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First order optimality condition for constrained set-valued optimization

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Abstract

A constrained optimization problem with set-valued data is considered. Different kind of solutions are defined for such a problem. We recall weak minimizer, efficient minimizer and proper minimizer. The latter are defined in a way that embrace also the case when the ordering cone is not pointed. Moreover we present the new concept of isolated minimizer for set-valued optimization. These notions are investigated and appear when establishing first-order necessary and sufficient optimality conditions derived in terms of a Dini type derivative for set-valued maps. The case of convex (along rays) data is considered when studying sufficient optimality conditions for weak minimizers.

Key words: Vector optimization, Set-valued optimization, First-order optimality conditions.

Math. Subject Classification: 90C29, 90C30, 90C46, 49J52.

1 Introduction

In this paper we focus on the following set-valued constrained optimization problem (in the sequel svp)

$$\min_C F(x), \quad x \in G(x) \cap (-K) \neq \emptyset, \quad (1)$$

where $F : X \rightsquigarrow Y$, and $G : X \rightsquigarrow Z$ are set-valued functions (svf) defined from a Banach Space X into Banach spaces Y and Z respectively and both F and G are nonempty-valued over X . We suppose that a ordering relation on Y and Z is induced by the closed convex cones $C \subset Y$ and $K \subset Z$ respectively. We will assume that in general these cones are not pointed, since pointedness is too restrictive assumption, when constrained problems are concerned. As for notation, throughout the paper we denote set-valued functions with capital letters and squiggled arrow, while, when single-valued examples are considered, the lower case letter and straight arrow will identify the functions. When we say a point $x \in X$ is feasible we mean, throughout the paper, that $G(x) \cap (-K) \neq \emptyset$.

The svp (1) can be regarded as a generalization of a single-valued vector optimization problem. As it is well known (see e.g. [19]) the notion of a solution to the latter problem can be defined in different ways. We shall distinguish efficient, weakly efficient, strict efficient, properly efficient and isolated solutions of a vector (constrained or unconstrained) optimization problem. Some of these notions have already been considered elsewhere as related to (1). The notion of weak-minimizer (w -minimizer) or efficient-minimizer (e -minimizer) has been widely studied and some first order necessary and sufficient conditions have been presented in terms of suitable notions of derivatives (see e.g. [1, 5, 14] for the unconstrained case, [18, 21] for the constrained problem).

Here we concentrate on the notion of i -minimizer (isolated minimizer) which extends the concept of isolated minimizer of order 1 introduced for scalar problems by Auslender in [2]. Notions of optimality

for problem (1) are presented in Section 2, together with some basic notations and some preliminary facts on set-valued functions. The case of unconstrained set-valued optimization has been studied in [5]. Here, for the sake of completeness, we recall in Section 3 some of the results from [5]. Sections 4 and 5 present the main results for the constrained problem. In Section 4 optimality conditions for svp with Lipschitz type functions are considered, while Section 5 deals with convexity type assumptions.

2 Preliminary

We denote by \mathbb{R} the set of the reals and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ its two point extension with the infinite elements. For the norm and the dual pairing in a normed space we write $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$. From the context it should be clear to exactly which spaces these notations are applied. We denote by $B_X = \{x \in X \mid \|x\| < 1\}$ and $B_Y = \{y \in Y \mid \|y\| < 1\}$ the open unit balls respectively in X and Y . Similarly, the notations $S_X = \{x \in X \mid \|x\| = 1\}$ and $S_Y = \{y \in Y \mid \|y\| = 1\}$ are used for the unit spheres. When X or Y are finite dimensional, of dimension n and m respectively, we identify them with the Euclidean spaces \mathbb{R}^m and \mathbb{R}^n respectively.

The notion of the positive polar cone is used in the sequel. We recall that for a closed convex cone $A \subset Y$ its positive polar cone is defined by $A' = \{\xi \in Y \mid \langle \xi, y \rangle \geq 0 \text{ for all } y \in A\}$. For a subset B of a normed space and an element y^0 of the same space, we set $\text{cone}(A - y^0) := \{\lambda(y - y^0) \mid \lambda \geq 0, y \in A\}$.

We introduce the following concepts of solutions for problem (1). The pair (x^0, y^0) , $y^0 \in F(x^0)$, x^0 feasible, is said to be w -minimizer (respectively e -minimizer) if there exists a neighbourhood U of x^0 such that if $x \in U \cap X_0$, x feasible, then $F(x) \cap (y^0 - \text{int } C) = \emptyset$ (respectively $F(x) \cap (y^0 - (C \setminus \{0\})) = \emptyset$). Obviously, if $C \neq Y$, each e -minimizer is w -minimizer.

Define the weakly efficient frontier (w -frontier) $w\text{-Min}_C A$ and efficient frontier (e -frontier) $e\text{-Min}_C A$ of a set $A \subset Y$ with respect to the cone C by $w\text{-Min}_C A = \{y \in A \mid A \cap (y - \text{int } C) = \emptyset\}$ and $e\text{-Min}_C A = \{y \in A \mid A \cap (y - (C \setminus \{0\})) = \emptyset\}$. If $C \neq Y$ it holds $\text{int } C \subset C \setminus \{0\}$, whence $w\text{-Min}_C A \supset e\text{-Min}_C A$ (for vector optimization theory based on notions of efficient frontiers see Luc [19]).

Putting $x = x^0$ in the above definitions we see that if (x^0, y^0) is a w -minimizer (respectively e -minimizer) for svp (1) then y^0 belongs to the w -frontier (respectively e -frontier) of the set $F(x^0)$. Thus, in order that (x^0, y^0) , $y^0 \in F(x^0)$, be a minimizer of certain type for svp (1) necessary some frontier-type limitations for the point y^0 do occur.

For a set $A \subset Y$ the distance from $y \in Y$ to A is given by $d(y, A) = \inf\{\|a - y\| \mid a \in A\}$. It is convenient to allow also value $+\infty$ of the distance function putting $d(y, \emptyset) = +\infty$.

