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Well-posed Vector Optimization Problems and Vector Variational Inequalities

Matteo Rocca*

Abstract

In this paper we introduce notions of well-posedness for a vector optimization problem and for a vector variational inequality of differential type, we study their basic properties and we establish the links among them. The proposed concept of well-posedness for a vector optimization problem generalizes the notion of well-setness for scalar optimization problems, introduced in [2]. On the other side, the introduced definition of well-posedness for a vector variational inequality extends the one given in [13] for the scalar case.

Keywords: vector optimization, vector variational inequality, well-posedness .

Mathematics Subject Classification (2000): 90C29, 90C31

1 Introduction

Well-posedness of a scalar minimization problem is a classical notion (see e.g. [5] and references therein) and plays a crucial role in the stability theory for optimization problems. The notion of well-posedness has been deeply studied in different areas of scalar optimization, such as mathematical programming, calculus of variations and optimal control (see e.g. [5]). In particular, we wish to recall the approach proposed by A.N. Tykhonov [18] in the 60's.
On the other hand, scalar variational inequalities provide a very general and suitable model for a wide range of problems, in particular equilibrium problems (see e.g. [10]). The links between variational inequalities of differential type (i.e. in which the operator involved is the gradient of a given function) and optimization problems have also been studied (see e.g. [10] and more recently [3, 4]). Furthermore, by means of Ekeland variational principle [6] a notion of well-posed scalar variational inequality has been introduced (see [13]) and its links with the concept of well-posed optimization problem have been investigated.

The notion of well-posedness for vector valued problems is less developed. However some definitions have been proposed for a vector minimization problem (see e.g. the survey by P. Loridan [11]) and some comparisons have been made between the definitions themselves

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and their scalar counterparts (see e.g. [14]).
Moreover, vector variational inequalities have been introduced in [7, 8] and developed in the last decades as a tool for vector optimization. Also a generalization of Ekeland variational principle has been proposed for the vector case (see e.g. [17]).

In this paper, we present a new notion of well-posedness for a vector optimization problem and we investigate its basic properties, showing in particular that analogously to the scalar case, optimization problems enjoing convexity properties are well-posed, according to the proposed definition. Further, we introduce a notion of well-posedness for vector variational inequalities of differential type and we investigate some links between this notion and the well-posedness of a vector optimization problem. The outline of the paper is the following. In Section 2 we recall some basics on Tykhonov well-posedness of a scalar optimization problem and well-posedness of a scalar variational inequality. In Section 3, we introduce the proposed concept of well-posedness for a vector optimization problem. Finally, Section 4 is devoted to the notion of well-posed vector variational inequality and its relations with the well-posedness of a vector optimization problem.

2 Well-posedness of scalar optimization problems and variational inequalities

Consider the scalar optimization problem:

$P(f,K) \qquad \min f(x), \qquad x \in K$

where $f : \mathbb{R}^n \to \mathbb{R}$ and K is a nonempty closed convex subset of $\mathbb{R}^n$. Recall that a sequence $\{x^k\}_{k\geq 0} \subseteq K$ is said to be minimizing for $P(f,K)$ when $f(x^k) \to \inf_K f(x)$ as $k \to +\infty$. The following definition is classical (see for references [5]):

Definition 1. *$P(f,K)$ is said to be* Tykhonov well-posed *when:*

i) $x^0 \in K$ is the unique solution of $P(f,K)$;

ii) every minimizing sequence converges to x^0.

For the sake of completeness we recall the following classical example of ill posed problem:

Example 1. *Consider problem $P(f,K)$, with $f(x) = x^2 e^{-x}$ and $K = \mathbb{R}$. Then, $P(f,K)$ is not Tykhonov well posed, since the sequence $\{x_k\} = \{k\}$ is minimizing but it does not converge to the unique minimum $x_0 = 0$.*

The next result is known (see e.g. [5]).

