0 \text{ and a differentiable injective function} \\ \varphi : [0,\, \delta] \to -K \text{ such that } \varphi(0) = x^0 \text{ and } \varphi'(0) = g'(x^0)u^0\,. \end{array}$$

In the case of a polyhedral cone K in $\mathbb{Q}_1(x^0)$ the requirement $\varphi : [0,\, \delta] \to -K$ can be replaced by $\varphi : [0,\, \delta] \to -K(x^0)$. This condition coincides with the classical Kuhn-Tucker constraint qualification (compare with Mangasarian [30, p. 102]).
The next theorem is a reformulation of Theorem 1 for C^1 problems, that is problems with f and g being C^1 functions.

Theorem 3 *Consider problem (1) with f, g being C^1 functions and C and K closed convex cones.*

(Necessary Conditions) *Let x^0 be w-minimizer of problem (1). Then for each $u \in S$ the following condition is satisfied:*

$$\mathbb{N}'_1: \quad \begin{array}{c} \exists (\xi^0,\, \eta^0) \in C' \times K' \setminus \{(0,\, 0)\} : \\ \langle \eta^0,\, g(x^0)\rangle = 0 \quad \text{and} \quad \langle \xi^0,\, f'(x^0)u\rangle + \langle \eta^0,\, g'(x^0)u\rangle \geq 0\,. \end{array}$$

(Sufficient Conditions) *Let $x^0 \in \mathbb{R}^n$.*
Suppose that for each $u \in S$ the following condition is satisfied:

$$\mathbb{S}'_1: \quad \begin{array}{c} \exists (\xi^0,\, \eta^0) \in C' \times K' \setminus \{(0,\, 0)\} : \\ \langle \eta^0,\, g(x^0)\rangle = 0 \quad \text{and} \quad \langle \xi^0,\, f'(x^0)u\rangle + \langle \eta^0,\, g'(x^0)u\rangle > 0\,. \end{array}$$

Then x^0 is an isolated minimizer of first order for problem (1).

Conversely, if x^0 is an isolated minimizer of first order for problem (1) and let the constraint qualification $\mathbb{Q}_1(x^0)$ have place, then condition $\mathbb{S}'_1$ is satisfied.

We underline without proof, that Theorem 3 remains true assuming for f and g only Fréchet differentiable at x^0, instead of being C^1.

The pairs of vectors $(\xi^0,\, \eta^0)$ are usually referred to as the Lagrange multipliers. Here we have different Lagrange multipliers to different $u \in S$ (and different $(y^0,\, z^0) \in (f(x^0), g(x^0))'_u$). The natural question arises, whether a common pair $(\xi^0,\, \eta^0)$ can be chosen to all directions. The next example shows that the answer is negative even for C^1 problems.

Example 4 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1,\, x_1^2 + x_2^2)$, and $g : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1,\, x_2)$. Define $C = \{y \in (y_1,\, y_2) \in \mathbb{R}^2 \mid y_2 = 0\}$, $K = \mathbb{R}^2$. Then f and g are C^1 functions and the point $x^0 = (0,\, 0)$ is w-minimizer of problem (1) (in fact x^0 is also isolated minimizer of second order, but not isolated minimizer of first order). At the same time the only pair $(\xi^0,\, \eta^0) \in C' \times K'$ for which $\langle \xi^0,\, f'(x^0)u\rangle + \langle \eta^0,\, g'(x^0)u\rangle \geq 0$ for all $u \in S$ is $\xi^0 = (0,\, 0)$ and $\eta^0 = (0,\, 0)$.*

The point x^0 is w-minimizer, since $\operatorname{int} C = 0$, whence each feasible point is w-minimizer. We have $f'(x)u = (u_1,\, 2x_1u_1+2x_2u_2)$, where from $f'(x^0)u = (u_1,\, 0)$, and $g'(x^0)u = u$. The positive polar cones are $C' = \{\xi \in \mathbb{R}^2 \mid \xi_1 = 0\}$ and $K' = \{0\}$. If $\xi^0 = (\xi_1^0,\, \xi_2^0) \in C'$ and $\eta^0 = (\eta_1^0,\, \eta_2^0) \in K'$ satisfy the desired inequality, then $\eta^0 = (0,\, 0)$, $\xi^0 = (\xi_1^0,\, 0)$ and the inequality turns into $\xi_1^0 u_1 \geq 0$, which should be true for all $u_1 \in \mathbb{R}$. This gives $\xi_1^0 = 0$ and finally $\xi^0 = (0,\, 0)$ and $\eta^0 = (0,\, 0)$.