The oriented distance from y to A is defined by $D(y, A) = d(y, A) - d(y, Y \setminus A)$. It takes values in $\overline{\mathbb{R}}$ and in particular $D(y, \emptyset) = +\infty$ and $D(y, Y) = -\infty$. The function D is introduced in Hiriart-Urruty [12, 13] and since then is often used in vector optimization. In [10], when A is a convex set, the authors prove that $D(y, A) = \sup_{\|\xi\|=1} (\langle \xi, y \rangle - \sup_{a \in A} \langle \xi, a \rangle)$ and apply this characterization to approximate set-valued functions by single valued ones. Let us underline that this formula works also for $A = \emptyset$ or $A = Y$. From this representation, if C is a convex cone and taking into account

$$\inf_{a \in C} \langle \xi, a \rangle = \begin{cases} 0 & , \quad \xi \in C', \\ -\infty & , \quad \xi \notin C', \end{cases}$$

we get easily

$$D(y, -C) = \sup_{\|\xi\|=1, \xi \in C'} (\langle \xi, y \rangle) , \quad (2)$$

where $C' = \{\xi \mid \langle \xi, y \rangle \geq 0\}$ is the positive polar cone of C (further we use similar notation also for other positive polar cones).

We define next the oriented distance $D(M, A)$ from a set $M \subset Y$ to the set $A \subset Y$ putting $D(M, A) = \inf\{D(y, A) \mid y \in M\}$.

Let $C \subset Y$ be a cone and let a be a real number. The set

$$C(a) = \{y \in Y \mid D(y, C) \leq a \|y\|\}.$$

is a closed (but not necessarily convex) cone, which is a consequence of the positive homogeneity of the oriented distance $D(\cdot, C)$ and the norm $\|\cdot\|$.

We define, for a the vector optimization problem (vvp)

$$\min_C f(x), \quad g(x) \in -K. \quad (3)$$

with $f : X \rightarrow Y, g : X \rightarrow Z$ a feasible point x^0 (i.e. $g(x^0) \in -K$) is a p -minimizer (proper minimizer) if there exists $a, 0 < a < 1$, and a neighbourhood U of x^0 such that $f(x) - f(x^0) \notin -\text{int } C(a)$ for $x \in U, x$ feasible. Clearly, when C is pointed closed convex cone, Y is finite dimensional, and $a > 0$ is sufficiently small, then $C(a)$ is also a pointed closed convex cone and p -minimizers coincides with Henig proper efficient points.

The notion of proper minimizer can be applied also to svp. We say that the point $(x^0, y^0), y^0 \in F(x^0), x^0$ feasible, is a p -minimizer for (1) if there exists $a, 0 < a < 1$, and a neighbourhood U of x^0 , such that $x \in U, x$ feasible, and $y \in F(x)$ imply $y - y^0 \notin -\text{int } C(a)$.

For a given a set $A \subset Y$ the set $p\text{-Min}_C A = \{y \in A \mid A \cap (y - C(a)) = \{y\} \text{ for some } a, 0 < a < 1\}$ is the properly efficient frontier (p -frontier) of A with respect to C . Obviously $e\text{-Min}_C A \supset p\text{-Min}_C A$.

For $x = x^0$ the definition of a p -minimizer for svp (1) gives now that if $(x^0, y^0), y^0 \in F(x^0)$, is a p -minimizer for svp (1) then $y^0 \in p\text{-Min}_C F(x^0)$.

Another concept of optimality is the concept of an isolated minimizer (i -minimizer). We say that $(x^0, y^0), y^0 \in F(x^0)$, is a i -minimizer for svp (1) if x^0 is feasible and there is a neighbourhood U of x^0 and a constant $A > 0$ such that $D(F(x) - y^0, -C) \geq A \|x - x^0\|$ and $y^0 \in p\text{-Min}_C F(x^0)$ for $x \in U \cap X_0, x$ feasible. In [5] under Lipschitz type conditions it has been shown that the i -minimizers are also p -minimizers.

The notion of isolated minimizer has been popularised by Auslender [2]. For vector functions it has been extended in [6, 7, 8, 9] and under the name of strict efficiency in [15, 16, 17].

In the definition of a i -minimizer for svp appears explicitly the inclusion $y^0 \in p\text{-Min}_C F(x^0)$, which deserves some explanation. For vvp (3) with locally Lipschitz function f each i -minimizer is also a p -minimizer, see [7]. In order that similar relation occurs for svp (1), we need to insert explicitly this assumption. It is necessary satisfied for a p -minimizer and does not follow from inequality $D(F(x) - y^0, -C) \geq A \|x - x^0\|$ being used in the definition of a i -minimizer for svp (1). It should be clear that p -minimizers are not necessarily i -minimizers.

Example 2.1 Let $X = Y = Z = \mathbb{R}$, F be given as $F(x) = [x^2, 2x^2]$, that is the image of $x \in \mathbb{R}$ is the interval $[x^2, 2x^2]$, $G(x) = [-x^2 - 1, -x^2]$ and $C = K = \mathbb{R}_+$. Note that for all $a > 0$ it holds $C(a) = C$. Then it is easy to check that $(x_0, y_0) = (0, 0)$ is a p -minimizer, but not i -minimizer.

The notion of isolated minimizer for vector optimization is frequently studied under assumption of Lipschitz data. We recall [1] that the svf $F : X \rightsquigarrow Y$ is locally Lipschitz at $x^0 \in X$, if there exists a neighbourhood U of x^0 and a constant $L > 0$, such that for $x^1, x^2 \in U$ it holds $F(x^2) \subset$

$F(x^1) + L \|x^2 - x^1\| B_Y$. The svf $F : X \rightsquigarrow Y$ is locally Lipschitz, if it is locally Lipschitz at each $x^0 \in X$. The property can be analogously defined with respect to the closed convex cone C (the ordering cone in the image space). The svf $F : X \rightsquigarrow Y$ is locally Lipschitz w.r.t. C at $x^0 \in X$, or locally C -Lipschitz at x^0 , if there exists a neighbourhood U of x^0 and a constant $L > 0$ such that it holds

$$F(x^2) \subset F(x^1) + C + L \|x^2 - x^1\| \text{cl } B_Y \quad \text{for all } x^1, x^2 \in U \cap X_0.$$

We say that svf $F : X_0 \rightsquigarrow Y$ is locally C -Lipschitz if it is locally C -Lipschitz at each $x^0 \in X_0$. For a review of the properties of locally C -Lipschitz functions we refer to [5].

