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Box-constrained vector optimization: a steepest descent method without "a priori" scalarization

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Summary. In this paper a notion of descent direction for a vector function defined on a box is introduced. This concept is based on an appropriate convex combination of the "projected" gradients of the components of the objective functions. The proposed approach does not involve an "apriori" scalarization since the coefficients of the convex combination of the projected gradients are the solutions of a suitable minimization problem depending on the feasible point considered. Subsequently, the descent directions are considered in the formulation of a first order optimality condition for Pareto optimality in a box-constrained multiobjective optimization problem. Moreover, a computational method is proposed to solve box-constrained multiobjective optimization problems. This method determines the critical points of the box constrained multiobjective optimization problem following the trajectories defined through the descent directions mentioned above. The convergence of the method to the critical points is proved. The numerical experience shows that the computational method efficiently determines the whole local Pareto front.

Keywords: Multi-objective optimization problems, path following methods, dynamical systems, minimal selection.

1 Introduction

Let $\mathbf{R}^n$ be the n-dimensional real Euclidean space, $\underline{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}^n$ be a generic vector, where the superscript T means transpose. Let $\underline{x}$, $\underline{y} \in \mathbf{R}^n$, we denote by $\underline{y}^T\underline{x}$ the Euclidean scalar product, by $\|\underline{x}\| = (\underline{x}^T\underline{x})^{1/2}$ the Euclidean norm and by the symbols $\underline{x} < \underline{y}$ and $\underline{x} \leq \underline{y}$, $\underline{x}$, $\underline{y} \in \mathbf{R}^n$ the componentwise inequalities, that is: $x_i < y_i$, $x_i \leq y_i$, $i = 1, 2, \ldots, n$.

Let $B \subset \mathbf{R}^n$ be the following box:

$$B = \{ \underline{x} \in \mathbf{R}^n \mid \underline{l} \leq \underline{x} \leq \underline{u} \} \tag{1}$$

where $\underline{l} \in \mathbf{R}^n$, $\underline{u} \in \mathbf{R}^n$, $\underline{l} < \underline{u}$ are two given vectors. Finally, let $E \subset \mathbf{R}^n$ be an open set with $B \subset E$ and let $\operatorname{int} B$ denote the interior of B.

Let $\underline{F} = (F_1, F_2, \ldots, F_s)^T \in \mathbf{R}^s$, $F_i : E \subseteq \mathbf{R}^n \longrightarrow \mathbf{R}$, $i = 1, 2, \ldots, s$, be continuously differentiable functions defined on the open set E, we consider the following box-constrained multiobjective optimization problem:

$$\min_{\underline{x} \in B} \underline{F}(\underline{x}). \tag{2}$$

We consider two distinct classes of solutions for problem (2): the global ones and the local ones.

Definition 1. *A point $\underline{x}^* \in B$ is a globally Pareto optimal point when*

$$\nexists\, \underline{x} \in B \text{ with } \underline{F}(\underline{x}) \leq \underline{F}(\underline{x}^*) \text{ and } \underline{F}(\underline{x}) \neq \underline{F}(\underline{x}^*). \tag{3}$$

Definition 2. *A point $\underline{x}^*$ is a locally Pareto optimal point if there exists a neighborhood $U \subseteq \mathbf{R}^n$ of $\underline{x}^*$ such that*

$$\nexists\, \underline{x} \in B \cap U \text{ with } \underline{F}(\underline{x}) \leq \underline{F}(\underline{x}^*) \text{ and } \underline{F}(\underline{x}) \neq \underline{F}(\underline{x}^*). \tag{4}$$

Naturally every globally Pareto optimal point is also a locally Pareto optimal point. In this paper we study the locally Pareto optimal points of problem (2). The relevance of the Pareto optimal fronts is well known in economics (see, e.g., welfare theorems), finance (see, e.g., portfolio selection), engineering (see, e.g., design problems, network problems).

In this paper we formulate a necessary condition for local Pareto optimality in a box constrained optimization problem following the approach outlined in [21] for a problem without constraints. This condition allows us to introduce the notion of critical point of a function $\underline{F}$ on the box B. We construct a system of first order differential equations which is analogous to the "projected" gradient system introduced in [12],[15],[16] and [18] in the framework of box constrained scalar optimization problems. The main properties of the solution of this system are that $\underline{F}$ strictly decreases (with respect to the componentwise order) along this trajectory and that the limit points of the trajectory itself are critical points for $\underline{F}$ on the box B.

In our setting, a special convex combination of the projected gradients of the components of the objective function $\underline{F}$ plays the same role as the "projected" gradient of the objective function in the scalar case. This convex combination is chosen as the element of minimal norm of the convex hull of the "projected" gradients of the components of the objective function $\underline{F}$. In the unconstrained case this convex combination is the element of minimal norm of the set of pseudogradients as proved in [14].

The first order necessary condition proposed allows us to formulate a suitable computational method (we will refer to it as Algorithm A) based on sequences $\{\underline{x}^k\}$, $k = 0, 1, \ldots$, $\underline{x}_0 \in \operatorname{int} B$ of feasible points. This computational method arises from a suitable numerical integration of the system of differential equations introduced. We prove that the sequences $\{\underline{x}^k\}$ have limit points that are critical points for problem (2). The algorithm developed here is a kind of interior point method for vector optimization problems that does not require any "a priori" scalarization of the objectives F_i, $i = 1, 2, \ldots, s$.

The computational method presented here is based on some ideas introduced in [19] in the context of the vector optimization problems and in [12], [15], [16], [18] in the context of scalar optimization problems. We remind that several scalarization procedures have been introduced to solve multi-objective optimization problems, see for example [9], [4], [5], [22], and, more recently, [20].

Only in the last years we can find papers (see [8], [11], [13], [3], [14], [10], [19],[6], [7]) that propose computational methods to solve vector optimization problems without using any "a priori" scalarization of the original vector function. Some of these papers introduce adaptations to vector optimization of some well-known methods of scalar optimization, such as steepest descent methods [8], [10], [19], proximal methods [3], differential inclusion techniques [14], genetic algorithms [6], [7].

We note that Algorithm A is highly parallelizable since the computations of the sequences $\{\underline{x}^k\}$ starting from several initial guesses $\underline{x}_0 \in \operatorname{int} B$ are independent one from the others. Hence, by suitably choosing a set of starting points, we can approximate the whole local Pareto front of problem (2) (see Section 4).

In Section 2, we give necessary conditions for Pareto minimal points of problem (2) and we study the properties of a system of differential equations that plays a role similar to the gradient system in scalar optimization. In Section 3, we derive the computational method from the theoretical results formulated in Section 2. In Section 4, we show some numerical results obtained by applying the computational method introduced in Section 3 to solve some test problems.

2 The dynamics of a box-constrained vector optimization problem.

Let $\underline{F} = (F_1, F_2, \ldots, F_s)^T \in \mathbf{R}^s$ be a vector valued function, whose components F_i, $F_i : E \subseteq \mathbf{R}^n \longrightarrow \mathbf{R}$, $i = 1, 2, \ldots, s$, are assumed continuously differentiable on the open set E.

