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Some Positive Dependence Orderings involving Tail Dependence*

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Abstract

In this paper we discuss the properties of the orderings of positive dependence introduced by Hollander et al. (1990) as generalizing the bivariate positive dependence concepts of left-tail decreasing (LTD) and right-tail increasing (RTI) studied by Esary and Proschan (1972). We show which of the postulates proposed by Kimeldorf and Sampson (1987) for a reasonable positive dependence ordering are satisfied and how the orders can be studied by restricting them to copulas, and we give some examples. We also investigate the relationship of these orders with some other orderings which have appeared in the literature and generalize the same notions of positive dependence.

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Key words and phrases: Copula; Fréchet class; positive dependence stochastic ordering; right-tail decreasing (RTI); left-tail decreasing (LTD).

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1 Introduction

In recent years, the statistical literature has reserved much attention to the study of positive dependence between random variables, intended as their tendency to assume concordant values, and many notions have been introduced to formally describe such concept.

One of the aspects researchers have been focusing on is the definition of stochastic orderings capable to compare, with respect to some criteria, the strength of dependence of two different bivariate random vectors with the same univariate marginal distributions, with interesting applications in reliability theory, multivariate data analysis, finance and many other related fields. In many cases, bivariate positive dependence orderings arose as generalizations of positive dependence notions, as such orderings reduce to the corresponding notions when the joint probability law of the random variables is compared with the distribution that the pair would have if they were independent. For instance, this is the case for the orderings PQD, SI and TP_2; see Yanagimoto and Okamoto (1969), Tchen (1980), Kimeldorf and Sampson (1987), Fang and Joe (1992) and the references therein.

The purpose of this paper is to discuss the properties of the orderings of positive dependence which were introduced by Hollander et al. (1990), generalizing the bivariate concepts of left-tail decreasing (LTD) and right-tail increasing (RTI) studied by Esary and Proschan (1972). We will then investigate their relationship with some orderings proposed by Averous and Dortet-Bernadet (2000) and Colangelo et al. (2006) which generalize the same notions.

The paper is organized as follows. In Section 2 we define the concept of bivariate positive dependence ordering and we also present some postulates as well as a copula representation that any reasonable such ordering should satisfy; these postulates were introduced by Kimeldorf and Sampson (1987). In Section 3 we discuss the orderings introduced by Hollander et al. (1990); since, to the best of our knowledge, no discussion can be found in the literature about their properties, we present them in detail, showing the strict relationship between these orders and how they can be studied by restricting them to copulas. A number of examples will also be provided. In Section 4 we introduce the positive dependence orderings which were studied by Averous and Dortet-Bernadet (2000) and, after discussing their properties, we show that although they generalize the same positive dependence notions as the orderings of Hollander et al. (1990), no implications exist between them; finally, we briefly present the properties of the positive dependence orderings introduced by Colangelo et al. (2006), and we review their results with regard to the relationship with the other orderings presented in the paper.

Some conventions that are used in this paper are the following. By "increasing" and "decreasing," we mean "non-decreasing" and "non-increasing," respectively. For any two bivariate vectors $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$, the notation $\boldsymbol{x} \leq \boldsymbol{y}$ means $x_i \leq y_i$ for $i = 1, 2$. Given a set $A \subseteq \mathbb{R}^2$, its closure will be denoted by $\overline{A}$. For any distribution function F of a random variable X, we denote by $\mathrm{Ran}(F)$ its range and by F^{-1} its left-continuous inverse, that is the function defined by $F^{-1}(u) = \inf\{t \in \mathbb{R} : F(t) \geq u\}$ for all $u \in [0, 1]$.

For every distribution function F of a bivariate random vector $\boldsymbol{X} = (X_1, X_2)$, let $\overline{F}$ and F^π respectively denote the corresponding survival function (i.e. $\overline{F}(\boldsymbol{x}) = \mathrm{P}(\boldsymbol{X} > \boldsymbol{x})$) and the distribution of $\boldsymbol{X}^\pi = (X_2, X_1)$. Define also the conditional distributions $F^L_{x_1}(x_2) = \mathrm{P}(X_2 \leq x_2 | X_1 \leq x_1)$ and $F^R_{x_1}(x_2) = \mathrm{P}(X_2 \leq x_2 | X_1 > x_1)$ for all $x_1 \in \mathbb{R}$ for which the conditional probabilities are well defined. Finally, let $\xrightarrow{\mathrm{d}}$ denote convergence in distribution.

2 Some preliminaries

We denote by Δ_2 the class of all bivariate distribution functions on $\mathbb{R}^2$ and by $\Gamma(F_1, F_2)$ the Fréchet class with marginal distribution functions F_1 and F_2, i.e., the subclass of Δ_2 containing the distribution functions with the univariate marginals F_1 and F_2. The Fréchet upper and lower bounds in each class $\Gamma(F_1, F_2)$ are defined as $F^+(x_1, x_2) = \min\{F_1(x_1), F_2(x_2)\}$ and $F^-(x_1, x_2) = \max\{F_1(x_1) + F_2(x_2) - 1, 0\}$ for all $\boldsymbol{x} \in \mathbb{R}^2$. These bounds are pointwise sharp and lie in the corresponding Fréchet Class. In each Fréchet class $\Gamma(F_1, F_2)$, we denote by $F^\perp$ the distribution function corresponding to the independence case, that is the function defined by $F^\perp(\boldsymbol{x}) = F_1(x_1)F_2(x_2)$ for all $\boldsymbol{x} \in \mathbb{R}^2$. Notice that, given a binary relation $\preceq$ on Δ_2, for any pair of random vectors $\boldsymbol{X} = (X_1, X_2)$ and $\boldsymbol{Y} = (Y_1, Y_2)$ with distributions F and G we will equivalently write $\boldsymbol{X} \preceq \boldsymbol{Y}$ or $F \preceq G$.

For any $F, G \in \Gamma(F_1, F_2)$, F is said to be smaller than G in the positive quadrant dependence order (and we write $F \leq_{\mathrm{PQD}} G$) if $F(\boldsymbol{x}) \leq G(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^2$. Letting $F = F^\perp$ it is easy to see that the ordering $\leq_{\mathrm{PQD}}$ generalizes the positive quadrant dependence notion introduced by Lehmann (1966).

