Goal-Based Investing with Cumulative Prospect Theory and Satisficing Behavior

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Abstract

This paper presents a time-continuous goal-based portfolio selection model with cumulative prospect theory preferences and satisficing behavior, where investors optimally split their wealth among several investment goals at different horizons. The paper extends the model of Berkelaar, Kouwenberg and Post (2004) to account for multiple-goals. We show that when the discounted values of all target wealths is not too high relative to the initial wealth (i.e., goals are not too ambitious), investors mainly invest to reach short-term investment goals and adopt safe investment strategies for this purpose. However, when goals are very ambitious, they put a high proportion of their wealth in long-term goals and adopt aggressive investment strategies with high leverage to reach short-term goals and the overall investment strategy also displays high leverage. High incentives to reach ambitious short-term goals (high target returns) and the consequent excessive leverage have been identified as causes for the global financial crisis erupted in 2008.

Keywords:
behavioral finance, portfolio selection, mental accounting, narrow framing, cumulative prospect theory, satisficing, loss aversion, goal-based approach.

JEL Classification:
D10, G11.
1 Introduction

Portfolio selection models typically assume that investors derive utility from consumption or total wealth. In this paper we consider the case of investors with several investment goals, possibly at different time horizons, who derive utility from portfolio payoffs’ relative to goals’ specific target payoffs. Additionally, they narrowly frame their investment goals, i.e., consider each investment goal separately from the others.

This way of structuring the portfolio decision problem derives from how people tend to organize and evaluate their investments, that is mental accounting as defined by Thaler (1985). From this perspective, investors do not consider the overall portfolio performance, but evaluate investment decisions relative to whether or not they have been able to reach their investment goals. Since goals are often related to specific achievements in people’s life (e.g., retirement, own house) investor tend to keep them into different mental accounts (e.g., “I have invested for my retirement”, “I have invested to buy an house”). Accordantly, each investment goal defines a unique decision problem and a goal’s specific portfolio is determined.

1There is also considerable evidence that managers do think in terms of goals as given by target returns; Mao (1970), for instance, interviews a number of executives for their definition of risk, and one representative response from these executives is “Risk is the prospect of not meeting the target rate of return.”

2Full separation of goals’ specific portfolios is a rather extreme assumption on how mental accounting might affect investors’ decisions. In the model presented in this paper, goals are partially integrated since investors decide how to optimally split wealth among them. Accordantly, if one goal is more ambitious (or more important) than other goals, investors might put a higher proportion of their wealth on it and use a safe investment strategy for it. On the other hand, less wealth will be available for the other goals,
There is a growing consensus among wealth managers, that mental accounting should be addressed when structuring the advisory process of financial institutions. Adherent to this line of thoughts, advocate the use of goal-based approaches to the asset allocation problem; see Brunel (2006), Chhabra (2005), Nevin (2004) and Brunel (2003). They argue that an investment strategy is only useful when investors are able to follow it even under unfavorable market conditions. Goal-based models are inspired by how individuals look at their investment strategies. Goal-based approaches are two-step approaches. First, the investor decides how to split her wealth among the different investment goals. Second, each investment goal is treated separately and a specific portfolio decision problem is solved. This considerably simplifies the portfolio decision process and facilitates the interaction between investors and financial advisors, so that investors are more likely to understand and follow their strategies. By contrast, traditional portfolio models might force investors into pre-defined modeling frameworks which do not necessarily reflect their goals, biases and risk perception. Without accounting for these characteristics, it is hard to believe that investors will stick at their asset allocation also during unfavorable market conditions. Indeed, individual investors displays high trading levels, which causes huge performance penalties (Barber and Odean 2000).

This paper presents a time-continuous goal-based model that uses cumulative prospect theory (CPT) of Tversky and Kahneman (1992) to describe investors’ preferences. The paper extends to a multiple-goal setting the model presented by Berkelaar, Kouwenberg, and Post (2004). Cumulative prospect theory is one of the pillars of behavioral economics which might force investors to adopt aggressive investment strategies.
and behavioral finance and has successfully been applied in finance to address several puzzles arising from standard economic models based on expected utility theory, both concerning individual portfolio choices and asset prices; see Barberis and Thaler (2003) for an overview. Being a referent-dependent theory of choice, cumulative prospect theory provides an adequate setting for integrating investment goals into a portfolio theory.

Cumulative prospect theory assumes that investors display a risk seeking behavior on losses (e.g., payoffs below the reference point): investors are willing to take risk in order to avoid missing their investment goals for sure. This behavior has been documented in several experimental works. Recently, the risk attitude of fund managers has also been related to their contractual incentives. Dass, Massa, and Patgiri (2008) found that mutual fund managers with high contractual incentives to rank at the top (i.e., those with more ambitious investment goals) adopted riskier investment strategies.

Cumulative prospect theory also assumes that investors are loss averse. Loss aversion is the observation that people usually require $n > 1$ units of payoff above the reference point in order to be compensated for one unit of payoff below the reference point, where $n$ can be seen as the degree of loss aversion. However, when the investors’ only objective is to reach several predefined goals, it seems plausible to assume that any payoff above the target payoff is considered as fully satisfactory, and additional units of payoff above the reference point in scenarios where they already reach their goals do not compensate them for scenarios where they fail their goals. When investors are not fully satisfied with a portfolio’s payoff that is (just slightly) above their target payoff, then they should bet-
ter re-define their goals and aim to a higher target payoff. By contrast, we believe that investors strictly prefer being just slightly below the target payoff to being well-below it. The behavior we just described is strictly related to the concept of satisficing introduced by Simon (1955) and we thus call it *satisficing behavior*.

Simon (1955) argues that people’s computational abilities are limited and we should better model their preferences using simple value functions, which only take, e.g., three values -1, 0 or 1, with the interpretation that the outcome is unsatisfactory, neutral, or satisfactory, respectively. Simon (1955) also states that whether or not a given outcome is considered satisfactory depends on a pre-defined aspiration level (or reference point in case of monetary alternatives). We point out that satisficing behavior does not mean that people’s final objective is not to optimize. People who display satisficing behavior might converge to optimal solutions as they redefine their investment goals. First, they define goals and are fully satisfied when they reach them. Second, they define new goals which are more ambitious than the previous ones. As in Simon (1955) we assume that payoffs above the reference point are satisfactory; in our model this means that they all have value zero, as the reference point. Differently from Simon (1955) we allow investors to attribute different values to payoffs below the reference point.

In this paper we use a slightly different specification of the CPT-value function if compared to Tversky and Kahneman (1992). While we also adopt the piecewise-power and kinked value function, we use two different parameters to specify the degree of loss aversion. This doesn’t have any impact on investors optimal strategies, but it allows us to
also characterizes investors with satisficing behavior. This is the case when the degree of loss aversion is very high. The intuition for this is straightforward: when investors’ only objective is to reach their goals and all payoffs above the reference point are considered as fully satisfactory, then any payoff (even if large) in scenarios where goals will be reached do not compensate investors for failing the reference point in other scenarios.

The goal-based model presented in this paper is well-suited to study how investors allocate their wealth between short-, medium- and long-term investment goals. We address the question whether investors with several investment goals at different horizons mainly invest to reach short-, medium or long-term investment goals. It is well-known that cumulative prospect theory investors put a smaller proportion of their wealth in stocks when the investment horizon is short; see Benartzi and Thaler (1995) and Berkelaar, Kouwenberg, and Post (2004). By contrast, when the investment horizon is medium- or long, stocks are very attractive also to cumulative prospect theory investors. Benartzi and Thaler (1995) suggest that investors’ strategies are mainly determined by how often investors evaluate them (the so called evaluation period). Accordingly, while the investment horizon might be long, investors tend to make investment decisions based on short evaluation periods and consequently only put a small proportion of their wealth in stocks. This explains the observed low participation to equity markets (Mankiw and Zeldes 1991) and the equity premium puzzle (Mehra and Prescott 1985).

Short evaluation periods can be easily explained by the fact that investors usually receive their financial statements on yearly base. However, when investors with cumula-
tive prospect theory preferences possess multiple investment goals at different horizons, we show that it is optimal for investors to focus on long-term goals, unless their inter-temporal discount rate is very high. In this paper we show that the same is not true when investors display the satisficing behavior we described before. We see that when goals are not too ambitious, investors with satisficing behavior optimally puts a high proportion of their wealth into short-term investment goals, which are more difficult to reach given the smaller upside potential over shorter periods of time. We show that the overall investment strategy appears to be biased toward short-term goals in the sense that the probability to reach those goals is significantly higher than the probability to reach long-term goals. This result also holds when the equity premium is high. Moreover, when investment goals are not too ambitious, investors with satisficing behavior puts a small proportion of their wealth in risky assets.