Because of the convexity of C , svf F is locally C -Lipschitz if and only if the set-valued function $x \rightsquigarrow F(x) + C$ is locally Lipschitz.

Further we recall [1] that for svf $\Phi : T_0 \rightsquigarrow Y$ given on a subset T_0 of the topological space T the upper limit $\text{Limsup}_{t \rightarrow t_0} \Phi(t)$ is defined by

$$\text{Limsup}_{t \rightarrow t_0} \Phi(t) = \{y \in Y \mid \liminf_{t \rightarrow 0^+} d(y, \Phi(t)) = 0\}.$$

We shall now define the (upper) Dini-derivative of a svf $\Phi : X_0 \rightsquigarrow Y$ at (x^0, y^0) , $y^0 \in \Phi(x^0)$, in the direction $u \in X$, as the upper limit

$$\Phi'(x^0, y^0; u) = \text{Limsup}_{t \rightarrow +0} \frac{1}{t} (F(x^0 + tu) - y^0). \quad (4)$$

The definitions of a derivative of a set-valued map are introduced in different ways, see e.g. [1, 4, 14]. Many of them are defined geometrically. Among the others, because of its wide applications, we recall the *contingent derivative* in the following definition and we illustrate in the example below some calculus of such a derivative and its relation to the Dini type derivative used in this paper.

Definition 2.1 Let $F : X \rightsquigarrow Y$ be a set valued map. The contingent derivative $\mathcal{D}F((x, y); u)$ of F at (x, y) , $y \in F(x)$ in the direction u is the set valued map from X to Y such that its graph (recall $\text{Graph } H := \{(x, y) \in X \times Y \mid y \in H(x)\}$, for any $H : X \rightsquigarrow Y$) is the Bouligand tangent cone to the graph of F at (x, y) .

Example 2.2 Let $X = Y = \mathbb{R}$, and $\Phi : [-1, 2] \rightsquigarrow \mathbb{R}$ be given (outside this interval the svf is arbitrary) as

$$\Phi(x) = \begin{cases} [x^2, 4 - (x - 2)^2] & , \quad 1 < x \leq 2, \\ [-x + 2, 12 - (x + 2)^2] & , \quad -1 \leq x \leq 1. \end{cases}$$

It can be computed that $\Phi'(1, 1; +1) = [2, +\infty)$ and $\Phi'(1, 1; -1) = [1, +\infty)$. Moreover the contingent derivative is the svf $\mathcal{D}\Phi$ for which it can easily be seen that $\mathcal{D}\Phi((1, 1); +1) = \Phi'(1, 1; +1)$ and $\mathcal{D}\Phi((1, 1); -1) = \Phi'(1, 1; -1)$.

3 The unconstrained problem

In this section after [5] we recall first-order optimality conditions for the unconstrained problem

$$\min_C F(x). \quad (5)$$

In the next Sections we generalize these results for the constrained problem (1). We start with w -minimizers.

Theorem 3.1 (Necessary Conditions, w -minimizers) Consider svp (5) with $F : X \rightsquigarrow Y$ and C closed convex cone. Let $(x^0, y^0), y^0 \in F(x^0)$, be a w -minimizer. Then

$$\forall u \in X : F'(x^0, y^0; u) \cap (-\text{int } C) = \emptyset. \quad (6)$$

Remark 3.1 The dual form of condition (6) is

$$\forall u \in X : \forall \bar{y}^0 \in F'(x^0, y^0; u) : \exists \bar{\xi}^0 \in C' \setminus \{0\} : \langle \bar{\xi}^0, \bar{y}^0 \rangle \geq 0.$$

The next theorem characterizes the i -minimizers of unconstrained svp with locally C -Lipschitz svf.

Theorem 3.2 (Sufficient Conditions, i -minimizers) Consider svp (5) with $Y = \mathbb{R}^m$, $C \neq Y$ a closed convex cone and svf $F : X \rightsquigarrow Y$ being locally C -Lipschitz. Suppose that $(x^0, y^0), y^0 \in F(x^0)$, is such that $y^0 \in p\text{-Min}_C F(x^0)$ and

$$\forall u \in X \setminus \{0\} : F'(x^0, y^0; u) \cap (-C) = \emptyset. \quad (7)$$

Then (x^0, y^0) is a i -minimizer for (5).

Remark 3.2 Like for Theorem 3.1 sufficient condition (7) can be stated in dual form as

$$\forall u \in X \setminus \{0\} : \forall \bar{y}^0 \in F'(x^0, y^0; u) : \exists \bar{\xi}^0 \in C' \setminus \{0\} : \langle \bar{\xi}^0, \bar{y}^0 \rangle > 0.$$

The reversal of Theorem 3.2 can also be stated.

Theorem 3.3 (Necessary Conditions, i -minimizers) Consider svp (5) with C closed convex cone and svf $F : X \rightsquigarrow Y$. Suppose that $(x^0, y^0), y^0 \in F(x^0)$, is an i -minimizer for (5). Then $y^0 \in p\text{-Min}_C F(x^0)$ and condition (7) holds.

The following result proved in [5] offers sufficient conditions for w -minimizers under C -convexity assumptions. The result is stated for global solutions. We recall that the pair $(x^0, y^0), y^0 \in F(x^0)$, is said to be a global w -minimizer for svp (1) if for every $x \in X_0$ it holds $F(x) \cap (y^0 - \text{int } C) = \emptyset$. Similarly, one can define global versions of all the optimality concepts introduced in Section 2.

We say that the the nonempty-valued svf $F : X_0 \rightsquigarrow Y$ is C -convex-along-rays at (x^0, y^0) if the set X_0 is star shaped at x^0 and $(1-t)y^0 + tF(x) \subset F((1-t)x^0 + tx) + C$ for all $x \in X_0$ and $0 < t < 1$. Recall that X_0 is star shaped at x^0 if $(1-t)x^0 + tx \subset X_0$ for all $x \in U \cap X_0$ and $0 < t < 1$. For single-valued functions the concept of a convex-along-rays function is introduced in Rubinov [20] and studied in the framework of abstract convexity and global optimization.

