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Executive Summary

In any dynamic model of the economy with changing population, population should properly be one of the state variables of the system. It enters both in the maxim and, at least under total utilitarianism, and into the production function in one way or another. If population growth is exponential and there are constant returns to scale, then a simple transformation to *per capita* variables can be used to eliminate one state variable. However, this simple transformation cannot be made if growth is not exponential, as it obviously is not and cannot be. If the growth of population is exogenous, then introducing it into the system does not affect the optimal policy. However, if one asks whether the system is sustainable, in the sense of at least maintaining total welfare (integral of discounted utilities), then the criterion is that the value of the rates of change of the state variables is non-negative, so that the shadow price of population becomes relevant. In this paper, we derive explicit formulas in a simple model, showing that the rate of growth of *per capita* capital is not the correct formula but must have other terms added to it. We also study the question under an alternative criterion of long-run average utilitarianism.

The Genuine Savings Criterion and The Value of Population[1]

Kenneth J. Arrow, Partha Dasgupta, and Karl-Göran Mäler

The idea of systematic planning for the future (whether by individual economic agents or by the collectivity) was implicit in economic theory since the late nineteenth century. It had been given more explict though still not very usable form in the 1930s with the work of Erik Lindahl [1929, 1939] and John R. Hicks [1939]. But dynamic planning became a practical possibility with the nearly contemporaneous work of two mathematicians, Richard Bellman on dynamic programming [1957] and L. S. Pontryagin and associates on optimal control theory [1962]. The two approaches are equivalent; each has technical advantages and disadvantages of its own. However, in many ways, optimal control theory is closer to standard economic thinking, and it has been the preferred approach, particularly in theoretical work.

Optimal control theory started being applied by economists fairly soon after being published in book form. One of the earliest applications was the work of Kenneth J. Arrow and Mordecai Kurz [1970].[2] It discussed the criteria for optimal public investment policy using the tools of optimal control theory to clarify much of the existing literature and to introduce new concepts. The present paper continues the intellectual impetus of the Arrow-Kurz book and brings some new considerations to bear.

[1] An earlier version of this paper was presented at a celebration of Mordecai Kurz's 66th birthday at Stanford University, 1-3 August 2002.
[2] It was a pleasure and an educational experience to have the opportunity for this collaboration. (KJA)

I. INTRODUCTION

We deal with a set of ideas with regard to control of the economy. Much of the stimulus has come from an increasing attention to the role of the environment and ecological factors in general. The argument that there are many forms of capital supplied by nature beyond the reproducible capital usually emphasized in growth models recognizes the importance of environmental and ecological factors

Two traditions are drawn on, though our conclusions go beyond those in the literature. One is the study of the criteria for evaluating policies when population is varying. The other is the question, whether and to what extent a given policy is causing a gain in aggregate welfare, what has come to be called the measurement of "genuine savings." The term, "sustainability," has been much used, especially since its adoption in the Brundtland Commission report of 1987. One interpretation of sustainability is a positive value of genuine savings.

The aim of this paper is to give a rigorous analysis of the role of varying population in measuring genuine savings, i.e., in giving a criterion for improvement in welfare. We argue that the only consistent approach is to recognize population as another form of capital (state variable); this does not exclude *a priori* its having a negative value, as many have argued. This will hold even if we do not consider population policy and regard the evolution of population as exogenous to the economic and policy variables. The main aim of the paper is to derive the accounting price for population (costate variable).

It should be emphasized that we make no claim that our approach is relevant to choice of population policy. There are deep ethical problems in comparing alternative sizes and compositions of population, and we make no claim to having addressed them. As already stated, we restrict ourselves to cases in which population growth is not affected by any control variables.

In the existing literature, varying population is usually modeled by a constant rate of growth of population. In this case, it is easy to measure the various kinds of capital on a *per capita* basis. The implications of constant exponential growth are obviously absurd, and certainly the dramatic reductions in birth rates throughout the world make such an assumption a poor guide to analysis.

As an introduction to the subject, we confine ourselves to the case where an economy is following an optimal course according to the fairly conventional criterion of maximizing the sum of discounted utilities of consumption. Further, we confine ourselves to the case of one form of capital made of a good which can be used indifferently for consumption and capital formation. The extension to many forms of capital does not offer any essential difficulties. The extension to accounting for growth in non-optimal policies is probably more difficult in practice if not in theory, but we think the present work provides a beginning.