Proposition 1. *Let $f : K \subseteq \mathbb{R}^n \to \mathbb{R}$ be a convex function. If f has a unique global minimizer over K, then $P(f,K)$ is Tykhonov well-posed.*

The following Theorem gives an alternative characterization of Tykhonov well-posedness.

Theorem 1. *If $P(f, K)$ is Tykhonov well-posed on K, then:*

$$\operatorname{diam}\{\varepsilon - \operatorname{argmin}(f, K)\} \to 0 \quad \textit{for } \varepsilon \downarrow 0, \tag{1}$$

where $\varepsilon - \operatorname{argmin}(f, K) := \{x \in K | f(x) \leq \varepsilon + \inf_K f(x)\}$ is the set of ε-minimizers of f over K.
Moreover, if f is lower semicontinuous and bounded from below, then condition (1) implies Tykhonov well-posedness of $P(f, K)$.

The notion of Tykhonov well-posedness is strong, since one of the requirements is that problem $P(f, K)$ has a unique solution. In order to weaken this assumption, other more general notions of well-posedness have been introduced. Here we wish to recall the concept of well-setness introduced in [2]. Given a set $A \subseteq \mathbb{R}^n$, and a point $b \in \mathbb{R}^n$, we denote by $d(b, A) = \inf_{a \in A} \|b - a\|$, the distance of the point b from the set A.

Definition 2. *Problem $P(f, K)$ is said to be well-set when for every minimizing sequence $\{x^k\}_{k \geq 0} \subseteq K$ we have $d(x^k, \operatorname{argmin}(f, K)) \to 0$, where $\operatorname{argmin}(f, K)$ denotes the set of solutions of problem $P(f, K)$.*

Now, let us turn briefly our attention to scalar variational inequalities of differential type. Assume that f is differentiable on an open set containing K and denote by f' the gradient of f. We recall that a point $x^* \in K$ is a solution of a (Stampacchia) variational inequality of differential type when:

$$VI(f', K) \qquad \langle f'(x^*), y - x^* \rangle \geq 0, \qquad \forall y \in K.$$

Clearly $VI(f', K)$ is a necessary optimality condition for problem $P(f, K)$.
The following definition gives the notion of well-posed variational inequality of differential type (see e.g. [5]).

Definition 3. *The variational inequality $VI(f', K)$ is well-posed when:*

i) $T(\varepsilon) \neq \emptyset, \ \forall y \in K$;

ii) $\operatorname{diam} T(\varepsilon) \to 0, \quad \textit{if } \varepsilon \downarrow 0$;

where $T(\varepsilon) := \left\{x \in K | \langle f'(x), x - y \rangle \leq \varepsilon \|y - x\|, \ \forall y \in K\right\}$.

The link between Tykhonov well-posedness and well-posedness of $VI(f', K)$ is given by the following Theorem (see e.g. [5], [13]).

Theorem 2. *Let f be bounded from below and differentiable on an open set containing K. If $VI(f', K)$ is well-posed, then problem $P(f, K)$ is Tykhonov well-posed. The converse is true if f is convex.*

3 A notion of well-posedness in vector optimization

Consider a function $f : K \subseteq \mathbb{R}^n \to \mathbb{R}^l$ and a cone $C \subseteq \mathbb{R}^l$ that we assume to be closed, convex, pointed and with nonempty interior. In the following we deal with the vector optimization problem:

$VP(f, K)$ $\qquad$ v-min $_C f(x)$, $\qquad$ $x \in K$

where K is a nonempty closed, convex subset of $\mathbb{R}^n$.
We recall (see e.g. [12]) that a point $x^0 \in K$ is said to be an efficient (weakly efficient) solution of problem $VP(f, K)$, when $f(x) - f(x^0) \notin -C\backslash\{0\}$, $(f(x) - f(x^0) \notin -\operatorname{int} C)$, $\forall x \in K$. We will denote by $\operatorname{Eff}(f, K)$ ($\operatorname{WEff}(f, K)$) the set of efficient solutions (weakly efficient solutions) of problem $VP(f, K)$. In the sequel, we assume that $\operatorname{Eff}(f, K)$ is nonempty.