Theorem 3.4 (Sufficient Conditions, w -minimizers) Consider svp (1) with $Y = \mathbb{R}^m$ and $C \subset Y$ pointed closed convex cone. Suppose that $(x^0, y^0), y^0 \in F(x^0)$, is such that X_0 is star shaped at x^0 , $F : X_0 \rightsquigarrow Y$ is C -convex-along-rays at (x^0, y^0) , and condition (6) is satisfied. Suppose also that for each direction $u \in X_0(x^0)$ there exists a vector $g_u \in Y$ such that $F(x^0 + tu) \subset y^0 + tg_u + C$. Then (x^0, y^0) is a global w -minimizer for (1).

Examples show that the condition $F(x^0 + tu) \subset y^0 + tg_u + C$ cannot be avoided in the statement of Theorem 3.4 (see [5]).

4 Constrained optimization

In connection with svp (1) we consider the set-valued function $H : X \rightsquigarrow Y \times Z$

$$H(x) = (F, G)(x) = F(x) \times G(x)$$

We assume, unless otherwise specified, that F and G are (respectively) locally C -Lipschitz and locally K -Lipschitz functions. We will made also use of the closed convex cone $K(w) \subset Z$, $w \in G(x)$, defined as the cone whose polar is the set $K'(w) = \{\xi \in K' \mid \langle \xi, w \rangle = 0\}$. It can be shown, that $K(w)$ is the contingent cone of K at w .

Remark 4.1 It is easily seen that when F is C -Lipschitz and G is K -Lipschitz, then H is $(C \times K)$ -Lipschitz. Moreover, since $K \subset K(w)$, for any (x, w) , $x \in X$, $w \in G(x)$, any K -Lipschitz function G is also $K(w)$ -Lipschitz.

According to (4), for given $x^0 \in X$, $y^0 \in F(x^0)$, $w^0 \in G(x^0)$ and $u \in X$ the first-order Dini derivative of H is

$$H'(x^0, (y^0, w^0); u) = \text{Limsup}_{t \rightarrow +0} \frac{H(x^0 + tu) - (y^0, w^0)}{t}.$$

Theorem 4.1 (Necessary condition for w -minimizers) *Let $x^0 \in X$ be feasible for problem (1) and (x^0, y^0) be a w -minimizer. Then for all $w^0 \in (G(x^0) \cap (-K))$ and $u \in X$ it holds*

$$H'(x^0, (y^0, w^0); u) \cap (-\text{int } C \times (-\text{int } K(w^0))) = \emptyset, \quad (8)$$

and $y^0 \in w\text{-min}_C F(x)$.

Proof Assume, by contradiction that there exists some $(v^0, z^0) \in H'(x^0, (y^0, w^0); u) \cap -\text{int } (C \times K(w^0))$, for some $u \in X$ and $w^0 \in G(x^0) \cap -K$. Therefore one can write, for some sequence $y^n \in F(x^0 + t_n u)$ and $w^n \in G(x^0 + t_n u)$:

$$v^0 = \lim_{n \rightarrow +\infty} \frac{y^n - y^0}{t_n} \quad \text{and} \quad z^0 = \lim_{n \rightarrow +\infty} \frac{w^n - w^0}{t_n}.$$

We claim now that there exists some n_0 such that $G(x^0 + t_n u) \cap -\text{int } K \neq \emptyset$ for all $n > n_0$, that is $x^0 + t_n u$ is feasible for $n > n_0$. Set $\Gamma_{K'} := \{\xi \in K' \mid \|\xi\| = 1\}$. Let now $\bar{\xi} \in \Gamma_{K'}$, we show that there exists a positive integer $n(\bar{\xi})$ and a neighbourhood $V(\bar{\xi})$, such that $\langle \xi, w^n \rangle < 0$, for $n > n(\bar{\xi})$ and $\xi \in V(\bar{\xi})$. Recalling $K'(w^0) \subset K'$ we split the proof in two parts.

1. Let first assume $\bar{\xi} \in \Gamma_{K'(x^0)}$. We have $\langle \bar{\xi}, z^0 \rangle < -\delta < 0$, for some $\delta = \delta(\bar{\xi}) > 0$, and so

$$\lim_{n \rightarrow +\infty} \frac{1}{t_n} \langle \bar{\xi}, w^n - w^0 \rangle = \lim_{n \rightarrow +\infty} \frac{1}{t_n} \langle \bar{\xi}, w^n \rangle = \langle \bar{\xi}, z^0 \rangle < 0$$

Hence there exists $n(\bar{\xi})$ such that $\forall n > n(\bar{\xi})$ it holds $\langle \bar{\xi}, w^n \rangle < 0$.

Now let $\langle \bar{\xi}, w^n \rangle < -\varepsilon < 0$, for some $\varepsilon > 0$ and $n > n(\bar{\xi})$. Then

$$\begin{aligned} \langle \xi, w^n \rangle &= \langle \bar{\xi}, w \rangle + \langle \xi - \bar{\xi}, w^n \rangle < -\varepsilon + \|\xi - \bar{\xi}\| \|w^n - w^0 + w^0\| \leq \\ &\leq -\varepsilon + \|\xi - \bar{\xi}\| (\|w^n - w^0\| + \|w^0\|). \end{aligned}$$

Since clearly $w^n \rightarrow w^0$, we have that for every $\beta > 0$ there exists $n(\beta) > 0$ so that $\|w^n - w^0\| < \beta$. Now we consider $\bar{n} = \max\{n(\beta), n(\bar{\xi})\}$ and we get

$$\langle \xi, w^n \rangle < -\varepsilon + \|\xi - \bar{\xi}\| (\beta + \|w^0\|) < -\frac{1}{2}\varepsilon,$$

as far as $\|\xi - \bar{\xi}\| < \frac{\varepsilon}{2(\beta + \|w^0\|)}$, which defines $V(\bar{\xi})$.