In the next section, we review a broad model and the concept of genuine savings as expressed in it. Then we consider in particular the role of population, which enters both into the maxim and into the production function as labor. The evolution of accounting prices is then found.

The basic analysis is in the tradition of "total utilitarianism," i.e., the criterion is the sum of the utilities. In the last section, we see how the sustainability criterion is modified when a dynamic form of "average utilitarianism" is considered.

This work is part of a broader research program of the authors (see Arrow, Dasgupta, and Mäler [forthcoming]) on the measurement of genuine savings as a criterion for sustainability. The full program includes measurement in non-optimal paths and under non-convex environments.

II. GENUINE SAVINGS

A general class of models of the economy over time takes as its criterion for choice among alternative policies,

$$\text{Max! INT } (0, \text{infinity}) \exp(-\delta t)\, U(c_t)\, dt, \qquad (1)$$

subject to various constraints. Among these constraints is a conservation law for produced goods, output equals consumption plus capital formation,

$$dK/dt = F(.) - c, \qquad (2).$$

where *K* is produced capital, *c* is consumption; for the moment, we leave the arguments of the production function *F* unspecified, but they are all forms of capital, including produced capital. Let *V(t)* be the integral of utility from *t* on discounted to *t*.

$$V(t) = \text{INT } (t, \text{infinity}) \exp[-\delta(u-t)]\, U(c_u)\, du. \qquad (3)$$

In an autonomous system (where all the capital variables completely determine the future for any given policy, including the optimal policy), V(t) is completely determined by the values of the capital variables at time t. Then the costate variable (accounting price) for state variable K is,

$$P_K(t) = \text{partial } V/\text{partial } K,$$

and, for other capital variables (forms of natural capital, including mineral resources, human capital, knowledge, etc.), similarly, the shadow prices, p_i are the partial derivatives of V with respect to the corresponding capital variables, K_i. Then,

$$dV/dt = p_K (dK/dt) + \Sigma\, p_I (dK_i/dt). \qquad (4)$$

In this context, Pezzey [1992] proposed a reasonable definition of "sustainability."

DEFINITION 1. The path is sustainable at time *t* if dV/dt > 0 then.

From (4) and Definition 1, evaluation requires determining the net formation of each kind of capital and the corresponding accounting prices.

THEOREM 1. A path is sustainable at time *t* if and only if,

$p_K (dK/dt) + \Sigma\, p_I (dK_i/dt). > 0$ then.

Since this expression is a weighted sum of the net formations of all the different kinds of capital, it has come to be called, "genuine savings." It has been developed with varying degrees of formality by Hamilton [1994], Pearce, Hamilton, and Atkinson [1996], and Dasgupta and Mäler [2000]. Empirical estimates based on this concept have appeared in Hamilton and Clemens [1999].

Note that V is measured in utility units. A variation of (4), with the same sign, is obtained by dividing through by p_K. If we let $q_I = p_i/p_K$, then sustainability is defined by the condition that,

$$(1/p_K)(dV/dt) = (dK/dt) + \Sigma\, q_I (dK_i/dt) > 0, \qquad (5)$$

which is expressed in commodity terms. Also note that the current-value Hamiltonian is given (in part) by,

$$H = U(c) + p_K [F(.) - c] + \ldots,$$

where the omitted terms are based on the equations of motion of the other types of capital. If c does not occur in the equations of motion for the other capital stocks, we have, by the Maximum Principle, that,

$$U'(c) = p_K, \quad (6)$$

an occasionally useful relation.

III. POPULATION AS A VARIABLE

In our model, population is assumed to be independent of economic conditions but evolving according to some laws. The analysis will treat it as another form of capital. Let *N(t)* be population at time *t*, and also labor force (they can be distinguished in a more sophisticated model). Then, *N* enters both the maximand (though this has sometimes been disputed) and the production function.

As usual, we assume that individual consumption, *c(t),* is the same for all. There is a single good, the production function for which is *F(K, N)*, concave with constant returns to scale.

