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2005/10

UNIVERSITÀ DELL'INSUBRIA
FACOLTÀ DI ECONOMIA

http://eco.uninsubria.it

Printed in Italy in September 2005
Università degli Studi dell'Insubria
Via Monte Generoso, 71, 21100 Varese, Italy

An axiomatic approach to approximate solutions in vector optimization

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Abstract

In vector optimization many notions of approximate solution have been proposed in the literature. In this paper an axiomatic approach is introduced in order to study the approximate solution map of a vector optimization problem in the image space.

An impossibility result is proved in the sense that, whenever all of the axioms are satisfied, either the set of the approximate solutions is a subset of the exact solution of the problem (the weakly efficient frontier), or it coincides with the whole admissible set.

Moreover, the geometry of the approximate solution map is studied in the special case of polyhedral ordering cones generated by a base of $\mathbb{R}^n$.

Finally, we study the shape of the approximate solution map under the assumption of weak approximation consistency.

Key words: vector optimization, approximate solution, axiomatization.

MSC: 90C29, 91B06

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1 Introduction

An axiomatic approach to some specific topic in mathematical research is interesting whenever the aim is to clarify the framework where the main notions of that field are defined. The need to investigate the theoretical foundations of a discipline plays an important role when mathematical methods are applied to social sciences. Among the most relevant instances we may recall the axiomatic foundation of utility theory in the second edition of the book by von Neumann and Morgenstern ([14]) and the axiomatic approach to social choice by Arrow ([1]).

In cooperative game theory, it is a standard approach to introduce specific solutions via a list of desirable properties for solutions. Early papers in this tradition were written by Nash on bargaining games and by Shapley on cooperative games. In [13] the Nash bargainig solution on a class of two-person bargaining games is characterized with a collection of properties (in which the independence of irrelevant alternatives property is dominant). In [19] an axiomatic characterization of the (Shapley) value of a cooperative game is given (by means of the properties of additivity, efficiency, symmetry and dummy player). Later, many other solutions were proposed in cooperative game theory and a comparison of these solutions with the aid of different axiomatic characterizations might be helpful in practical situations to select a suitable solution concept. However, the list of the required properties is often so long that it leads to the so called "impossibility theorems".

In noncooperative game theory, the axiomatic approach starts with [18], where the Nash equilibrium solution concept is characterized by means of some consistency properties and of the one-person rationality property. This approach has been extended to the notion of approximate Nash equilibria in [17].

The notion of approximate solution plays a crucial role also in optimization problems with one objective function, where the assumptions that ensure the existence of exact optimal solutions are often heavy and do not fit well with the models where optimization is applied. Moreover, even if an exact optimal solution exists, it is not always easy to develop a numerical method to reach it. A fundamental issue in optimization theory is the development of direct methods where the notion of minimizing sequence is important. The elements of a minimizing sequence are just approximate solutions of the optimization problem that converge in value to the value of the exact solution. So approximate solutions are studied, whenever the well-posedness and the stability of the problem are involved. An axiomatic approach to approximate solutions in scalar optimization is developed in [15] and in [16].

Many different approaches are proposed in the literature to generalize to vector optimization the idea of approximate solution, where the uniqueness of the optimal value is generally not ensured. There are approaches that consider the efficient frontier as a whole (see, e.g., [11], [12] and the references therein) and solutions that approximate a fixed optimal value (see [5], [6], [7] and [9]). Here we will consider only the second type of approximate solutions, where a fixed optimal value is approximated.

In this work we study the vector optimization problem in its image space, i.e.

we consider the problem of finding the (weakly) efficient frontier of a suitable set S.

In Section 2 we try to enucleate the main properties of a notion of approximate solution in vector optimization suitable for applications in decision theory. We group our axioms into two distinct classes: in the first group we list the properties that we deem to be desirable for a rational decision maker - existence, independence from irrelevant alternatives, consistency with respect to the order structure. A second group of axioms - translation and multiplication invariance - is introduced in order to obtain the compatibility of the notion of approximate solution with the von Neumann-Morgenstern utility theory. We test the validity of our axioms on some existing definitions.

In Section 3 we prove that, if all the axioms hold, either the approximate solution is trivially coincident with the whole set or it is a subset of the weakly efficient frontier, i.e. it is a part of the exact solution of the vector optimization problem. We interpret this result as an impossibility result: it is not possible to define a reasonable notion of approximate solution in vector optimization that is compatible with the von Neumann-Morgenstern framework An analogous result was developed in [15] for the scalar case.

In Section 4 we study the geometrical structure of approximate solutions in the special case of comprehensive sets where the ordering cone is a polyhedral cone whose generators are a base for the whole space $\mathbb{R}^n$. We emphasize that the behaviour of the approximate solutions on a special subclass of sets, the translations of the nonpositive orthants, determines the behaviour of the approximate solutions on the whole class of comprehensive sets.

In Section 5 we study the property of approximation consistency. The results that we obtain allow us to characterize some important examples of approximate solutions (see [9] and [5]) by means of suitable uniformity properties with respect to order intervals.

2 Approximate solutions in vector optimization: axioms and examples

Let $P \subset \mathbb{R}^n$ be a pointed (i.e. $P \cap (-P) = \{0\}$), closed and convex cone with nonempty interior. It is known that the cone P induces a partial order relation in the space $\mathbb{R}^n$ in the following way:

$$x \leq_P y \Leftrightarrow y - x \in P.$$

We denote by $[a, b]_P$ the order interval defined as

$$[a, b]_P = (a + P) \cap (b - P).$$

Let us consider the vector optimization problem (S, P) where S is a subset of $\mathbb{R}^n$. We define, the set of Pareto maximal points as follows:

$$\text{Max}\,(S, P) = \{x \in S : (x + P) \cap S = \{x\}\}.$$

We also define a weaker notion of solution for the vector optimization problem, i.e. the weakly Pareto maximal points:

$$\text{WMax}\,(S, P) = \{x \in S : (x + \text{int}P) \cap S = \varnothing\}.$$

Let us consider the collection

$$\mathcal{S} := \{S \subset \mathbb{R}^n : S \text{ is non empty, closed, bounded from above}\}$$

where bounded from above means that there exists $a \in \mathbb{R}^n$ such that $S \subset a - P$.