2. Let now assume $\bar{\xi} \in \Gamma_{K'} \setminus \Gamma_{K'(w^0)}$. We have now $\langle \bar{\xi}, w^0 \rangle < -\varepsilon < 0$, for some $\varepsilon = \varepsilon(\bar{\xi}) > 0$. Then:

$$\begin{aligned} \langle \xi, w^n \rangle &= \langle \bar{\xi}, w^0 \rangle + \langle \xi, w^n - w^0 \rangle + \langle \xi - \bar{\xi}, w^0 \rangle < -\varepsilon + \|w^n - w^0\| + \|\xi - \bar{\xi}\| \|w^0\| < \\ &< -\varepsilon + o(n) + \|\xi - \bar{\xi}\| \|w^0\| < -\varepsilon + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < 0, \end{aligned}$$

for n large enough, i.e. $n > n(\bar{\xi})$ and $\|\xi - \bar{\xi}\| < -\frac{\varepsilon}{3\|w^0\|}$, which defines $V(\bar{\xi})$.

Since $\Gamma_{K'}$ is a compact set, we can find a finite number of elements $\xi_1, \dots, \xi_s \in \Gamma_{K'}$ such that $\Gamma_{K'} \subset \bigcup_{i=1}^s V(\xi_i)$. Let $n_0 = \max\{n(\xi_i), i = 1, \dots, s\}$. For $n > n_0$, it holds $\langle \xi, w^n \rangle < 0, \forall \xi \in \Gamma_{K'}$ and hence, $\forall \xi \in K'$. This shows that $w^n \in -\text{int } K \subset -K$ and so points $x^0 + t_n u$ are feasible for $n > n_0$.

From the assumptions, we have $v^0 \in -\text{int } C$, which implies the contradiction $y^n - y^0 \in -\text{int } C$, for n large enough. \square

Remark 4.2 It can be easily derived the following dual form of (8): For all $(v^0, z^0) \in H'(x^0, (y^0, w^0); u)$, there exist $\xi \in C'$ and $\eta \in K'(w^0)$, $(\xi, \eta) \neq (0, 0)$, such that

$$\langle \xi, z^0 \rangle + \langle \eta, v^0 \rangle \geq 0.$$

We present now sufficient conditions in terms of $H'(x^0, (y^0, w^0); u)$ to have (x^0, y^0) a i -minimizer for the constrained problem (1). To do this, we first need the following technical result.

Lemma 4.1 Let x^0 be feasible for problem (1). Suppose there exist vectors $y^0 \in F(x^0)$ and $w^0 \in G(x^0) \cap -K$, such that for some positive A and α it holds

$$D(H(x) - h^0, -(C \times K(w^0))) \geq A \|x - x^0\|^\alpha, \quad \forall x \in U(x^0) \setminus \{x^0\},$$

where $h^0 = (y^0, w^0)$. Then there exists $A' \in \mathbb{R}$ such that

$$D(F(x) - y^0, -C) \geq A' \|x - x^0\|^\alpha, \quad \forall x \in U(x^0) \setminus \{x^0\}.$$

Proof Assume there exists A such that

$$D((F(x), G(x)) - (y^0, w^0), -(C \times K(w^0))) \geq A \|x - x^0\|^\alpha, \quad \forall x \in U(x^0) \setminus \{x^0\}.$$

Set $\theta = (a, b) \in F(x) \times G(x)$ and $\xi = (\xi_1, \xi_2) \in C' \times K'(w^0) \cap S$ (S denotes the unit sphere in $Y \times Z$). Hence the latter means

$$\inf_{\theta} \max_{\xi} (\langle \xi_1, a - y^0 \rangle + \langle \xi_2, b - w^0 \rangle) \geq A \|x - x^0\|^\alpha$$

or equivalently

$$\max_{\xi} (\langle \xi_1, a - y^0 \rangle + \langle \xi_2, b - w^0 \rangle) \geq A \|x - x^0\|^\alpha, \quad \forall (a, b) \in F(x) \times G(x).$$

Let now x be any feasible point, that is there exists, eventually dependent on x , some $b(x) \in G(x) \cap -K$. We can now evaluate the previous inequality along any couple $(a, b(x))$, $a \in F(x)$. Then, certainly $\langle \xi_2, b(x) - w^0 \rangle = \langle \xi_2, b(x) \rangle \leq 0$. Moreover, by assumptions, the maximum should be attained at some $\hat{\xi}$, that is, for every feasible $x \in U(x^0)$, $x \neq x^0$, and for all $(a, b) \in H(x)$ fixed, there exist $\hat{\xi}_1$, $\hat{\xi}_2$, eventually dependent on x , a , b (respectively), such that

$$\langle \hat{\xi}_1, a - y^0 \rangle + \langle \hat{\xi}_2, b - w^0 \rangle \geq A \|x - x^0\|^\alpha.$$

When $b = b(x)$, we have $\langle \hat{\xi}_2, b(x) - w^0 \rangle \leq 0$. Therefore it holds

$$\langle \hat{\xi}_1, a - y^0 \rangle \geq A \|x - x^0\|^\alpha \quad \text{for all } x \in U(x^0) \setminus \{x^0\}, a \in F(x), \quad (9)$$

and $\hat{\xi}_1 \neq 0$. Note that if $\hat{\xi}_1 = 0$, then we would get the contradiction

$$0 \geq \langle \hat{\xi}_2, b(x) - w^0 \rangle \geq A \|x - x^0\|^\alpha > 0.$$

Now, since $(\hat{\xi}_1, \hat{\xi}_2) \in S$, recalling that $\hat{\xi}_1$ depends on x, a, b , we have

$$0 < \sup \left\{ \|\hat{\xi}_1\| \mid x \in U(x^0) \text{ feasible}, a \in F(x) \right\} < \tau < +\infty.$$

Hence for all $a \in F(x)$ from equation (9) one finally gets

$$\frac{1}{\|\hat{\xi}_1\|} \langle \hat{\xi}_1, a - y^0 \rangle \geq \frac{A}{\|\hat{\xi}_1\|} \|x - x^0\|^\alpha \geq \frac{A}{\tau} \|x - x^0\|^\alpha.$$

Putting $A' = \frac{A}{\tau}$ we complete the proof. \square

We characterize now the i -minimizers.

Theorem 4.2 (Sufficient condition for i -minimizer) *Let $C \subset \mathbb{R}^n$ be a closed convex cone, $F : X \rightsquigarrow \mathbb{R}^n$ and $G : X \rightsquigarrow \mathbb{R}^m$ be, respectively, C -Lipschitz and K -Lipschitz. Assume x^0 is feasible for svp (1) and $y^0 \in p\text{-Min}_{CF}(x^0)$. If for some $w^0 \in G(x^0) \cap (-K)$ holds*

$$H'(x^0, (y^0, w^0); u) \cap (-C \times K(w^0)) = \emptyset, \quad \forall u \in X \setminus \{0\}, \quad (10)$$

then (x^0, y^0) is i -minimizer.