By contrast, when investment goals are too ambitious, investors with satisficing behavior mainly invest to reach long-term investment goals, while they adopt aggressive investment strategies in order to achieve short-term goals. This is due to cumulative prospect theory preferences, which imply a risk seeking behavior on losses. Even if the long-term strategies are less aggressive, when investment goals are too ambitious the overall investment strategy displays a high leverage ratio. We note that high incentives to reach ambitious short-term goals (high target returns) and the consequent excessive leverage have been identified as causes for the global financial crisis erupted in 2008 (see Shefrin 2009).
The model presented in this paper is related to the behavioral portfolio theory (BPT) of Shefrin and Statman (2000). However, beside the fact that we use cumulative prospect theory to model investors’ preferences, while the BPT is build on the SP/A theory of Lopes (1987), our model differs from BPT in many directions. First, it is a continuous-time model, while BPT is a static model. This allows us to consider investment goals at different horizons. As far as we know, this paper is the first one which considers a portfolio model where mental accounting refers to investment goals at possibly different investment horizons. Second, in this paper how investors’ wealth is allocated among investment goals is determined endogenously, while in BPT this is exogenously given. Finally, our model also allow us to study investors who display satisficing behavior, as discussed above.

Mental accounting in a mean-variance framework has been studied by Das, Markowitz, Scheid, and Statman (2009). Also their model is static and how wealth is allocated among investment goals is exogenously given. The main result in Das, Markowitz, Scheid, and Statman (2009) is that in their setting portfolios with mental accounting belongs to the mean-variance efficient frontier if short-sale is allowed, i.e., mental accounting does not introduce inefficiency. We show that a similar result holds with CPT preferences and satisficing behavior.

As we discussed above, this paper also relates to Benartzi and Thaler (1995) and Berkelaar, Kouwenberg, and Post (2004). Differently from these papers, here we don’t address the question about how CPT strategies change as function of the time horizon (even if this is a by-product of the analysis), but we look at how CPT investors with several investments goals (at different horizons) split their wealth among them.
The remainder of the paper is structured as follows. Section 2 introduces the goal-based model while Section 3 presents numerical examples to illustrate our results. Section 4 concludes. All proofs are given in the Appendix.

2 Goal-based Investment Model

2.1 The Economy

We assume a standard, continuous-time financial market as described in Karatzas and Shreve (1998). There are $K + 1$ assets with price $S_k(t)$ at time $t$ for $k = 0, \ldots, K$. The zero-th asset is a risk-free asset $S_0(t)$:

\begin{equation}
    dS_0(t) = r(t) S_0(t) \, dt
\end{equation}

where $r(\cdot)$ is the interest rate process. The remaining assets are risky and follow an Ito process

\begin{equation}
    dS_k(t) = \mu_k(t) S_k(t) \, dt + S_k(t) \sum_{l=1}^{K} \sigma_{kl}(t) dB_l(t), \quad k = 1, \ldots, K
\end{equation}

with drift rates $\mu_k(\cdot)$ and volatility components $\sigma_{kl}(\cdot)$. The processes $r(\cdot)$, $\mu(\cdot) = (\mu_1, \ldots, \mu_K)$ and $\sigma(\cdot) = (\sigma_{kl})_{k,l=1,\ldots,K}$ are progressively measurable with respect to the filtration generated by the $K$-dimensional Brownian motion $B = (B_1, \ldots, B_K)'$ and satisfy the usual regularity conditions (Karatzas and Shreve 1998, Definition 1.3).

We assume a complete market. This is the case when the volatility matrix $\sigma(t)$ has full rank for all $t$ and there exists a $K$-dimensional progressively measurable process $\kappa$
such that $\sigma(t) \kappa(t) = \mu(t) - 1 r(t)$ almost surely for all $t$, where $1$ is a $K$-dimensional vector of 1. Under market completeness, there exists a unique pricing kernel $\xi$:

$$
\xi(t) = \exp\left(-\int_0^t (r(s) + \frac{1}{2} ||\kappa(s)||^2) \, ds - \int_0^t \kappa(s)' \, dB(s)\right).
$$

$\kappa$ is called the market price of risk process. The pricing kernel satisfies the following dynamics:

$$
\frac{d\xi_t}{\xi_t} = -r(t) \, dt - \kappa(t)' \, dB(t), \quad \xi(0) = \xi_0 = 1.
$$

### 2.2 Investor’s Preferences

The investor possesses $J$ different investment goals, which can be characterized by target payoffs $W_j$ the clients wants to obtain at time $T_j$, for $j = 1, \ldots, J$. At time $t$ she allocates a fraction $w_j(t)$ of her wealth $W(t)$ to goal $j$ and chooses goal-specific portfolios $\lambda_j(t) = (\lambda_{j,1}(t), \ldots, \lambda_{j,K})'$, where $\lambda_{j,k}(t)$ is the fraction of wealth $W_j(t) = w_j(t) W(t)$ allocated to the asset $k$ at time $t$. We put $W_0 = W(0)$ and $w_{j,0} = w_j(0)$.

The wealth dynamics for goal $j$ is given as follows:

$$
dW_j(t) = r(t) W_j(t) \, dt + (\mu(t) - 1 r(t))' \lambda_j(t) W_j(t) \, dt + \sigma(t)' \lambda_j(t) W_j(t) \, dB(t), \quad W_j(0) = w_{j,0} W_0.
$$

The investor derives utility from each goal separately at the corresponding horizons $T_j$. This reflects two kinds of mental accounting the investor has. First, each invest-

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$^3$Target payoffs are assumed to be exogenously given and fixed. However, we could easily extends our model to allow target payoffs to be endogenous and stochastic. For example, if the target wealth for goal $j$ is the value at time $T_j$ of a benchmark portfolio $\lambda_b(t)$, then we simply take the difference $W_j(t) - W_b(t)$ where $W_b(t)$ is the wealth level at time $t$ obtained the initial wealth $W_j(0)$ is invested in $\lambda_b(t)$. Since $W_j(t) - W_b(t)$ also follows an Ito process, we can apply the same technology to extend our model to the case where target payoffs are endogenous and stochastic; see Jin and Zhou (2008).
ment goal is considered as a separate account and the covariances between goal-specific portfolios are ignored. This is similar to the behavioral portfolio theory of Shefrin and Statman (2000) or the model with narrow framing introduced by Barberis and Huang (2001). Second, different time horizons are not integrated, i.e., in order to reach a given investment goal the investor does not take into account portfolios which are related to investment goals with longer horizons. We therefore assume that wealths $W_j(T_j)$ are fully consumed at time $T_j$, even if the investor is not able to reach her goal ($W_j(T_j) < W_j$) or there exists a surplus ($W_j(T_j) > W_j$). In our model we will see that in case $W_j(T_j) < W_j$, then $W_j(T_j) = 0$, i.e., there is nothing to consume relative to goal $j$ when goal $j$ is not reached.

Mental accounting with respect to the time horizon has been addressed by Benartzi and Thaler (1995), who show that a cumulative prospect theory investor with a investment horizon of one year or less will prefer bonds to stocks, while the preferences revert if the horizon is longer than one year. In order to explain the equity premium puzzle, Benartzi and Thaler (1995) distinguish between investment horizon and evaluation period. They introduce the concept of myopic loss aversion which refers to the fact that investors perceive risk (and behave accordingly) in correspondence of their evaluation period which is usually much shorter than the investment horizon. Therefore, even if their investment horizon is long, investors chooses their investment strategies according to their evaluation period, which is short. In this paper we don’t distinguish between investment horizon and evaluation period, but argue that investors might have investment goals at different horizons. We address the question about how a CPT investor optimally allocates wealth
between the different investment goals.

The value function for goal \( j \) corresponds to the cumulative prospect theory value function

\[
V_j(W; \overline{W}_j) = \int_{-\infty}^{\overline{W}_j} v_j(x - \overline{W}_j) d\pi_j^-(F_W(x)) - \int_{\overline{W}_j}^{\infty} v_j(x - \overline{W}_j) d\pi_j^+(1 - F_W(x))
\]

where \( v_j(x) \) is a piecewise-power value function:

\[
v_j(x) = \begin{cases} 
    \beta_j^+ x^{\alpha_j} & x > 0, \\
    -\beta_j^- (-x)^{\alpha_j} & x \leq 0,
\end{cases}
\]

and \( \pi_j^+, \pi_j^- \) are non-decreasing, continuous probability weighting functions from \([0, 1]\) into \([0, 1]\) with \( \pi_j^+(p) = p \) for \( p = 0, 1 \), \( \pi_j^+(p) > p \) for \( p \) small and \( \pi_j^+(p) < p \) for \( p \) large. Furthermore, \( \beta_j^- \geq \beta_j^+ \geq 0 \) and \( \alpha_j \in (0, 1) \). \( F_W \) denotes the cumulative distribution function of the random payoff \( W \).