Moreover, in what follows, we consider the subcollection $\mathcal{C}$ of $\mathcal{S}$ given by

$$\mathcal{C} := \{S \subset \mathbb{R}^n : S \text{ is non empty, closed, comprehensive, bounded from above}\}$$

where a set S is said to be comprehensive when $S - P \subset S$. We remark that we can always consider the comprehensive hull $S - P$ of a set S without changing the exact solutions of the related optimization problem, i.e. WMax$(S - P, P) \cap S =$WMax(S, P). Moreover we recall that the notion of comprehensive set was originally introduced by G. Debreu in his fundamental monograph on general economic equilibrium [4] as free disposal property. This property is widely used in vector optimization theory, for instance in order to study the topological structure of the efficient frontiers (see e.g. [2] and [3]).

In this work we study some properties of approximate solutions. From a general point of view, an approximate solution is a map σ which assigns to every set $S \in \mathcal{S}$ a subset $\sigma(S)$ of S. Now we specify the main properties that should be satisfied by a notion of approximate solution by means of a list of axioms concerning the behaviour of the map σ on the collection $\mathcal{S}$:

NEM (*non emptiness*) $\sigma(S) \neq \emptyset$ for every $S \in \mathcal{S}$.

WAC (*weak approximation consistency*) for every $S \in \mathcal{S}$ it holds

$$(x + P) \cap S \subset \sigma(S),$$

for every $x \in \sigma(S)$.

OC (*order consistency*)

$$\sigma(S) \subset \sigma(S - P),$$

for every $S \in \mathcal{S}$.

CCA (*Chernoff's Choice Axiom*) for every $S_1, S_2 \in \mathcal{S}$ such that $S_1 \subset S_2$ it holds

$$\sigma(S_2) \cap S_1 \subset \sigma(S_1).$$

Remark 2.1 *The last axiom deals with the comparison of the same solution map on two distinct problems. As in the scalar case (see [15]), also WAC can be enforced in order to obtain a formulation of approximation consistency that compares different problem with the same exact solutions (weakly efficient frontiers):*

For every $S_1, S_2 \in \mathcal{S}$ with $WMax(S_1, P) = WMax(S_2, P)$. If $y \in \sigma(S_1)$, $x \in S_2$ and $x \in y + P$ then $x \in \sigma(S_2)$.

We can easily observe that this property implies WAC.

The previous properties can be interpreted as rationality requirements on the preferences of the decision maker: they directly involve the order structure of the vector optimization problem.

When vector optimization is applied to decision theory, i.e. the objective function is a vector of utility functions, a natural question arises: is it possible to formulate a notion of approximate solution in vector optimization that is compatible with the properties of von Neumann-Morgenstern utility functions?

Since the preferences of a rational agent are represented by a utility function unique up to increasing affine transformations, we have to require some properties of invariance with respect to translation and to positive multiplication.

In order to deal with translation and multiplication invariance in the image set S, we have to consider a collection $\mathcal{S}$ with additional requirements. We say that the collection $\mathcal{S}$ is *closed under translation* if, for every $S \in \mathcal{S}$ and $x \in \mathbb{R}^n$, we have

$$x + S := \{x + s : s \in S\} \in \mathcal{S}.$$

The collection $\mathcal{S}$ is said to be *closed under multiplication* if, for every $S \in \mathcal{S}$ and $\lambda > 0$, we have

$$\lambda S := \{\lambda s : s \in S\} \in \mathcal{S}.$$

The following two axioms concern translation and multiplication invariance.

TI (*translation invariance*) $\sigma(x + S) = x + \sigma(S)$ for every $S \in \mathcal{S}$, where $\mathcal{S}$ is closed under translation and $x \in \mathbb{R}^n$.

MI (*multiplication invariance*) $\sigma(\lambda S) = \lambda\sigma(S)$ for every $S \in \mathcal{S}$, where $\mathcal{S}$ is closed under multiplication and $\lambda > 0$.

In the sequel of this section we consider some notions of approximate solutions known in the literature. Naturally, we begin to examine the concept of maximal and weakly maximal points of a set $S \in \mathcal{S}$.

1. Let $\sigma_M(S) = \text{Max}(S, P) = \{x \in S : (x + P) \cap S = \{x\}\}$. It is easy to see that σ_M satisfies NEM, TI, MI, WAC, OC and CCA.

2. Let $\sigma_W(S) = \text{WMax}(S, P) = \{x \in S : (x + \text{int}P) \cap S = \varnothing\}$. Also in this case σ_W satisfies NEM, TI, MI, WAC, OC and CCA.

Now we pass to examine some notions of approximate solution in vector optimization.

3. Let $\sigma_\Delta(S) = (\text{Max}\,(S, P) - \Delta) \cap S$ where $\Delta = \{x \in \mathbb{R}^n : \|x\| \leq \varepsilon\} \cap P$ and ε is a fixed positive real number. Here the solution map satisfies NEM, TI and OC, but the axioms MI and WAC are not valid. In order to show that WAC axiom is not satisfied, it is sufficient to consider $P = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq -x_1, x_1 \geq 0\}$ and $S = -\mathbb{R}^2_+$. Moreover, CCA does not hold (for any choice of ε), as it is apparent when we consider

$$\begin{aligned} S_1 &= (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) - \mathbb{R}^2_+, \\ S_2 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\} \cup ((1,0) - \mathbb{R}^2_+) \cup ((0,1) - \mathbb{R}^2_+) \end{aligned}$$

and $P = \mathbb{R}^2_+$.