The next Theorem 4 guarantees, that in the case when x^0 is rw-minimizer of the C^1 problem (1), a nonzero pair $(\xi^0,\, \eta^0)$ exists, which satisfies the Necessary Conditions of Theorem 1 and which is common for all directions. In order to prepare the proof, we need the following two lemmas.

Lemma 4 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be $C^{0,1}$ function and let $\mathbf{L} \subset \mathbb{R}^m$ be a subspace. Denote $\bar f = \pi_{\mathbf{L}} \circ f$. Then $\bar f$ is $C^{0,1}$ function and $\bar f'_u(x^0) = \pi_{\mathbf{L}} \circ f'_u(x^0)$. Similarly, if f is C^1 function, then $\bar f$ is C^1 function and $f'(x^0)u = \pi_{\mathbf{L}} \circ f'(x^0)u$.*

Proof The function $\bar f$ is locally Lipschitz, hence $C^{0,1}$, as a composition of a bounded linear function and a locally Lipschitz function.

Let $y^0 \in f'_u(x^0)$ and $y^0 = \lim_k (1/t_k)(f(x^0 + t_k u) - f(x^0))$. Since the projection commutes with the passing to a limit and with the linear operations, we see that

$$\pi_{\mathbf{L}} \circ y^0 = \lim_k \frac{1}{t_k}((\pi_{\mathbf{L}} \circ f)(x^0 + t_k u) - (\pi_{\mathbf{L}} \circ f)(x^0)) \in \bar f'_u(x^0)\,.$$

Conversely, let $\bar y^0 = \lim_k (1/t_k)(\bar f(x^0 + t_k u) - \bar f(x^0))$. From f locally Lipschitz, it follows that there exists a subsequence $\{t_{k'}\}$ of $\{t_k\}$, such that $\lim_{k'} (1/t_{k'})(f(x^0 + t_{k'} u) - f(x^0)) = y^0$. Now $y^0 \in f'_u(x^0)$ and $\bar y^0 = \pi_{\mathbf{L}} \circ y^0 \in \pi_{\mathbf{L}} \circ f'_u(x^0)$.

The case of $f \in C^1$ is treated similarly. $\square$

Lemma 5 *Consider problem (1) with f and g being $C^{0,1}$ functions and C and K closed convex cones. If x^0 is rw-minimizer and $(y^0,\, z^0) \in (\bar f(x^0),\, \bar g(x^0))'_u$ (here $\bar f = \pi_{\mathbf{L}_C} \circ f$ and $\bar g = \pi_{\mathbf{L}_K} \circ g$), then $(y^0,\, z^0) \notin -(\operatorname{ri} C \times \operatorname{ri}(K(x^0) \cap \mathbf{L}_K))$.*

The proof is omitted, since it nearly repeats that of Lemma 3, but relating the considerations to the phase space $\mathbf{L}_C \times \mathbf{L}_K$ instead of $\mathbb{R}^m \times \mathbb{R}^p$.

Theorem 4 (Necessary Conditions) *Consider problem (1) with f, g being C^1 functions and C and K closed convex cones. Let x^0 be rw-minimizer of problem (1). Then there exists a pair $(\xi^0,\, \eta^0) \in C'_{\mathbf{L}_C} \times K'_{\mathbf{L}_K} \setminus \{(0,\, 0)\}$ such that $\langle \eta^0,\, g(x^0)\rangle = 0$ and $\langle \xi^0,\, f'(x^0)u\rangle + \langle \eta^0,\, g'(x^0)u\rangle = 0$ for all $u \in \mathbb{R}^n$. The latter equality could be written also as $\xi^0 f'(x^0) + \eta^0 g'(x^0) = 0$.*