We denote by ∇F_i, $i = 1, 2, \ldots, s$, the gradient vector of F_i, that is:

$$\nabla F_i(\underline{x}) = \begin{pmatrix} \frac{\partial F_i}{\partial x_1} \\ \vdots \\ \frac{\partial F_i}{\partial x_n} \end{pmatrix}, \; i = 1, 2 \ldots, s, \tag{5}$$

with $J_{\underline{F}}(\underline{x})^T = (\nabla F_1(\underline{x}) | \nabla F_2(\underline{x}) | \cdots | \nabla F_s(\underline{x})) \in \mathbf{R}^{n \times s}$ the transposed matrix of the Jacobian matrix $J_{\underline{F}}(\underline{x}) \in \mathbf{R}^{s \times n}$ of $\underline{F}$ at the point $\underline{x}$.

Let $\underline{l} \in \mathbf{R}^n$ and $\underline{u} \in \mathbf{R}^n$ be the vector given in (1), we denote by $D_{\underline{l}}(\underline{x}) \in \mathbf{R}^{n \times n}$ and $D_{\underline{u}}(\underline{x}) \in \mathbf{R}^{n \times n}$, the diagonal matrices defined by:

$$\left(D_{\underline{l}}(\underline{x})\right)_{i,j} = \begin{cases} (x_i - l_i) \; i = j \\ 0 \qquad\quad i \neq j \end{cases} \quad \left(D_{\underline{u}}(\underline{x})\right)_{i,j} = \begin{cases} (u_i - x_i) \; i = j \\ 0 \qquad\quad i \neq j \end{cases}, \; \underline{l} \leq \underline{x} \leq \underline{u}, \tag{6}$$

In the following result we adapt to a box-constrained problem the known result concerning the classical first order optimality conditions (see for example [21]). In fact the necessary first order optimality condition for the unconstrained case can be obtained by replacing the product $D_{\underline{l}}(\underline{x}^*)\underline{D_{\underline{u}}}(\underline{x}^*)$ in (7) with the identity matrix.

Proposition 1. *Let $\underline{x}^* \in B$ be a local Pareto minimal point for problem (2), then there exist s nonnegative constants λ_1, λ_2,...,λ_s, with $\sum_{i=1}^{s} \lambda_i = 1$ such that we have:*

$$\sum_{i=1}^{s} \lambda_i D_{\underline{l}}(\underline{x}^*)\underline{D_{\underline{u}}}(\underline{x}^*)\nabla F_i(\underline{x}^*) = \underline{0}. \tag{7}$$

Proof. : Let $t^* > 0$ we define the following set:

$$\mathcal{F}_{t^*} = \{\, \underline{h} \in \mathbf{R}^n \mid \underline{x}^* + t\underline{h} \in B, \; \forall\, t, \, 0 < t < t^* \,\}. \tag{8}$$

Since $\underline{x}^* \in B$ is a local Pareto minimal point for problem (2) then there exists no vector $\underline{h} \in \mathcal{F}_{t^*}$, $t^* > 0$, such that $J_{\underline{F}}(\underline{x}^*)\underline{h} < \underline{0}$, that is the system of inequalities given by:

$$\begin{cases} \nabla F_1(\underline{x}^*)^T \underline{h} < 0 \\ \vdots \\ \nabla F_s(\underline{x}^*)^T \underline{h} < 0 \end{cases} \tag{9}$$

has no solution in the set $\mathcal{F}_{t^*}$ As a consequence the system

$$\begin{cases} \nabla F_1(\underline{x}^*)^T D_{\underline{l}}(\underline{x}^*) D_{\underline{u}}(\underline{x}^*)\underline{h} < 0 \\ \vdots \\ \nabla F_s(\underline{x}^*)^T D_{\underline{l}}(\underline{x}^*) D_{\underline{u}}(\underline{x}^*)\underline{h} < 0 \end{cases} \tag{10}$$

has no solutions in $\mathbf{R}^n$. By virtue of the Gordan's theorem we have that there exists $\underline{\lambda} = (\lambda_1, \ldots, \lambda_s)^T \in \mathbf{R}^n$, $\underline{\lambda} \geq \underline{0}$ and $\underline{\lambda} \neq \underline{0}$ such that the following equation holds:

$$\left[J_{\underline{F}}(\underline{x}^*) D_{\underline{l}}(\underline{x}^*) D_{\underline{u}}(\underline{x}^*)\right]^T \underline{\lambda} = \underline{0}\,. \tag{11}$$

Equation (11) implies equation (7).

A point $\underline{x} \in B$ that satisfies condition (7) will be said to be a critical point for problem (2). We denote by $K_{\underline{F},B}$ the set of critical points, i.e.,

$$K_{\underline{F},B} = \{\, \underline{x} \in B \,|\, \exists\, \lambda_i \geq 0,\ i = 1, 2, \ldots, s,\ \sum_{i=1}^{s} \lambda_i = 1, \\ \sum_{i=1}^{s} \lambda_i D_{\underline{l}}(\underline{x}) D_{\underline{u}}(\underline{x}) \nabla F_i(\underline{x}) = \underline{0} \,\}. \tag{12}$$

Let $\underline{x} \in B$. We denote by $H_{\underline{F},B}(\underline{x})$ the following set

$$H_{\underline{F},B}(\underline{x}) = \left\{ \sum_{i=1}^{s} \lambda_i D_{\underline{l}}(\underline{x}) D_{\underline{u}}(\underline{x}) \nabla F_i(\underline{x}),\ |\ \lambda_i \geq 0,\ i = 1, 2, \ldots, s,\ \sum_{i=1}^{s} \lambda_i = 1, \right\} \tag{13}$$

We denote by $m(H_{\underline{F},B}(\underline{x}))$ the minimal norm element belonging to $H_{\underline{F},B}(\underline{x})$ and finally let $s_{\underline{F},B} : B \subset \mathbf{R}^n \to R_+$ be the function defined as follows:

$$s_{\underline{F},B}(\underline{x}) = \|m(H_{\underline{F},B}(\underline{x}))\| = \tag{14}$$
$$\min \left\{ \left\| \sum_{i=1}^{s} \lambda_i D_{\underline{l}}(\underline{x}) \underline{D_{\underline{u}}}(\underline{x}) \nabla F_i(\underline{x}) \right\| \,:\, \lambda_i \in \mathbf{R},\ \lambda_i \geq 0,\ \sum_{i=1}^{s} \lambda_1 = 1 \right\}.$$

Now we study the regularity properties of the set-valued map $H_{\underline{F},B}(\underline{x})$.