We use a slightly different notation for the value function than suggested by Tversky and Kahneman (1992), i.e., we introduce two different parameters, \( \beta_j^+ \) and \( \beta_j^- \), to characterize the degree of loss aversion which then corresponds to \( \beta_j = \beta_j^- / \beta_j^+ \) (see also Berkelaar, Kouwenberg, and Post 2004). The reason for this notation, is that it allows to also describe investors who are extremely loss averse for some investment goals, i.e., \( \beta_j \to \infty \). This corresponds to the case \( \beta_j^+ \to 0 \), i.e., the investor doesn’t obtain any positive value for being above the target wealth, but only a negative value for being below it. The case \( \beta_j^+ = 0 \) for all \( j \) is important because it characterizes investors who
are only concerned about reaching their investment goals, while they don’t receive any additional value from payoffs which are strictly above their target payoff levels. This type of investors display satisficing behavior. Satisficing, as opposed to maximizing, assumes that investors consider all payoffs above the reference point as fully satisfactory; see Simon (1955). Recently, Brown and Sim (2009) and Brown, De Giorgi, and Sim (2009) presented an axiomatic foundation of preferences with satisficing behavior.

2.3 Investor’s Decision Problem

Given her preferences as described in the previous section, the investor determines at each time \( t \) how to optimally split wealth among the different investment goals and, additionally, how to optimally invest the wealth amounts allocated to the different investment goals. She solves the following decision problem:

\[
\max_{\lambda_1(t), \ldots, \lambda_J(t)} \sum_{j=1}^J \delta^{-T_j} V_j(W_j(T_j); W_j) \\
\text{such that for } j = 1, \ldots, J
\]

\[
(2.6) \quad dW_j(t) = r(t) W_j(t) \, dt + (\mu(t) - 1 \, r(t))^\prime \lambda_j(t) W_j(t) \, dt + \sigma(t)^\prime \lambda_j(t) W_j(t) \, dB(t),
\]

\[
W_j(t) \geq 0, \ t \in [0, T_j],
\]

\[
\sum_{j=1}^J W_j(0) = W_0.
\]

The constant \( \delta \geq 1 \) is the discount factor and characterizes the investor’s time preferences, i.e., her preference for immediate consumption relative to consumption far in the future. In order to avoid concentration in few asset classes, we could add additional constraints on investment strategies, e.g., \( \sum_{j=1}^J \lambda_{j,k}(t) \in (\lambda_k^{\min}, \lambda_k^{\max}) \).
To keep analytical tractability, we assume no probability weighting, i.e., \( \pi_j(p) = p \) for all \( j = 1, \ldots, J \) and \( p \in [0, 1] \). This is a common assumption in behavioral finance.\(^4\) For the discussion and the results in this paper probability weighting is not crucial.

We apply martingale methods and rewrite the dynamic decision Problem (2.6) as a static one:

\[
\max_{W_{j}(T_{j})} \sum_{j=1}^{J} \delta^{-T_j} \mathbb{E} \left[ v_j(W_j(T_j)) - W_j \right]
\]

such that

\[
\mathbb{E} \left[ \xi(T_j) W_j(T_j) \right] \leq \xi_0 w_{j,0} W_0
\]

\[
W_j(T_j) \geq 0, w_{j,0} \geq 0, \quad j = 1, \ldots, J,
\]

\[
\sum_{j=1}^{J} w_{j,0} \leq 1.
\]

The vector \( w_0 = (w_{1,0}, \ldots, w_{J,0})' \) corresponds to the wealth’s shares at time \( t = 0 \). At time \( t = 0 \) the investor decides how to split wealth among the \( J \) investment goals. Consequently, she allocates \( W_j(0) = w_{j,0} W_0 \) to goal \( j \), which corresponds to the budget constraint for this goal. Initial wealth shares and the corresponding goal-specific terminal wealths determine the investor’s global value she obtains from the \( J \) different investment goals.

We solve Problem (2.7) in two stages. First, for a given vector of initial wealth shares \( w_0 = (w_{1,0}, \ldots, w_{J,0})' \), we solve for each investment goal \( j \) the following goal-specific

problem:

$$\max_{W_j(T_j)} \mathbb{E} \left[ v_j(W_j(T_j) - W_j) \right]$$

such that

(2.8) $$\mathbb{E} \left[ \xi(T_j) W_j(T_j) \right] \leq \xi_0 w_{j,0} W_0$$

$$W_j(T_j) \geq 0.$$  

The optimal terminal wealths obviously depend on $w_0$ since wealth shares at time 0 define the budget constraints for all investment goals. Second, given optimal terminal wealths $W_j^*(T_j)$ for all goals as function of $w_0$, we find the optimal vector of shares $w_0^*$ that maximizes the investor’s value function:

$$\max_{w_{1,0}, \ldots, w_{J,0}} \sum_{j=1}^{J} \delta^{-T_j} \mathbb{E} \left[ v_j(W_j^*(T_j) - W_j) \right]$$

such that

(2.9) $$w_{j,0} \geq 0, \ j = 1, \ldots, J$$

$$\sum_{j=0}^{J} w_{j,0} \leq 1.$$  

We point out that in our model, goals’ specific portfolios are somehow related, even if the portfolio decision problem for each goal does not directly account for the investment strategies for the other goals. This can be seen as follows: if one goal is very ambitious or very important so that it is optimal for the investor to put a high proportion of her wealth on it, then less wealth will be available for other goals and the investors might use aggressive investment strategies (given her risk seeking behavior on losses) for them. Therefore, the investor might finally end up with aggressive investment strategies for almost all goals, while she safely invest to reach few important goals.
The following Proposition gives the solution to Problem (2.8):

**Proposition 2.1.** Let \( w_0 = (w_{1,0}, \ldots, w_{J,0})' \) be a vector of initial wealth shares. Then for \( j = 1, \ldots, J \) the optimal terminal wealth for investment goal \( j \) is given by:

\[
W_j^*(T_j) = \begin{cases} \mathbb{W}_j + \left( \frac{y_j \xi(T_j)}{\beta_j^+} \right)^{1/(\alpha_j - 1)}, & \text{if } \xi(T_j) < \xi_j^*(y_j) \\ 0, & \text{if } \xi(T_j) \geq \xi_j^*(y_j) \end{cases}
\]

(2.10)

where \( \xi_j^*(y_j) \) solves \( g_j(\xi_j^*(y_j), y_j) = 0 \) and \( y_j \geq 0 \) satisfies \( \mathbb{E} [\xi(T_j) W_j(T_j)] = \xi_0 w_{J,0} \mathbb{W}_0 \).

The function \( g_j \) is defined as follows:

\[
g_j(x, y) = \frac{1 - \alpha_j}{\alpha_j} \left( \frac{1}{y} \right)^{\alpha_j/(1 - \alpha_j)} (\beta_j^+ \alpha_j)^{1/(1 - \alpha_j)} - \mathbb{W}_j y x + \beta_j^- \mathbb{W}_j^{y_j}
\]

(2.11)

where \( x, y > 0 \).

Optimal terminal wealths \( W_j^*(T_j) \) present the following characteristic. In good states of the world at time \( T_j (\xi(T_j) < \xi_j^*(y_j)) \) the investor is able to reach her investment goal \( \mathbb{W}_j \). In this case there is a strictly positive surplus \( (\beta_j^+ \alpha_j/(y_j \xi(T_j)))^{1/(1 - \alpha_j)} \) which increases as \( \xi_j(T_j) \) becomes smaller. By contrast, in bad states of the world at time \( T_j (\xi(T_j) < \xi_j^*(y_j)) \) the investor fails to reach her goal and her terminal wealth is zero. The probability of beating the investment goal corresponds to the probability that \( \xi(T_j) \leq \xi_j^*(y_j) \). We will provide an explicit expression for this probability under further assumptions on the dynamics of \( \xi_t \).
In order to derive optimal initial wealth shares \( w_0^* \), we need to understand how \( W_j^*(T_j) \) depends on \( w_0 \). For sake of simplicity, we drop the index \( j \) in our discussion below, since the results apply to all investment goals. When it is not confusing we denote by \( w_0 \) the wealth share allocated to one specific investment goal. The following Lemma provides an explicit characterization of \( \xi^*(y) \).

**Lemma 2.1.** Let \( x, y > 0 \) and \( g : \mathbb{R}_+^2 \to \mathbb{R} \) be defined as in Equation (2.11). Then for \( y > 0 \), \( g(x, y) = 0 \) possesses a unique solution \( \xi^*(y) = a/y \) where \( a > 0 \) solves:

\[
1 - \frac{\alpha}{\alpha} a^{-\alpha/(1-\alpha)} (\beta + \alpha)^{1/(1-\alpha)} - a \bar{W} + \beta^{-\frac{\alpha}{1-\alpha}} = 0.
\]

We impose additional conditions on the dynamics of the stochastic discount factor \( \xi_t \). We assume that the interest rate process \( r \), the drift process \( \mu \) and the volatility matrix \( \sigma \) are constant. Let \( m = -(r + (1/2) ||\kappa||^2) \) and \( s^2 = ||\kappa||^2 \). Then \( \xi(T) \) is log-normally distributed with parameters \( m_T = mT \) and \( s_T = s \sqrt{T} \). Under these conditions, we can easily determine the probability of reaching an investment goal at the time horizon \( T \), which corresponds to \( \Phi((\log(\xi^*(y) - m_T)/s_T)) \). We also obtain an explicit characterization of the parameter \( y \) which satisfies the budget constraint for \( W^*(T) \) in Proposition 2.1.