Proof Put $\bar f = \pi_{\mathbf{L}_C} \circ f$ and $\bar g = \pi_{\mathbf{L}_K} \circ g$. According to Lemma 5, $(\bar f'(x^0)u,\, \bar g'(x^0)u) \notin -(\operatorname{ri} C \times \operatorname{ri}(K(x^0) \cap \mathbf{L}_K)) \neq \emptyset$ for all $u \in \mathbb{R}^n$. Therefore the convex set $M = \{(\bar f'(x^0)u,\, \bar g'(x^0)u) \mid u \in \mathbb{R}^n\} \subset \mathbf{L}_C \times \mathbf{L}_K$ (the convexity is implied from the properties of the Fréchet derivative) does not intersect the non-empty interior (relative to $\mathbf{L}_C \times \mathbf{L}_K$) of the convex set $-C \times (K(x^0) \cap \mathbf{L}_K)$. From the Separation Theorem there exists a nonzero pair $(\xi^0,\, \eta^0) \in C'_{\mathbf{L}_C} \times K'_{\mathbf{L}_K}$ such that $\langle \xi^0,\, \bar f'(x^0)u\rangle + \langle \eta^0,\, \bar g'(x^0)u\rangle \geq 0$ for all $u \in \mathbb{R}^n$. This leads to an equality, since

$$0 \leq \langle \xi^0,\, \bar f'(x^0)(-u)\rangle + \langle \eta^0,\, \bar g'(x^0)(-u)\rangle = -\big(\langle \xi^0,\, \bar f'(x^0)u\rangle + \langle \eta^0,\, \bar g'(x^0)u\rangle\big) \leq 0\,.$$

Since $\xi^0 \in \mathbf{L}_C$ we have $\langle \xi^0,\, \bar{f}'(x^0)u\rangle = \langle \xi^0,\, f'(x^0)u\rangle$. Indeed, applying Lemma 4, we get

$$\langle \xi^0,\, \bar{f}'(x^0)u\rangle = \langle \xi^0,\, (\pi_{\mathbf{L}_C} \circ f)'(x^0)u\rangle = \langle \xi^0,\, \pi_{\mathbf{L}_C} \circ f'(x^0)u\rangle = \langle \xi^0,\, f'(x^0)u\rangle\,.$$

Similarly, since $\eta^0 \in \mathbf{L}_K$, we get $\langle \xi^0,\, \bar{g}'(x^0)u\rangle = \langle \xi^0,\, g'(x^0)u\rangle$. Finally $\eta^0 \in (K(x^0) \cap \mathbf{L}_K)'_{\mathbf{L}_K}$ gives $0 = \langle \eta^0,\, \bar{g}(x^0)\rangle = \langle \eta^0,\, \pi_{\mathbf{L}_K} \circ g(x^0)\rangle = \langle \eta^0,\, g(x^0)\rangle$. □

The established in Theorem 4 common for all directions $u \in \mathbb{R}^n$ multipliers are the reason to come back to Theorem 1 and to investigate more carefully this situation. We discover a relation to lw-minimizers.

Remark 1 *Consider problem (1) with f and g being $C^{0,1}$ functions and C and K closed convex cones. Suppose that x^0 is such that*

$$\begin{gathered}\exists\, (\xi^0,\, \eta^0) \in C' \times K'(x^0) \setminus \{(0,\,0)\} : \forall\, u \in \mathbb{R}^n : \\ \forall\, (y^0,\, z^0) \in (f(x^0), g(x^0))'_u : \langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle \geq 0\,.\end{gathered}$$

Then obviously $(\xi^0,\, \eta^0)$ separates the cone $-(C \times K(x^0)) \subset \mathbb{R}^{m+p}$ from the set

$$F' = \{(y,z) \in \mathbb{R}^{m+p} \mid (y,z) \in (f(x^0), g(x^0))'_u \text{ for some } u \in \mathbb{R}^n\} \tag{16}$$

in the sense that

$$\begin{gathered}\langle \xi^0,\, y\rangle + \langle \eta^0,\, z\rangle \leq 0 \text{ for all } (y,z) \in -(C \times K(x^0)) \\ \langle \xi^0,\, y\rangle + \langle \eta^0,\, z\rangle \geq 0 \text{ for all } (y,z) \in F'\,.\end{gathered}$$

The latter inequality is valid also for $(y,z) \in \operatorname{co} F'$.