Lemma 1. *The set-valued map $\underline{x} \to H_{\underline{F},B}(\underline{x})$ is a continuous map for $\underline{x} \in B$.*

Proof. We first prove the upper semi-continuity of the map. Let us denote by $\mathrm{Graph}(H_{\underline{F},B})$ the following set:

$$\mathrm{Graph}(H_{\underline{F},B}) = \{\, (\underline{x}, \underline{v}) \in B \times \mathbf{R}^n \mid \underline{v} \in H_{\underline{F},B}(\underline{x}) \,\}. \tag{15}$$

Since $\underline{F}$ is a continuously differentiable function, for any sequence $\{\, (\underline{x}^j, \underline{v}^j) \,\} \subset \mathrm{Graph}(H_{\underline{F},B})$ such that $(\underline{x}^j, \underline{v}^j) \to (\hat{\underline{x}}, \hat{\underline{v}})$ as $j \to +\infty$, we have $\hat{\underline{v}} \in H_{\underline{F},B}(\underline{x})$. The proof of the upper semi-continuity follows using Corollary 1, p. 42, [1]. Now we prove the lower semi-continuity. We have to show that for any $\epsilon > 0$ and $\underline{v} \in H_{\underline{F},B}(\underline{x})$ there exists $\delta > 0$ such that we have:

$$H_{\underline{F},B}(\underline{y}) \cap U_\epsilon(\underline{v}) \neq \emptyset,\ \forall \underline{y} \in U_\delta(\underline{x}), \tag{16}$$

where $U_\epsilon(\underline{v})$ and $U_\delta(\underline{x})$ denote neighborhoods of $\underline{v}$ and $\underline{x}$ respectively. Let $\underline{v} = \sum_{j=1}^{m} \lambda_j D_{\underline{u}}(\underline{x}) D_{\underline{l}}(\underline{x}) \nabla F_j(\underline{x})$. Since $\underline{F}$ is continuously differentiable and $D_{\underline{u}}$, $D_{\underline{l}}$ are continuous, there exists $\delta > 0$ such that we have:

$$\|\underline{v} - \sum_{j=1}^{m} \lambda_j D_{\underline{u}}(\underline{y}) D_{\underline{l}}(\underline{y}) \nabla F_j(\underline{y})\| < \epsilon,\ \ \forall\, \underline{y} \in U_\delta(\underline{x}). \tag{17}$$

This concludes the proof.

The previous result allows us to study the continuity of the function $m(H_{\underline{F},B}(\underline{x}))$.

Lemma 2. *The function $m(H_{\underline{F},B}(\underline{x}))$ is a continuous function in B.*

Proof. The set-valued map $H_{\underline{F},B}(\underline{x})$ is a continuous map with closed convex values. By the minimal selection theorem (see [1], Theorem 1 p. 70), the thesis follows.

Now we use the tools introduced above in order to obtain a system of differential equation that rules the dynamics of the underlying multiobjective optimization problem. We begin to characterize the critical points as the zeroes of the function $s_{\underline{F},B}(\underline{x})$. The following Lemma follows immediately from the definition of critical point (see (12)).

Lemma 3. *Let $\underline{x} \in B$. The point $\underline{x}$ is a critical point for problem (2) if and only if $s_{\underline{F},B}(\underline{x}) = 0$.*

The next result shows how the image of the point $\underline{x}$ through the function $\underline{w} = -m(H_{\underline{F},B}(\underline{x}))$ can be considered as a descent direction for the function $\underline{F}$ at the point $\underline{x}$.

Lemma 4. *Let $\underline{w} = -m(H_{\underline{F},B}(\underline{x}))$ then we have:*

$$J_{\underline{F}}(\underline{x}) D_{\underline{l}}(\underline{x}) D_{\underline{u}}(\underline{x}) \underline{w} < \underline{0},\ \underline{x} \in B. \tag{18}$$

Proof. Since $-\underline{w}$ is a minimal norm vector in $H_{\underline{F},B}(\underline{x})$ the Best Approximation Theorem ensures that

$$-\underline{w}^T(-\underline{w} - D_{\underline{l}}(\underline{x}) D_{\underline{u}}(\underline{x}) \nabla F_i(\underline{x})) \leq 0,\ i = 1, 2, \ldots, s, \tag{19}$$

that is:

$$\underline{w}^T D_{\underline{l}}(\underline{x}) D_{\underline{u}}(\underline{x}) \nabla F_i(\underline{x}) < -\left[s_{\underline{F},B}(\underline{x})\right]^2,\ i = 1, 2, \ldots, s. \tag{20}$$

Rewriting equation (20) in vectorial form we obtain:

$$J_{\underline{F}}(\underline{x}) D_{\underline{l}}(\underline{x}) D_{\underline{u}}(\underline{x}) \underline{w} < -\left[s_{\underline{F},B}(\underline{x})\right]^2 \underline{e},\ \underline{x} \in B, \tag{21}$$

where $\underline{e} = (1, 1, \ldots, 1)^T \in \mathbf{R}^s$. Equation (21) concludes the proof.

Let us consider the following Cauchy problem:

$$\begin{cases} \frac{dx}{dt} = D_{\underline{u}}(\underline{x}(t)) D_{\underline{l}}(\underline{x}(t)) \underline{w}(\underline{x}(t)), \\ \underline{x}(0) = \underline{x}_0 \in int\, B, \end{cases} \tag{22}$$

where $\underline{w}(\underline{x}) = -m(H_{\underline{F},B}(\underline{x}))$. In the sequel of this section we will study the trajectories solutions of (22). We will prove that their limit values are critical points of $\underline{F}$ in B.

Theorem 1. *Let $\underline{x}(t)$ be a solution of the Cauchy problem (22). Then we have:*

(a) $\underline{x}(t)$ exists for $t \in [0, +\infty)$;
(b) $\underline{x}(t) \in B$, $t \in [0, +\infty)$;

(c) $\underline{F}$ *is strictly decreasing along the trajectory solution of (22) that does not cross a critical point in* $K_{\underline{F},B}$.
(d) Any limit value of $\underline{x}(t)$ *when* $t \to +\infty$ *is a critical point of* $\underline{F}$.

Proof. Assertion (a) follows from the regularity of the function $\underline{F}$, Lemma 1, Lemma 2, the standard existence theorem for the Cauchy problem and the fact that the trajectory $\underline{x}(t)$ belongs to a compact set for each t (see [17], Corollary 2, p. 91). Assertion (b) follows from the fact that $\underline{x}_0 \in \operatorname{int} B$ and $dx_i(t)/dt = 0$ when $x_i = l_i$ or $x_i = u_i$ for some index i. An easy computation shows that the following relation holds:

$$\frac{d\underline{F}(\underline{x}(t))}{dt} = J_{\underline{F}}(\underline{x}(t))\frac{d\underline{x}(t)}{dt} = J_{\underline{F}}(\underline{x}(t))D_{\underline{u}}(\underline{x}(t))D_{\underline{l}}(\underline{x}(t))\underline{w}(\underline{x}(t)), \tag{23}$$

hence using Lemma 4 assertion (c) follows.

Finally, we prove assertion (d). Let Λ be the limit class of $\underline{x}(t)$ when $j \to +\infty$. Let $\underline{x}^* \in \Lambda$, then there exists a subsequence $\{ t_j \}_{j \in \mathbf{N}}$ such that $t_j \to +\infty$ when $j \to +\infty$ and $\lim_{j \to +\infty} \underline{x}(t_j) = \underline{x}^*$.