**Lemma 2.2.** Let \( W^*(T) \) be the optimal wealth from Proposition 2.1. Then \( y > 0 \) solves

\[
E[\xi(T) W^*(T)] = \xi_0 w_0 W_0.
\]

if and only if \( y > 0 \) solves \( h(y) = w_0 \) where:

\[
h(y) = b \Phi \left( \frac{\log(a/y) - m_T - s_T^2}{s_T} \right) + c y^{-\gamma-1} \Phi \left( \frac{\log(a/y) - m_T - \frac{\alpha}{\alpha-1} s_T^2}{s_T} \right).
\]
The constant $a > 0$ solves Equation (2.12) from Lemma 2.1, $b = \frac{W}{\xi_0 W_0} \exp(-r T)$ and $c = \frac{1}{\xi_0 W_0} (\beta^+ \alpha)^{-1} \exp \left( \frac{\alpha m_T}{\pi - 1} + \frac{1}{2} \frac{\alpha^2 s_T^2}{(\pi - 1)^2} \right)$.

The function $h$ is strictly decreasing. Therefore $y$ decreases as the initial share allocated to one investment goal increases. Consequently, $\xi^*$ becomes larger, i.e., the probability of reaching the investment goal is higher as more wealth is allocated to that investment goal. Moreover, for $y$ and $\beta^-$ fix, $h(y)$ strictly increases as $\beta^+$ increases, i.e., as loss aversion decreases. Consequently, $y$ decreases with loss aversion, i.e., as loss aversion increases the probability of reaching the investment goal increases, but the surplus becomes smaller. The probability of reaching the investment goal is therefore maximal when $\beta^+ = 0$, which implies satisficing behavior.

When $h(y) = w_0$ possesses a solution, then it is unique since $h$ is strictly decreasing. For $\beta^+ > 0$, $h(y) = w_0$ possesses a solution for all $w_0$. However, for $\beta^+ = 0$, $h(y) = w_0$ cannot be solved for $w_0 > b$ since $h(y) \in [0, b]$ for all $y \geq 0$. For the case $\beta^+ = 0$ we will impose some further conditions on $b$ when we solve for the vector of wealth share $(w_1, 0, \ldots, w_J)$.

If $\beta^+ = 0$ and $w_0 \leq b$ we obtain an explicit solution for $y$ as function of $w_0$:

**Corollary 2.1.** Let $h$ be as given by Equation (2.13). Let $\beta^+ = 0$ and $w_0 < b$. Then $h(y) = w_0$ if and only if

\[
y = a \exp \left( -s_T \Phi^{-1}(w_0/b) - m_T - s_T^\beta \right),
\]

where $b = \frac{W}{\xi_0 W_0} \exp(-r T)$.  

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Note that for $\beta^+ = 0$, the optimal terminal wealth $W^*_T$ does not depend on $\beta^-$. Indeed, $\beta^-$ only enters into $\xi^*(y)$ through the constant $a$. However, since $\xi^*(y) = a/y$ we have $\xi^*(y) = \exp(s_T \Phi^{-1}(w_0/b) + m_T + s_T^2)$, which is independent from $a$. More generally, the following results holds:

**Corollary 2.2.** The optimal terminal wealth $W^*(T)$ depends on $\beta^+$ and $\beta^-$ only through the ratio $\beta^-/\beta^+$, i.e., the degree of loss aversion.

The following Lemma gives an explicit characterization of the optimal utility level $E[W^*(T)]$ as function of $y$:

**Lemma 2.3.** We have $E[W^*(T)] = k(y)$ where

\begin{equation}
(2.15) \quad k(y) = \frac{\log(a/y) - m_T}{s_T} \Phi \left( \frac{\log(a/y) - m_T + s_T^2}{s_T} \right) + dy^{1/\alpha - 1} \Phi \left( \frac{\log(a/y) - m_T + s_T^2}{s_T} \right)
\end{equation}

and $d = \left( \beta^+ \alpha \right)^{-1} \exp \left( \frac{m_T}{\alpha - 1} + \frac{1}{2} \frac{s_T^2}{(\alpha - 1)^2} \right)$. The function $k$ is continuous, strictly decreasing and $\lim_{y \to \infty} k(y) = 0$.

We now rewrite Problem (2.9) as follows. Optimal wealths $W_1(T_1), \ldots, W_J(T_J)$ are given by Proposition 2.1 where $y_j = h^{-1}(w_{j,0})$ for $j = 1, \ldots, J$ and $w_0 = (w_{1,0}, \ldots, w_{J,0})'$ solves

\begin{align*}
\max_{w_{1,0}, \ldots, w_{J,0}, \delta_T} & \sum_{j=1}^{J} \delta^{-T_j} k_j(h_j^{-1}(w_{j,0})) \\
\text{such that} & \sum_{j=1}^{J} w_{j,0} = 1.
\end{align*}

(2.16)
In general, Problem (2.16) must be solved numerically, since no explicit expression for $h_j^{-1}$ is available. We present some numerical exercises in Section 3. Before we discuss a special case where solutions to Problem (2.16) can be derived analytically, we report here optimal wealths and optimal strategies for all investment goals and at any time $t \in [0, T_j]$: 

**Proposition 2.2.** The optimal wealth $W_j(t)$ for investment goal $j$ and at time $t \in [0, T_j]$ corresponds to

\[(2.17)\]

\[W_j^*(t) = \overline{W}_j e^{-r(T_j - r)} \Phi (d_1(\xi^*(y_j), T_j)) + \left(\frac{y_j \xi(t)}{\beta^+_j \alpha_j}\right)^{1/(\alpha_j - 1)} e^{\Gamma(t, T_j)} \Phi (d_2(\xi^*(y_j), T_j))\]

where

\[(2.18)\]

\[d_1(\xi, T) = \log(\xi/\xi(t)) + (r - \frac{1}{2} ||\kappa||^2) (T - t),\]

\[(2.19)\]

\[d_2(\xi, T) = d_1(\xi, T) + ||\kappa|| \sqrt{T - t},\]

\[(2.20)\]

\[\Gamma(t, T) = \frac{1 - \alpha}{\alpha} \left(1 + \frac{1}{2} ||\kappa||^2 (T - t) + \frac{1}{2} \left(\frac{\alpha}{1 - \alpha}\right)^2 ||\kappa||^2 (T - t)\right).\]

The optimal wealth share for goal $j$ at time $t \in [0, T_j]$ is given by

\[w_j^*(t) = \frac{W_j^*(t)}{\sum_{j=1}^{J} W_j^*(t)}\]

and $w_j^*(t) = 0$ for $t > T_j$.

The optimal strategy $\lambda_j(t) = (\lambda_{j,1}, \ldots, \lambda_{j,K}(t))'$ for investment goal $j$ at any time $t \in [0, T_j]$ corresponds to

\[(2.21)\]

\[
\lambda_j(t) = \frac{(\sigma')^{-1} \kappa}{\overline{W}_j(t)} \left(\frac{\xi^*(y_j)}{\beta^+_j \alpha_j}\right)^{1/(\alpha_j - 1)} e^{\Gamma(t, T_j)} \left(\frac{\varphi(d_2(\xi^*(y_j), T_j))}{||\kappa|| \sqrt(T_j - t)} + \frac{\Phi(d_2(\xi^*(y_j), T_j))}{1 - \alpha_j}\right)
\]
where \( \varphi = \Phi' \) is the density function of the standard normal distribution.

### 2.4 Optimal Wealth Shares with Satisficing Behavior

We now consider investors who display satisficing behavior, i.e., \( \beta_j^+ = 0 \) for all \( j \). In this case Problem (2.16) is analytically tractable since we have an explicit expression for \( h_j^{-1} \).

As discussed before, satisficing behavior describes investors who are only concerned about reaching their investment goals, while a surplus above their target wealth does not deliver any additional value to them. In our opinion this is the typical case when investment goals have been clearly specified.

When \( \beta_j^+ = 0 \) for all \( j \), we can prove the following proposition:

**Proposition 2.3.** Let \( \beta_j^+ = 0 \) for all \( j \). Let \( b_j = \frac{W_j \exp(-r T_j)}{\xi_0 W_0} \) and assume that \( \sum_{j=1}^{J} b_j \geq 1 \). Then

\[
    w^*_{j,0} = \begin{cases} 
    b_j \Phi \left( -\frac{1}{\sigma_j^2} \log \left( \frac{\nu}{\xi_0 W_0} \right) - \frac{1}{\sigma_j^2} \left( T_j \log(\delta) - r T_j + \frac{1}{2} \sigma_j^2 T_j \right) \right) & \text{if } \nu > 0 \\
    b_j & \text{else} 
    \end{cases}
\]

where \( \nu \) solves \( \sum_{j=1}^{J} w^*_{j,0} = 1 \).