Proposition 9 *The feasible point x^0 is lw-minimizer for problem (1) if and only if there exists a pair $(\xi^0,\, \eta^0) \in C' \times K'(x^0) \setminus \{(0,\,0)\}$ and a neighbourhood U of x^0, such that $(\xi^0,\, \eta^0)$ separates the cone $-(C \times K(x^0))$ from the set*

$$F = \{(y,z) \in \mathbb{R}^{m+p} \mid y = f(x) - f(x^0),\ z = g(x),\ x \in U\}$$

(and also from $\operatorname{co} F$), in the sense that

$$\begin{gathered}\langle \xi^0,\, f(x) - f(x^0)\rangle + \langle \eta^0,\, g(x)\rangle \geq 0 \text{ for all } x \in U, \\ \langle \xi^0,\, y\rangle + \langle \eta^0,\, z\rangle \leq 0 \text{ for all } (y,z) \in -(C \times K(x^0))\,.\end{gathered} \tag{17}$$

Proof Let x^0 be lw-minimizer and let U be the neighbourhood of x^0 for which $\varphi^0(x) \geq \varphi^0(x^0)$, $u \in U$, where φ^0 is the function in (6). This inequality with account of $\varphi^0(x^0) = 0$ and $(\xi^0,\, \eta^0) \in C' \times K(x^0) \setminus \{(0,\,0)\}$ gives (17).

Conversely, if $(\xi^0,\, \eta^0) \in C' \times K(x^0) \setminus \{(0,\,0)\}$ separates $-(C \times K(x^0))$ from F for some neighbourhood U of x^0, then for the function φ^0 in (6) and $x \in U$ we have

$$\varphi^0(x) = \langle \xi^0,\, f(x) - f(x^0)\rangle + \langle \eta^0,\, g(x)\rangle \geq 0 = \varphi^0(x^0)\,.$$

Thus, x^0 is a minimizer of φ^0, and therefore lw-minimizer of problem (1). □

Corollary 2 *Consider problem (1) with f, g being $C^{0,1}$ functions and C and K closed convex cones. If x^0 is lw-minimizer, then there exists $(\xi^0,\, \eta^0) \in C' \times K(x^0) \setminus \{(0,\,0)\}$, such that the inequality*

$$\langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle \geq 0 \tag{18}$$

holds for each $u \in \mathbb{R}^n$ and $(y^0,\, z^0) \in (f(x^0), g(x^0)'_u$. This implies that $-(\operatorname{int} C \times \operatorname{int} K(x^0))$ does not intersect the set $\operatorname{co} F'$, where F' is given by (16).

Proof Let $(y^0,\, z^0) \in (f(x^0), g(x^0)'_u$ are determined by (13). Then (17) gives for k sufficiently large

$$\langle \xi^0,\, \frac{1}{t_k}\left(f(x^0+t_k u) - f(x^0)\right)\rangle + \langle \eta^0,\, \frac{1}{t_k}\left(g(x^0+t_k u) - g(x^0)\right)\rangle \ge 0\,,$$

whence passing to a limit we get (18). Further $-(\mathrm{int}\, C \times \mathrm{int}\, K(x^0))$ and F' are separated, which cannot have place it $-(\mathrm{int}\, C \times \mathrm{int}\, K(x^0))$ and $\mathrm{co}\, F'$ intersect. $\square$

Restricting the considerations to the phase space $\mathbf{L}_C \times \mathbf{L}_K$ instead to $\mathbb{R}^m \times \mathbb{R}^p$ and replacing problem (1) by (4), we can introduce as in Definition 2 the concept of a relatively linearly scalarized weakly efficient point, for short *rlw*-minimizer (and similarly *rlp*-minimizer). Now under the assumptions of Corollary 2 and assuming also x^0 *rlw*-minimizer, we get that $-(\mathrm{ri}\, C \times \mathrm{ri}\,(K(x^0) \cap \mathbf{L}_K))$ and $\mathrm{co}\, F'$ do not intersect.