4. (See [9]). Let $\sigma_L(S) = \{x \in S : x + \epsilon + (P \backslash \{0\}) \cap S = \varnothing\}$ where ϵ is a fixed element belonging to $P\backslash\{0\}$. This solution map satisfies NEM, TI, WAC, OC and CCA, but it does not satisfy MI.

5. (see [5]). Let $\sigma_D(S) = \{x \in S : (x + \epsilon + (\text{int}P)) \cap S = \varnothing\}$ where ϵ is a fixed element belonging to $P\backslash\{0\}$. This solution map satisfies NEM, TI, WAC, OC and CCA, but it does not satisfy MI. This solution coincides with the closure of the solution map defined in point 4. above.

6. (See [7] and [6]). Let

$$\sigma_{DH}(S) = \{x \in S : h(x) \geq h(y) - \varepsilon \text{ for every } y \in (x+P) \cap S\}$$

where ε is a fixed positive real number and h is a fixed function, monotone with respect to the order induced by P (i.e. if $x \in y-P$ then $h(x) \leq h(y)$). This solution map satisfies NEM and OC. It satisfies also WAC. Indeed, let $x \in \sigma_{DH}(S)$ and $z \in (x+P) \cap S$. It is $h(x) \geq h(y) - \varepsilon$ for every $y \in (x+P) \cap S$. By the monotonicity of h, we have $h(z) \geq h(x)$, hence $h(z) \geq h(u) - \varepsilon$, $\forall u \in (z+P) \cap S$, i.e. $z \in \sigma_{DH}(S)$. Moreover, also CCA holds. Indeed, let $S_1 \subset S_2$ and let $y \in \sigma_{DH}(S_2) \cap S_1$. Then $h(y) \geq h(x) - \varepsilon$, $\forall x \in (y+P) \cap S_2 \subset (y+P) \cap S_1$, i.e. it is $y \in \sigma_{DH}(S_1)$. The axioms TI, MI do not hold for a generic choice of the scalarizing function h. We can observe that TI is satisfied when $h(x) =< p^*, x >$, where p^* belongs to the positive polar cone $P^*$of P (where $P^* = \{y \in \mathbb{R}^n : \langle p, y \rangle \geq 0, \forall p \in P\}$.

3 Is there an approximate solution compatible with the axioms ?

The aim of this section is to prove that the axioms stated in the previous section imply that either the solution map $\sigma(S)$ is included in the exact solution WMax(Q, P) or it coincides with the whole set S. This can be interpreted as an impossibility result: no effective notion of approximate solution of a vector optimization problem can be formulated without a conflict with at least one of the axioms. This result generalizes to vector optimization the analogous impossibility result for the scalar case contained in [15].

In order to prove our result, we consider a subcollection $\mathcal{Q}$ of $\mathcal{S}$, whose elements are translations of the opposite of the ordering cone.

Let $\mathcal{Q}$ be the subcollection of $\mathcal{S}$ given by

$$\mathcal{Q} := \{Q \subset \mathbb{R}^n : Q = q - P,\ q \in \mathbb{R}^n\}.$$

We can immediately see that $\mathcal{Q}$ is closed under translation and multiplication.

We begin to prove the above mentioned impossibility result for the subcollection $\mathcal{Q}$.

Lemma 3.1 *If* σ *satisfies* NEM, TI, MI *and* WAC *on* $\mathcal{Q}$ *then, either*

$\sigma(Q) \subset \mathrm{WMax}(Q, P)$, *for every* $Q \in \mathcal{Q}$,

or

$\sigma(Q) = Q$, *for every* $Q \in \mathcal{Q}$.

Proof. Since $\mathrm{WMin}(Q, P) = \mathrm{bd}\ (Q)$, if $\sigma(Q) \not\subset \mathrm{WMax}(Q, P)$ we have that there exists a quadrant $\hat{Q} \in \mathcal{Q}$ such that

$$\sigma(\hat{Q}) \cap \ \mathrm{int}\hat{Q} \neq \emptyset.$$

>From TI, it follows immediately that

$$\sigma(Q) \cap \ \mathrm{int}Q \neq \emptyset$$

for every $Q = q - P \in \mathcal{Q}$. Indeed, let $\hat{Q} = \hat{q} - P$ and $\hat{x} \in \sigma(\hat{Q}) \cap \ \mathrm{int}\hat{Q}$. Then it holds $\hat{x} + q - \hat{q} \in \sigma(Q) \cap \ \mathrm{int}Q$. Hence, without loss of generality, we can suppose that $\hat{Q} = -P$. Let $\tilde{x} \in \mathrm{int}(-P)$. MI implies that

$$\lambda\tilde{x} \in \sigma(-P)$$

for every $\lambda > 0$. Now, by WAC, we obtain that

$$(\lambda\tilde{x} + P) \cap (-P) \subset \sigma(-P)$$

for every $\lambda > 0$. The last relation implies that $-P = \sigma(-P)$. By TI, it holds $\sigma(Q) = Q$, for every $Q \in \mathcal{Q}$. ■

Now we pass to consider the case of a generic set S in $\mathcal{S}$.

We need the following simple result.

Lemma 3.2 *Let* $S \in \mathcal{S}$, *then* $\text{bd}(S - P) \cap S = \text{WMax}(S, P)$.