Proof The assumptions guarantee that $(x^0, (y^0, w^0)) \in p\text{-Min}_{C \times K(w^0)} H(x^0)$. Then Lemma 4.1 ensures that $(x^0, (y^0, w^0))$ is i -minimizer for the unconstrained problem $\min_{C \times K(w^0)} H(x)$, $x \in X$. Applying Theorem 3.2 we complete the proof. \square

Remark 4.3 The latter condition (10) can be expressed also in a dual form, namely: for all $(v^0, z^0) \in H'(x^0, (y^0, w^0); u)$, and for all $u \in X$, there exists a couple $(\xi^0, \eta^0) \in (C' \times K'(w^0))$, $(\xi^0, \eta^0) \neq (0, 0)$, such that

$$\langle \xi^0, z^0 \rangle + \langle \eta^0, v^0 \rangle > 0.$$

Remark 4.4 The assumption G to be K -Lipschitz in Theorem 4.2 can be replaced with G to be $K(w^0)$ -Lipschitz. Since $K(w^0) \supset K$, the latter assumption is weaker.

Dealing with isolated minimizers of svp (1) we can also prove a reversal of the previous sufficient conditions under the following constraint qualification of Kuhn-Tucker type.

Definition 4.1 We say that the constraint qualification \mathbb{Q} holds for svp (1) at (x^0, w^0) if for any $z^0 \in -K(w^0)$, $z^0 = \lim_{t_k \rightarrow 0^+} \frac{w^k - w^0}{t_k}$, where $w^0 \in G(x^0) \cap -K$, $t_k \rightarrow 0^+$, $w^k \in G(x^0 + t_k u)$, $u \in X$, there exist sequences $u^k \in X$, $\gamma^k \in G(x^0 + t_k u^k) \cap -K$ such that $\gamma^k \rightarrow w^0$ and $u^k \rightarrow u$.

The proof of the necessary condition for i minimizers is based on the following lemmas.

Lemma 4.2 Let E_k be a sequence of sets in Y such that $D(E_k, -C) \geq A$, for all k , and $u^k \in X$ be a sequence converging to some $u^0 \in X$. Then, for any positive number L there exists a positive number A' , such that

$$D(E_k + L \|u^k - u^0\| B_Y, -C) \geq A' \quad \text{for } k \text{ large enough.}$$

Proof Assume ab absurdo that there exists a sequence $\varepsilon_k \downarrow 0$ such that

$$D(E_k + L \|u^k - u^0\| B_Y, -C) \leq \varepsilon_k.$$

Recall that, by definition

$$D(E_k + L \|u^k - u^0\| B_Y, -C) = \inf \left\{ D(y, -C) \mid y \in E_k + L \|u^k - u^0\| B_Y \right\}.$$

Therefore for every fixed k there exists $y^k \in E_k + L \|u^k - u^0\| B_Y$ such that

$$D(y^k, -C) \leq D(E_k + L \|u^k - u^0\| B_Y, -C) + \frac{1}{k},$$

that is

$$\max_{\xi \in C' \cap S_Y} \langle \xi, y^k \rangle \leq D(E_k + L \|u^k - u^0\| B_Y, -C) + \frac{1}{k} \leq \varepsilon_k + \frac{1}{k}.$$

We get

$$\max_{\xi \in C' \cap S_Y} \langle \xi, y^k \rangle \leq \varepsilon_k + \frac{1}{k}$$

for $y^k = e^k + L \|u^k - u^0\| b^k$, $e^k \in E_k$, $b^k \in B_Y$. Hence, by trivial estimations we obtain

$$\begin{aligned} \max_{\xi \in C' \cap S_Y} \langle \xi, e^k \rangle &= \max_{\xi \in C' \cap S_Y} \langle \xi, e^k + L \|u^k - u^0\| b^k - L \|u^k - u^0\| b^k \rangle \\ &\leq \max_{\xi \in C' \cap S_Y} \langle \xi, e^k + L \|u^k - u^0\| b^k \rangle + \max_{\xi \in C' \cap S_Y} \langle \xi, -L \|u^k - u^0\| b^k \rangle \\ &\leq \varepsilon_k + \frac{1}{k} + \max_{\xi \in C' \cap S} \langle \xi, -L \|u^k - u^0\| b^k \rangle \rightarrow 0. \end{aligned}$$

The latter contradicts $D(E_k, -C) \geq A$. □

Lemma 4.3 For any subset $A \subset Y$ it holds

$$D(A, -C) = D(A + C, -C).$$

Theorem 4.2 can be reverted under constraint qualification \mathbb{Q} from Definition 4.1.

Theorem 4.3 *Let x^0 be feasible for svp (1). Assume that the constraint qualification \mathbb{Q} holds for svp (1) at (x^0, w^0) , $w^0 \in G(x^0) \cap (-K)$. Assume the couple (x^0, y^0) , $y^0 \in F(x^0)$, is a i -minimizer for problem (1) and F be C -Lipschitz. Then $y^0 \in p\text{-Min}_C F(x^0)$ and the condition*

$$H'(x^0, (y^0, w^0); u) \cap (-C \times K(w^0)) = \emptyset, \quad \forall u \in X \setminus \{0\}, \quad (11)$$

is satisfied.