Kimeldorf and Sampson (1987, 1989) proposed a few postulates that any binary relation $\preceq$ on Δ_2 should satisfy to define a reasonable bivariate positive dependence ordering. Here is a slight variation of the postulate list in Kimeldorf and Sampson (1987).

- P.1 The relation $\preceq$ is a partial order (reflexive, transitive and antisymmetric).
- P.2 If $F \preceq G$, then $F \leq_{\mathrm{PQD}} G$.
- P.3 For any $F \in \mathcal{M}(F_1, F_2)$, $F^- \preceq F \preceq F^+$.
- P.4 If $(X_1, X_2) \preceq (Y_1, Y_2)$, then $(\phi_1(X_1), \phi_2(X_2)) \preceq (\phi_1(Y_1), \phi_2(Y_2))$ for all increasing functions $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$.
- P.5 If $(X_1, X_2) \preceq (Y_1, Y_2)$, then $(\phi_1(X_1), \phi_2(X_2)) \preceq (\phi_1(Y_1), \phi_2(Y_2))$ for all decreasing functions $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$.
- P.6 If $(X_1, X_2) \preceq (Y_1, Y_2)$, then $(\phi(Y_1), Y_2) \preceq (\phi(X_1), X_2)$ for all decreasing functions $\phi : \mathbb{R} \to \mathbb{R}$.
- P.7 If $(X_1, X_2) \preceq (Y_1, Y_2)$, then $(Y_1, \phi(Y_2)) \preceq (X_1, \phi(X_2))$ for all decreasing functions $\phi : \mathbb{R} \to \mathbb{R}$.
- P.8 If $F \preceq G$, then $F^\pi \preceq G^\pi$.

P.9 If $\{F_n, n \geq 1\}, \{G_n, n \geq 1\}$ are such that $F_n \preceq G_n$ for all n, $F_n \xrightarrow{\text{d}} F$ and $G_n \xrightarrow{\text{d}} G$, then $F \preceq G$.

Many bivariate positive dependence orderings are known to satisfy these properties. For instance, this is the case for the orderings $\leq_{\text{PQD}}$ and $\leq_{\text{TP}_2}$; see Kimeldorf and Sampson (1987). For a generalization of the axioms to the multivariate setting and a detailed discussion of some multivariate positive dependence orderings, we refer to Joe (1997) and Müller and Stoyan (2002).

Let (X_1, X_2) be a random vector with distribution function $F \in \Gamma(F_1, F_2)$; by Sklar Theorem (see, for instance, Schweizer and Sklar (1983)) there exists a function $C_F : [0,1]^2 \to [0,1]$ such that, for all $\boldsymbol{x} \in \mathbb{R}^2$, we have

$$F(x_1, x_2) = C_F(F_1(x_1), F_2(x_2)). \tag{1}$$

The function C_F is called copula of F and it is a bivariate distribution function with uniform marginals. In addition, for all $\boldsymbol{u} \in \mathcal{R}_{F_1,F_2} = \text{Ran}(F_1) \times \text{Ran}(F_2)$, it satisfies $C_F(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2))$, so that C_F is uniquely defined on $\mathcal{R}_{F_1,F_2}$ and thus it is unique whenever F is continuous.

As the dependence structure of any distribution function is completely summarized by the corresponding copulas, it follows that an interesting property that any positive dependence ordering $\preceq$ should fulfill is that, for any two distribution functions F and G in the same Fréchet class, $F \preceq G$ if, and only if, F and G admit two copulas C_F and C_G satisfying $C_F \preceq C_G$. This property can be shown to be satisfied by several bivariate positive dependence orderings and it is strictly connected to postulate P.4. In fact, for fixed $F, G \in \Gamma(F_1, F_2)$, let C_F and C_G be their corresponding copulas which satisfy $C_F \preceq C_G$; letting $\boldsymbol{U} \sim C_F$ and $\boldsymbol{V} \sim C_G$, $F \preceq G$ thus follows by postulate P.4 since $(F_1^{-1}(U_1), F_2^{-1}(U_2)) \sim F$ and $(F_1^{-1}(V_1), F_2^{-1}(V_2)) \sim G$, where F_1^{-1} and F_2^{-1} are increasing transformations. On the converse, let $F, G \in \Gamma(F_1, F_2)$ and suppose that $\boldsymbol{X} \sim F$, $\boldsymbol{Y} \sim G$ and $F \preceq G$; it is easy to see that the distribution functions of the random vectors $(F_1(X_1), F_2(X_2))$ and $(F_1(Y_1), F_2(Y_2))$ respectively coincide with any copula C_F of F and C_G of G on the set $\mathcal{R}_{F_1,F_2}$, and then postulate P.4 entails that any such C_F and C_G do satisfy the definition of $\preceq$ on $\mathcal{R}_{F_1,F_2}$. Therefore, whenever F_1 and F_2 are continuous, it will follow that the uniquely defined C_F and C_G satisfy $C_F \preceq C_G$ while, in general, it will have to be proven the existence of two such copulas satisfying the definition of $\preceq$ on $([0,1]^2 - \mathcal{R}_{F_1,F_2})$.

We close the section recalling some positive dependence notions which will be central in the sequel. Given a random vector $\boldsymbol{X} = (X_1, X_2)$, X_2 is said to be left-tail decreasing (LTD) [right-tail increasing (RTI)] in X_1 if $F^L_{x_1}(x_2) \geq F^L_{x_1'}(x_2)$ $[F^R_{x_1}(x_2) \geq F^R_{x_1'}(x_2)]$ whenever $x_1 \leq x_1'$. For a general treatment of bivariate positive dependence notions and the corresponding generalizations to the multivariate setting, additional references are Kimeldorf and Sampson (1989) and Colangelo et al. (2005).