Proof. The inclusion $\text{bd}(S) \supset \text{WMax}(S, P)$ holds, so the inclusion

$$\text{WMax}(S, P) \subset \text{bd}(S - P) \cap S$$

follows immediately. We have to prove that $\text{bd}(S - P) \cap S \subset \text{WMax}(S, P)$. Let $x \in \text{bd}(S - P) \cap S$. By contradiction, we suppose that $x \notin \text{WMax}(S, P)$. Then there exists $y \in S \cap (x + \text{int}P)$. This relation implies that $x \in (y - \text{int}P) \cap S$, hence $x \in \text{int}(S - P) \cap S$, a contradiction. ■

Now we prove the main result of this section.

Theorem 3.3 *If* σ *satisfies* NEM, WAC, OC *and* CCA *on* $\mathcal{S}$ *and* σ *satisfies* MI *and* TI *on* $\mathcal{Q}$, *then, either*

$\sigma(S) \subset \text{WMax}(S, P)$ *for every* $S \in \mathcal{S}$

or

$\sigma(S) = S$ *for every* $S \in \mathcal{S}$.

Proof. If there exists $\hat{S} \in \mathcal{S}$ such that $\sigma(\hat{S}) \not\subset \text{WMax}(\hat{S}, P)$, by Lemma 3.2, there exists $x \in \sigma(\hat{S}) \cap \text{int}\left(\hat{S} - P\right)$. We can always find a quadrant $\hat{Q} \in \mathcal{Q}$

such that $\hat{Q} \subset \hat{S} - P$ and $x \in \text{int}\hat{Q}$. By OC, $x \in \sigma\left(\hat{S} - P\right)$, then, using CCA we have that $x \in \sigma(\hat{Q})$. Therefore, by Lemma 3.1, we have that $\sigma(Q) = Q$, for every $Q \in \mathcal{Q}$. Now, since the sets of the collection $\mathcal{S}$ is bounded from above, for every $S \in \mathcal{S}$ there exists a quadrant $Q_S \in \mathcal{Q}$ such that $S \subset Q_S$. Again by CCA, we have that

$$S = Q_S \cap S = \sigma(Q_S) \cap S \subset \sigma(S),$$

hence $\sigma(S) = S$. ■

For a comprehensive set, the OC property trivially holds, hence we have the following corollary.

Corollary 3.4 *If σ satisfies* NEM, WAC *and* CCA *on $\mathcal{C}$ and σ satisfies* MI *and* TI *on $\mathcal{Q}$ then, either*

$\sigma(S) \subset \text{WMax}(S, P)$ *for every* $S \in \mathcal{C}$

or

$\sigma(S) = S$ *for every* $S \in \mathcal{C}$.

4 Geometry of solution set

In this section we study the geometrical structure of the solution map σ when the partial order is given by a special polyhedral cone that can be interpreted as a generalization of the nonnegative orthant. Let us consider the cone

$$P = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq 0, i = 1, ..., n\}$$

where $\{a_i\}_{i=1,...n}$ is a base for $\mathbb{R}^n$. It is easy to observe that int $P \neq \emptyset$.

A f-*dimensional face* of P is a subset of P defined as follows:

$$F = \{x \in P : \langle a_i, x \rangle = 0, i \in I\}$$

where $I \subset \{1, ..., p\}$ and $\#I = n - f$.

Clearly, every face F of the cone P is itself a polyhedral pointed cone. We can observe that P has $\binom{n}{n-f}$ faces of dimension f $(f = 0, 1, ..., n-1)$. We denote by T_F the number of faces of P. We can easily compute T_F as follows:

$$T_F = \sum_{f=0}^{n-1} \binom{n}{n-f}.$$

Now we consider again the subcollection $\mathcal{Q}$ of $\mathcal{S}$.

Proposition 4.1 *Let* σ *satisfy* NEM, TI, MI *and* WAC *on* $\mathcal{Q}$. *If there exists a set* $\hat{Q} \in \mathcal{Q}$ *such that* $\sigma(\hat{Q}) \neq \hat{Q}$, *for every* $Q \in \mathcal{Q}$, *then*

$$\sigma(Q) = \bigcup_{j \in J} (q - F_j),$$

where $\{F_l\}_{l \in \{1,...T_F\}}$ *and* $J \subset \{1, ..., T_F\}$.

Proof. By TI, without loss of generality, we can consider only $Q = -P$. Let $x \in \sigma(Q)$. Then, by Lemma 3.1, $x \in WMax(Q, P)$, hence $x \in F_{\hat{l}}$ where $\hat{l} \in \{1, ...T_F\}$. By MI, $\lambda x \in F_{\hat{l}}$ for every $\lambda > 0$, then, by WAC, $F_{\hat{l}} \subset \sigma(Q)$. ■

Now we can use the geometry of the solution map on the subcollection $\mathcal{Q}$ in order to describe the structure of the solution map σ on the subcollection $\mathcal{C}$.

We need the following lemma.

Lemma 4.2 *Let* $C \in \mathcal{C}$. *If there exists* $h \in \mathbb{R}^n$ *such that* $(h - F_{\hat{j}}) \subset \mathrm{bd}(C)$ *for* $\hat{j} \in J$, *then there exists* $\hat{h} \in \mathbb{R}^n$ *such that* $C \subset \hat{h} - P$ *and* $(\hat{h} - F_{\hat{j}}) \supset (h - F_{\hat{j}})$.