The next definition can be found in [9, 17] and extends to the vector case the notion of ε-minimizer.

Definition 4. *i) A point $x^\varepsilon \in K$ is said to be an* approximately efficient *solution of $VP(f, K)$ with respect to $c^0 \in \operatorname{int} C$ and $\varepsilon \geq 0$, when, $\forall x \in K$ it holds:*

$$f(x) - f(x^\varepsilon) + \varepsilon c^0 \notin -C\backslash\{0\}.$$

ii) A point $x^\varepsilon \in K$ is said to be a weakly approximately efficient *solution of $VP(f, K)$ with respect to $c^0 \in \operatorname{int} C$ and $\varepsilon \geq 0$, when, $\forall x \in K$ it holds:*

$$f(x) - f(x^\varepsilon) + \varepsilon c^0 \notin -\operatorname{int} C.$$

The set of solutions which fulfill Definition 4 i) is denoted by $\operatorname{Eff}_{\varepsilon c^0}(f, K)$. From the definition it follows that, for every $\varepsilon \geq 0$ and $c^0 \in \operatorname{int} C$ we have $\operatorname{Eff}(f, K) \subseteq \operatorname{Eff}_{\varepsilon c^0}(f, K)$, with equality holding if $\varepsilon = 0$. Analogously, the set of solutions that satisfy Definition 4 ii) is denoted by $\operatorname{WEff}_{\varepsilon c^0}(f, K)$.

In order to define a notion of well-posedness for problem $VP(f, K)$, we need also the concept of Hausdorff convergence of sequences of sets. Let E, F be subsets of $\mathbb{R}^n$ and define:

$$\delta(E, F) = \max\{e(E, F), e(F, E)\}$$

where $e(E, F) = \sup\limits_{a \in E} d(a, F)$.

Definition 5. *Let A_k be a sequence of subsets of $\mathbb{R}^n$. We say that A_k* converges to $A \subseteq \mathbb{R}^n$ in the sense of Hausdorff *and we write $A_k \longrightarrow A$, when $\delta(A_k, A) \to 0$.*

We can similarly define upper and lower convergence of sets in the sense of Hausdorff.

Definition 6. *A sequence of sets $A_k \subseteq \mathbb{R}^n$ is said to be* upper (resp. lower) Hausdorff convergent *to a set $A \subseteq \mathbb{R}^n$, and we write $A_k \rightharpoonup A$ ($A_k \rightharpoondown A$) when:*

$$e(A_k, A) \to 0 \quad \Big(e(A, A_k) \to 0\Big).$$

Clearly if both $A_k \rightharpoonup A$ and $A_k \rightharpoondown A$, then $A_k \longrightarrow A$.

Observe that the previous definitions can be given analogously when we consider a family of sets A_ε, $\varepsilon > 0$, instead of a sequence of sets. For a deeper exposition on the notions of set-convergence, see e.g. [16].

The next definition introduces a notion of well-posedness for vector optimization problems by means of Hausdorff upper convergence of the sets of approximately efficient solutions of problem $VP(f, K)$.

Definition 7. *The vector optimization problem $VP(f, K)$ is* Hausdorff well-posed *when:*

$$\text{Eff}_{\,\varepsilon c^0}(f, K) \rightharpoonup \text{Eff}\,(f, K), \quad \text{as } \varepsilon \downarrow 0,$$

for every $c^0 \in \text{int}\, C$.

The previous definition can be rephrased by means of appropriate minimizing sequences. If $c^0 \in \text{int}\, C$, we say that a sequence $x^k \in K$ is a c^0-minimizing sequence for $VP(f, K)$ when there exists a sequence $\varepsilon_k \downarrow 0$, such that $x^k \in \text{Eff}_{\varepsilon_k c^0}(f, K)$.
The proof of the following result is easy and we omit it.