3 The orderings of Hollander, Proschan and Sconing

Let (X_1, X_2) and (Y_1, Y_2) be two random vectors with distributions F and G lying in $\Gamma(F_1, F_2)$. Hollander et al. (1990) define Y_2 to be more LTD in Y_1 than X_2 is in X_1 (and we write $(X_1, X_2) \leq_{\mathrm{LTD}} (Y_1, Y_2)$ or $F \leq_{\mathrm{LTD}} G$) if, for all $x_1 < x_1'$,

$$F^L_{x_1}(x_2) - F^L_{x_1'}(x_2) \leq G^L_{x_1}(x_2) - G^L_{x_1'}(x_2) \qquad \text{for any } x_2 \in \mathbb{R}. \tag{2}$$

Analogously, they define Y_2 to be more RTI in Y_1 than X_2 is in X_1 (and we write $(X_1, X_2) \leq_{\mathrm{RTI}} (Y_1, Y_2)$ or $F \leq_{\mathrm{RTI}} G$) if, for all $x_1 < x_1'$,

$$F^R_{x_1}(x_2) - F^R_{x_1'}(x_2) \leq G^R_{x_1}(x_2) - G^R_{x_1'}(x_2) \qquad \text{for any } x_2 \in \mathbb{R}. \tag{3}$$

Letting $F = F^\perp$, it is easy to see that the relations $\leq_{\mathrm{LTD}}$ and $\leq_{\mathrm{RTI}}$ are indeed generalizations of the LTD and the RTI positive dependence notions. Hollander et al. (1990) introduced these orderings in relation to the evaluation of the degree of dependence in the randomly censored models, but, to the best of our knowledge, their properties have never been studied in the literature.

Simple arguments can be used to show that the binary relations $\leq_{\mathrm{LTD}}$ and $\leq_{\mathrm{RTI}}$ are partial orders, so that postulate P.1 is satisfied. In order to see that also postulate P.2 holds, it suffices to respectively let $x_1' \to \infty$ and $x_1 \to -\infty$ in conditions (2) and (3); the result would then follow by noticing that the distributions lie in the same Fréchet class.

We now discuss whether the orderings satisfy postulate P.3. Notice that, in any given Fréchet class $\Gamma(F_1, F_2)$, for any $\boldsymbol{x} \in \mathbb{R}^2$,

$$F^{+,L}_{x_1}(x_2) = \min\left(1, \frac{F_2(x_2)}{F_1(x_1)}\right), \quad F^{-,L}_{x_1}(x_2) = \max\left(0, \frac{F_1(x_1) + F_2(x_2) - 1}{F_1(x_1)}\right)$$

and

$$F^{+,R}_{x_1}(x_2) = 1 - \min\left(1, \frac{\overline{F}_2(x_2)}{\overline{F}_1(x_1)}\right), \quad F^{-,R}_{x_1}(x_2) = 1 - \max\left(0, \frac{\overline{F}_1(x_1) + \overline{F}_2(x_2) - 1}{\overline{F}_1(x_1)}\right).$$

Therefore, $F \leq_{\mathrm{LTD}} F^+$ if, and only if, for all $x_1 < x_1'$ and $x_2 \in \mathbb{R}$,

$$F^L_{x_1}(x_2) - F^L_{x_1'}(x_2) \leq \min\left(1, \frac{F_2(x_2)}{F_1(x_1)}\right) - \min\left(1, \frac{F_2(x_2)}{F_1(x_1')}\right), \tag{4}$$

and then, whenever x_1' is such that $F_1(x_1') \leq F_2(x_2)$, condition (4) cannot hold with a strict inequality if F is LTD. Analogously, $F^- \leq_{\mathrm{LTD}} F$ if, and only if, for all $x_1 < x_1'$ and $x_2 \in \mathbb{R}$,

$$\max\left(0, \frac{F_1(x_1) + F_2(x_2) - 1}{F_1(x_1)}\right) - \max\left(0, \frac{F_1(x_1') + F_2(x_2) - 1}{F_1(x_1')}\right) \leq F^L_{x_1}(x_2) - F^L_{x_1'}(x_2), \tag{5}$$

so that it's easy to see that condition (5) is always satisfied if F is LTD but it needn't hold in general.

A symmetric argument applies for the ordering $\leq_{\text{RTI}}$; in particular, if F is RTI then condition (3) cannot hold with a strict inequality for all $x_1 < x_1'$ and $x_2 \in \mathbb{R}$ with $G^R_{x_1} = F^{+,R}_{x_1}$, while condition (3) is satisfied with $F^R_{x_1} = F^{-,R}_{x_1}$ if G is RTI, but it needn't hold in general. Numerical counterexamples are provided in Example 2 below.

The following result establishes some important properties of the orderings under consideration; in particular, it implies that postulates P.4 and P.7 are satisfied and it also shows the close relationship between the orderings $\leq_{\text{LTD}}$ and $\leq_{\text{RTI}}$. The proof of the theorem is omitted as it is based on standard arguments.

Theorem 1. *Let (X_1, X_2) and (Y_1, Y_2) be two random vectors with distribution functions $F, G \in \Gamma(F_1, F_2)$.*

(a) *If $(X_1, X_2) \leq_{\text{LTD}} [\leq_{\text{RTI}}](Y_1, Y_2)$ then $(\phi_1(X_1), \phi_2(X_2)) \leq_{\text{LTD}} [\leq_{\text{RTI}}](\phi_1(Y_1), \phi_2(Y_2))$ for all increasing functions $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$. Conversely, if $(\phi_1(X_1), \phi_2(X_2)) \leq_{\text{LTD}} [\leq_{\text{RTI}}] (\phi_1(Y_1), \phi_2(Y_2))$ for some strictly increasing functions $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$ then $(X_1, X_2) \leq_{\text{LTD}} [\leq_{\text{RTI}}](Y_1, Y_2)$.*

(b) *If $(X_1, X_2) \leq_{\text{LTD}} [\leq_{\text{RTI}}](Y_1, Y_2)$ then $(\phi_1(X_1), \phi_2(X_2)) \leq_{\text{RTI}} [\leq_{\text{LTD}}](\phi_1(Y_1), \phi_2(Y_2))$ for all decreasing functions $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$. Conversely, if $(\phi_1(X_1), \phi_2(X_2)) \leq_{\text{RTI}} [\leq_{\text{LTD}}] (\phi_1(Y_1), \phi_2(Y_2))$ for some strictly decreasing functions $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$ then $(X_1, X_2) \leq_{\text{LTD}} [\leq_{\text{RTI}}](Y_1, Y_2)$.*

(c) *If $(X_1, X_2) \leq_{\text{LTD}} [\leq_{\text{RTI}}](Y_1, Y_2)$ then $(\phi(X_1), X_2) \geq_{\text{RTI}} [\geq_{\text{LTD}}](\phi(Y_1), Y_2)$ for all decreasing functions $\phi : \mathbb{R} \to \mathbb{R}$. Conversely, if $(\phi(X_1), X_2) \geq^*_{\text{RTI}} [\geq^*_{\text{LTD}}](\phi(Y_1), Y_2)$ for some strictly decreasing function $\phi : \mathbb{R} \to \mathbb{R}$ then $(X_1, X_2) \leq_{\text{LTD}} [\leq_{\text{RTI}}](Y_1, Y_2)$.*