Proof. We recall that using the invertible linear transformation defined by the matrix $A = [a_i]_{i=1,...,n}$,without loss of generality, we can consider the problem $(D, \mathbb{R}^n_+)$ where $D = A^{-1}(C)$ is bounded from above and comprehensive with respect to the ordering cone $\mathbb{R}^n_+$. Since D is bounded from above, there exists an element $\hat{d} \in \mathbb{R}^n$ such that $D \subset \hat{d} - \mathbb{R}^n_+$. Now, by the assumption, there exists $d = A^{-1}(h)$ such that $(d - A^{-1}(F_{\hat{\jmath}})) \subset \mathrm{bd}(D)$ for $\hat{\jmath} \in J$. Since $A^{-1}(F_{\hat{\jmath}})$ is a face of $\mathbb{R}^n_+$, we can find an integer m, with $1 \leq m \leq n$, such that $A^{-1}(F_{\hat{\jmath}}) = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_k = 0, k = 1, ..., m\}$. Now, we consider the translation defined by the vector $t = (d_1 - \hat{d}_1, ..., d_m - \hat{d}_m, 0, ..., 0)$. Since D is a comprehensive set and $(d - A^{-1}(F_{\hat{\jmath}})) \subset \mathrm{bd}(D)$, it holds $D \subset \hat{d} + t - \mathbb{R}^n_+$ and $(\hat{d} + t - A^{-1}(F_{\hat{\jmath}})) \supset (d - A^{-1}(F_{\hat{\jmath}}))$. ■

Proposition 4.3 *Let σ satisfy* NEM, TI, MI, WAC *and* CCA *on $\mathcal{C}$.*

1. *If $\sigma(-P) = \{0\}$, then $\sigma(C) \subset \mathrm{Max}(C, P)$, for every $C \in \mathcal{C}$.*

2. *If $\sigma(-P) = \bigcup_{j \in J}(-F_j)$ (with $F_j \neq 0$ for every $j \in J$) and if there exist $h \in \mathbb{R}^n$ and $\hat{\jmath} \in J$ such that $(h - F_{\hat{\jmath}}) \subset \mathrm{bd}(C)$, then $(h - F_{\hat{\jmath}}) \subset \sigma(C)$, for every $C \in \mathcal{C}$.*

3. *If $\sigma(-P) = -P$, then $\sigma(C) = C$, for every $C \in \mathcal{C}$.*

Proof.

1. Let $\sigma(-P) = \{0\}$. By TI, $\sigma(Q) = q$ for every $Q \in \mathcal{Q}$. Let $\hat{c} \in \sigma(C)$. By contradiction, there exists $\bar{c} \in C$ such that $\hat{c} \in \bar{c} - P \backslash \{0\}$. Since C is comprehensive, $\bar{c} - P \subset C$. Then, using CCA, it holds

$$\hat{c} \in \sigma(C) \cap (\bar{c} - P) \subset \sigma(\bar{c} - P) = \bar{c}.$$

2. Let $\sigma(-P) = \bigcup_{j \in J}(-F_j)$ (with $F_j \neq 0$ for every $j \in J$). Since an element $h \in \mathbb{R}^n$ exists such that $(h - F_{\hat{j}}) \subset \mathrm{bd}(C)$, by Lemma 4.2 we can always find $\hat{h} \in \mathbb{R}^n$ such that $C \subset \hat{h} - P$ and $(\hat{h} - F_{\hat{j}}) \supset (h - F_{\hat{j}})$. By TI, we have that $\hat{h} - F_{\hat{j}} \subset \sigma\left(\hat{h} - P\right)$. Since $C \subset \hat{h} - P$, from CCA, we have that $(\hat{h} - F_{\hat{j}}) \cap C \subset \sigma(C)$, hence $(h - F_{\hat{j}}) \subset \sigma(C)$.

3. Let $\sigma(-P) = -P$. Since C is bounded from above, we can always find an element $u \in \mathbb{R}^n$ such that $C \subset u - P$. The thesis follows immediately from TI and CCA.

■

The previous theorem shows that the behaviour of the solution map σ on the whole family of comprehensive sets is completely determined by the behaviour of σ on the opposite of the ordering cone.

5 The property of weak approximation consistency

In this section we study the property of weak approximation consistency through the existence of an appropriate map a_σ that assigns to every set $S \in \mathcal{S}$ a subset $a_\sigma(S)$ of S such that $a_\sigma(S)$ represents a "lower boundary" to the solution $\sigma(S)$. Let us consider the map a_σ defined by

$$a_\sigma(S) = \begin{cases} \{x \in \mathrm{cl}\,(\sigma(S)) : (x - (\mathrm{int}P)) \cap \sigma(S) = \varnothing\} & \text{if } \sigma(S) \neq S \\ S & \text{if } \sigma(S) = S \end{cases}.$$

We begin to state a sufficient condition for WAC.

Proposition 5.1 *The solution map σ satisfies WAC if*

$$(a_\sigma(S) + P\backslash\{0\}) \cap S \subset \sigma(S) \subset (a_\sigma(S) + P) \cap S. \qquad (1)$$

for every $S \in \mathcal{S}$.

Proof. Let $x \in \sigma(S)$ and $y \in (x + P) \cap S$. If $x \in a_\sigma(S) + P\backslash\{0\}$, then we have

$$y \in (a_\sigma(S) + P\backslash\{0\}) \cap S \subset \sigma(S).$$

Otherwise $x \in a_\sigma(S)$, hence $y = x \in \sigma(S)$ or $y \in (x + P\backslash\{0\}) \cap S \subset \sigma(S)$. ■

We can observe that the second inclusion by itself in (1) is not sufficient to obtain WAC.

Example 5.2 *Let $\sigma = \sigma_L$ be defined as in 4. of section 2. Let us consider the solution map $\tilde{\sigma}$ defined by*

$$\tilde{\sigma}(S) = \sigma(S) \cup ((cl(\sigma(S))) \cap (\mathbb{R}^n \backslash B))$$

where B is the closed unit ball of $\mathbb{R}^n$. Let $S = -\mathbb{R}^n_+$. It is easy to see that $\tilde{\sigma}$ does not satisfy WAC even if $\tilde{\sigma}(S) \subset (a_{\tilde{\sigma}}(S) + P) \cap S$.