Proposition 2. *Problem $VP(f, K)$ is Hausdorff well-posed if and only if for every $c^0 \in \text{int}\, C$ we have $d(x^k, \text{Eff}(f, K)) \to 0$, whatever the c^0-minimizing sequence x^k.*

Remark 1. Assume $l = 1$, $C = \mathbb{R}_+$ and let $f : \mathbb{R}^n \to \mathbb{R}$. Then the previous definition reduces to the notion of well-setness (Definition 2). If if particular, f admits a unique minimizer over K, then Definition 7 collapses into the notion of Tykhonov well-posedness. The idea behind the notion of Hausdorff well-posedness is to extend the characterization of Tykhonov well-posedness in Theorem 1 to the vector case, but to avoid the requirement of the uniqueness of the solution. Indeed the latter is quite unusual for vector optimization.

Another rephrasing of Definition 7 can be given in terms of upper Hausdorff continuity of a set-valued map (see e.g. [15]). Denote by $S : X \subseteq \mathbb{R}^m \rightsquigarrow \mathbb{R}^n$ a set valued map. We recall that S is said to be upper Hausdorff continuous at $x^0 \in X$ when for every neighborhood V of 0 in $\mathbb{R}^n$, there exists a neighborhood W of $x^0 \in X$, such that $S(x) \subseteq S(x^0) + V$, for every $x \in W \cap X$.
Now consider the map $S_{c^0} : \mathbb{R}_+ \rightsquigarrow \mathbb{R}^n$, defined as:

$$S_{c^0}(\varepsilon) = \{x^\varepsilon \in K : f(x) - f(x^\varepsilon) + \varepsilon c^0 \notin -C\backslash\{0\}\} = \text{Eff}_{\varepsilon c^0}(f, K).$$

Observe that clearly $S_{c^0}(0) = \text{Eff}(f, K)$.

Proposition 3. *Problem $VP(f, K)$ is Hausdorff well-posed if and only if for every $c^0 \in \text{int}\, C$, the set valued map $S_{c^0}(\varepsilon)$ is upper Hausdorff continuous at $\varepsilon = 0$.*

Proof: It follows readily from the definitions and hence is omitted. □

We now show that under convexity assumptions on the function f and compactness of the set $\text{Eff}(f, K)$, problem $VP(f, K)$ is Hausdorff well-posed.

Definition 8. *A function $f : \mathbb{R}^n \to \mathbb{R}^l$ is said to be C-*convex *over the convex set $K \subseteq \mathbb{R}^n$ (respectively* $\operatorname{int} C$*-convex) when, $\forall x, y \in K$ and $\forall t \in [0,1]$, it holds:*

$$f(tx + (1-t)y) - tf(x) - (1-t)f(y) \in -C$$
$$\Big(f(tx + (1-t)y) - tf(x) - (1-t)f(y) \in -\operatorname{int} C\Big)$$

Theorem 3. *Let f be* $\operatorname{int} C$*-convex. If* $\operatorname{Eff}(f,K)$ *is compact, then $VP(f,K)$ is Hausdorff well-posed.*

Proof: By contradiction, assume that for some $c^0 \in \operatorname{int} C$ it holds $\operatorname{Eff}_{\varepsilon c^0}(f,K) \not\to \operatorname{Eff}(f,K)$. Then $\exists \delta > 0$ and sequences $\varepsilon_k \downarrow 0$ and $x^k \in \operatorname{Eff}_{\varepsilon_k c^0}(f,K)$, such that $x^k \notin \operatorname{Eff}(f,K) + \delta\mathcal{B}$ (here $\mathcal{B}$ denotes the closed unit ball in $\mathbb{R}^n$).
For some arbitrarily chosen $x^0 \in \operatorname{Eff}(f,C)$, consider the points $tx^0 + (1-t)x^k$. For all k there exists some $t_k \in (0,1)$ such that $y^k = t_k x^0 + (1-t_k)x^k \in \operatorname{bd}\big[\operatorname{Eff}(f,K) + \delta\mathcal{B}\big]$. Hence we have:

$$f(x) - f(x^k) \notin -\varepsilon_k c^0 - C\backslash\{0\}, \quad \forall x \in K$$

and $f(x) - f(x^0) \notin -C\backslash\{0\}$. By the $\operatorname{int} C$-convexity of f, we also have:

$$-f(y^k) \in -t_k f(x^0) - (1-t_k)f(x^k) + \operatorname{int} C.$$

Hence:

$$\begin{aligned}
f(x) - f(y^k) &\in t_k f(x) - t_k f(x^0) + (1-t_k)f(x) - (1-t_k)f(x^k) + \operatorname{int} C = \\
&= t_k\big(f(x) - f(x^0)\big) + (1-t_k)\big(f(x) - f(x^k)\big) + \operatorname{int} C \subseteq \\
&\subseteq t_k\Big[-C\backslash\{0\}\Big]^c + (1-t_k)\Big[-\varepsilon_k c^0 - C\backslash\{0\}\Big]^c + \operatorname{int} C = \\
&= -\Big[C\backslash\{0\}\Big]^c + (1-t_k)\Big[-\varepsilon_k c^0 - \big(C\backslash\{0\}\big)^c\Big] + \operatorname{int} C.
\end{aligned}$$

Since $t_k \in \big[0,1\big]$, we can assume, without loss of generality, that $t_k \to \bar{t} \in \big[0,1\big]$ and also $y^k \to \bar{y} \in \operatorname{bd}\big[\operatorname{Eff}(f,C) + \delta\mathcal{B}\big]$, since this last set is compact.
We have now:

$$f(x) - f(y^k) = c^k + (1-t_k)\Big[-\varepsilon_k c^0 - \beta^k\Big] - \gamma^k$$

where $c^k \in \operatorname{int} C$, $\beta^k \in \Big[C\backslash\{0\}\Big]^c$ and $\gamma^k \in \Big[C\backslash\{0\}\Big]^c$. Therefore:

$$f(x) - f(\bar{y}) \notin -\operatorname{int} C,$$

which together with the $\operatorname{int} C$-convexity of f, leads to the conclusion that $\bar{y}$ is an efficient solution to $VP(f,K)$. This contradicts to $y \in \operatorname{bd}\big[\operatorname{Eff}(f,K) + \delta\mathcal{B}\big]$ and completes the proof. □

The $\operatorname{int} C$-convexity assumption in the previous Theorem cannot be weakened to C-convexity, as shown by the following example.

Example 2. *Let* $f : [-2, 2] \to \mathbb{R}^2$ *be defined as* $f(x) = \begin{bmatrix} x \\ \varphi(x) \end{bmatrix}$*, where:*

$$\varphi(x) = \begin{cases} (x+1)^2, & -2 \le x \le -1 \\ 0, & -1 < x < 0 \\ x^2, & 0 \le x \le 2 \end{cases}$$

and let $K = \big[-2, 2\big]$ *and* $C = \mathbb{R}^2_+$*. Then* f *is* C*-convex, but not* $\mathrm{int}\, C$*-convex and we have* $\mathrm{Eff}\,\big(f, \mathbb{R}^2_+\big) = \big[-2, -1\big]$ *and* $\mathrm{Eff}_{\,\varepsilon c^0}\,(f, K) = \big[-2, 0\big]$*,* $\forall \varepsilon > 0$*,* $\forall c^0 \in \mathrm{int}\,\mathbb{R}^2_+$*.*
It follows that the vector optimization problem corresponding to f *is not Hausdorff well-posed.*