(d) *If $(X_1, X_2) \leq_{\text{LTD}} [\leq_{\text{RTI}}](Y_1, Y_2)$ then $(X_1, \phi(X_2)) \geq_{\text{LTD}} [\geq_{\text{RTI}}](Y_1, \phi(Y_2))$ for all decreasing functions $\phi : \mathbb{R} \to \mathbb{R}$. Conversely, if $(X_1, \phi(X_2)) \geq^*_{\text{LTD}} [\geq^*_{\text{RTI}}](Y_1, \phi(Y_2))$ for some strictly decreasing function $\phi : \mathbb{R} \to \mathbb{R}$ then $(X_1, X_2) \leq_{\text{LTD}} [\leq_{\text{RTI}}](Y_1, Y_2)$.*

Simple approximation arguments easily establish that also postulate P.9 must hold for both orderings, while the following example shows that $\leq_{\text{LTD}}$ doesn't need to admit the lower and upper Fréchet bounds as minimal and maximal elements and that postulates P.5, P.6 and P.7 are also not satisfied. In view of part (b) of Theorem 1 it is easy to see that neither $\leq_{\text{RTI}}$ satisfies postulates P.3, P.5, P.6 and P.7.

Example 2. Let (X_1, X_2) be a random vector with probability mass function

$x_2 \backslash x_1$	1	2	3
3	0	0	1/5
2	0	1/5	0
1	2/5	0	1/5

and denote by F the distribution function of (X_1, X_2) and by F^- the Fréchet lower bound in its corresponding Fréchet class. Letting (Y_1, Y_2) denote a random vector with F^- as its distribution, a simple calculation shows that $(Y_1, Y_2) \leq_{\text{LTD}} (X_1, X_2)$.

Consider now the random vector $(-X_1, -X_2)$ and let G be its corresponding distribution function. Denoting by G_1 and G_2 the marginals of G, it is easy to notice that the Fréchet lower bound G^- in $\Gamma(G_1, G_2)$ is the distribution function of $(-Y_1, -Y_2)$. Letting $x_1 = -3$, $x_1' = -2$ and $x_2 = -2$, it holds $G^L_{x_1}(x_2) = 1/2$, $G^L_{x_1'}(x_2) = 2/3$, $G_1(x_1) = 2/5$, $G_1(x_1') = 3/5$ and $G_2(x_2) = 2/5$, so that inequality (5) fails. Hence $(-Y_1, -Y_2) \not\leq_{\text{LTD}} (-X_1, -X_2)$, showing that the Fréchet lower bound needn't be a minimal element with respect to the ordering $\leq_{\text{LTD}}$ and that postulate P.5 is not satisfied.

Analogously, consider the random vector $(-X_1, X_2)$ and let H be its corresponding distribution function. Denoting by H_1 the marginal distribution of $-X_1$, it is easy to notice that the Fréchet upper bound H^+ in $\Gamma(H_1, F_2)$ is the distribution function of $(-Y_1, Y_2)$. Letting $x_1 = -3$, $x_1' = -2$ and $x_2 = 1$, it holds $H^L_{x_1}(x_2) = 1/2$, $H^L_{x_1'}(x_2) = 1/3$, $H_1(x_1) = 2/5$, $H_1(x_1') = 3/5$ and $F_2(x_2) = 3/5$, so that inequality (4) fails. Hence $(-X_1, X_2) \not\leq_{\text{LTD}} (-Y_1, Y_2)$, showing that the Fréchet upper bound needn't be a maximal element with respect to the ordering $\leq_{\text{LTD}}$ and that postulate P.6 is also not satisfied.

Finally, let G^π be the distribution of $(-X_2, -X_1)$ and, as above, notice that the Fréchet lower bound G^{π^-} in $\Gamma(G_2, G_1)$ is the distribution function of $(-Y_2, -Y_1)$. Lehmann (1966) shows that G^π is LTD and then, in view of the previous discussion, $G^{\pi^-} \leq_{\text{LTD}} G^\pi$; hence, as $G^- \not\leq_{\text{LTD}} G$, postulate P.8 is not met. ◀

We now prove that the orderings $\leq_{\text{LTD}}$ and $\leq_{\text{RTI}}$ admit the copula representation discussed in Section 2.

Theorem 3. *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *have, respectively, distribution functions* $F, G \in \Gamma(F_1, F_2)$. *Then* $\boldsymbol{X} \leq_{\text{LTD}} \boldsymbol{Y}$ $[\boldsymbol{X} \leq_{\text{RTI}} \boldsymbol{Y}]$ *if, and only if, there exist copulas* C_F *and* C_G (*as in* (1)) *such that* $C_F \leq_{\text{LTD}} C_G$ $[C_F \leq_{\text{RTI}} C_G]$.

Proof. We give only the proof for the ordering $\leq_{\text{LTD}}$; the proof for the other ordering is similar. In view of the discussion above, postulate P.4 implies the sufficiency part of the theorem and that the functions $C_F, C_G : \mathcal{R}_{F_1,F_2} \to [0,1]$ defined by $C_F(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2))$ and $C_G(u_1, u_2) = G(F_1^{-1}(u_1), F_2^{-1}(u_2))$, satisfy

$$\frac{C_F(u_1, u_2)}{u_1} - \frac{C_F(u_1', u_2)}{u_1'} \leq \frac{C_G(u_1, u_2)}{u_1} - \frac{C_G(u_1', u_2)}{u_1'} \tag{6}$$

for any $u_2 \in \text{Ran}(F_2)$ and $u_1, u_1' \in \text{Ran}(F_1)$ such that $u_1 < u_1'$. Hence, it remains to prove that C_F and C_G can be extended to two copulas satisfying equation (6) on the set $[0,1]^2 - \mathcal{R}_{F_1,F_2}$.