The condition \( \sum_{j=1}^{J} b_j \geq 1 \) is equivalent to \( \sum_{j=1}^{J} \frac{W_j \exp(-r T_j)}{\xi_0 W_0} \geq W_0 \), i.e., the discounted value of all target wealths must be larger than or equal to the initial wealth. If the discounted value of all target wealths is strictly larger than the initial wealth
(\sum_{j=1}^J b_j > 1) then we have \(\nu > 0\) and the investor invests some of her wealth into the risky assets. Therefore, in this case, the volatility of the market price of risk \(s\) and the inter-temporal discount factor \(\delta\) enter into the expression for wealth shares \(w_{j,0}^*\). If \(\sum_{j=1}^J W_j \exp(-rT_j) = W_0\), then \(\nu \leq 0\) and the investor can reach all investment goals with probability one by simply putting all her wealth into the risk-free asset. Therefore, in this case, \(w_{j,0}^*\) simply corresponds to \(b_j\), that is the ratio between the discounted value of the target wealth for goal \(j\) and the initial wealth.

The ratio \(b_j\) can be interpreted as a measures of how ambitious an investment goal is relative to the initial wealth. Obviously, we expect investors with satisficing behavior to put a higher proportion of their wealth into goals with a higher discounted target wealth. Indeed, as we discussed above, if the initial wealth is high enough, then using the risk-free strategy will ensure that all investment goals will reached with probability one, and investors put a higher proportion of their wealth into goals with higher \(b_j\). However, when \(\sum_{j=1}^J b_j > 1\), the risk-free strategy causes investors to fails some of their investment goals for sure. Therefore in this case investors might prefer having some risky assets into their portfolios and optimal wealth shares will then deviate from \(b_j\). We also point out that when the risk-free strategy fails, then wealth shares \(w_{j,0}^*\) are strictly smaller than \(b_j\) for all investment goals. This means that investors decrease the proportion of wealth put into goal \(j\) relative to \(b_j\) for all investment goals, i.e., instead of using the risk-free strategy form some goals and risky strategies for others, they prefer to invests into risky strategies for all investment goals. This is due to their risk-seeking behavior, which is implied by CPT preferences.
The question now is how investors decide to split their wealth among investment goals when the risk-free strategy fails. In other words, on which goals do investors take more risk and put less wealth if we also account for how ambitious an investment goal is? In order to take into consideration the importance of one investment goal relative to the others, we consider the ratio $w_{j,0}/b_j$, which only depends on the characteristics of the market and the time horizon. We therefore analyze how the ratio $w_{j,0}/b_j$ changes as function of the time horizon. The results are reported in the following corollary:

**Corollary 2.3.** Let $\nu > 0$ such that $\sum_{j=1}^{J} w^*_j = 1$ and $w^*_j$ is given in Proposition 2.3. Then

$$
\frac{w^*_{j,0}}{b_j} = \Phi \left( -\frac{1}{sT_j} \log \left( \frac{\nu}{\xi_0 W_0} \right) - \frac{1}{sT_j} \left( T_j \log(\delta) - r T_j + \frac{1}{2} s^2 T_j \right) \right)
$$

and the following holds:

1. If $r - \log(\delta) - \frac{1}{2} s^2 < 0$, then the ratio $w_{j,0}/b_j$ is maximal for

   $$
   T_j = \hat{T}_j = \frac{-\log \left( \frac{\nu}{\xi_0 W_0} \right)}{r - \log(\delta) - \frac{1}{2} s^2},
   $$

   increasing for $T_j < \hat{T}_j$ and decreasing for $\hat{T}_j > T_j$ with $\lim_{T_j \to \infty} w^*_{j,0}/b_j = 0$.

2. If $r - \log(\delta) - \frac{1}{2} s^2 > 0$, then the ratio $w_{j,0}/b_j$ is minimal for

   $$
   T_j = \hat{T}_j = \frac{-\log \left( \frac{\nu}{\xi_0 W_0} \right)}{r - \log(\delta) - \frac{1}{2} s^2},
   $$

   strictly increasing for $T_j > \hat{T}_j$ with $\lim_{T_j \to \infty} w^*_{j,0}/b_j = 1$, and strictly decreasing for $T_j < \hat{T}_j$. 

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(3) If \( r - \log(\delta) - \frac{1}{2} s^2 = 0 \), then the ratio \( w_{j,0}/b_j \) is strictly increasing for all \( T_j \) if \( \log(\nu/(\xi_0 W_0)) > 0 \), strictly decreasing for all \( T_j \) if \( \log(\nu/(\xi_0 W_0)) < 0 \) and constant if \( \log(\nu/(\xi_0 W_0)) = 0 \).

(4) Moreover,

\[
\lim_{T_j \to 0} \frac{w_{j,0}}{b_j} = \begin{cases} 
1 & \text{if } \nu < W_0 \\
0.5 & \text{if } \nu = W_0 \\
0 & \text{if } \nu > W_0 
\end{cases}
\]

We define the wealth ratio \( WR_0 = W_0/\sum_{j=1}^{J} W_j \exp(-r T_j) \) as the ratio between the initial wealth and the discounted value of all target payoffs. Before we discuss Corollary 2.3, we briefly present here how \( WR_0 \) is linked to \( \nu \). First, we notice that \( 1/WR_0 = \sum_{j=1}^{J} b_j \). Therefore, when \( WR_0 = 1 \) then also \( \sum_{j=1}^{J} b_j = 1 \), and all investment goals can be reached with probability one by simply adopting the risk-free strategy for all goals. When \( WR_0 \) is smaller than 1, then \( \sum_{j=1}^{J} b_j \) is larger than 1 and optimal wealth shares differ from \( b_j \). If \( WR_0 \) is very small, then \( \nu \) must be very large in order to have \( \sum_{j=1}^{J} w_{j,0} = 1 \). Indeed, \( w_{j,0} \) strictly decreases with \( \nu \) and a large \( \nu \) is required in order to satisfies \( \sum_{j=1}^{J} w_{j,0} = 1 \) when \( \sum_{j=1}^{J} b_j \) is much larger than 1. On the other hand, when \( WR_0 \) is slightly smaller than 1, then \( \sum_{j=1}^{J} b_j \) is slightly higher than 1 and a small \( \nu > 0 \) is sufficient to have \( \sum_{j=1}^{J} w_{j,0} = 1 \). Therefore, whether \( \nu \) is large or small depends on how ambitious the investments goals are relative to the initial wealth, i.e., on whether the wealth ratio \( WR_0 \) is very small or near to 1. We now use this observation to discuss
The quantity $r - \log(\delta) - (1/2) s^2$ can be written as $-(-(r + (1/2) s^2) + \log(\delta) + s^2)$, where $-(r + (1/2) s^2)$ is the growth rate of the pricing kernel, $s^2$ its volatility, and $\delta$ is the inter-temporal discount factor. Therefore, $r - \log(\delta) - (1/2) s^2$ is negative (positive), when the pricing kernel displays small (high) absolute growth rate, high (low) volatility, and, additionally, the inter-temporal discount factor is high (small). When these conditions hold, long-term investing appears less (more) attractive. The results in Corollary 2.3 are consistent with this observation, as will become clear from the following discussion.

Let us first consider the case $r - \log(\delta) - (1/2) s^2 < 0$, i.e., long-term investing is less attractive. When the wealth ratio $WR_0$ is small enough such that $\nu$ is larger than $W_0$, then $\log(\nu/(\xi_0 W_0))$ is strictly positive and an intermediate horizon $\hat{T}_j$ exists where the corresponding ratio $w^*_j,0/b_j$ is maximal, while it decreases as the horizon increases. Note that $\hat{T}_j$ can be large when $WR_0$ is very small, i.e., when the initial wealth is very small relative to the discounted sum of target wealths the ratio $w^*_j,0/b_j$ is maximal for a long-term goals. On the other hand, when $WR_0$ is slightly smaller than 1 such that $\nu$ is smaller than $W_0$, then $\log(\nu/(\xi_0 W_0))$ is strictly negative and the ratio $w^*_j,0/b_j$ strictly decreases with the time horizon. This means that in this case $w^*_j,0/b_j$ is maximal for very short-term goals, while it is small for long-term goals. Summarizing, when long-term investing is less attractive, investors put a higher proportion of their wealth (after accounting for how important the goal is) to long-term goals when the initial wealth is very small relative to the current value of target payoffs (goals are too ambitious), while they put a higher
proportion of their wealth into short-term goals when goals are not too ambitious.