Proof There exists a neighbourhood U of x^0 such that, for every feasible $x \in U$ it holds

$$D(F(x) - y^0, -C) \geq A \|x - x^0\|.$$

Assume, by contradiction, that condition (11) does not hold. Thus there exists $(v^0, z^0) \in H'(x^0, (y^0, w^0); u)$ such that $(v^0, z^0) \in -(C \times K(w^0))$. Hence $z^0 \in -K(w^0)$ and it can be written as

$$z^0 = \lim_{k \rightarrow +\infty} \frac{w^k - w^0}{t_k}$$

for some $w^k \in G(x^0 + t_k u)$. Since the constraint qualification \mathbb{Q} holds, then it follows that there exists some suitable sequence $u^k \rightarrow u$, such that for k large enough it holds $G(x^0 + t_k u^k) \cap -K \neq \emptyset$. It follows

$$D(F(x^0 + t_k u^k) - y^0, -C) \geq A t_k \|u^k\|,$$

whence

$$D\left(\frac{1}{t_k} (F(x^0 + t_k u^k) - y^0, -C)\right) \geq A \|u^k\|.$$

Since F is assumed to be C -Lipschitz, we have

$$\frac{1}{t_k} (F(x^0 + t_k u^0) - y^0) \subset \frac{1}{t_k} (F(x^0 + t_k u^k) - y^0) + L \|u^k - u^0\| B_Y + C.$$

It follows

$$\begin{aligned} & D\left(\frac{1}{t_k} (F(x^0 + t_k u^0) - y^0, -C)\right) \geq \\ & \geq D\left(\frac{1}{t_k} (F(x^0 + t_k u^k) - y^0 + L \|u^k - u^0\| B_Y + C, -C)\right) \\ & = D\left(\frac{1}{t_k} (F(x^0 + t_k u^k) - y^0 + L \|u^k - u^0\| B_Y, -C)\right) \geq A' \end{aligned}$$

(the last inequality follows from Lemma 4.2, using also Lemma 4.3). Hence, we have also

$$D\left(\frac{1}{t_k} (y^k - y^0), -C\right) \geq A',$$

where $y^k \in F(x^0 + t_k u^0)$ is such that

$$v^0 = \lim_{k \rightarrow +\infty} \frac{y^k - y^0}{t_k},$$

and by continuity of $D(\cdot, -C)$ also

$$D(v^0, -C) \geq A' > 0.$$

Therefore $v^0 \notin -C$, which completes the proof. \square

The condition of Theorem 4.3 is not true if the constraint qualification is not assumed. The next example gives an illustration even for the case of a single-valued problem.

Example 4.1 Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$, $C = \mathbb{R}_+$, $K = \mathbb{R}_-$. Assume $f(x) = x^3$ and $g(x) = -x^2$, $(x^0, y^0) = (0, 0)$. Since $g(0) = 0 = w^0$, $K(w^0) = \mathbb{R}_-$ and $g'(0) = 0$, we have $g'(0)u = 0 \in \mathbb{R}$ for all $u \in \mathbb{R}$. Therefore constraint qualification 4.1 does not hold. Clearly (x^0, y^0) is the only feasible point and hence an i -minimizer. However condition (11) is not satisfied as one can not find any couple $(\xi^0, \eta^0) \in (C' \times K'(w^0))$ such that

$$\langle \xi^0, f'(x^0)u \rangle + \langle \eta^0, g'(x^0)u \rangle > 0.$$

5 Optimality under convexity type conditions

In general we cannot state the reversal of Theorem 4.1.

Example 5.1 Consider the svf $F : \mathbb{R} \rightsquigarrow \mathbb{R}$, $F(x) = [-x^2, x^2]$ and the point $(x^0, y^0) = (0, 0)$. Assume that the constraint is given by the function $G : \mathbb{R} \rightsquigarrow \mathbb{R}$ defined as $G(x) = [x^2 - 1, x^2 + 1]$, with the point $(x^0, w^0) = (0, -1)$. It is easy to calculate that condition (8) is fulfilled at $(x^0, (y^0, w^0); u)$, $u \in \mathbb{R}$, but the couple (x^0, y^0) is not a w -minimizer.

However, similarly to known results in scalar optimization, a reversal of Theorem 4.1 holds under some convexity type properties of the involved functions. First we state the following lemmas, which we quote from [5].

Lemma 5.1 Let $C \subset \mathbb{R}^m$ be a closed convex cone and $a_1, a_2 > 0$ be two positive numbers. Then $C(a_1)(a_2) \subset C(a_1 + a_2)$.

Lemma 5.2 Let $Y = \mathbb{R}^m$ and $C \subset Y$ be a closed convex cone. Assume the svf $F : X \rightsquigarrow Y$ is C -Lipschitz with constant L in the neighbourhood U of some x^0 , and $y^0 \in F(x^0)$. Suppose that for some $\sigma > 0$ it holds $C(2\sigma) \neq Y$ and $F(x^0) \cap (y^0 - C(2\sigma)) = \{y^0\}$. Then for each $x \in U \cap X_0$ and each $y \in F(x) \cap (y^0 - C(\sigma))$ it holds $\|y - y^0\| \leq (L/\sigma) \|x - x^0\|$.

Lemma 5.3 Let $C \subset \mathbb{R}^m$ be pointed closed convex cone. Then for any $a^1, a^2 \in \mathbb{R}^m$ the set $(a^1 - C) \cap (a^2 + C)$ is bounded.

Theorem 5.1 Let $Y = \mathbb{R}^m$, $Z = \mathbb{R}^l$, and let $C \subset Y$ and $K \subset Z$ be closed convex and pointed cones. Suppose that x^0 is feasible for svp (1), $y^0 \in F(x^0)$, $w^0 \in G(x^0) \cap (-K)$. Assume that F is C -convex along-rays starting at (x^0, y^0) and G is K -convex along-rays starting at (x^0, w^0) . Assume also that for all $u \in X \setminus \{0\}$ there exists $f_u \in Y$ and $g_u \in Z$ such that

$$\begin{aligned} F(x^0 + tu) &\subset y^0 + tf_u + C, \\ G(x^0 + tu) &\subset w^0 + tg_u + K. \end{aligned} \tag{12}$$

Then if condition (8) is satisfied, (x^0, y^0) is a w -minimizer for svp (1).