To construct such extensions, we use the method illustrated in Schweizer and Sklar (1983), namely considering linear interpolations along each variable. First define C_F and C_G on all boundary points of $\mathcal{R}_{F_1,F_2}$ by taking the limit (and this implies that condition (6) holds on the closure of $\mathcal{R}_{F_1,F_2}$, i.e. $\overline{\mathcal{R}_{F_1,F_2}} = \overline{\text{Ran}(F_1)} \times \overline{\text{Ran}(F_2)}$). Then, if C_F is defined in (l_1, u_2) and (m_1, u_2) but not for $u_1 \in (l_1, m_1)$, let $C_F(u_1, u_2) =$

$\beta_{u_1} C_F(l_1, u_2) + (1-\beta_{u_1}) C_F(m_1, u_2)$ with $\beta_{u_1} = \frac{m_1 - u_1}{m_1 - l_1}$. Analogously, if C_F is defined in (u_1, l_2) and (u_1, m_2) but not for $u_2 \in (l_2, m_2)$, let $C_F(u_1, u_2) = \beta_{u_2} C_F(u_1, l_2) + (1-\beta_{u_2}) C_F(u_1, m_2)$ with $\beta_{u_2} = \frac{m_2 - u_2}{m_2 - l_2}$. The same contruction is applied to C_G. Five cases need to be considered.

Case 1. Let $u_2 \in \overline{\mathrm{Ran}(F_2)}$, $u_1 \notin \overline{\mathrm{Ran}(F_1)}$ and $u_1' \in \overline{\mathrm{Ran}(F_1)}$, with $u_1 < u_1'$; then there exist l_1, m_1 boundary points of $\mathrm{Ran}(F_1)$ such that C_F and C_G are not defined on $(l_1, m_1) \times [0,1]$ and $l_1 < u_1 < m_1 < u_1'$. Noting that $\gamma_{u_1} = \beta_{u_1} \frac{l_1}{u_1} \in (0,1)$, it holds

$$\begin{aligned}
&\frac{C_F(u_1,u_2)}{u_1} - \frac{C_F(u_1',u_2)}{u_1'} \\
&\qquad = \gamma_{u_1} \left[\frac{C_F(l_1,u_2)}{l_1} - \frac{C_F(u_1',u_2)}{u_1'} \right] + (1-\gamma_{u_1}) \left[\frac{C_F(m_1,u_2)}{m_1} - \frac{C_F(u_1',u_2)}{u_1'} \right] \\
&\qquad \le \gamma_{u_1} \left[\frac{C_G(l_1,u_2)}{l_1} - \frac{C_G(u_1',u_2)}{u_1'} \right] + (1-\gamma_{u_1}) \left[\frac{C_G(m_1,u_2)}{m_1} - \frac{C_G(u_1',u_2)}{u_1'} \right] \\
&\qquad = \frac{C_G(u_1,u_2)}{u_1} - \frac{C_G(u_1',u_2)}{u_1'},
\end{aligned}$$

where the inequality follows by condition (6) as l_1 and m_1 belong to $\overline{\mathrm{Ran}(F_1)}$.

Case 2. Let $u_2 \in \overline{\mathrm{Ran}(F_2)}$, $u_1 \in \overline{\mathrm{Ran}(F_1)}$ and $u_1' \notin \overline{\mathrm{Ran}(F_1)}$, with $u_1 < u_1'$. The same reasoning as above can be applied to show that (6) is satisfied also in this case.

Case 3. Let $u_2 \in \overline{\mathrm{Ran}(F_2)}$ and $u_1, u_1' \notin \overline{\mathrm{Ran}(F_1)}$ with $u_1 < u_1'$, and suppose that there exist l_1, m_1 boundary points of $\mathrm{Ran}(F_1)$ such that C_F and C_G are not defined on $(l_1, m_1) \times [0,1]$ and $l_1 < u_1 < u_1' < m_1$. Using the construction outlined above, it clearly holds $C_F(u_1', u_2) = \epsilon_{u_1'} C_F(u_1, u_2) + (1-\epsilon_{u_1'}) C_F(m_1, u_2)$ and $C_G(u_1', u_2) = \epsilon_{u_1'} C_G(u_1, u_2) + (1-\epsilon_{u_1'}) C_G(m_1, u_2)$, with $\epsilon_{u_1'} = \frac{m_1 - u_1'}{m_1 - u_1}$; hence, noting that $\gamma_{u_1'} = \epsilon_{u_1'} \frac{u_1}{u_1'} \in (0,1)$,

$$\begin{aligned}
\frac{C_F(u_1,u_2)}{u_1} - \frac{C_F(u_1',u_2)}{u_1'} &= (1-\gamma_{u_1'}) \left[\frac{C_F(u_1,u_2)}{u_1} - \frac{C_F(m_1,u_2)}{m_1} \right] \\
&\le (1-\gamma_{u_1'}) \left[\frac{C_G(u_1,u_2)}{u_1} - \frac{C_G(m_1,u_2)}{m_1} \right] \\
&= \frac{C_G(u_1,u_2)}{u_1} - \frac{C_G(u_1',u_2)}{u_1'},
\end{aligned}$$

where the inequality follows by Case 1 as $u_1 \notin \overline{\mathrm{Ran}(F_1)}$ and $m_1 \in \overline{\mathrm{Ran}(F_1)}$.

Case 4. Let $u_2 \in \overline{\mathrm{Ran}(F_2)}$ and $u_1, u_1' \notin \overline{\mathrm{Ran}(F_1)}$ with $u_1 < u_1'$, and suppose that there exist two pairs $\{l_1, m_1\}$ and $\{l_1', m_1'\}$ of boundary points of $\mathrm{Ran}(F_1)$ such that C_F and C_G are not defined on $(l_1, m_1) \times [0,1]$ and $(l_1', m_1') \times [0,1]$, and $l_1 < u_1 < m_1 < l_1' < u_1' < m_1'$. Noting that $\gamma_{u_1'} = \beta_{u_1'} \frac{l_1'}{u_1'} \in (0,1)$, it holds

$$\begin{aligned}
&\frac{C_F(u_1,u_2)}{u_1} - \frac{C_F(u_1',u_2)}{u_1'} \\
&\qquad = \gamma_{u_1'} \left[\frac{C_F(u_1,u_2)}{u_1} - \frac{C_F(l_1',u_2)}{l_1'} \right] + (1-\gamma_{u_1'}) \left[\frac{C_F(u_1,u_2)}{u_1} - \frac{C_F(m_1',u_2)}{m_1'} \right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \gamma_{u'_1}\left[\frac{C_G(u_1,u_2)}{u_1}-\frac{C_G(l'_1,u_2)}{l'_1}\right]+(1-\gamma_{u'_1})\left[\frac{C_G(u_1,u_2)}{u_1}-\frac{C_G(m'_1,u_2)}{m'_1}\right]\\
&=\frac{C_G(u_1,u_2)}{u_1}-\frac{C_G(u'_1,u_2)}{u'_1},
\end{aligned}
$$

where the inequality follows by Case 1 as $u_1 \notin \overline{\mathrm{Ran}(F_1)}$ and $l'_1, m'_1 \in \overline{\mathrm{Ran}(F_1)}$.