Unfortunately the condition (1) is not a necessary condition for WAC. Indeed the first inclusion does not hold even if the solution map σ satisfies WAC, as shown in the following example.

Example 5.3 *Let $S = -\mathbb{R}^n_+$ and let the solution map $\sigma = \sigma_L$ be defined as in 4. in section 2.*

On the other hand, also the second inclusion in (1) is not a necessary condition, as shown in the following example.

Example 5.4 *Let*

$$S = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = e^{x_1} \text{ for } x_1 < 0, x_2 = 1 - x_1 \text{ for } x_1 \geq 0\}$$

and let the solution map σ be defined by

$$\sigma(S) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\} \cap S.$$

It is $a_\sigma(S) = \{(1,0)\}$, so $\sigma(S)$ is not included in $(a_\sigma(S) + P) \cap S$.

If we suppose that $\sigma(S)$ is a bounded set, we can prove that the second inclusion in (1) is a necessary condition for WAC.

Proposition 5.5 *If the solution map σ satisfies WAC and $\sigma(S)$ is bounded, then*

$$\sigma(S) \subset (a_\sigma(S) + P) \cap S$$

where $S \in \mathcal{S}$.

Proof. If $\sigma(S) = S$, the thesis follows immediately.If $\sigma(S) \neq S$, it holds

$$\text{WMax}(\text{cl}\,(\sigma(S), -P) \subset a_\sigma(S).$$

Since $\sigma(S)$ is a bounded set, from Prop. 4.10, Ch. 2 in [10], we have

$$\sigma(S) \subset cl\,(\sigma(S)) \subset (\text{WMax}(\text{cl}\,(\sigma(S), -P) + \text{int}P) \cap S \subset (a_\sigma(S) + P) \cap S.$$

■

Moreover, the same inclusion is a necessary condition for WAC (without any bondedness assumption) when we deal with the subcollection $\mathcal{C}$ of comprehensive sets.

Proposition 5.6 *If the solution map σ satisfies WAC and $C \in \mathcal{C}$, then*

$$\sigma(C) \subset (a_\sigma(C) + P) \cap C.$$

Proof. If $\sigma(C) = C$, the thesis follows immediately. If $\sigma(C) \neq C$, by contradiction we suppose that there is $x \in \sigma(C)$ such that $x \notin (a_\sigma(C) + P) \cap C$. Since the set C is comprehensive $x - P \subset C$. Now we show that $x - P \subset \sigma(C)$. By contradiction, let us suppose that there exists $q \in P$ such that $x - q \notin \sigma(C)$. By WAC, it easily follows that $(x - q - P) \cap \sigma(C) = \varnothing$. Let $\bar{\lambda} := \sup\{\lambda \in [0,1] : x - \lambda q \in \sigma(C)\}$. If $\left(x - \bar{\lambda}q - \text{int}P\right) \cap \sigma(C) = \varnothing$, then $x - \bar{\lambda}q \in a_\sigma(C)$, a contradiction against $x \notin (a_\sigma(C) + P) \cap C$. Otherwise, if $\left(x - \bar{\lambda}q - \text{int}P\right) \cap \sigma(C) \neq \varnothing$, then there exists an element $q' \in \text{int}P$ such that $x - \bar{\lambda}q - q' \in \sigma(C)$. By WAC, it is $\left(x - \bar{\lambda}q - q' + P\right) \cap C \subset \sigma(C)$. Since $q' \in \text{int}P$, it holds $x - \bar{\lambda}q \in \left(x - \bar{\lambda}q - q' + \text{int}P\right)$. Hence there exists a positive real number ε, such that $x - \left(\bar{\lambda} + \varepsilon\right) q \in \sigma(C)$, a contradiction against the definition of $\bar{\lambda}$.

Now we recall that $\text{int}P \neq \varnothing$ implies that the cone P is reproducing (i.e. $P - P = \mathbb{R}^n$; see, e.g. [8], p. 17). By WAC it is $(p + P) \cap C \subset \sigma(C)$ for every $p \in x - P$, hence $\sigma(C) = C$, a contradiction. ■

We remark that Example 5.4 shows that this proposition does not hold for a non comprehensive set.

We observe that the idea of a "lower bound" of the solution set, implicitely given here by the map a, completely characterizes the property of approximation consistency in the scalar case (see Prop. 4.1 in [15]). When we deal with vector optimization problems we have shown that the map a_σ introduced above does not completely characterize the approximation consistency. Nevertheless, this map allows us to study in more depth the properties of some relevant notions of approximate solution.

One of the most common notions of approximate solution in the field of vector optimization is due to P. Loridan (see [9]). In this special case of approximate solution, we characterize the solution map σ by means of its associate "lower bound" map a_σ.

The following result shows that if the shape of the map a_σ implies some uniformity property on the map σ, then the approximate solution is precisely the approximate solution in the sense of Loridan.

Proposition 5.7 *Let $S \in \mathcal{S}$. If there exist an element $z \in \mathrm{Max}(S,P)$ and an element $w \in a_\sigma(S) \cap (z - P\setminus\{0\})$ such that $\bigcup_{y\in a_\sigma(S)} (y + [0, z-w]_P) = \mathrm{cl}(\sigma(S))$ and $y + z - w \in \mathrm{Max}(S,P)$, for every $y \in a_\sigma(S)$, then the solution map σ is defined by*

$$\sigma(S) = \sigma_L(S) = \{x \in S : (x + \epsilon + (P\setminus\{0\})) \cap S = \varnothing\},$$

where $\epsilon = z - w$.