In Example 2 the point $x^0 = 0$ is w-minimizer, and even p-minimizer. We show that x^0 is not lw-minimizer. Indeed, in this case we have $f'_1(x^0) = (1,\, -2)$, $f'_{-1}(x^0) = (-2,\, 1)$ and $\frac{1}{2} f'_1(x^0) + \frac{1}{2} f'_{-1}(x^0) = (-\frac{1}{2},\, -\frac{1}{2})$ belongs both to $-\mathrm{int}\, C = -\mathrm{int}\, \mathbb{R}^2_+$ and to $\mathrm{co}\, F'$, where F' is the set (16).

The considered in Example 2 problem is $C^{0,1}$ but not C^1. In connection with Theorem 4 the following question arises. Is it true, that each w-minimizer of a C^1 problem is lw-minimizer? The next Example 5 gives a negative answer.

Example 5 *Consider the unconstrained problem (2) with*

$$f : \mathbb{R} \to \mathbb{R}^2, \quad f(x) = \begin{cases} (x^2,\, -2x^2) & ,\quad x \ge 0, \\ (2x^2,\, -x^2) & ,\quad x < 0, \end{cases}$$

optimized with respect to $C = \mathbb{R}^2_+$. The function f is C^1. Then the point $x^0 = 0$ is w-minimizer (it is also s-minimizer though not p-minimizer), but not lw-minimizer.

To establish that f is C^1 is quite easy. The function φ_0 in Corollary 1 is $\varphi_0(x) = x^2$. Since x^0 is an isolated minimizer of second order, but not an isolated minimizer of first order for φ_0, it is w-minimizer and s-minimizer for the initial problem, but not p-minimizer.

To show that $x^0 = 0$ is not lw-minimizer, observe that the function φ^0 in (6) is

$$\varphi^0(x) = \begin{cases} (\xi_1^0 - 2\xi_2^0)x^2 & ,\quad x \ge 0, \\ (-2\xi_1^0 + \xi_2^0)x^2 & ,\quad x < 0. \end{cases}$$

Then $\varphi^0(x) \ge \varphi^0(0)$, $x > 0$, implies $\xi_1^0 - 2\xi_2^0 \ge 0$ and $\varphi^0(x) \ge \varphi^0(0)$, $x < 0$, implies $-2\xi_1^0 + \xi_2^0 \ge 0$. Adding the two inequalities, we get $-\xi_1^0 - \xi_2^0 \ge 0$. At the same time $\xi^0 \in C' \setminus \{0\}$ gives $\xi_1^0 \ge 0$, $\xi_2^0 \ge 0$, where the two inequalities are not simultaneously satisfied. This however contradicts to the obtained above inequality.

In the following proposition, as an application of Theorem 1, we find a relation of lp-minimizers and p-minimizers.

Proposition 10 *Let in problem (1) f and g be locally Lipschitz functions. If x^0 is lp-minimizer, then x^0 is p-minimizer.*

Proof Let $u \in S$ and the pair $(y^0,\, z^0) \in (f(x^0), g(x^0))'_u$ is determined by (13). From x^0 isolated minimizer of first order for the scalar function $\varphi^0 : \mathbb{R}^n \to \mathbb{R}$ in (6), there exists $A > 0$, such that $\varphi^0(x^0 + t_k u) - \varphi(x^0) \ge A t_k$, whence

$$\langle \xi^0,\, \frac{1}{t_k}\left(f(x^0+t_k u) - f(x^0)\right)\rangle + \langle \eta^0,\, \frac{1}{t_k}\left(g(x^0+t_k u) - g(x^0)\right)\rangle \ge A > 0\,.$$

A passing to a limit gives $\langle \xi^0, y^0\rangle + \langle \eta^0, z^0\rangle \ge A > 0$. Now the Sufficient Condition in Theorem 1 gives that x^0 is an isolated minimizer of first order for problem (1), and according to Proposition 4 it is also p-minimizer. □

If $\operatorname{int} C = \emptyset$ each feasible point of problem (1) is w-minimizer and the Necessary Conditions are trivially satisfied. In this case a more essential information is that x^0 is rw-minimizer. The next Theorem 5 generalizes the Necessary Conditions part of Theorem 1 to relative concepts. Obviously, the Sufficient Conditions part admits also a generalization, which is not given here.