Case 5. The cases so far discussed prove that equation (6) must hold for all $u_1, u'_1 \in [0,1]$ with $u_1 < u'_1$ and $u_2 \in \overline{\mathrm{Ran}(F_2)}$. Therefore, suppose that $u_2 \notin \overline{\mathrm{Ran}(F_2)}$; then there exist l_2, m_2 boundary points of $\mathrm{Ran}(F_2)$ such that C_F and C_G are not defined on $[0,1]\times(l_2,m_2)$ and $l_2 < u_2 < m_2$. Hence

$$
\begin{aligned}
&\frac{C_F(u_1,u_2)}{u_1}-\frac{C_F(u'_1,u_2)}{u'_1}\\
&\qquad=\beta_{u_2}\left[\frac{C_F(u_1,l_2)}{u_1}-\frac{C_F(u'_1,l_2)}{u'_1}\right]+(1-\beta_{u_2})\left[\frac{C_F(u_1,m_2)}{u_1}-\frac{C_F(u'_1,m_2)}{u'_1}\right]\\
&\qquad\leq\beta_{u_2}\left[\frac{C_G(u_1,l_2)}{u_1}-\frac{C_G(u'_1,l_2)}{u'_1}\right]+(1-\beta_{u_2})\left[\frac{C_G(u_1,m_2)}{u_1}-\frac{C_G(u'_1,m_2)}{u'_1}\right]\\
&\qquad=\frac{C_G(u_1,u_2)}{u_1}-\frac{C_G(u'_1,u_2)}{u'_1},
\end{aligned}
$$

where the inequality follows by the previous cases as $l_2, m_2 \in \overline{\mathrm{Ran}(F_2)}$. □

A useful corollary of Theorem 1 that will be used in the sequel is the following.

Corollary 4. *Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be two random vectors whose distribution functions are copulas. Then*

1. $\boldsymbol{U} \leq_{\mathrm{LTD}} \boldsymbol{V}$ *if, and only if,* $\mathbf{1}-\boldsymbol{U} \leq_{\mathrm{RTI}} \mathbf{1}-\boldsymbol{V}$*;*
2. $\boldsymbol{U} \leq_{\mathrm{RTI}} \boldsymbol{V}$ *if, and only if,* $\mathbf{1}-\boldsymbol{U} \leq_{\mathrm{LTD}} \mathbf{1}-\boldsymbol{V}$*.*

Recall that if the distribution function of $\boldsymbol{U}$ is the copula C, then the distribution of $\mathbf{1}-\boldsymbol{U}$ is also a copula; the latter is called the *survival copula* corresponding to C (see Nelsen (1999, p.28)).

We close the section by discussing some parametric bivariate distributions which are ordered with respect to $\leq_{\mathrm{LTD}}$ and $\leq_{\mathrm{RTI}}$.

Example 5 (*Gumbel-Barnett*). The family $\{C_\alpha, \alpha \in (0,1]\}$, where

$$C_\alpha(u,v) = uv\exp\{-\alpha \ln u \ln v\}$$

for all $(u,v) \in [0,1]^2$ and $\alpha \in (0,1]$, is said to be the Gumbel-Barnett family of copulas. The functions in this class are the survival copulas associated with Gumbel's bivariate exponential distributions, whose importance in reliability theory is well known; see Gumbel (1960) and Kotz et al. (2000) for the properties of such family of distributions. Barnett (1980) first considered this class as a family of copulas. Notice that the conditional copulas of a Gumbel-Barnett copula remain in the family; for more details on this property and its importance in applications, see Charpentier (2003) and the references therein.

This family is negatively ordered in α with respect to $\leq_{\text{LTD}}$. In fact, $C_\beta \leq_{\text{LTD}} C_\alpha$ for $\alpha \leq \beta$ if, and only if, the function $g(\alpha) = \exp\{-\alpha \ln u \ln v\} - \exp\{-\alpha \ln u' \ln v\}$ is decreasing in α for all $u < u'$ and $v \in [0,1]$, which fact is not difficult to prove by differentiation. From Corollary 4 it also follows that the family $\{D_\alpha, \alpha \in (0,1]\}$ of survival copulas associated to the Gumbel-Barnett family is negatively ordered in α with respect to $\leq_{\text{RTI}}$. ◀

Example 6 (*Ali-Mikhail-Haq*). The family $\{C_\alpha, \alpha \in [-1,1)\}$, where

$$C_\alpha(u,v) = \frac{uv}{1-\alpha(1-u)(1-v)}$$

for all $(u,v) \in [0,1]^2$ and $\alpha \in [-1,1)$, is said to be the Ali-Mikhail-Haq family of copulas. Ali et al. (1978) obtained this family of distributions as the solutions of a functional equation involving the so-called bivariate survival odds ratio, which is a natural quantity to consider in reliability theory; the interested reader is referred to Nelsen (1999, Section 3.3.2) for a simple treatment of the subject.

This family is positively ordered in α with respect to $\leq_{\text{LTD}}$. In fact, $C_\alpha \leq_{\text{LTD}} C_\beta$ for $\alpha \leq \beta$ if, and only if, the function $g(\alpha) = (1-\alpha(1-u)(1-v))^{-1} - (1-\alpha(1-u')(1-v))^{-1}$ is increasing in α for all $u < u'$ and $v \in [0,1]$, which fact is not difficult to prove by differentiation. From Corollary 4 it also follows that the family $\{D_\alpha, \alpha \in [-1,1)\}$ of survival copulas associated to the Ali-Mikhail-Haq family is positively ordered in α with respect to $\leq_{\text{RTI}}$. ◀

Example 7 (*Farlie-Gumbel-Morgenstern*). The family $\{C_\alpha, \alpha \in [-1,1]\}$, where

$$C_\alpha(u,v) = uv(1+\alpha(1-u)(1-v))$$

for all $(u,v) \in [0,1]^2$ and $\alpha \in [-1,1]$, is called the Farlie-Gumbel-Morgenstern family of bivariate copulas. We refer to Kotz et al. (2000) for a detailed discussion on the properties of the distributions in this family; we only stress here that the Farlie-Gumbel-Morgenstern copulas coincide with their corresponding survival copulas. The simple analytical form of the family has made these copulas very appealing in many fields of application; for instance, the interested reader is referred to Shaked (1975), who describes their usefulness in reliability theory and in Bayesian survey sampling, and to Conway (1984), who briefly reviews some applications in quality control and in medical studies.

