View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Research Papers in Economics

Rock-Paper-Scissors; A New and Elegant Proof

by

A. van den Nouweland*

September 12, 2007

Abstract

I provide an elegant proof identifying the unique mixed Nash equilibrium of the Rock-Paper-Scissors game. The proof is based on intuition rather than elimination of cases. It shows that for any mixed strategy other than the one that puts equal probability on each of a player's actions, it holds that this strategy is not a best response to any mixed strategy that is a best response to it.

1 Introduction

The game of Rock-Paper-Scissors is a popular example in textbooks (such as, for example, Osborne 2004, Page 141). It is well-known that this game

*Department of Economics, University of Oregon, Eugene, OR 97403-1285, USA, and Department of Economics, the University of Melbourne, Australia. E-mail: annev@uoregon.edu. I thank my students for motivating me to find the proof presented in this note, Myrna Wooders for encouragement to submit it for publication, and Van Kolpin for helpful suggestions.

has a unique mixed Nash equilibrium in which each player plays each of the actions Rock, Paper, and Scissors with equal probability.[1] The proof of this fact consists of two parts – one in which it is shown that the strategy profile mentioned is a Nash equilibrium and another in which it is shown that no other strategy profile is a Nash equilibrium.The proof of the second part usually proceeds by distinguishing various cases (for example, the number of actions in the support of a player's mixed strategy) and showing that in each of these cases no Nash equilbrium can be found. In my opinion, this is a fairly messy and not very intuitive approach.

In this note I provide an elegant proof identifying the unique mixed Nash equilibrium of the Rock-Paper-Scissors game. The proof is based on intuition and shows that for any mixed strategy other than the one that puts equal probability on each of a player's actions, it holds that this strategy is not a best response to any mixed strategy that is a best response to it. Therefore, such a strategy cannot be part of a Nash equilbrium.

2 The game of Rock-Paper-Scissors

The well-known game of Rock-Paper-Scissors (RPS from now on) is the 2-player zero-sum game in which players simultaneously each choose either Rock, Paper, or Scissors. The game ends in a draw when players' action choices are the same. If players choose different actions, then one wins and the other loses according to Rock beats Scissors, Scissors beats Paper, and Paper in turn beats Rock. The winning player's payoff is 1 and the losing player's payoff is -1. The game is represented in the following table, where

[1] Of course, this fact follows easily from general results for zero-sum games, such as can be found in, for example, Raghavan 1994. However, in this note I am concentrating on proofs that are specifically for RPS.

I follow the usual convention that player 1 is the row player and player 2 is the column player.

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

3 Mixed Nash Equilibrium in RPS

Each player's set of possible actions is denoted $A = \{Rock, Paper, Scissors\}$ and $\Delta(A) = \{(p(R), p(P), p(S)) \in \mathbb{R}^3 \mid (p(R), p(P), p(S)) \geq (0,0,0)$ and $p(R) + p(P) + p(S) = 1\}$ denotes the set of probability distributions on A. A *mixed strategy* for player i is a $p_i = (p_i(R), p_i(P), p_i(S)) \in \Delta(A)$, whose interpretation is that the player plays actions Rock, Paper, and Scissors with probabilities $p_i(R)$, $p_i(P)$, and $p_i(S)$, respectively.[2] Actions are special cases of mixed strategies because if player i plays an action a_i, this is equivalent to player i playing the mixed strategy p_i that puts probability 1 on action a_i and probability 0 on all other actions of player i.

Denoting a player i's payoff when the action pair $(a_i, a_j) \in A \times A$ is played by $U_i(a_i, a_j)$, player i's *expected payoff* for a pair of mixed strategies $(p_i, p_j) \in \Delta(A) \times \Delta(A)$ equals $EU_i(p_i, p_j) = \sum_{(a_i,a_j) \in A \times A} p_i(a_i) p_j(a_j) U_i(a_i, a_j)$.

A pair of mixed strategies (p_i, p_j) is a *mixed Nash equilibrium* if for every player i and every alternative mixed strategy $p_i' \in \Delta(A)$ of player i it holds that $EU_i(p_i', p_j) \leq EU_i(p_i, p_j)$.

In the proof that follows below, I use best responses and properties of Nash equilibria based on best responses. The facts mentioned below are all

[2] Throughout this note, whenever I use i, it is implicitly understood that this refers to a player and that $i \in \{1, 2\}$. Also, whenever j is used in addition to i, it is implicitly understood that this refers to the other player and that $j \in \{1, 2\}$ and $j \neq i$.

well-known, but for the sake of making this note self-contained I list them.

When player j plays a specific mixed strategy p_j, a strategy p_i by player i is a *best response* to p_j if $EU_i(p_i, p_j) \geq EU_i(p'_i, p_j)$ for every $p'_i \in \Delta(A)$.

Fact 1. A pair of mixed strategies (p_i, p_j) is a mixed Nash equilibrium if and only if player i's strategy is a best response to player j's strategy and vice versa.

Fact 2. If (p_i, p_j) is a strategy profile and every action $a_i \in A_i$ that player i plays with positive probability ($p_i(a_i) > 0$) is at least as good a response to p_j as every other action (i.e. $EU_i(a_i, p_j) \geq EU_i(a'_i, p_j)$ for all $a'_i \in A$), then p_i is a best response to p_j.