Theorem 5 (First-order conditions) *Consider problem (1) with f, g being $C^{0,1}$ functions and C and K closed convex cones.*

(Necessary Conditions) *Let x^0 be rw-minimizer of problem (1). Then for each $u \in S$ the following condition is satisfied:*

$$r\text{-}\mathbb{N}'_{0,1}: \quad \begin{array}{c} \forall (y^0, z^0) \in (f(x^0), g(x^0))'_u : \exists (\xi^0, \eta^0) \in \mathbf{L}_C{}' \times \mathbf{L}_K{}' : \\ (\xi^0, \eta^0) \neq (0, 0), \quad \langle \eta^0, g(x^0)\rangle = 0 \quad \textit{and} \quad \langle \xi^0, y^0\rangle + \langle \eta^0, z^0\rangle \ge 0 \,. \end{array}$$

We omit the proof. In principle it repeats the proof of the Necessary Conditions of Theorem 1 replacing the phase space from $\mathbb{R}^m \times \mathbb{R}^p$ to $\mathbf{L}_C \times \mathbf{L}_K$, replacing the considered problem from (1) to (4) and making use of Lemma 4.

4 Isolated minimizers and proper efficiency

Consider the unconstrained problem (2) with $C^{0,1}$ function f. According to Proposition 4 if x^0 is an isolated minimizer of first order, then x^0 is p-minimizer. It is natural to ask, whether the converse is true. The next example gives a negative answer of this question.

Example 6 *Let $t_k \to +0$, $k = 0, 1, \dots,$ be a strictly decreasing sequence with $t_0 = +\infty$. Define the function $h : \mathbb{R} \to \mathbb{R}$,*

$$h(t) = \begin{cases} \min\,(t_{k-1} - |t|,\, |t| - t_k) & , \quad t_k \le |t| \le t_{k-1}, \\ 0 & , \quad t = 0\,. \end{cases}$$

Consider the unconstrained problem (2) with $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (h(x), h(x))$ and $C = \mathbb{R}^2_+$. Then $x^0 = 0$ is p-minimizer, but not an isolated minimizer of first order.

The function f is $C^{0,1}$, since h is $C^{0,1}$. The latter follows by the easy-to-prove inequality $|h(t') - h(t'')| \le |t' - t''|$, $t', t'' \in \mathbb{R}$.

According to Proposition 8 and Corollary 1,if x^0 is an isolated minimizer of first order for (2), we should have, that x^0 is an isolated minimizer of first order for the function $\varphi_0(x) = \min(f_1(x) - f_1(x^0),\, f_2(x) - f_2(x^0)) = h(x)$. However, this is not the case, since for $x^k = t_k \to x^0 = 0$ we have $\varphi_0(x^k) = h(t_k) = 0$. Indeed, assuming in the contrary, that x^0 is an isolated minimizer of first order, we should have for some $A > 0$ and all sufficiently large k

$$0 = \varphi_0(x^k) = \varphi_0(x^k) - \varphi_0(x^0) \ge A\,\|x^k - x^0\| = A\,t_k > 0\,,$$

a contradiction.

The point x^0 is p-minimizer. Indeed, let $\tilde{C} = \{y \in \mathbb{R}^2 \mid y_1 + y_2 \ge 0\}$. Then $\operatorname{int}\tilde{C} = \{y \in \mathbb{R}^2 \mid y_1 + y_2 > 0\} \supset C \setminus \{0\} = \mathbb{R}^2_+ \setminus \{(0, 0)\}$ and x^0 is w-minimizer of the problem

$$f(x) \to \min\nolimits_{\tilde{C}}\,. \tag{19}$$

The latter follows from $f(x) = (h(x),\, h(x)) \in \mathbb{R}^2_+ = C$ and $\mathbb{R}^2_+$ disjoint from $-\mathrm{int}\,\tilde{C} = \{y \in \mathbb{R}^2 \mid y_1 + y_2 < 0\}$.

By a slight modification of this example we can see, that even the additional assumption x^0 strong e-minimizer does not guarantee that x^0 is an isolated minimizer of first order.