This family is positively ordered in α with respect to $\leq_{\text{LTD}}$ and $\leq_{\text{RTI}}$. In fact it is not difficult to see that $g(\alpha) = C^L_{\alpha,u}(v) - C^L_{\alpha,u'}(v) = C^R_{\alpha,u}(v) - C^R_{\alpha,u'}(v) = \alpha v(1-v)(u'-u)$ increases in α. ◀

4 Relationships to other orders involving tail dependence

In this section we discuss the relationship between the orderings $\leq^*_{\text{LTD}}$ and $\leq^*_{\text{RTI}}$ and other orderings generalizing the LTD and the RTI positive dependence notions, namely the orderings proposed by Averous and Dortet-Bernadet (2000) and by Colangelo et al. (2006).

Let Λ_2 be the subclass of Δ_2 containing all bivariate distributions F for which the conditional distributions F_x^L and F_x^R are continuous and strictly increasing on their support for all $x \in \mathbb{R}$ (in particular, notice that they are continuous for all $x \in \mathbb{R}$ if, and only if, F_2 is continuous). Averous and Dortet-Bernadet (2000) introduced the following bivariate positive dependence orders. Let (X_1, X_2) and (Y_1, Y_2) be random vectors with distribution functions F and G in $\Gamma(F_1, F_2) \cap \Lambda_2$; then (X_1, X_2) is said to be smaller than (Y_1, Y_2) in the left tail decreasing order if

$$G_{x_1'}^L\left((G_{x_1}^L)^{-1}(u)\right) \leq F_{x_1'}^L\left((F_{x_1}^L)^{-1}(u)\right), \quad u \in [0,1], \tag{7}$$

whenever $x_1 \leq x_1'$, and we denote this by $(X_1, X_2) \leq^*_{\mathrm{LTD}} (Y_1, Y_2)$ or $F \leq^*_{\mathrm{LTD}} G$. If F and G satisfy

$$G_{x_1'}^R\left((G_{x_1}^R)^{-1}(u)\right) \leq F_{x_1'}^R\left((F_{x_1}^R)^{-1}(u)\right), \quad u \in [0,1], \tag{8}$$

whenever $x_1 \leq x_1'$, then (X_1, X_2) is said to be smaller than (Y_1, Y_2) in the right tail increasing order and we denote this by $(X_1, X_2) \leq^*_{\mathrm{RTI}} (Y_1, Y_2)$ or $F \leq^*_{\mathrm{RTI}} G$. Letting $F = F^\perp$, a simple calculation shows that the relations $\leq^*_{\mathrm{LTD}}$ and $\leq^*_{\mathrm{RTI}}$ are indeed generalizations of the LTD and the RTI positive dependence notions.

Averous and Dortet-Bernadet (2000) proved that $\leq^*_{\mathrm{LTD}}$ and $\leq^*_{\mathrm{RTI}}$ define reasonable positive dependence orderings as they are stronger than the PQD ordering, so that postulate P.2 is satisfied, and they also claim that postulates P.1, P.3 and P.9 hold, while the conditional nature of the orderings makes impossible to meet P.8; under some regularity conditions on the nature of the transformations, an obvious version of Theorem 1 can be stated, and this implies that P.4 and P.7 are satisfied (under such conditions, at least). It is possible to find counterexamples showing that P.5 and P.6 do not hold. Whether these orderings admit the copula representation mentioned in Section 2 and whether the assumptions on the form of the conditional distributions can be relaxed, letting conditions (7) and (8) keep defining meaningful dependence orderings, seem to be interesting open problems.

The following example proves that

$$(X_1, X_2) \leq^*_{\mathrm{LTD}} (Y_1, Y_2) \quad \not\Rightarrow \quad (X_1, X_2) \leq_{\mathrm{LTD}} (Y_1, Y_2);$$

it also shows another instance where the upper Fréchet bound fails to be the maximal element with respect to $\leq_{\mathrm{LTD}}$. Part (b) of Theorem 1 and its analogous for the orderings $\leq^*_{\mathrm{LTD}}$ and $\leq^*_{\mathrm{RTI}}$ also shows that

$$(X_1, X_2) \leq^*_{\mathrm{RTI}} (Y_1, Y_2) \quad \not\Rightarrow \quad (X_1, X_2) \leq_{\mathrm{RTI}} (Y_1, Y_2).$$

Example 8. Let C_α be a Farlie-Gumbel-Morgenstern copula with $\alpha > 0$ and C^+ be the Fréchet upper bound in the family of all copulas. Clearly $C_\alpha \in \Lambda_2$ and $C_\alpha \leq^*_{\mathrm{LTD}} C^+$. Let now $u, u', v \in [0,1]$ be such that $0 < u < u' < v < 1$; for such points, condition (4) becomes $\alpha(u' - u) \leq 0$, which is naturally false, so that $C_\alpha \not\leq_{\mathrm{LTD}} C^+$. ◀

The following example is borrowed from Colangelo et al. (2006, Example 2.16); it shows that

$$(X_1,X_2) \leq_{\mathrm{LTD}} (Y_1,Y_2) \quad \not\Rightarrow \quad (X_1,X_2) \leq^*_{\mathrm{LTD}} (Y_1,Y_2).$$