Let us now consider the case $r - \log(\delta) - \frac{1}{2} s^2 > 0$, i.e., long-term investing is more attractive. If $WR_0$ is small enough such that $\nu > W_0$ and $\log(\nu/(\xi_0 W_0))$ is strictly positive, then $\hat{T}_j$ is negative, i.e., the ratio $w_{j,0}^*/b_j$ strictly increases with $T_j$. It is therefore maximal for very long-term goals. On the other hand, when $WR_0$ is slightly smaller than 1 such that $\nu$ is smaller than $W_0$ and $\log(\nu/(\xi_0 W_0))$ is negative, then $\hat{T}_j$ is positive. Therefore, there exists an intermediate horizon $\tilde{T}_j$ where the ratio $w_{j,0}^*/b_j$ is minimal, while it strictly increases for $T_j > \tilde{T}_j$. Moreover, when $\nu < W_0$, the ratio is also maximal equals to 1 at $T_j = 0$. Summarizing, when long-term investing is more attractive, the ratio $w_{j,0}^*/b_j$ is maximal for very long-term goals and, when goals are not too ambitious, also for very short-term investment goals. Table 1 summarizes the results in Corollary 2.3.

[Table 1 about here.]

How does the investment strategy just discussed impact the probability of reaching the investment goals? The answer to this question is given in the following corollary:

**Corollary 2.4.** Let $\nu > 0$ such that $\sum_{j=1}^J w_{j,0}^* = 1$ and $w_{j,0}^*$ is given in Proposition 2.3. Then

$$P_j(T_j) = \mathbb{P}[W_j^*(T_j) \geq \overline{W}_j] = \Phi \left( \frac{-\log \left( \frac{\nu}{\xi_0 W_0} \right) - T_j \log(\delta) + r T_j + \frac{1}{2} s^2 T_j}{s T_j} \right).$$

and the following holds:
(1) If \( r - \log(\delta) + \frac{1}{2} s^2 < 0 \), then \( P_j(T_j) \) is maximal for

\[
T_j = \hat{T}_j = \frac{-\log \left( \frac{\nu}{\xi_0 W_0} \right)}{r - \log(\delta) + \frac{1}{2} s^2},
\]

increasing for \( T_j < \hat{T}_j \) and decreasing for \( T_j > \hat{T}_j \), with \( \lim_{T_j \to \infty} P_j(T_j) = 0 \).

(2) If \( r - \log(\delta) + \frac{1}{2} s^2 > 0 \), then \( P_j(T_j) \) is minimal for

\[
T_j = \hat{T}_j = \frac{-\log \left( \frac{\nu}{\xi_0 W_0} \right)}{r - \log(\delta) + \frac{1}{2} s^2},
\]

decreasing for \( T_j < \hat{T}_j \) and increasing for \( T_j > \hat{T}_j \) with \( \lim_{T_j \to \infty} P_j(T_j) = 1 \).

(3) If \( r - \log(\delta) + \frac{1}{2} s^2 = 0 \), then \( P_j(T_j) \) is constant as function of \( T_j \).

(4) Moreover,

\[
\lim_{T_j \to 0} P_j(T_j) = \begin{cases} 
1 & \text{if } \nu < W_0 \\
0.5 & \text{if } \nu = W_0 \\
0 & \text{if } \nu > W_0
\end{cases}
\]

The quantity \( r - \log(\delta) + \frac{1}{2} s^2 \) can be written as \( -[-(r + (1/2) s^2) + \log(\delta)] \). Thus \( r - \log(\delta) + \frac{1}{2} s^2 \) is negative (positive) when the absolute growth rate of the pricing kernel is small (high) and the inter-temporal discount factor is high (small). If the absolute growth rate of the pricing kernel is small, bad states of the world are more likely to occur. Therefore, the probability to reach an investment goal decreases, especially for long-term goals. On the other hand, if the absolute growth rate of the pricing kernel is large, good states of the world are more likely to occur and the probability to reach an investment goal increases.
goal is higher, especially for long-term goals. Finally, we also see that when the initial wealth is high enough, (very) short-term goals will be reached almost surely. This is due to the fact that for very short horizons wealth shares almost corresponds to $b_j$ (the risk-free strategy), as reported in Corollary 2.3. Table 2 summarizes these findings.

[Table 2 about here.]

We conclude this section with a brief discussion on whether mental accounting is inefficient from the perspective of an investor who integrates investment goals. We restrict ourselves to the case where all investment goals have the same time horizon, as in Das, Markowitz, Scheid, and Statman (2009). In this case goal-based investing delivers the same optimal terminal total wealth as when investors consider a unique goal with target wealth equals to the sum of goals’ specific target wealths. This result is reported in the following corollary.

**Corollary 2.5.** Let $\beta_j^+ = 0$ for all $j$ and $T_j = T$ for all $j$. Then the optimal terminal total wealth is

$$W^*(T) = \begin{cases} 
\sum_{j=1}^{J} W_j, & \text{if } \xi(T) < \xi^*(y) \\
0, & \text{if } \xi(T) \geq \xi^*(y)
\end{cases}$$

where $\xi^*(y) = \xi^*(y_j)$ is independent from $j$. This corresponds to the terminal wealth for a unique investment goal with time horizon $T$ and target wealth $\sum_{j=1}^{J} W_j$. 

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Corollary 2.5 implies that goal-based investing with cumulative prospect theory preferences and satisficing behavior is not inefficient from the perspective of an investor who integrates investment goals, when all goals are at same horizon. In other words, investors who do not integrate investment goals will have strategies that are also optimal for an investor who integrates investment goals. This result is similar to that reported by Das, Markowitz, Scheid, and Statman (2009) for the case of goal-based investing with mean-variance preferences.

3 Numerical Examples

To illustrate the implications of the goal-based model presented in the previous section, we derive the optimal investment strategy for an investor with cumulative prospect theory preferences and satisficing behavior.

We assume that the investor has three investment goals at different time horizons. We assume that \( W_j = $50,000 \exp(r T_j) \) for \( j = 1, 2, 3 \) where \( T_1 = 1 \) year (short-term), \( T_2 = 5 \) years (medium term) and \( T_3 = 20 \) years (long-term), i.e., all investment goals have the same discounted value equal to $50,000. Under this assumption, \( b_j \) in Proposition 2.3 is identical for all investment goals and thus wealth shares are not affected by the how ambitious an investment goal is relative to the others.

We specify the investor’s preferences assuming \( \beta_j^- = 2.25 \) and \( \alpha_j = 0.88 \) for all \( j = 1, 2, 3 \) and \( \delta = \exp(r) \), while \( \beta_j^+ \) is not fixed and will determine the degree of loss aversion \( \beta_j = \beta_j^- / \beta_j^+ \). In our numerical examples we further assume that there is one
risky asset, that is the market portfolio, with drift $\mu$ and volatility $\sigma$. The Sharpe ratio corresponds to $\kappa = (\mu - r)/\sigma$, where $r$ is the risk-free rate of return.

When $\beta_j^+ \neq 0$ and the degree of loss aversion $\beta_j = \beta_j^- / \beta_j^+$ is in the range of 2-4 for all $j$, as calibrated by Tversky and Kahneman (1992) (but also if it is much larger), we see that the investor optimally puts almost all her wealth in the long-term goal. This happens also if the degree of loss aversion is much higher for the short-term goal than for the long-term goal. This is because for cumulative prospect theory investors long-term investing is very attractive. Figure 1 displays optimal wealth shares as function of $\beta_j^+$, under the assumption that $\beta_j^- = 2.25$ for all $j$. For $\beta_j^+$ in the range from 0.6 to 1, which implies a degree of loss aversion from 2.25 to 3.75, almost all wealth is invested in the long-term goal. This result is robust with respect to different values for the wealth ratio (0.4 in the top panel and 0.75 in the bottom panel). It follows that in case that the degree of loss aversion is in the range 2-4, the overall investment strategy almost corresponds to the long-term investment strategy, as shown in Figure 2.

We now discuss more in detail the case with $\beta_j^+ = 0$ for all $j$, i.e., the investor displays the statisficing behavior. Figure 3 shows optimal wealth shares at time 0 as function of the wealth ratio $WR_0 = W_0 / \sum_{j=1}^{3} \bar{W}_j \exp(-rT_j)$. We assumed $r = 0.03$, $\sigma = 0.2$, while $\kappa$ takes values 0.2 (top panel) and 0.4 (bottom panel), which imply an equity premium of 4% and 8%, respectively. With $r = 0.03$ we have $\bar{W}_1 = \$51,523$, $\bar{W}_2 = \$58,092$.
and $W_3 = $91,106. When the wealth ratio is small (i.e., goals are too ambitious), the investor puts the largest proportion of her wealth to the long-term goal. However, as the wealth ratio increases (i.e., goals are less ambitious) the investor mainly invests to reach short-term goals, i.e., the probability to reach this goal is significantly higher, as shown in Figure 4. We also see that when the wealth ratio is small, the probability to reach the long-term goal is higher. By contrast, in this case, the probability to reach the short-term goal is small, and the investor adopts aggressive investment strategies for all investment goals, as shown in Figure 5. Moreover, short- and medium-term goals display a high leverage and this is true also for the overall investment strategy. We also note that these results do not qualitatively depend on the equity premium.

[Figure 3 about here.]