The objective (felicity) for a single period will be taken to be N U(c). The literature on this subject is vast and will not be reviewed here; the dispute goes back to the pioneers of utilitarianism, Henry Sidgwick and Francis Y. Edgeworth. In determining the optimal accumulation policy, it seems hard to deny something like this. Any idea of treating people more or less equally implies that if tomorrow's population is bigger, it should get proportionately more weight. We are still weighing people according to their futurity (discounting) but not according to the numbers of their contemporaries. Then (3) becomes,

$V(t) = INT (t, infinity) [exp (-\delta(u-t)] N(u) U(c_u) du$, (7)

and the criterion of optimality is,

Max! V(0). (8)

The equation of motion for produced capital is the obvious modification of (2),

$dK/dt = F(K, N) – Nc$, (9)

We need to make an assumption about the evolution of population. Since we want to exclude the dependence of population on economic conditions, clearly population growth must be a function of N only.

$dN/dt = \phi(N)$. (10)

Some but not all formulas will simplify if we write,

$\phi(N) = \nu(N) N$, (11)

where $\nu(N)$ is the (relative) rate of growth. As far as we know, virtually all models which have introduced changing population have assumed $\nu(N)$ constant. A somewhat more acceptable formulation is that giving rise to the logistic curve,

$\phi(N) = A N (N^* - N)$. (12)

Then, the genuine increase in wealth in commodity units, (5), is,

$dK/dt + q dN/dt$, (13)

where *q* is the ratio of the costate variable for *N* to that for *K*.

Expression (15) takes a slightly simpler form when divided by *N* (*per capita* genuine savings); the sign is unaltered. Let k = K/N.

$(1/N) [(dK/dt) + q (dN/dt)] = dk/dt + (q+ k) \nu(t)$. (14)

Before giving an explicit expression for *q(t) + k(t)*, it is worth thinking about (14). It would strike most students and laymen as reasonable to look at the increase in *per capita* capital as a measure of sustainability. This would hold if, $q+k = 0$. This has an intuitive basis; indeed, the demographic literature (see Leibenstein [1971]) has noted that faster population growth is costly (apart from Malthusian effects) because it requires higher capital accumulation (and therefore lower consumption) to maintain the same capital-labor ratio. This is precisely and more accurately captured by the second term in (14). A careful analysis shows that we cannot take q(t) + k(t) to be zero. In fact, we have,

$$q(t) + k(t)$$

$$= \text{INT } (t, \text{infinity}) [R(u)/R(t)]\{\phi[N(u)]/\phi[N(t)]\}\{L(c_u) - \nu'[N(u)]\, K(u)\}\, du, \qquad (15)$$

where,

$$R(t) = \exp\{ - \text{INT } (0, t)\, F_K [K(u), N(u)]\, du\}, \qquad (16)$$

$$L(c) = U(c)/ U'(c), \qquad (17)$$

Thus the "benefit term" is discounted at the marginal productivity of capital. Note that if we assume, as is natural, that the rate of growth of population, ν, decreases as population grows (at least for large populations), then, $q(t) + k(t) > 0$, so that genuine savings exceed

increases in *per capita* capital. This does not mean that population itself is a good; that depends on the sign of *q*, which may itself easily be negative.

We have not succeeded in making the terms and factors in (15) entirely intuitive. However, the important term defined in (17) does have an interpretation as being, in a sense, the "value of life." As is commonly done, this is interpreted to mean the (compensated) willingness to pay for a marginal increase in the probability of survival. The indifferences curves between consumption and probability, *p*, of survival are the curves on which *p U(c)* is constant, so that, p dc/dp = - U(c)/U'(c). If we start from a situation where the probability of survival is 1, then indeed *L(c)* is the value of life, and its presence in the accounting price for population is natural.

The computations leading to (15-17) are relegated to the Appendix.

THEOREM 2. The optimal path for the model with varying population is sustainable if and only if,

$dk/dt + (q+ k)\, n(t) > 0,$

where q(t) + k(t) is defined by (15-17). If the rate of growth of population decreases as population increases, then q(t) + k(t) > 0, so that sustainability is possible even if *per capita* capital is decreasing.

The only example simple enough to be illustrative is the case of constant population growth. Then, $\nu'(N) = 0$, and also,

$$\phi[N(u)]/\phi[N(t) = N(u)/N(t),$$

so that, q(t) + k(t) is the discounted value of the total value of life.for the entire population.

Note that the value of life is evidenced by willingness to spend on avoiding death (e.g., medical expenditures) and on raising children. Thus, (1) it reflects a revealed preference, and (2) it is capable of measurement from observed quantities.