Example 9. Let (X_1,X_2) and (Y_1,Y_2) be two random vectors with distribution functions in $\Gamma_2(F_1,F_2)$, where X_1 and Y_1 are discrete random variables taking on the values 0, 1, and 2, with respective probabilities 1/8, 3/40, and 4/5, while F_2 is uniform on $[0,2]$. Define the distribution functions F and G of (X_1,X_2) and (Y_1,Y_2) through the following conditional distribution functions of X_2 given X_1, and of Y_2 given Y_1, as follows:

$$P(X_2 \leq y \big| X_1 = 0) = \begin{cases} \frac{2}{3}y, & y \in [0,.6) \\ \frac{3}{2}y - \frac{1}{2}, & y \in [.6,1) \\ 1, & y \geq 1 \end{cases}$$

$$P(Y_2 \leq y \big| Y_1 = 0) = \begin{cases} \frac{3}{2}y, & y \in [0,.4) \\ \frac{2}{3}y + \frac{1}{3}, & y \in [.4,1) \\ 1, & y \geq 1 \end{cases}$$

$$P(X_2 \leq y \big| X_1 = 1) = \begin{cases} \frac{10}{9}y, & y \in [0,.6) \\ \frac{5}{6}y + \frac{1}{6}, & y \in [.6,1) \\ 1, & y \geq 1 \end{cases}$$

$$P(Y_2 \leq y \big| Y_1 = 1) = \begin{cases} \frac{3}{2}y, & y \in [0,.4) \\ 2y - \frac{1}{5}, & y \in [.4,.6) \\ 1, & y \geq .6 \end{cases}$$

$$P(X_2 \leq y \big| X_1 = 2) = \begin{cases} \frac{5}{12}y, & y \in [0,.6) \\ \frac{5}{16}y + \frac{1}{16}, & y \in [.6,1) \\ \frac{5}{8}y - \frac{1}{4}, & y \in [1,2) \\ 1, & y \geq 2 \end{cases}$$

$$P(Y_2 \leq y \big| Y_1 = 2) = \begin{cases} \frac{1}{4}y, & y \in [0,.4) \\ \frac{1}{3}y - \frac{1}{30}, & y \in [.4,.6) \\ \frac{25}{48}y - \frac{7}{48}, & y \in [.6,1) \\ \frac{5}{8}y - \frac{1}{4}, & y \in [1,2) \\ 1, & y \geq 2 \end{cases}$$

Note that $F_0^L(y) = P(X_2 \leq y \big| X_1 = 0)$ and $G_0^L(y) = P(Y_2 \leq y \big| Y_1 = 0)$, given above. A straightforward computation gives

$$F_1^L(y) = \begin{cases} \frac{5}{6}y, & y \in [0,.6) \\ \frac{5}{4}y - \frac{1}{4}, & y \in [.6,1) \\ 1, & y \geq 1 \end{cases} \qquad G_1^L(y) = \begin{cases} \frac{3}{2}y, & y \in [0,.4) \\ \frac{7}{6}y + \frac{2}{15}, & y \in [.4,.6) \\ \frac{5}{12}y + \frac{7}{12}, & y \in [.6,1) \\ 1, & y \geq 1, \end{cases}$$

while F_2^L and G_2^L coincide with F_2. It is easy to verify that $F, G \in \Lambda_2$.

Colangelo et al. (2006) show that $F \not\leq^*_{\mathrm{LTD}} G$. On the converse, to prove $F \leq_{\mathrm{LTD}} G$, notice that it suffices to verify condition (2) for $x_1 = 0$ and $x_1' = 1$. Clearly

$$F_0^L(y) - F_1^L(y) = \begin{cases} -\frac{1}{6}y & y \in [0, .6), \\ \frac{1}{4}y - \frac{1}{4} & y \in [.6, 1), \\ 0 & \text{otherwise}, \end{cases}$$

and

$$G_0^L(y) - G_1^L(y) = \begin{cases} -\frac{1}{2}y + \frac{1}{5} & y \in [.4, .6), \\ \frac{1}{4}y - \frac{1}{4} & y \in [.6, 1), \\ 0 & \text{otherwise}, \end{cases}$$

so that the result follows by noticing that $-\frac{1}{6}y \leq -\frac{1}{2}y + \frac{1}{5}$ on the interval $[.4, .6]$. ◄

Applying again part (b) of Theorem 1 and the analogous result for the orderings $\leq^*_{\mathrm{LTD}}$ and $\leq^*_{\mathrm{RTI}}$, we obtain that

$$(X_1, X_2) \leq_{\mathrm{RTI}} (Y_1, Y_2) \quad \not\Rightarrow \quad (X_1, X_2) \leq^*_{\mathrm{RTI}} (Y_1, Y_2).$$

Colangelo et al. (2006) proposed two pairs of multivariate positive dependence stochastic orderings which, in the bivariate case, both generalize the LTD and RTI notions. They are respectively called the *lower orthant decreasing ratio order* and the *upper orthant increasing ratio order* (denoted by $\leq_{\mathrm{lodr}}$ and $\leq_{\mathrm{uoir}}$), and the *strong lower orthant decreasing ratio order* and the *strong upper orthant increasing ratio order* (denoted by $\leq_{\mathrm{slodr}}$ and $\leq_{\mathrm{suoir}}$).

We will not reproduce the definition of these orders here. We simply stress that $\leq_{\mathrm{lodr}}$ and $\leq_{\mathrm{uoir}}$ fulfill most of the postulates proposed in Section 2 for a reasonable bivariate positive dependence order. In particular, postulates P.1, P.2, P.8 and P.9 are satisfied, while P.3 partially fails as, although the lower Fréchet bound is a minimal element with respect to the orderings, the upper Fréchet bound needn't be a maximal element, so that also P.6 and P.7 cannot be met; finally, Theorems 2.1 and 2.2 in Colangelo et al. (2006) imply that postulate P.4 is fulfilled but P.5 isn't. Similarly, the orderings $\leq_{\mathrm{slodr}}$ and $\leq_{\mathrm{suoir}}$ satisfy postulates P.2, P.8 and P.9, while P.1 and P.3 fail as the orderings are not reflexive and they don't admit the Fréchet bounds as minimal and maximal elements; simple arguments, in view of Proposition 3.6 and Theorem 3.1 in Colangelo et al. (2006), respectively imply that postulates P.6 and P.7 are not fulfilled, and that postulate P.4 is satisfied but P.5 isn't. In addition, all these orderings admit the copula representation discussed in Section 2.

We close the section outlining the relationship between these orderings and the orderings discussed above. In particular, the ordering $\leq_{\mathrm{slodr}}$ is stronger than $\leq_{\mathrm{lodr}}$, $\leq_{\mathrm{LTD}}$ and $\leq^*_{\mathrm{LTD}}$ and, correspondingly, $\leq_{\mathrm{suoir}}$ is stronger than $\leq_{\mathrm{uoir}}$, $\leq_{\mathrm{RTI}}$ and $\leq^*_{\mathrm{RTI}}$. In addition, no other relationships exist between these ordering. For a proof of these statements and other related results, we refer again to Colangelo et al. (2006).

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