By (c), the set $\{ \underline{F}(\underline{x}(t)) : t \geq 0 \}$ is totally ordered and $\underline{F}(\underline{x}^*)$ is its greatest lower bound. Let us consider the orbit of $\underline{x}^*$, that is the image in $\mathbf{R}^n$ of the solution $\underline{y}(s)$ of the Cauchy problem given by:

$$\begin{cases} \frac{d\underline{y}}{ds} = -D_{\underline{u}}(\underline{y}(s))D_{\underline{l}}(\underline{y}(s))\underline{w}(\underline{y}(s)), \\ \underline{y}(0) = \underline{x}^*. \end{cases} \tag{24}$$

It is easy to see that every point of the orbit belongs to the limit class Λ. Indeed we have:

$$\lim_{j \to +\infty} \underline{x}(t_j + s) = \underline{y}(s), \, \forall s \geq 0. \tag{25}$$

Moreover, since $\underline{F}(\underline{z}) = \underline{F}((\underline{x}^*))$ for every $\underline{z} \in \Lambda$, by (24) we obtain that $\underline{F}$, restricted to the orbit of $\underline{x}^*$, is constant. Hence we have

$$J_{\underline{F}}(\underline{x}^*)D_{\underline{u}}(\underline{x}^*)D_{\underline{l}}(\underline{x}^*)\underline{w}(\underline{x}^*) = 0. \tag{26}$$

Assertion (d) follows by Lemma 3.

3 A computational method to find locally Pareto optimal fronts

In order to develop a computational method we focus here on the main steps of our approach to the box constrained problem obtained in the previous section. Starting from the interior of the box B, a descent trajectory can be found as a solution of the following system of differential equations:

$$\frac{d\underline{x}}{dt} = -\sum_{i=1}^{s} \hat{\lambda}_i(\underline{x}(t)))D_{\underline{l}}(\underline{x}(t))^2 D_{\underline{u}}(\underline{x}(t))^2 \nabla F_i(\underline{x}(t)), \tag{27}$$

where $\hat{\lambda}_i(\underline{x}(t))$, $i = 1, 2, \ldots, s$ are theirselves the solutions of the scalar optimization problem involved in the definition of $s_{\underline{F},B}(\underline{x})$. Indeed, for a fixed $\underline{x} \in \operatorname{int} B$, the quantities $\hat{\lambda}_i$, $i = 1, 2, \ldots, s$ are the solutions of the following problem:

$$\min_{\substack{\lambda_i \in \mathbf{R} \\ i=1,2,\ldots,s}} \left\| \sum_{i=1}^{s} \lambda_i D_{\underline{u}}(\underline{x}) D_{\underline{l}}(\underline{x}) \nabla F_i(\underline{x}) \right\|, \tag{28}$$

subject to

$$\begin{aligned} &\lambda_i \geq 0,\ i = 1, 2, \ldots, s, \\ &\textstyle\sum_{i=1}^{s} \lambda_i = 1. \end{aligned} \tag{29}$$

The notation $\hat{\lambda}_i(\underline{x})$ is used to underline the dependence of these nonnegative coefficients on the point $\underline{x}$.

As proved in Theorem 1, when $\underline{x}(0) \in \operatorname{int} B$, the accumulation points of the trajectories solution of (27) belong to $K_{\underline{F},B}$, i.e. they are critical points of the box-constrained multi-objective optimization problem (2). Here we develop a computational method to determine these accumulation points. In order to provide a practical tool to find the locally Pareto optimal solutions of problem (2), we approximate the trajectories of the initial value problem (27) and the solutions of the scalar optimization problem (28).

Let us introduce some further notation. We denote by $\underline{\lambda}$ the vector $(\lambda_1, \lambda_2, \ldots, \lambda_s)^T$, by Σ and Γ the sets $\Sigma = \{\, \underline{\lambda} \in \mathbf{R}^s \mid \sum_{i=1}^{s} \lambda_i = 1 \,\}$ and $\Gamma = \{\, \underline{\lambda} \in \mathbf{R}^s \mid \lambda \geq 0 \,\}$, by $M_{\underline{F}}(\underline{x}) \in \mathbf{R}^{s \times s}$, the matrix:

$$M_{\underline{F}}(\underline{x}) = J_{\underline{F}}(\underline{x}) D_{\underline{l}}(\underline{x})^2 D_{\underline{u}}(\underline{x})^2 J_{\underline{F}}(\underline{x})^T,\ \underline{x} \in B, \tag{30}$$

by $D(\underline{\lambda}) \in \mathbf{R}^{s \times s}$ the diagonal matrix whose diagonal entries are λ_1, $\lambda_2, \ldots, \lambda_s$ and by $I \in \mathbf{R}^{s \times s}$ the s-dimensional identity matrix. Furthermore, we consider the subset of the cartesian product $\mathbf{R}^n \times \mathbf{R}^s$ defined by:

$$\tilde{K}_{\underline{F},B} = \{\, (\underline{x}, \underline{\lambda}) \in B \times (\Sigma \cap \Gamma),\ \mid\ \sum_{i=1}^{s} \lambda_i \nabla F_i(\underline{x}) = \underline{0} \,\}. \tag{31}$$

We will consider the vector functions defined by:

$$\underline{v}(\underline{x}, \underline{\lambda}) = \sum_{i=1}^{s} \lambda_i \nabla F_i(\underline{x}), \tag{32}$$

$$\underline{z}(\underline{x}, \underline{\lambda}) = (I - \underline{e}\underline{e}^T D(\underline{\lambda})) M_{\underline{F}}(\underline{x}) \underline{\lambda}, \tag{33}$$

where $\underline{e} = (1, 1, \ldots, 1)^T$ and $M_{\underline{F}}(\underline{x})$ is the matrix defined in (30). Finally, we introduce the quantity h given by:

$$h = \min\{\delta, \min_{(\underline{x},\underline{\lambda}) \in B \times (\Sigma \cap \Gamma)} \psi(\underline{x}, \underline{\lambda})\} \tag{34}$$

where δ is a positive constant sufficiently small and ψ is the function given by:

$$\psi(\underline{x},\underline{\lambda}) = \min\left\{ \min_{i=1,\ldots,n} \frac{1}{(x_i-l_i)(u_i-x_i)^2|v_i(\underline{x},\underline{\lambda})|}\ , \right.$$
$$\left. \min_{i=1,\ldots,n} \frac{1}{(x_i-l_i)^2(u_i-x_i)|v_i(\underline{x},\underline{\lambda})|}, \min_{i=1,\ldots,s} \frac{1}{|z_i(\underline{x},\underline{\lambda})|} \right\}, \quad (35)$$

Note that the quantity h is well defined since $\underline{F}$ is a continuously differentiable vector function in E with $B \subset E$ and the set $B \times (\Sigma \cap \Gamma)$ is a compact set. Moreover it is easy to see that $\underline{x} \in K_{\underline{F},B}$ if and only if there exists $\lambda \in \Sigma \cap \Gamma$ such that $(\underline{x},\underline{\lambda}) \in \tilde{K}_{\underline{F},B}$. Now we can define the map $\mathcal{A} : B \times (\Sigma \cap \Gamma) \to \mathbf{R}^n \times \mathbf{R}^s$ as $(\underline{y},\underline{\xi})^T = \mathcal{A}(\underline{x},\underline{\lambda})$ where

$$\begin{array}{l} \underline{y} = \underline{x} - h \sum_{i=1}^{s} \lambda_i D_{\underline{u}}(\underline{x})^2 D_{\underline{l}}(\underline{x})^2 \nabla F_i(\underline{x}), \\ \underline{\xi} = \underline{\lambda} - h^2\, D(\underline{\lambda})(I - \underline{e}\underline{e}^T D(\underline{\lambda})) M_{\underline{F}}(\underline{x})\underline{\lambda}, \end{array} \quad (36)$$

where h is given by (34).