[Figure 4 about here.]

[Figure 5 about here.]

Figure 6 shows the optimal wealth shares as function of the equity premium. We fixed the volatility $\sigma = 0.20$ and the wealth ratio takes values 0.40 (top panel) and 0.75 (bottom panel), which correspond to $W_0 = $60,000 and $W_0 = $112,500, respectively (the discounted value of all target payoffs is $150,000). We see that investors put a higher proportion of initial wealth into short-term goals as the equity premium increases. Nevertheless, as show in Figure 7 the probability to reach short-term goals is the lowest when the wealth ratio is low. By contrast, it is the highest when the wealth ratio is high. Figure 8 reports the investment strategies for all goals and globally (red line). When the wealth ratio is low, the short-term strategy becomes less aggressive as the equity premium
increases, while the opposite holds for the long-term strategy. However, in this case, the overall investment strategy is almost unaffected by the equity premium. When the wealth ratio is high, goals’ specific strategies displays similar patterns as when the wealth ratio is low. However, the overall investment strategy now becomes less aggressive as the equity premium increases. This is because in this case the long-term goal can be reached with higher probability and the investor can allocates less wealth to it. Consequently, she can allocate more wealth to short- and medium-term goals and adopts safer strategies for those goals. Finally, the impact of short- and medium-term strategies on the overall investment strategy is stronger and thus the overall investment strategy also becomes safer as the equity premium increases.

4 Conclusion

In this paper we applied cumulative prospect theory to obtain a goal-based portfolio selection model, where investors possess different investment goals at different time horizons. Our model assumes that investors mentally organize each investment goal as a separate account and derive optimal investment strategies for each investment goal, ignoring covariances between goal-specific portfolios. We derived optimal wealth shares allocated to each investment goal and optimal investment strategies, when investors additionally
display the satisficing behavior, i.e., are fully satisfied when they reach their investment goals.

We showed that investors mainly invest too reach short-term goals when they are not too ambitious. However, when goals are very ambitious, they mainly invest to reach long-term goals and adopt aggressive investment strategies for their short-term goals. In this case, the overall investment strategy displays a high leverage. Therefore, our model explains high leverage ratios for investors, who have high incentive to reach ambitious short-term investment goals.

References


A  Proofs

A.1 Proof of Proposition 2.1

See Berkelaar, Kouwenberg, and Post (2004). We additionally point out that $g_j(\cdot, y)$ is a continuous, strictly decreasing function. For $y > 0$ fix, we have $\lim_{x \to 0} g_j(x, y) = \infty$ and $\lim_{x \to \infty} g_j(x, y) = -\infty$. So for each $y > 0$ we find a unique $x$ such that $g_j(x, y) = 0$. For $y > 0$ fix, we denote by $\xi_j^*(y)$ the solution to $g_j(x, y) = 0$.

A.2 Proof of Lemma 2.1

Assume that $\xi^*(y) = a/y^b$ for some $a, b \in \mathbb{R}$. We have:

$$g(\xi^*(y), y) = \frac{1 - \alpha}{\alpha} \left( \frac{y^b}{a x} \right)^{\alpha/(1-\alpha)} (\beta^+ \alpha)^{1/(1-\alpha)} - a \frac{y a}{y} y^{-b} + \beta^{-} \bar{W}^\alpha$$

$$= \frac{1 - \alpha}{\alpha} a^{-\alpha/(1-\alpha)} y^{(b-1)\alpha/(1-\alpha)} (\beta^+ \alpha)^{1/(1-\alpha)} - a \bar{W} y^{-b+1} + \beta^{-} \bar{W}^\alpha.$$ 

This expression is constant if and only if $b = 1$. In this case we have:

$$g(\xi^*(y), y)) = \frac{1 - \alpha}{\alpha} a^{-\alpha/(1-\alpha)} (\beta^+ \alpha)^{1/(1-\alpha)} - a \bar{W} + \beta^{-} \bar{W}^\alpha$$

and thus $g(\xi^*(y), y)) = 0$ if and only if

$$\frac{1 - \alpha}{\alpha} a^{-\alpha/(1-\alpha)} (\beta^+ \alpha)^{1/(1-\alpha)} - a \bar{W} + \beta^{-} \bar{W}^\alpha = 0.$$ 

We still have to show that $a > 0$ and is unique. Note that

$$\frac{1 - \alpha}{\alpha} a^{-\alpha/(1-\alpha)} (\beta^+ \alpha)^{1/(1-\alpha)} - a \bar{W} + \beta^{-} \bar{W}^\alpha = g(a, 1)$$

and $g(\cdot, 1)$ is strictly decreasing, $\lim_{a \to 0} g(a, 1) = \infty$ and $\lim_{a \to \infty} g(a, 1) = -\infty$. So, $g(a, 1) = 0$ possesses a unique solution $a > 0$ and the statement in the Lemma follows.
A.3 Proof of Lemma 2.2

From Proposition 2.1 it follows:

\[
\mathbb{E}[\xi(T) W^*(T)] = \mathbb{E}\left[\mathcal{W} \xi(T) 1_{\xi(T) \leq \xi^*(y)} + \left(\frac{y}{\beta^+ \alpha}\right)^{\frac{1}{\alpha^+}} \xi(T)^{\frac{\alpha}{\alpha^+}} 1_{\xi(T) < \xi^*(y)}\right]
\]

\[
= \mathcal{W} \mathbb{E}\left[\xi(T) 1_{\xi(T) \leq \xi^*(y)}\right] + \left(\frac{y}{\beta^+ \alpha}\right)^{\frac{1}{\alpha^+}} \mathbb{E}\left[\xi(T)^{\frac{\alpha}{\alpha^+}} 1_{\xi(T) < \xi^*(y)}\right].
\]

Since \(\xi(T)\) is log-normally distributed with parameters \(m_T = m T\) and \(s_T = s \sqrt{T}\), then

\[
\mathbb{E}\left[\xi(T)^{\frac{\alpha}{\alpha^+}} 1_{\xi(T) < \xi^*(y)}\right] = \exp\left(m_T + \frac{1}{2} s_T^2\right) \Phi\left(\frac{\log(\xi^*(y)) - m_T - s_T^2}{s_T}\right).
\]

Moreover, \(\xi(T)^{\alpha/(\alpha-1)}\) is also log-normally distributed with parameters \(\alpha m_T / (\alpha - 1)\) and \(\alpha s_T / (1 - \alpha)\). It follows

\[
\mathbb{E}\left[\xi(T)^{\frac{\alpha}{\alpha^+}} 1_{\xi(T) < \xi^*(y)}\right] = \exp\left(\frac{\alpha m_T}{\alpha - 1} + \frac{1}{2} \frac{\alpha^2 s_T^2}{(\alpha - 1)^2}\right) \Phi\left(\frac{\log(\xi^*(y)) - m_T - \frac{s_T^2}{\alpha - 1}}{s_T}\right).
\]

Let \(b = \frac{\mathcal{W}}{\xi_0 W_0} \exp(m_T + \frac{1}{2} s_T^2) = \frac{\mathcal{W}}{\xi_0 W_0} \exp(-r T)\) and \(c = \frac{1}{\xi_0 W_0} (\beta^+ \alpha)^{\frac{1}{\alpha^+}} \exp\left(\frac{m_T}{\alpha - 1} + \frac{1}{2} \frac{\alpha^2 s_T^2}{(\alpha - 1)^2}\right)\)

then

\[
\mathbb{E}[\xi(T) W^*(T)] = b \Phi\left(\frac{\log(\xi^*(y)) - m_T - s_T^2}{s_T}\right) + c y^{\frac{1}{\alpha^+}} \Phi\left(\frac{\log(a/y) - m_T - \frac{s_T^2}{\alpha - 1}}{s_T}\right).
\]

Now let

\[
h(y) = b \Phi\left(\frac{\log(\xi^*(y)) - m_T - s_T^2}{s_T}\right) + c y^{\frac{1}{\alpha^+}} \Phi\left(\frac{\log(a/y) - m_T - \frac{s_T^2}{\alpha - 1}}{s_T}\right).
\]

then

\[
\mathbb{E}[\xi(T) W^*(T)] = \xi_0 W_0 \iff h(y) = w_0.
\]

Putting \(\xi^*(y) = a/y\) from Lemma 2.1 the statement in the Lemma follows.

A.4 Proof of Corollary 2.1

Follows directly from Equation (2.13) when \(\beta^+ = 0\).
A.5 Proof of Corollary 2.2

Let $a$ and $y$ be solutions to Equation (2.12) and $h(y) = w_0$, respectively, given parameters $\beta^+$ and $\beta^-$. Let $\tilde{\beta}^+ = u \beta^+$ and $\tilde{\beta}^- = u \beta^-$, while the other parameters are fix. Denote by $\tilde{a}$ and $\tilde{y}$ the solutions to Equation (2.12) and $h(y) = w_0$, respectively, given parameters $\tilde{\beta}^+$ and $\tilde{\beta}^-$. It can be easily shown that $\tilde{a} = u a$ and $\tilde{y} = u y$. Therefore, $\xi^*(\tilde{y}) = a/y = \xi^*(y)$. Moreover, since the surplus in the good scenario depends on the ratio $y/\beta^+$, it is also independent from $u$.