Proof. Let $x \in \sigma(S)$. By contradiction, we suppose that there exists u such that

$$u \in (x + z - w + (P\setminus\{0\})) \cap S.$$

By the assumption, there exist $y \in a_\sigma(S)$ and $p \in P$ such that $x = y + p$ and $y + z - w \in Max(S, P)$. Then there exists an element $p' \in P \setminus \{0\}$ such that $u = y + z - w + p' \in S$, against the assumption $y + z - w \in \text{Max}(S, P)$. ■

We remark that the assumption $y + z - w \in \text{Max}(S, P)$ in Proposition 5.7. cannot be weakened to $y + z - w \in \text{WMax}(S, P)$, as shown in the following example.

Example 5.8 *Let* $S = -\mathbb{R}^2_+$, *let* $\sigma(S) = -\mathbb{R}^2_+ \setminus (-int\mathbb{R}^2_+ - (1,1))$. *Then let* $z = (0,0) \in \text{Max}(S, P)$. *The only element* $w \in a_\sigma(S) \cap (z - P \setminus \{0\})$ *such that* $y + [0, z - w]_P \subset \text{cl}(\sigma(S))$ *for every* $y \in a_\sigma(S)$ *is* $w = (-1, -1)$ *but* $y + z - w \in \text{WMax}(S)$, *for every* $y \in a_\sigma(S)$.

We cannot prove exactly the converse of the previous result, as it is shown in the following example where we cannot obtain $y + z - w \in \text{Max}(S, P)$, for every $y \in a_\sigma(S)$.

Example 5.9 *Let* $S = -\mathbb{R}^2_+$, $\sigma = \sigma_L$ *and* $\epsilon = (1, 1)$. *Then let* $z = (0,0) \in \text{Max}(S, P)$. *The only element* $w \in a_\sigma(S) \cap (z - P \setminus \{0\})$ *such that* $y + [0, z - w]_P \subset \text{cl}(\sigma(S))$ *is* $w = (-1, -1)$ *but* $y + z - w \notin \text{Max}(S, P)$, *for every* $y \in a_\sigma(S) \setminus \{w\}$.

In order to obtain a sort of converse implication for the subcollection $\mathcal{C}$ of comprehensive sets, we have to prove the following preliminary result that relates the shape of $a_\sigma(S)$ to the shape of the weakly maximal frontier.

Lemma 5.10 *Let* $C \in \mathcal{C}$ *and*

$$\sigma(C) = \sigma_L(C) = \{x \in C : (x + \epsilon + (P \setminus \{0\})) \cap C = \varnothing\}$$

where $\epsilon \in P\backslash\{0\}$. *If* $w \in a_\sigma(C)$, *then* $w + \epsilon \in \mathrm{WMax}(C, P)$.

Proof. We begin to prove that $w + \epsilon \in C$. Indeed, let us suppose, by contradiction, that $w + \epsilon \notin C$. Since C is a closed set, we can choose an element $p \in \mathrm{int}P$ such that $w + \epsilon - p \notin C$. Then $w - p \in \sigma(C)$, because C is a comprehensive set, against the assumption $w \in a_\sigma(C)$.

Since $w \in \mathrm{cl}(\sigma(C))$, there exists a sequence $\{x_n\} \subset \sigma(C)$ such that $x_n \to w$ and

$$(x_n + \epsilon + (P\backslash\{0\})) \cap C = \varnothing \qquad \text{for every } n \in \mathbb{N}.$$

Hence we have $(w + \epsilon + (\mathrm{int}P)) \cap C = \varnothing$. ■

The following result can be interpreted as a converse of Proposition 5.7, where we can prove only that the element $y + z - w$ is a weakly maximal point of the set S.

Proposition 5.11 *Let* $C \in \mathcal{C}$. *If*

$$\sigma(C) = \{x \in C : (x + \epsilon + (P\backslash\{0\})) \cap C = \varnothing\},$$

where $\epsilon \in P\backslash\{0\}$, *then there exist an element* $z \in$WMax(C, P) *and an element* $w \in a_\sigma(C) \cap (z - P\backslash\{0\})$ *such that* $\bigcup_{y \in a_\sigma(S)} (y + [0, z - w]_P) =cl(\sigma(C))$ *and* $y + z - w \in \mathrm{WMax}(C, P)$ *for every* $y \in a_\sigma(C)$.

Proof. Let $w \in a_\sigma(C)$ and let $z = w + \epsilon$. It is $w \in a_\sigma(C) \cap (z - P\backslash\{0\})$. By Lemma 5.10, we have $z \in$WMax(C, P). Now let $y \in a_\sigma(C)$, it is

$$(y + P) \cap C \subset \mathrm{cl}\,(\sigma(C)).$$

By Lemma 5.10, $y + z - w \in$WMax(C, P). Since C is a comprehensive set, $y + z - w - P \subset C$. We conclude that

$$y + [0, z - w]_P \subset \mathrm{cl}\,(\sigma(C))\,,$$

hence it holds

$$\bigcup_{y \in a_\sigma(S)} (y + [0, z - w]_P) \subset \mathrm{cl}\,(\sigma(C))\,.$$

Now, since the solution map σ satisfies WAC, by Proposition 5.6 we have

$$\sigma(C) \subset \bigcup_{y \in a_\sigma(S)} (y + [0, z - w]_P)\,.$$

Let $z \in \mathrm{cl}(\sigma(C)) \setminus (\sigma(C))$. If $z \in a_\sigma(C)$ then $z \in \bigcup_{y \in a_\sigma(S)} (y + [0, z - w]_P)$. Otherwise, if $z \notin a_\sigma(C)$ there exists an element $p' \in \mathrm{int}P$ such that $z - p' \in \sigma(C)$. By WAC we have that $z \in \sigma(C)$. Therefore we can conclude that

$$\mathrm{cl}\,(\sigma(C)) \subset \bigcup_{y \in a_\sigma(S)} (y + [0, z - w]_P)\,.$$

■

Now we consider the solution map $\sigma = \sigma_D$ defined in 5 of section 2.