We consider the pair of sequences $\{\,\underline{\lambda}^k\,\}$ $\{\,\underline{x}^k\,\}$ defined as follows

$$\begin{array}{l} \underline{x}^{k+1} = \underline{x}^k - h_k \sum_{i=1}^{s} \lambda_i^k D_{\underline{u}}(\underline{x}^k)^2 D_{\underline{l}}(\underline{x}^k)^2 \nabla F_i(\underline{x}^k),\ k = 0, 1, \ldots, \\ \underline{\lambda}^{k+1} = \underline{\lambda}^k - \mu_k\, D(\underline{\lambda}^k)(I - \underline{e}\underline{e}^T D(\underline{\lambda}^k)) M_{\underline{F}}(\underline{x}^k)\underline{\lambda}^k, k = 0, 1, \ldots, \end{array} \quad (37)$$

where $\mu_k = h^2$ and $h_k = h$ are step sizes. We can immediately observe that $(\underline{x}^{k+1},\underline{\lambda}^{k+1}) = \mathcal{A}(\underline{x}^k,\underline{\lambda}^k)$.

We propose a computational method that can be described by the following steps:

Step 1. Choose a set of initial points $\underline{x}^0$ in the interior of the box B and set $\underline{\lambda}^0 = \frac{1}{s}\underline{e}$ and $k = 0$.

Step 2. If $s_{\underline{F}}(\underline{x}^k) \leq \epsilon$ stop (i.e.: there exists $\hat{\underline{x}} \in K_{\underline{F},B}$ such that $\|\underline{x}^k - \hat{\underline{x}}\| < \eta(\epsilon)$ where $\eta(\epsilon)$ is a positive function such that $\lim_{\epsilon\to 0} \eta(\epsilon) = 0$), otherwise go to Step 3.

Step 3. Compute $\underline{x}^{k+1}$ and $\underline{\lambda}^{k+1}$ using (37) where $h_k = h$ and $\mu_k = h^2$

Step 4. go to Step 2.

In the sequel we refer to this procedure as Algorithm A. The following lemma shows some properties of the map $\mathcal{A}$ necessary to prove the convergence of Algorithm A.

Lemma 5. *The map $\mathcal{A}$ defined in (36) has the following properties:*

(a) $\mathcal{A}$ maps $B \times (\Sigma \cap \Gamma)$ to $B \times (\Sigma \cap \Gamma)$;

(b) $\mathcal{A}$ is a closed map at every $(\underline{x},\underline{\lambda}) \in B \times (\Sigma \cap \Gamma)$ that is, if the relations $\lim_{k\to+\infty}(\underline{x}^k,\underline{\lambda}^k) = (\underline{x},\underline{\lambda})$, $(\underline{y}^k,\underline{\xi}^k)^T = \mathcal{A}(\underline{x}^k,\underline{\lambda}^k)$ and $\lim_{k\to+\infty}(\underline{y}^k,\underline{\xi}^k) = (\underline{y},\underline{\xi})$ hold they imply that $(\underline{y},\underline{\xi})^T = \mathcal{A}(\underline{x},\underline{\lambda})$;

(c) let $(\underline{x},\underline{\lambda}) \notin \tilde{K}_{\underline{F},B}$ *and* $(\underline{y},\underline{\xi})^T = \mathcal{A}(\underline{x},\underline{\lambda})$ *we have:*

$$\underline{\xi}^T \underline{F}(\underline{y}) < \underline{\lambda}^T \underline{F}(\underline{x}). \tag{38}$$

Proof. The proof of assertion (a) follows noting that when $(\underline{x},\underline{\lambda}) \in B \times (\Sigma \cap \Gamma)$ using (36) and (34) it is easy to see that we have:

$$\begin{aligned} &u_i - y_i = (u_i - x_i)\left(1 - h(u_i - x_i)(x_i - l_i)^2 v_i(\underline{x},\underline{\lambda})\right) \geq \\ &(u_i - x_i)\left(1 - h(u_i - x_i)(x_i - l_i)^2 |v_i(\underline{x},\underline{\lambda})|\right) \geq 0,\ i = 1,2,\ldots,n, \end{aligned} \tag{39}$$

$$\begin{aligned} &y_i - l_i = (y_i - l_i)\left(1 - h(u_i - x_i)^2 (x_i - l_i) v_i(\underline{x},\underline{\lambda})\right) \geq \\ &(x_i - l_i)\left(1 - h(u_i - x_i)^2 (x_i - l_i) |v_i(\underline{x},\underline{\lambda})|\right) \geq 0,\ i = 1,2,\ldots,n, \end{aligned} \tag{40}$$

and

$$\xi_i = \lambda_i(1 - h^2 z_i(\underline{x},\underline{\lambda})) \geq \lambda_i(1 - h^2 |z_i(\underline{x},\underline{\lambda})|) \geq 0,\ i = 1,2,\ldots,s. \tag{41}$$

Finally, we note that when $\underline{\lambda} \in \Sigma$ then $\underline{e}^T D(\underline{\lambda})\underline{e} = \underline{e}^T \underline{\lambda} = 1$ and we have:

$$\underline{e}^T D(\underline{\lambda})(I - \underline{e}\underline{e}^T D(\underline{\lambda})) M_{\underline{F}}(\underline{x})\underline{\lambda} = 0. \tag{42}$$

This concludes the proof of assertion (a).

The proof of assertion (b) is a consequence of definition (36) and of the regularity of the function $\underline{F}$.

Let us prove assertion (c). Using equations (36) we obtain

$$\begin{aligned} &\underline{\xi}^T \underline{F}(\underline{y}) = \\ &\left(\underline{\lambda} - h^2\, D(\underline{\lambda})(I - \underline{e}\underline{e}^T D(\underline{\lambda})) M_{\underline{F}}(\underline{x})\underline{\lambda}\right)^T F\left(\underline{x} - h D_{\underline{u}}(\underline{x})^2 D_{\underline{l}}(\underline{x})^2 J_{\underline{F}}(\underline{x})^T \underline{\lambda}\right) = \\ &\underline{\lambda}^T \underline{F}(\underline{x}) - h \| D_{\underline{l}}(\underline{x}) D_{\underline{u}}(\underline{x}) J_F(\underline{x})^T \underline{\lambda} \|^2 + o(h),\ h \to 0, \end{aligned} \tag{43}$$

where $o(\cdot)$ is the Landau symbol. The proof of (33) follows from equation (43) since

$$\| D_{\underline{l}}(\underline{x}) D_{\underline{u}}(\underline{x}) J_F(\underline{x})^T \underline{\lambda} \| > 0 \tag{44}$$

when $(\underline{x},\underline{\lambda}) \notin \tilde{K}_{\underline{F},B}$ and

$$\| D_{\underline{l}}(\underline{x}) D_{\underline{u}}(\underline{x}) J_F(\underline{x})^T \underline{\lambda} \| = 0 \tag{45}$$

when $(\underline{x},\underline{\lambda}) \in \tilde{K}_{\underline{F},B}$.