Example 7 *Let h be like in Example 6. Consider problem (2) with $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = (h(x) + x^2,\, h(x) + x^2)$ and $C = \mathbb{R}^2_+$. Then f is $C^{0,1}$, $x^0 = 0$ is both strong e-minimizer and p-minimizer, but not an isolated minimizer of first order.*

Here $\varphi_0(x) = h(x) + x^2$ has $x^0 = 0$ as a strong minimizer, but not as an isolated minimizer of first order.

We can strengthen the property x^0 is p-minimizer in a way, that we get x^0 is an isolated minimizer of first order.

For the constrained problem (1) we introduce the property

$\mathbb{P}(x^0, u)$: $\qquad (y^0,\, z^0) \in (f(x^0),\, g(x^0))'_u \Rightarrow (y^0,\, z^0) \neq (0,\, 0)\,.$

For the unconstrained problem (2) this property transforms into $y^0 \in f'_u(x^0) \Rightarrow y^0 \neq 0\,.$ In the next Proposition 11 we show, that this property, together with x^0 p-minimizer implies that x^0 is an isolated minimizer of first order.

Proposition 11 *Consider the unconstrained problem (2) with f being $C^{0,1}$ function. Let x^0 be p-minimizer, which satisfies property $\mathbb{P}(x^0, u)$ for each $u \in S$. Then x^0 is an isolated minimizer of first order.*

Proof Since x^0 is p-minimizer, therefore there exists a closed convex cone $\tilde{C}$, such that $\mathrm{int}\,\tilde{C} \supset C \setminus \{0\}$ and x^0 is w-minimizer for problem (19). According to the Necessary Conditions of Theorem 1 (and Theorem 2), this means, that for each $u \in S$ and $y^0 \in f'_u(x^0)$, there exists $\tilde{\xi}^0 \in \tilde{C}' \setminus \{0\}$, such that $\langle \tilde{\xi}^0,\, y^0 \rangle \geq 0$. This inequality, together with property $\mathbb{P}(x^0, u)$, shows that $y^0 \notin -\mathrm{int}\,\tilde{C} \cup \{0\}$. Since $C \subset \mathrm{int}\,\tilde{C} \cup \{0\}$, we see that $y^0 \notin C$. This implies, that there exists $\xi^0 \in C'$, such that $\langle \xi^0,\, y^0 \rangle > 0$. According to the Sufficient Conditions of Theorem 1 (and Theorem 2), the point x^0 is an isolated minimizer of first order. $\square$

In the next section we discuss similar reversal of Proposition 4 for the constrained problem (1).

5 The related unconstrained problem

We relate to the constrained problem (1) and the feasible point x^0 the unconstrained problem

$$(f(x),\, g(x)) \to \min\nolimits_{C \times K(x^0)}\,. \tag{20}$$

In Section 2 we defined the concept of a p-minimizer of problem (1), which from here on is called p-minimizer in sense I. The same definition determines the p-minimizers of problem (20), which seem to have a closer link to the isolated minimizers of first order. This justifies the following definition: We say, that the feasible for problem (1) point x^0 is a p-minimizer in sense II of the constrained problem (1), if it is a p-minimizer for the unconstrained problem (20). Similarly, to each defined in Section 2 type of minimizer of the constrained problem (1) (we call it a minimizer in sense I), we juxtapose the respective type of minimizer of the related unconstrained problem (20) (we call it a minimizer in sense II).

The next proposition illustrates, that there is a relation between the two type of minimizers.

Proposition 12 *Let x^0 be a feasible point for problem (1). If x^0 is w-minimizer in sense II, then x^0 is w-minimizer in sense I.*

Proof If x^0 is w-minimizer in sense II, then x^0 is a minimizer of the function

$$\varphi_{II}(x) = \max\{\langle \xi,\, f(x) - f(x^0)\rangle + \langle \eta,\, g(x) - g(x^0)\rangle \mid \xi \in C',\, \eta \in K'(x^0),\, \|(\xi,\, \eta)\| = 1\}\,.$$