In this case we can explicitly compute the map $a_\sigma(C)$, whenever $C \in \mathcal{C}$.

Proposition 5.12 *Let $C \in \mathcal{C}$. If $\sigma(C) = \sigma_D(C) = \{x \in C : (x + \epsilon + (intP)) \cap C = \varnothing\}$ where $\epsilon \in P \setminus \{0\}$, then*

$$a_\sigma(C) = \mathrm{WMax}(C, P) - \epsilon$$

for every $C \in \mathcal{C}$.

Proof. First we show that $a_\sigma(C) \subset WMax(C, P) - \epsilon$. Let $w \in a_\sigma(C)$. We begin to prove that $w + \epsilon \in C$. Indeed, let us suppose, by contradiction, that

$w+\epsilon \notin C$. Since C is a closed set, we can choose an element $p \in \text{int}P$ such that $w+\epsilon-p \notin C$. Then $w-p \in \sigma(C)$, because C is a comprehensive set, against the assumption $w \in a_\sigma(C)$.

Since $w \in \text{cl}(\sigma(C))$, there exists a sequence $\{x_n\} \subset \sigma(C)$ such that $x_n \to w$ and

$$x_n + \epsilon + (\text{int}P) \cap C = \varnothing, \qquad \text{for every } n \in \mathbb{N}.$$

Hence we have $(w+\epsilon+(\text{int}P)) \cap C = \varnothing$, that is $w+\epsilon \in$WMax(C,P).

Now we have to prove that WMax$(C,P) - \epsilon \subset a_\sigma(C)$. Let $z \in$WMax(C,P). We suppose, by contradiction, that $z-\epsilon \notin a_\sigma(C)$. We have two distinct cases.

1. $z-\epsilon \notin$cl$(\sigma(C))$,then there exists $u \in (z+(\text{int}P)) \cap C$, against the weak minimality of z.

2. $z-\epsilon \in$cl$(\sigma(C))$. Since $z-\epsilon \notin a_\sigma(C)$, there exists $v = z-\epsilon-p \in \sigma(C)$ where $p \in$int$P$. Since $z \in (z-p+(\text{int}P)) \cap C = (v+\epsilon+(\text{int}P)) \cap C$, we have a contradiction.

■

This notion of approximate solution on the subcollection $\mathcal{C}$ is completely characterized by the following result.

Theorem 5.13 *The solution map σ is defined by*

$$\sigma(C) = \sigma_D(C) = \{x \in C : (x+\epsilon+(\text{int}P)) \cap C = \varnothing\}$$

for every set $C \in \mathcal{C}$, where $\epsilon \in P\backslash\{0\}$ if and only if there exist an element $z \in$WMax(C,P) and an element $w \in a_\sigma(C) \cap (z-P\backslash\{0\})$ such that

$\bigcup_{y\in a_\sigma(S)}(y+[0,z-w]_P)$ =cl$(\sigma(C))$ *and* $y+z-w \in$ WMax(C,P), *for every* $y \in a_\sigma(C)$.

Proof. First we prove the if-part. We will show that

$$\sigma(C)=\{x\in C:(x+\epsilon+(\text{int}P))\cap C=\varnothing\}$$

with $\epsilon = z-w \in P\setminus\{0\}$. Let $x \in \sigma(C)$. By contradiction, we suppose that

$$u\in(x+z-w+(\text{int}P))\cap C.$$

By the assumption $\bigcup_{y\in a_\sigma(S)}(y+[0,z-w]_P)=cl(\sigma(C))$, there exist $y \in a_\sigma(C)$ and $p \in P$ such that $x = y+p$. Then there exists an element $p' \in$int$P$ such that $u = y+z-w+p' \in C$, a contradiction.

Now we prove the only if-part. Let $w \in a_\sigma(C)$ and let $z = w+\epsilon$. It is $w \in a_\sigma(C)\cap(z-P\setminus\{0\})$. By Proposition 5.12, $z \in$WMax(C,P). By the same argument we obtain that $y+z-w \in$WMax(C,P) for every $y \in a_\sigma(C)$. Moreover, by Proposition 5.1, it is

$$(y+P)\cap C\subset \text{cl}\,(\sigma(C))\,.$$

Since C is a comprehensive set, $y+z-w-P\subset C$. We conclude that

$$y+[0,z-w]_P\subset \text{cl}\,(\sigma(C))\,,$$

hence

$$\bigcup_{y\in a_\sigma(S)}(y+[0,z-w]_P)\subset \text{cl}\,(\sigma(C))\,.$$

Now since the solution map σ satisfies WAC, by the proposition 5.6 we have

$$\sigma(C)\subset\bigcup_{y\in a_\sigma(S)}(y+[0,z-w]_P)\,.$$

Now let $z \in \mathrm{cl}(\sigma(C)) \setminus (\sigma(C))$. If $z \in a_\sigma(C)$ then $z \in \bigcup_{y \in a_\sigma(S)} (y + [0, z-w]_P)$. Otherwise, if $z \notin a_\sigma(C)$ there exists an element $p' \in \mathrm{int}P$ such that $z-p' \in \sigma(C)$. By WAC we have that $z \in \sigma(C)$. Therefore we can conclude that

$$\mathrm{cl}\,(\sigma(C)) \subset \bigcup_{y \in a_\sigma(S)} (y + [0, z-w]_P).$$

■

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