The following Lemma shows that the sequences $\{\, \underline{x}^k \,\}$ and $\{\, \underline{\lambda}^k \,\}$ belong, respectively, to the interior of B and to the intersection of Σ with the interior of Γ, for every positive integer k.

Lemma 6. *Let* $\{\underline{x}^k\}_{k=0,1,\ldots}$ *and* $\{\underline{\lambda}^k\}_{k=0,1,\ldots}$ *be the sequences generated by Algorithm A. Then we have* $\underline{x}^k \in \operatorname{int} B$, $k = 1, 2, \ldots$, $\underline{\lambda}^k \in \Sigma \cap \operatorname{int}\Gamma$, $k = 0, 1, \ldots$.

Proof. As proved in Lemma 5, since $\underline{x}^0 \in \operatorname{int} B$ and $(D_{\underline{l}}(\underline{x}^k))_{i,i} = 0$ when $x_i^k = l_i$ and $(D_{\underline{u}}(\underline{x}^k))_{i,i} = 0$ when $x_i^k = u_i$, from equations (37) we have that $\underline{x}^k$ belongs to B . Similarly, since $\underline{\lambda}^0 \in \operatorname{int}\Gamma$ and $(D(\underline{\lambda}^k))_{i,i} = 0$ when $\lambda_i^k = 0$, $\underline{\lambda}^k$ belongs to Γ . Let $\underline{\lambda} \in \Sigma$, then $\underline{e}^T D(\underline{\lambda}^k)\underline{e} = \underline{e}^T \underline{\lambda}^k = 1$ and we have:

$$\underline{e}^T D(\underline{\lambda}^k)(I - \underline{e}\underline{e}^T D(\underline{\lambda}^k))M_{\underline{F}}(\underline{x}^k)\underline{\lambda}^{=}0. \tag{46}$$

From equation (37) and (46) we have:

$$\underline{e}^T \underline{\lambda}^{k+1} = \underline{e}^T \underline{\lambda}^k - h_k \underline{e}^T D(\underline{\lambda}^k)(I - \underline{e}\underline{e}^T D(\underline{\lambda}^k))M_{\underline{F}}(\underline{x}^k)\underline{\lambda}^k = \underline{e}^T \underline{\lambda}^k, \tag{47}$$

hence $\underline{\lambda}^k \in \Sigma$ for $k = 1, 2, \ldots$ when $\underline{\lambda}^0 \in \Sigma$.

Using the previous lemmas, we can prove the convergence of algorithm A to a critical point for problem (2).

Theorem 2. *Let* $\{\underline{x}^k, \underline{\lambda}^k\}$, $k = 0, 1, \ldots$ *be an infinite sequence generated by Algorithm A with* $\underline{x}^0 \notin K_{\underline{F},B}$. *We have:*

(a_1) *the sequence* $\{\underline{x}^k, \underline{\lambda}^k\}$ *has at least a feasible accumulation point* $\hat{\underline{x}}$, $\hat{\underline{\lambda}}$;

(b_1) $\{\underline{\lambda}^{k^T} \underline{F}(\underline{x}^k)\}$, $k = 0, 1, \ldots$ *is monotonically strictly decreasing sequence, that is:*

$$\underline{\lambda}^{k+1^T} \underline{F}(\underline{x}^{k+1}) < \underline{\lambda}^{k^T} \underline{F}(\underline{x}^k); \tag{48}$$

(c_1) *each limit point* $(\hat{\underline{x}}, \hat{\underline{\lambda}})$ *of* $\{\underline{x}^k, \underline{\lambda}^k\}$ *belongs to* $\tilde{K}_{\underline{F},B}$.

Proof. The proof of assertion (a_1) follows from Lemma 6 and the fact that B and $\Lambda \cap \Sigma$ are compact sets.

The proof of assertion (b_1) is a consequence of definition (37) and the fact that the map $\mathcal{A}$ is a closed map satisfying assertion (c) of Lemma 5.

Now we prove assertion (c_1). Let $(\hat{\underline{\lambda}}, \hat{\underline{x}})$ be an accumulation point of the sequence $\{\underline{x}^k, \underline{\lambda}^k\}$, that is there exists a subsequence k_j such that we have:

$$\lim_{j \to +\infty} x^{k_j} = \hat{\underline{x}}, \quad \lim_{j \to +\infty} \underline{\lambda}^{k_j} = \hat{\underline{\lambda}}. \tag{49}$$

Using equations (37), assertion (b_1), Lemma 5 and Theorem 7.2.3 p. 249 [2](with descent function given by $\underline{\lambda}^T \underline{F}(\underline{x})$) the proof follows.

4 Numerical experiments

In this section we validate the numerical method named Algorithm A proposed in Section 3 on several bi-criteria optimization problems.

The numerical method has been implemented in Matlab on a Pentium M 1.6GHz in double precision arithmetic. For each test problem we consider N_{tot} sequences starting from N_{tot} initial guesses $\underline{x}_{0,i} \in int\, B$, $i = 1, 2, \ldots, N_{tot}$. In particular, the starting points $\underline{x}_{0,i}$, $i = 1, 2 \ldots, N_{tot}$ are chosen equally spatially distributed or randomly uniformly distributed on the box B. We note that Algorithm A is well suited for parallel computing since we can compute the N_{tot} sequences independently one from another.

The first two test problems belong to a class of two-objective optimization problems proposed by K. Deb in [6].
Test 1) We have two objective functions (i.e.: $s = 2$) and two spatial variables (i.e.: $n = 2$). The objective functions are given by:

$$f_1(\underline{x}) = x_1 \qquad\qquad f_2(\underline{x}) = \psi(x_2)/x_1\ , \tag{50}$$

where

$$\psi(x_2) = 2 - 0.8\ e^{-\left(\frac{x_2-0.6}{0.4}\right)^2} -\ e^{-\left(\frac{x_2-0.2}{0.04}\right)^2}, \tag{51}$$

and the box constraint is given by:

$$B = \{\ \ \underline{x} = (x_1, x_2), \quad |\ \ 0.1 \le x_1 \le 1, \quad 0 \le x_2 \le 1\ \ \}. \tag{52}$$

The function ψ has a global minimizer at $x_2 \approx 0.2$ and has a local minimizer at $x_2 \approx 0.6$. The global minimizer of the function ψ has a narrow attraction region when compared with the attraction region of its local minimizer. This feature makes it a very interesting test problem.
We have a convex global Pareto optimal front corresponding to the global minimizer of the function ψ, that is the set given by:

$$Global : \mathcal{P}_g = \{(x_1, x_2)\ |\, x_2 \approx 0.2, \quad 0.1 \le x_1 \le 1\}\,. \tag{53}$$

Moreover we have a convex local Pareto optimal front whose elements are those local Pareto optimal points that are not included in $\mathcal{P}_g$. This set corresponds to the local minimizer of the function ψ, that is the set given by:

$$Local : \mathcal{P}_l = \{(x_1, x_2)\ |\ x_2 \approx 0.6,\ \ 0.1 \le x_1 \le 1\}\,. \tag{54}$$

Figure 2(a) shows the numerical approximations of the local and global Pareto fronts determined by Algorithm A starting from $N_{tot} = 400$ points distributed in the interior of the box B as shown in Figure 1(a). Figures 1(b) and 2(b) show the values of the objective functions in the f_1-f_2 plane.