Proof By convexity along-rays of F and G with respect to the ordering cones, it follows that the map $H(x) = (F(x), G(x))$ is $C \times K$ -convex along-rays starting at (x^0, y^0, w^0) . Set, for simplicity, $h^0 = (y^0, w^0)$. We have

$$(1-t)h^0 + tH(x) \subset H((1-t)x^0 + tx) + (C \times K)$$

for all $x \in X$, $t \in (0, 1)$. Therefore

$$tH(x) - th^0 \subset H(x^0 + t(x - x^0)) - h^0 + (C \times K),$$

and hence, for $u = x - x^0$ it holds

$$H(x) - h^0 \subset \frac{1}{t} (H(x^0 + tu) - h^0) + (C \times K).$$

Hence for all $h \in H(x)$ and $t \in (0, 1)$ there exists $h^t \in H(x^0 + tu)$ and $(c^t, k^t) \in (C \times K)$ such that

$$\frac{1}{t} (h^t - h^0) = h - h^0 - (c^t, k^t).$$

By (12) there exists $h_u = (f_u, g_u)$ such that

$$\frac{1}{t} (H(x^0 + tu) - h^0) \subset th_u + (C \times K).$$

Therefore,

$$\frac{1}{t} (h^t - h^0) \in (h - h^0 - (C \times K)) \cap (h_u + (C \times K)).$$

The latter according to Lemma 5.3 is a bounded set. Hence there exists some sequence $t_k \rightarrow 0^+$ such that

$$\frac{1}{t} (h^t - h^0) \rightarrow \psi^0 \in H'(x^0, h^0, u), \quad \psi^0 = (v^0, z^0),$$

where $v^0 \in F'(x^0, y^0, u)$ and $z^0 \in G'(x^0, y^0, u)$. For $h \in H(x)$, it holds, by convexity

$$h - h^0 = \frac{1}{t_k} (h^{t_k} - h^0) + (c^{t_k}, k^{t_k}),$$

and for all $(\xi, \eta) \in C' \times K'$ it holds

$$\begin{aligned} & \langle \xi, y - y^0 \rangle + \langle \eta, w - w^0 \rangle \\ &= \frac{1}{t_k} (\langle \xi, y^{t_k} - y^0 \rangle + \langle \eta, w_{t_k} - w^0 \rangle) + \langle \xi, c^{t_k} \rangle + \langle \eta, k^{t_k} \rangle \\ &\geq \frac{1}{t_k} (\langle \xi, y^{t_k} - y^0 \rangle + \langle \eta, w_{t_k} - w^0 \rangle). \end{aligned}$$

Passing to the limit as $t_k \rightarrow 0^+$ and by (8) we see that for all $h \in H(x)$, $\xi \in C'$, $\eta \in K'(w^0) \subset K'$ it holds

$$\langle \xi, y - y^0 \rangle + \langle \eta, w - w^0 \rangle \geq 0.$$

Moreover $\langle \eta, w^0 \rangle = 0$ and one can choose $w \in G(x) \cap (-K)$ so that $\langle \eta, w \rangle \leq 0$. So for all feasible $x \in X$ and $y \in F(x)$ it holds

$$\langle \xi, y - y^0 \rangle \geq \langle \eta, -w \rangle \geq 0,$$

that is $y - y^0 \notin -\text{int } C$. Finally, $F(x) \cap (y^0 - \text{int } C) = \emptyset$ for every feasible x . \square

We can test the assumption of the previous theorem by few examples. First we note on an unconstrained example, that the existence of f_u is essential.

Example 5.2 Consider vvp (3) with $X = X_0 = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}_+^2$, and

$$f : \mathbb{R} \rightarrow \mathbb{R}^2, \quad f(x) = (x, -\sqrt{|x|}).$$

Then f is C -convex-along-rays at (x^0, y^0) , where $x^0 = 0$ and $y^0 = (0, 0)$. Condition (6) is satisfied. At the same time (x^0, y^0) is not a w -minimizer. To prove the convexity property of f we must check the inclusion $(1-t)y^0 + tf(x) \in f((1-t)x^0 + tx) + \mathbb{R}_+^2$, for each $x \in \mathbb{R}$ and $0 < t < 1$. This follows from $tf(x) - f(tx) = (0, (\sqrt{t} - t)\sqrt{|x|}) \in \mathbb{R}_+^2$. For the derivative of f we have

$$f'(x^0, u) = \begin{cases} \{(0, 0)\} & , \quad u = 0, \\ \emptyset & , \quad u \neq 0. \end{cases}$$

The second row follows from $(1/t)(f(x^0 + tu) - y^0) = (u, -\sqrt{|u|/t})$. From here obviously for $u \neq 0$ it holds $f'(x^0, u) \cap (-\text{int } \mathbb{R}_+^2) = \emptyset$. For each $x < 0$ we have $f(x) = (x, -\sqrt{|x|}) \in -\text{int } \mathbb{R}_+^2$. Therefore x^0 is not (even local) w -minimizer.

Finally, as an illustration of an application of Theorem 5.1 we present the next example.

Example 5.3 Let $X = X_0 = \mathbb{R}$, $Y = \mathbb{R}^2$, $Z = \mathbb{R}$, and let $C = \mathbb{R}_+^2$, $K = \mathbb{R}_+$. Suppose that $F : X_0 \rightsquigarrow \mathbb{R}^2$ is given by

$$F(x) = \begin{cases} [0, 1] \times [0, 1] & , \quad x \neq 0, \\ ([-1, 0] \times \{0\}) \cup (\{0\} \times [-1, 0]) & , \quad x = 0, \end{cases}$$

and $G(x) = |x| - 1$. Put $x^0 = 0$ and $y^0 = (0, 0)$, $w^0 = -1$. It can be easily checked that G fulfill all the assumptions of Theorem 5.1. To show the C -convexity-along-rays of F at (x^0, y^0) we must check that $tF(x) \subset F(tx) + \mathbb{R}_+^2$ for $0 < t < 1$. For $x \neq x^0$ this is the true inclusion $[0, 1] \times [0, 1] \subset ([0, 1] \times [0, 1]) + \mathbb{R}_+^2$. For $x = x^0$ the validity follows from the true inclusion $[-t, 0] \subset [-1, 0]$. Easy calculations give that

$$F'(x^0, y^0; u) = \begin{cases} \mathbb{R}_+^2 & , \quad u \neq 0, \\ (\mathbb{R}_- \times \{0\}) \cup (\{0\} \times \mathbb{R}_-) & , \quad u = 0, \end{cases}$$

and $G'(x^0, w^0; u) = |u|$, whence it is obvious that condition (8) is satisfied. Further, for $u \neq 0$ the vectors $f_u = (0, 0)$ and $g_u = 0$ satisfy conditions (12). Then (x^0, y^0) is a global w -minimizer, which follows from Theorem 5.1.

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