4 Well-posed vector variational inequalities

The scalar variational inequality of differential type introduced in Section 2 has been extended to the vector case in [7]. If $f : \mathbb{R}^n \to \mathbb{R}^l$ is a function differentiable on an open set containing the closed convex set $K \subseteq \mathbb{R}^n$, we denote its Jacobian with f' and the components of f by f_i. The vector variational inequality problem (of differential type) consists in finding a point $x^0 \in K$ such that:

$$VVI(f', K) \qquad \langle f'(x^0), y - x^0 \rangle_l \notin -\mathrm{int}\, C, \qquad \forall y \in K,$$

where $\langle f'(x^0), y - x^0 \rangle_l$ stands for the vector whose components are the inner products $\langle f'_i(x^0), y - x^0 \rangle$.
It is well known that $VVI(f', K)$ is a necessary optimality condition for x^0 to be an efficient solution of problem $VP(f, K)$ (see e.g. [7]). Furthermore, if f is $\mathrm{int}\, C$-convex (resp. C-convex), $VVI(f', K)$ is a sufficient condition for x^0 to be an efficient solution (resp. weakly efficient solution) of $VP(f, K)$.
In this section, our aim is to introduce a notion of well-posedness for the vector variational inequality problem $VVI(f', K)$ and to give some links between this notion and the Hausdorff well-posedness of problem $VP(f, K)$.
To this extent we need to recall the following result that can be deduced from Theorem 1 in [9] (see also [17]) and regarded as an extension of the classical Ekeland's variational principle.

Theorem 4 ([9]). *Let* $c^0 \in \mathrm{int}\, C$*. For every* $\varepsilon > 0$ *and any element* $x^0 \in \mathrm{Eff}_{\,\varepsilon c^0}(f, K)$*, there exists* $x^\varepsilon \in \mathbb{R}^n$ *such that:*

α) $x^\varepsilon \in \mathrm{WEff}_{\varepsilon c^0}(f, K)$*;*

β) $\|x^\varepsilon - x^0\| \le \sqrt{\varepsilon}$*;*

γ) $x^\varepsilon \in \mathrm{WEff}(f_{\varepsilon c^0}, K)$.

where $f_{\varepsilon c^0}(x) := f(x) + \sqrt{\varepsilon}\|x - x^\varepsilon\| c^0$.

The next result follows from Theorem 4 and can be wieved as as an extension of Corollary 11 in [6].

Theorem 5 ([9]). *Let* $f : \mathbb{R}^n \to \mathbb{R}^l$ *be differentiable and* $c^0 \in \operatorname{int} C$*. Then* $\forall \varepsilon > 0$ *and* $x^0 \in \operatorname{Eff}_{\varepsilon c^0}(f, K)$ *there exists* $x^\varepsilon \in K$ *with:*

α') $x^\varepsilon \in \operatorname{WEff}_{\varepsilon c^0}(f, K)$;

β') $\|x^\varepsilon - x_0\| \leq \sqrt{\varepsilon}$;

γ') $\langle f'(x^\varepsilon), y - x^\varepsilon \rangle_l \notin -\sqrt{\varepsilon}\|y - x^\varepsilon\|c^0 - \operatorname{int} C$.

Now we define the following sets:

$$Z_\varepsilon(c^0) := \left\{x \in K \,:\, f(y) - f(x) \notin -\sqrt{\varepsilon}\|y - x\|c^0 - \operatorname{int} C\right\}$$

and

$$T_\varepsilon(c^0) := \left\{x \in K \,:\, \langle f'(x), y - x \rangle_l \notin -\sqrt{\varepsilon}\|y - x\|c^0 - \operatorname{int} C,\ \forall y \in K\right\}.$$

Remark 2. *Observe that when* $l = 1$ *and* $C = \mathbb{R}_+$*, then the set* $T_\varepsilon(c^0)$ *reduces to the set* $T(\varepsilon)$ *of Definition 3.*

From Theorem 4 we easily get the inclusions:

$$\operatorname{Eff}(f, K) \subseteq \operatorname{Eff}_{\varepsilon c^0}(f, K) \subseteq Z_\varepsilon(c^0) + \sqrt{\varepsilon}\mathcal{B},$$

and from Theorem 5 also:

$$\operatorname{Eff}(f, K) \subseteq T_\varepsilon(c^0) + \sqrt{\varepsilon}\mathcal{B}.$$

The next result gives a sufficient condition for Hausdorff well-posedness of $VP(f, K)$.