Test 2 We have $s = 2$, $n = 2$ and the following objective functions:

$$F_1(\underline{x}) = f_1(x_1) \qquad\qquad F_2(\underline{x}) = \psi(x_2) r(x_1, x_2),$$

$$r(x_1, x_2) = 1 - \left(\frac{f_1(x_1)}{\psi(x_2)}\right)^{\alpha} - \frac{f_1(x_1)}{\psi(x_2)} \sin(2\pi q f_1(x_1)),$$

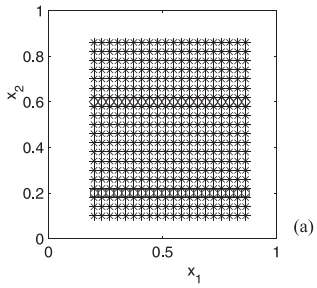

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Fig. 1. Starting points (a); Objective functions values (b)

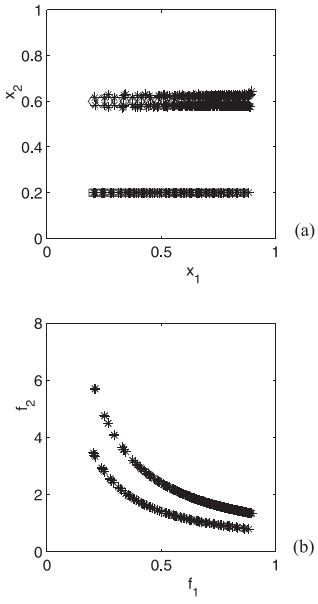

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Fig. 2. Accumulation points (a); Objective functions values (b)

and we choose $f_1(x_1) = x_1$, $\psi(x_2) = 1 + 10x_2$, $\alpha = 2$, $q = 4$. The box constraints are given by

$$B = \{ \ \underline{x} = (x_1, x_2), \quad | \ 0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 1 \ \}.$$

This test problem is interesting since the Pareto-optimal front is not a connected set.

Figure 3 shows a part of the graph $r(x_1, 0)$ versus x_1, $x_1 \in [0, 1]$. In particular Figure 3 shows the parts of the graph $r(x_1, 0)$ versus x_1 where $r(x_1, 0)$ is a non-increasing function of x_1. A point $(x_1, 0)$ belongs to the global Pareto optimal front when the point $(x_1, r(x_1, 0))$ belongs to the dashed line represented in Figure 3. A point $(x_1, 0)$ is a local (non global) Pareto optimal point front when the point $(x_1, r(x_1, 0))$ belongs to the solid line represented in Figure 3. Hence we have a non connected Pareto optimal front. Furthermore note that the points $(0, x_2)$, $x_2 \in [0, 1]$ are global minimizers of the function F_1, but only the point $(0, 0)$ belongs to the global Pareto-optimal front. Figure 4

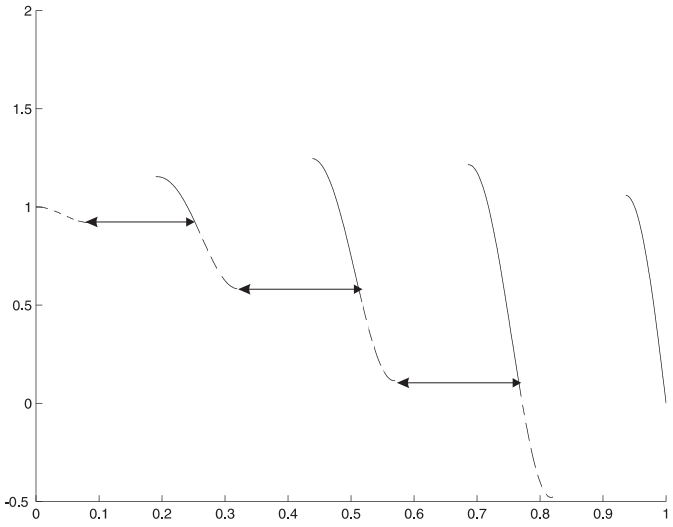

Fig. 3. $r(x_1, 0)$ versus x_1

and Figure 5 show the numerical results obtained applying Algorithm A to solve *Test 2*. Figure 4 shows the starting points ($N_{tot} = 1600$) and Figure 5 shoes the accumulation points of the sequence defined by Algorithm A. Note that Figures 4(a) and 5(a) show the points in the x_1-x_2 plane and Figure 5(a) and 5(b) show the values of the objective functions in the plane f_1-f_2.

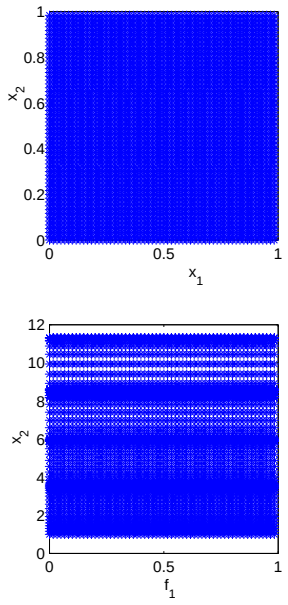

Fig. 4. Starting points (a); Objective functions values (b)

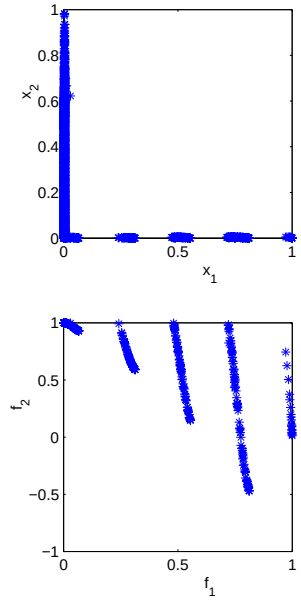

Fig. 5. Accumulation points (a); Objective functions values (b)

Test 3) We consider two objective functions (i.e.: $s = 2$) and two spatial variables (i.e.: $n = 2$). The objective functions are given by:

$$f_1(\underline{x}) = x_1^3, \qquad f_2(\underline{x}) = (x_2 - x_1)^3, \tag{55}$$

and the box is given by:

$$B = \{\ \underline{x} = (x_1, x_2) \quad | \quad -1 \leq x_1 \leq 1, \quad -1 \leq x_2 \leq 1\ \ \}. \tag{56}$$

This test problem is very difficult to deal with since there are two sets of points where the gradient vectors of the objective functions are identically null but these points do not belong to the local Pareto optimal front. In fact we have the global Pareto front given by:

$$\mathit{Global} : \mathcal{P}_g = \{(-1, 1)\}, \tag{57}$$

and the local (non global) Pareto front given by:

$$\mathit{Local} : \mathcal{P}_l = \{(-1, -1)\}. \tag{58}$$

Moreover we have the following two sets of points that belong to $K_{\underline{f},B}$ (where $\underline{f} = (f_1, f_2)$) but do not belong to the Pareto local front:

$$\mathcal{B} = \{(x_1, x_2) \in B \,|\, x_1 = x_2,\ x_1 \neq -1\}, \tag{59}$$

$$\mathcal{Z} = \{(x_1, x_2) \in B \mid x_1 = 0\}. \tag{60}$$

Figure 6 and Figure 7 show the numerical results obtained applying Algorithm A_1 to solve *Test 3*. Figure 6 shows the starting points ($N_{tot} = 100$) and Figure 7 shows the accumulation points of the sequence defined by Algorithm A_1. As in the previous cases, Figures 6(a) and 7(a) show the points in the x_1-x_2 plane and Figure 6(b) and 7(b) show the values of the objective functions in the plane f_1-f_2.