IV. AN ALTERNATIVE CRITERION

Dasgupta [2001, pp. 258-9] presents an alternative criterion for measuring welfare and consequently genuine savings in a world of changing population. Rather than the total, we take the expected utility of a random individual out of the present and future chosen with probabilities weighted by futurity, i.e., the probability density of an individual *t* years hence is proportional to exp (- δt). This may be regarded as a dynamic version of average utilitarianism. Define, then,

$$V^*(t) = V(t)/N^*(t), \qquad (18)$$

where

$$N^*(t) = \text{INT } (t, \text{infinity}) [\exp(-\rho(u-t)] N(u) du, \qquad (19)$$

and *V(t)* is defined by (7). He shows that with this criterion, production under constant returns to scale, and exponential rate of growth of population, then genuine savings are measured by *per capita* wealth.

Here, we reexamine the issue for our more general assumptions about population growth. Since we are taking the time path *N(t)* to be determined exogenously, the optimal policy is unaltered. However, the accounting prices for *K* and *N* become,

$$p_K^* = (\text{partial } V^*)/(\text{partial } K), \; p_N^* = (\text{partial } V^*)/(\text{partial } N).$$

If we define,

$$q^* = p_K^*/ p_N^*,$$

the sustainability criterion becomes,

$$dK/dt + q^* (dN/dt) > 0,$$

or, equivalently, as in (14),

$$dk/dt + (q^* + k)\, n(t) > 0. \qquad (20)$$

It will be shown in the Appendix that,

$$q^* = q - (V^*/p_K)\, [\delta N^*(t) - N(t)]/\phi[N(t)]. \qquad (21)$$

DEFINITION 2. The Dynamic Average Utilitarian criterion, at any time *t* is,

$$V^*(t) = V(t)/N^*(t),$$

where *V(t* and *N*(t)* are defined in (7) and (18), respectively,

THEOREM 3. Under the Dynamic Average Utilitarian criterion, sustainability is defined by (20) and (21).

It is important to note that the sustainability criterion of Theorem 2 is not invariant under an additive shift in the utility function, even though the optimal path is. If one adds a constant *h* to the utility function, then *V(t)* is increased by *h N*(t)*, which depends on *N(t)*, so that the accounting price of *N* is altered. However, *V*(t)* is increased by the constant *h*, so that the accounting prices are unaltered.

APPENDIX

As promised, we here sketch the derivation of equations (15) and (21).

For equation (15), the Hamiltonian for the total utilitarian criterion (7) with the dynamic equations (8) and (9) is.

$H = N\,U(c) + p_K\,[F(K, N) – c\,] + p_N\,\phi(N).$

Hence, the equations of motion for the accounting prices are,

$$dp_K/dt = p_K\,(\delta - F_K), \qquad (22)$$

$$dp_N/dt = p_N\,[\delta - \phi'(N)] – U(c) - p_K\,(F_N - c). \qquad (23)$$

Since $q = p_N/\,p_K$,

$$(1/q)\,(dq/dt) = (1/p_N)\,(dp_N dt) – (1/p_K)\,(dp_K/dt). \qquad (24)$$

Divide through in (22) and (23) by p_K and p_N, respectively, substitute into (24), use the definition of *q*, and multiply both sides by *q*. Then,

$$dq/dt = (F_K - \phi')q – [U(c)/] + c- F_N\,.$$

In the accumulation equation for capital, (9), use Euler's theorem to replace *F(K, N)* by $F_K K + F_N N$; then we can deduce,

$$dk/dt = F_K\,k + F_N – c - \nu(N)\,k.$$

Adding the last two equations and setting $p_K = U'(c)$, by (6), yields,

$$d(q + k)/dt = (F_K - \phi')\,(q + k) + \nu'\,K – [U(c)/\,U'(c)].$$

Replace *t* by *u*, integrate from *t* to infinity, and use the transversality conditions. Then (15-17) follow.

To deduce (21), first take the partial derivatives of (18) with respect to *K* and *N*.

$$p_K^* = p_K/N^*,$$

$p_N^* = (p_N/N^*) – (V^*/N^*) (dN^*/dN).$

Then,

$q^* = q - (V^*/ p_K) (dN^*/dN).$

It remains to compute the last factor. Since *N(u)* is completely determined by *N(t)* for all u >=t, *N*(t)* is determined by *N(t)*. It follows that,

$$dN^*/dN = (dN^*/dt)/(dN/dt). \quad (25)$$

But, from the definition (19), it follows immediately that,

$dN^*/dt = \delta N^* - N,$

while,

$dN/dt = \phi(N),$

by (10), so that (21) follows from (25).

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