Theorem 6. *If, for every* $c^0 \in \operatorname{int} C$, $Z_\varepsilon(c^0) \rightharpoonup \operatorname{Eff}(f, K)$*, as* $\varepsilon \downarrow 0$*, then* $VP(f, K)$ *is Hausdorff well-posed.*

Proof: It follows from the chain of inclusions:

$$\operatorname{Eff}(f, K) \subseteq \operatorname{Eff}_{\varepsilon c^0}(f, K) \subseteq Z_\varepsilon(c^0) + \sqrt{\varepsilon}\mathcal{B}$$

□

We can now, analogously to the scalar case, state the following definition of well-posedness for the vector variational inequality $VVI(f', K)$.

Definition 9. *The variational inequality* $VVI(f', K)$ *is* Hausdorff well-posed *when for every* $c^0 \in \operatorname{int} C$*, it holds:*

$$T_\varepsilon(c^0) \rightharpoonup \operatorname{Eff}(f, K)$$

The notion in Definition 9 is motivated by the next result, which relates it to Hausdorff well-posedness of $VP(f, K)$.

Theorem 7. *If the variational inequality* $VVI(f', K)$ *is Hausdorff well-posed, then problem* $VP(f, K)$ *is Hausdorff well-posed.*

Proof: From the chain of inclusions:

$$\operatorname{Eff}(f, K) \subseteq \operatorname{Eff}_{\varepsilon c^0}(f, K) \subseteq T_\varepsilon(c^0) + \sqrt{\varepsilon}\mathcal{B}$$

the thesis follows. □

Remark 3. When $l = 1$ and f is a function which admits a unique minimizer over K, then we recover the first part of Theorem 2.

Theorem 7 can be reverted under convexity assumptions on f and a compactness hypothesis on the set K.

Theorem 8. *Let $K \subseteq \mathbb{R}^n$ be a compact convex set and $f : \mathbb{R}^n \to \mathbb{R}^l$ be a continuously differentiable* $\operatorname{int} C$*-convex function such that* $\text{Eff}\,(f, K)$ *is nonempty and compact. If $VP(f, K)$ is Hausdorff well-posed, then $VVI(f, K)$ is Hausdorff well-posed.*

Proof: By assumption we have:

$$\text{Eff}_{\,\varepsilon c^0}(f, K) \rightharpoonup \text{Eff}\,(f, K), \quad \forall c^0 \in \operatorname{int} C.$$

By contradiction, assume that $T_\varepsilon(c^0) \not\rightharpoonup \text{Eff}\,(f, K)$, for some $c^0 \in \operatorname{int} C$. Then it can always be found some positive δ and suitable sequences $\varepsilon_k \downarrow 0$ and $x^k \in T_{\varepsilon_k}(c^0)$, such that $x^k \notin \text{Eff}\,(f, K) + \delta\mathcal{B}$. Since K is compact, we can assume, without loss of generality, that $x^k \to \bar{x} \in K$. Therefore:

$$\langle f'(x^k), y - x^k\rangle_l \notin -\sqrt{\varepsilon_k}\|y - x^k\|c^0 - \operatorname{int} C, \quad \forall y \in K.$$

We can now consider the limit as $k \to +\infty$ to get:

$$\langle f'(\bar{x}), y - \bar{x}\rangle_l \notin -\operatorname{int} C, \quad \forall y \in K.$$

Since f is $\operatorname{int} C$-convex, we get $\bar{x} \in \text{WEff}(f, K) = \text{Eff}\,(f, K)$ and the latter contradict the assumption $x^k \notin \text{Eff}\,(f, K) + \delta\mathcal{B}$. □

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