Fact 3. If $p_i \in \Delta(A)$ is a best response to $p_j \in \Delta(A)$ and player i plays action $a_i \in A_i$ with positive probability, i.e. $p_i(a_i) > 0$, then a_i is at least as good a response to p_j as every other action (i.e. $EU_i(a_i, p_j) \geq EU_i(a'_i, p_j)$ for all $a'_i \in A$).

Theorem 1 *The game of Rock-Paper-Scissors has a unique mixed Nash equilibrium. In this equilibium, both players play the mixed strategy that puts equal probabilities on all three actions.*

Proof. Part 1. First I prove that the strategy profile $\big((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\big)$ is a mixed Nash equilibrium. If player j plays $p_j = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, then player i's expected payoff from each of the three actions equals 0; $EU_i(a_i, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})) = \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times (-1) = 0$ for all $a_i \in A$. Hence, it follows using Fact 2 that $p_i = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a best response to p_j. By Fact 1 I can then conclude that $\big((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\big)$ is a mixed Nash equilibrium.

Part 2. I now prove that a strategy profile in which a player plays a mixed strategy different from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is not a mixed Nash equilibrium. Suppose player i plays mixed strategy

$$p_i = (p_i(R), p_i(P), p_i(S)) \neq (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

Without loss of generality, I assume that player i plays action Rock with higest probability, i.e. $p_i(R) \geq p_i(P)$ and $p_i(R) \geq p_i(S)$. It then necessarily holds that

$$p_i(R) > \frac{1}{3}.$$

Now consider player j's expected payoffs from his actions Paper and Scissors, the actions that beat and are beaten by Rock, respectively. $EU_j(p_i, P) = p_i(R) \times 1 + p_i(P) \times 0 + p_i(S) \times (-1) = p_i(R) - p_i(S)$ and $EU_j(p_i, S) = p_i(R) \times (-1) + p_i(P) \times 1 + p_i(S) \times 0 = -p_i(R) + p_i(P)$. Using that $p_i(S) = 1 - p_i(R) - p_i(P)$, I derive $EU_j(p_i, P) - EU_j(p_i, S) = 3p_i(R) - 1 > 0$. Hence, by Fact 3 I know that

$$p_j(S) = 0 \text{ for every mixed strategy } p_j \text{ that is a best response to } p_i.$$

Now, suppose $p_j(S) = 0$ and consider player i's expected payoffs from his actions Rock and Paper, the actions that beat and are beaten by Scissors, respectively. $EU_i(R, p_j) = p_j(R) \times 0 + p_j(P) \times (-1) + p_j(S) \times 1 = -p_j(P)$ and $EU_i(P, p_j) = p_j(R) \times 1 + p_j(P) \times 0 + p_j(S) \times (-1) = p_j(R)$. Because $p_j(S) = 0$, it holds that $p_j(R) + p_j(P) = 1$, so that either $p_j(R) > 0$ or $p_j(P) > 0$ or both. In either case, $EU_i(R, p_j) < EU_i(P, p_j)$. Therefore, by Fact 3 I know that

$$p_i(R) = 0 \text{ for every mixed strategy } p_i \text{ that is a best response to } p_j.$$

I have now shown that a mixed strategy other than $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is not a best response to any mixed strategy that is a best response to it. Therefore, using Fact 1, I derive that there is no mixed Nash equilibrium in which a player plays a strategy different from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Q.E.D.

4 Applicability

The proof technique that I have used, namely to prove that a strategy cannot be a best response to any strategy that is a best response to it, can in principle be applied to any game in strategic form. That is not to say that it will provide a less messy and more intuitive answer for each game, as that will depend on the characteristics of the specific game. But I think this proof technique is a valuable tool in a game theorist's tool box and it can be used in combination with other techniques as well.

For example, in the Bertrand price-setting game with two firms producing homogeneous goods with the same constant marginal costs of production, the proof technique in this note can be used to show that any pair of prices in which one firm sets a price below the marginal cost cannot possibly be a Nash equilibrium. This is so because if firm i sets a price $p_i < c$, where c denotes the marginal cost of production, then a best response by firm j is to set any price $p_j > p_i$. Firm i will then satisfy demand for it's product at a price below cost and make a loss, whereas it can have no loss if it sets a price equal to c. Hence, $p_i < c$ is not a best response to any price p_j that is a best response to it. If there is a smallest unit of money, then the proof technique can also be used to show that any pair of prices in which one firm sets a price more than one unit above the marginal cost cannot possibly be a Nash equilibrium. This is so because if firm i sets a price $p_i > c + e$, where e denotes the smallest unit of money, then a best response by firm j is to set a price $p_j = \min\{p^m, p_i - e\}$, where p^m denotes the monopoly price in the market. Firm i can then increase its profit by setting a price equal to p_j so that demand for it's product will be positive. Hence, $p_i > c + e$ is not a best response to any price p_j that is a best response to it. Now that it has been determined that in a Nash equilibrium player i has only two possible prices, namely $p_i = c$ and $p_i = c + e$, the Nash equilibria can be easily identified by finding the best responses to these two prices as $p_j = c$ and $p_j = c + e$, respectively.

5 References

Osborne, M.J. "An Introduction to Game Theory," Oxford University Press, 2004.

Raghavan, T.E.S. "Zero-sum two-person games." In: R. Aumann and S. Hart (Eds.) "Handbook of Game Theory," Volume 2, Elsevier North-Holland, 1994, pp. 735-759.