Let $\varphi_{II}(x) \geq \varphi_{II}(x^0)$, for $x \in U$, where U is some neighbourhood of x^0. Choose $x \in U \cap g^{-1}(-K)$. From $K \subset K(x^0)$ we have $g(x) \in -K \subset -K(x^0)$, whence

$$\max\{\langle \eta,\, g(x)\rangle \mid \xi \in C',\, \eta \in K'(x^0),\, \|(\xi,\, \eta)\| = 1\} \leq 0\,.$$

Therefore $\varphi_{II}(x) \geq \varphi_{II}(x^0) = 0$ implies

$$\max\{\langle \xi,\, f(x) - f(x^0)\rangle \mid \xi \in C',\, \eta \in K'(x^0),\, \|(\xi,\, \eta)\| = 1\} \geq 0\,,$$

whence for the function (3) we have

$$\varphi(x) = \max\{\langle \xi,\, f(x) - f(x^0)\rangle \mid \xi \in C',\, \eta \in K'(x^0),\, \|(\xi,\, \eta)\| = 1\} \geq 0 = \varphi(x^0)\,.$$

Therefore x^0 is w-minimizer of problem (1). □

Next we write Theorem 2 for the unconstrained problem (20) and on this base we compare the isolated minimizers in sense I and II.

Theorem 6 *Consider problem (20) with f and g being $C^{0,1}$ functions and C and K closed convex cones.*

(Necessary Conditions) *Let x^0 be w-minimizer of (20), i. e. w-minimizer in sense II of (1). Then for each $u \in S$ and $(y^0,\, z^0) \in (f(x^0),\, g(x^0))'_u$ there exists $(\xi^0,\, \eta^0) \in C' \times K'(x^0) \setminus \{(0,\, 0)\}$, such that $\langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle \geq 0$.*

(Sufficient Conditions) *Let x^0 be feasible for (1). Suppose that for each $u \in S$ and $(y^0,\, z^0) \in (f(x^0),\, g(x^0))'_u$ there exists $(\xi^0,\, \eta^0) \in C' \times K'(x^0) \setminus \{(0,\, 0)\}$, such that $\langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle > 0$. Then x^0 is an isolated minimizer of first order of (20), i. e. an isolated minimizer of first order in sense II of (1).*

Conversely, the given condition is not only sufficient, but also necessary the point x^0 to be an isolated minimizer of first order in sense II of (1).

We obtain the next proposition as a corollary of Theorem 6.

Proposition 13 *Let x^0 be a feasible point for problem (1). If x^0 is an isolated minimizer of first order in sense I, then x^0 is an isolated minimizer of first order in sense II.*

Proof Let x^0 be an isolated minimizer in sense I. According to Theorem 1, for any $u \in S$ and $(y^0,\, z^0) \in (f(x^0),\, g(x^0))'_u$ there exists $(\xi^0,\, \eta^0) \in C' \times K'(x^0) \setminus \{(0,\, 0)\}$, such that $\langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle > 0$. Now the reversal of the Sufficient Conditions of Theorem 6 gives that x^0 is an isolated minimizer of first order in sense II. □

Let us say, that by a symmetry it does not follow, that if x^0 is an isolated minimizer of first order in sense II, then x^0 is an isolated minimizer of first order in sense I. The obstacle is, that the reversal of the Sufficient Conditions of Theorem 1 is proved only under the assumption that the constrained qualification $\mathbb{Q}_{0,1}(x^0)$ has place.

Perhaps it is not trivial to find a relation between p-minimizers in sense I and II. However, concerning p-minimizers in sense II, we can apply the results for the unconstrained problem obtained in Sections 2 and 4.

Proposition 14 *Let in problem (1) f, g be $C^{0,1}$ functions and let x^0 be a feasible point. If x^0 is an isolated minimizer of first order in sense II, then x^0 is p-minimizer of (1) in sense II.*

The proof is an immediate application of Proposition 4.

Proposition 15 *Let in problem (1) f and f be $C^{0,1}$ functions and x^0 be a feasible point. Let x^0 be p-minimizer in sense II, which satisfies Property $\mathbb{P}(x^0, u)$ for each $u \in S$. Then x^0 is an isolated minimizer of first order in sense II of (1).*

The proof is an immediate application of Proposition 11.

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