Test 4) We consider two objective functions (i.e.: $s = 2$) and three spatial variables (i.e.: $n = 3$). The objective functions are given by:

$$f_1(\underline{x}) = x_1 - 2x_2 - x_3 - \frac{36}{2x_1 + x_2 + 2x_3 + 1}, \qquad f_2(\underline{x}) = -3x_1 + x_2 - x_3, \tag{61}$$

and the box is given by:

$$B = \{\ \underline{x} = (x_1, x_2, x_3) \quad | \ 0 \leq x_1 \leq 4,\ 0 \leq x_2 \leq 4, \quad 0 \leq x_3 \leq 4\ \}. \tag{62}$$

Figure 8 and Figure 9 show the numerical results obtained applying Algorithm A to solve *Test 4*. As in the previous experiments Figure 8 shows the starting points ($N_{tot} = 1000$) and Figure 9 shows the accumulation points of the

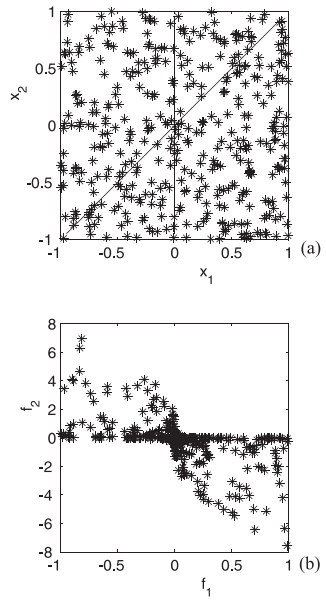

Fig. 6. Starting points (a); Objective functions values (b)

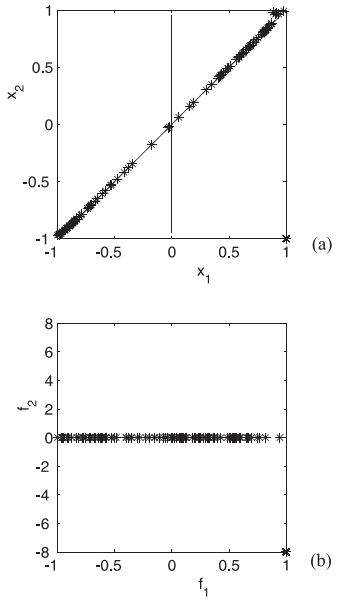

Fig. 7. Accumulation points (a); Objective functions values (b)

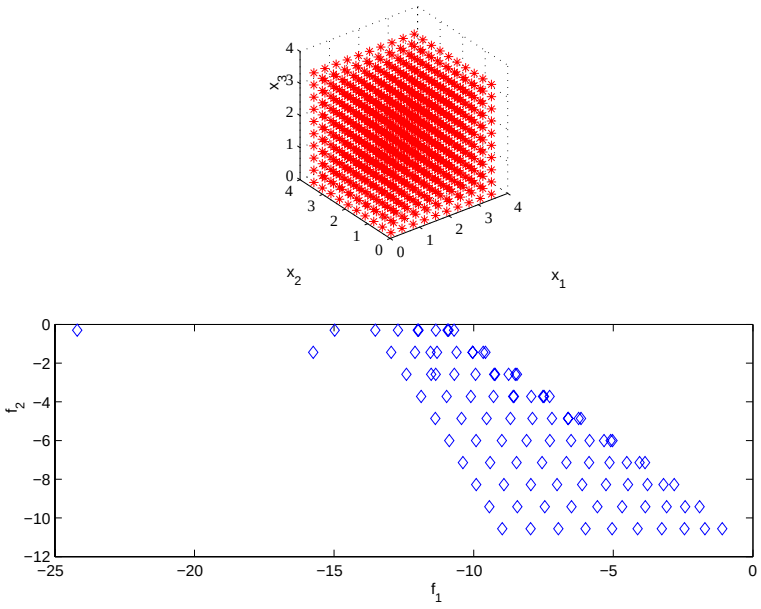

Fig. 8. Starting points (a); Objective functions values (b)

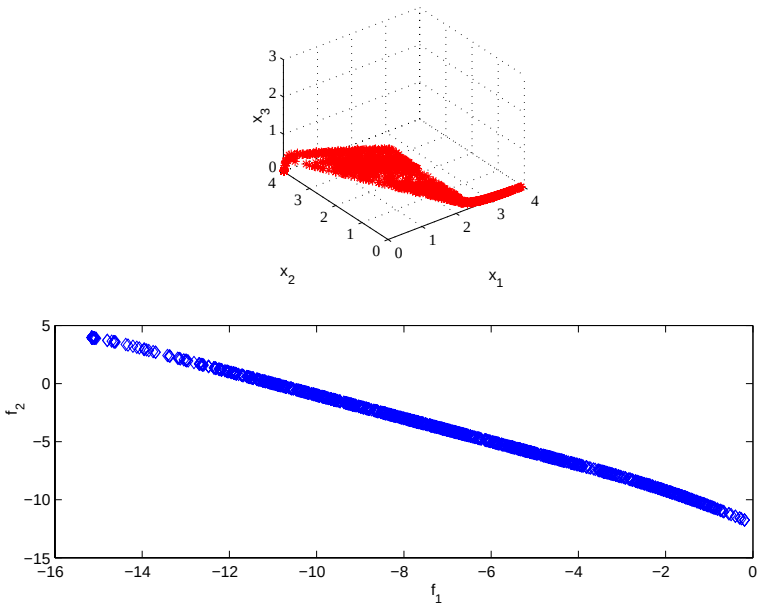

Fig. 9. Accumulation points (a); Objective functions values (b)

sequence defined by Algorithm A. Note that Algorithm A determines all the local Pareto optimal front.

We conclude noting that the computational method is very efficient with respect to the usual numerical methods that solve a multi-objective optimization problem using a scalarization of the objective functions. In fact, when the objective functions have not desirable properties (convexity or some other property) these last methods need to make several choices of the scalarization parameters in order to determine the entire local Pareto front, and, at our knowledge, no efficient strategies to make these choices are known. Algorithm A is able to approximate the entire Pareto front following a sufficient number of trajectories. This fact is not computationally demanding, since Algorithm A can be easily implemented with a parallel code. Indeed, by dividing the number of the trajectories to be computed among the processors used, we divide the execution time by a factor exactly equal to the number of processors used, with great savings in computation time.

We note that we have chosen the initial points on a rectangular lattice of the box, however several other choices can be made. For example, we can distribute the initial points taking into account the features of the objective functions or we can choose the initial points randomly distributed according with a probability density chosen in relation with the objective functions.

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