A.6 Proof of Lemma 2.3

From Proposition 2.1, it follows:

$$
\mathbb{E}[W^*(T)] = \mathbb{E} \left[ \bar{W} 1_{\xi(T) \leq \xi^*(y)} + \left( \frac{y}{\beta^+ \alpha} \right)^{\frac{1}{\alpha-t}} \xi(T)^{\frac{1}{\alpha-t}} 1_{\xi(T) \leq \xi^*(y)} \right]
$$

$$
= \bar{W} \mathbb{P}[\xi(T) \leq \xi^*(y)] + \left( \frac{y}{\beta^+ \alpha} \right)^{\frac{1}{\alpha-t}} \mathbb{E} \left[ \xi(T)^{\frac{1}{\alpha-t}} 1_{\xi(T) \leq \xi^*(y)} \right].
$$

Since $\xi(T)$ is log-normally distributed with parameters $m_T = m T$ and $s_T = s \sqrt{T}$, then

$$
\mathbb{P}[\xi(T) \leq \xi^*(y)] = \Phi \left( \frac{\log(\xi^*(y)) - m_T}{s_T} \right).
$$

Moreover, $\xi(T)^{1/(\alpha-1)}$ is also log-normally distributed with parameters $m_T/(\alpha - 1)$ and $s_T/(1 - \alpha)$.

It follows

$$
\mathbb{E} \left[ \xi(T)^{\frac{1}{\alpha-t}} 1_{\xi(T) \leq \xi^*(y)} \right] = \exp \left( \frac{m_T}{\alpha - 1} + \frac{1}{2} \frac{s_T^2}{(\alpha - 1)^2} \right) \Phi \left( \frac{\log(\xi^*(y)) - m_T + \frac{s_T^2}{1 - \alpha}}{s_T} \right).
$$

Let $d = (\beta^+ \alpha)^{\frac{1}{1-\alpha}} \exp \left( \frac{m_T}{\alpha - 1} + \frac{1}{2} \frac{s_T^2}{(\alpha - 1)^2} \right)$ then

$$
\mathbb{E}[W^*(T)] = \bar{W} \Phi \left( \frac{\log(\xi^*(y)) - m_T}{s_T} \right) + d y^{\frac{1}{\alpha-t}} \Phi \left( \frac{\log(\xi^*(y)) - m_T + \frac{s_T^2}{1 - \alpha}}{s_T} \right).
$$
We define
\[ k(y) = W \Phi \left( \frac{\log(\xi^*(y)) - m_T}{s_T} \right) + dy^{\frac{1}{\alpha - 1}} \Phi \left( \frac{(1 - \alpha) \log(\xi^*(y)) + m_T - \frac{s_T^2}{1 - \alpha}}{s_T} \right). \]

A.7 Proof of Proposition 2.2

See Berkelaar, Kouwenberg, and Post (2004).

A.8 Proof of Proposition 2.3

When \( \beta_j^+ = 0 \), we know from Corollary 2.1 that \( y_j = a_j \exp \left( -s_{T_j} \Phi^{-1} \left( w_{j,0}/b_j \right) - m_{T_j} - s_{T_j}^2 \right) \)
solves \( h_j(y) = w_{j,0} \). We put \( y_j \) into \( k_j \) and obtain
\[ f_j(w_{0,j}) = k_j(h_j^{-1}(w_{j,0})) = W_j \Phi \left( \Phi^{-1}(w_{j,0}/b_j) + s_{T_j} \right). \]

The function \( f_j \) is strictly increasing and strictly concave with
\[ \frac{d}{dw_{j,0}} f(w_{j,0}) = \frac{W_j}{b_j} \exp \left( -s_{T_j} \Phi^{-1}(w_{j,0}/b_j) - (1/2) s_{T_j}^2 \right). \]

The Karush-Kuhn-Tucker conditions for the convex optimization Problem (2.16) with the additional constraint \( w_{j,0} \leq b_j \) for all \( j \) are as follows:

(A.22) \( \eta_{1,j} \geq 0, \eta_{2,j} \geq 0, w_{j,0} \geq 0, \)

(A.23) \( \eta_{1,j} w_{j,0} = 0, \eta_{2,j} (w_{0,j} - b_j) = 0, \)

(A.24) \( \sum_{j=1}^J w_{j,0} = 0, \)

(A.25) \( -\frac{\delta^{-T_j}}{b_j} W_j \exp \left( -s_{T_j} \Phi^{-1}(w_{j,0}/b_j) - (1/2) s_{T_j}^2 \right) - \eta_{1,j} + \eta_{2,j} + \nu = 0. \)

From Equations (A.22) and (A.25) we obtain:

(A.26) \( \eta_{1,t} = \nu - \frac{\delta^{-T_j}}{b_j} W_j \exp \left( -s_{T_j} \Phi^{-1}(w_{j,0}/b_j) - (1/2) s_{T_j}^2 \right) + \eta_{2,j} \geq 0. \)
We multiply \( \eta_{1,j} \) with \( w_{0,j} \) and obtain:

(A.27) \[ 0 = w_{j,0} \left( \nu - \frac{\delta - T_j}{b_j} \exp \left( -s_{T_j} \Phi^{-1}(w_{j,0}/b_j) - (1/2) s_{T_j}^2 \right) + \eta_{2,j} \right). \]

Using that \( \eta_{2,j} (w_{j,0} - b_j) = 0 \) we can solve the latter equation for \( \eta_{2,j} \) and we obtain:

(A.28) \[ \eta_{2,j} = -\frac{w_{j,0}}{b_j} \left( \nu - \frac{\delta - T_j}{b_j} \exp \left( -s_{T_j} \Phi^{-1}(w_{j,0}/b_j) - (1/2) s_{T_j}^2 \right) \right) \geq 0 \]

which implies

(A.29) \[ \nu \leq \frac{\delta - T_j}{b_j} \exp \left( -s_{T_j} \Phi^{-1}(w_{j,0}/b_j) - (1/2) s_{T_j}^2 \right). \]

since \( w_{j,0} \geq 0 \). Finally, using \( \eta_{2,j} (w_{j,0} - b_j) = 0 \) we have

(A.30) \[ 0 = -\frac{w_{j,0}}{b_j} (w_{0,j} - b_j) \left( \nu - \frac{\delta - T_j}{b_j} \exp \left( -s_{T_j} \Phi^{-1}(w_{j,0}/b_j) - (1/2) s_{T_j}^2 \right) \right). \]

If \( \nu > 0 \), then \( w_{j,0} < b_j \). Indeed, if \( w_{j,0} = b_j \) and \( \nu > 0 \) then condition (A.29) is violated. Moreover, since \( w_{j,0} < b_j \), then \( \eta_{j,2} = 0 \) by the second Slater’s condition in (A.23).

Therefore, \( w_{0,j} > 0 \), else condition (A.26) is violated. It follows from Equation (A.30) that \( w_{0,j} \) must solve

\[ \nu - \frac{\delta - T_j}{b_j} \exp \left( -s_{T_j} \Phi^{-1}(w_{j,0}/b_j) - (1/2) s_{T_j}^2 \right) = 0 \]

and after some re-arrangements we obtain

\[ w_{0,j} = b_j \Phi \left( -\frac{1}{s_{T_j}} \log \left( \frac{\nu}{\xi_0 W_0} \right) - \frac{1}{s_{T_j}} \left( T_j \log(\delta) - r T_j + \frac{1}{2} s_{T_j}^2 \right) \right). \]

If \( \nu < 0 \), then \( \eta_{j,2} > 0 \) else condition (A.26) is violated since the inequality

\[ \nu - \frac{\delta - T_j}{b_j} \exp \left( -s_{T_j} \Phi^{-1}(w_{j,0}/b_j) - (1/2) s_{T_j}^2 \right) < 0 \]

holds for all \( w_{0,j} \in [0, b_j] \). Therefore, \( w_{0,j} = b_j \) by the second Slater’s condition in (A.23).

If \( \nu = 0 \) and \( w_{0,j} \neq b_j \) then \( w_{0,j} = 0 \) by Equation (A.30). However, when \( w_{0,j} = 0 \) then Equation (A.26) is violated. Thus, also in this case we must have \( w_{0,j} = b_j \).
A.9 Proof of Corollary 2.3

Straightforward implication of Proposition 2.3.

A.10 Proof of Corollary 2.4

Follow directly from Proposition 2.3, Proposition 2.1 and Corollary 2.1.
<table>
<thead>
<tr>
<th>Wealth ratio</th>
<th>Small</th>
<th>High</th>
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</thead>
<tbody>
<tr>
<td>$\nu &gt; W_0$</td>
<td>$\hat{T}_j$</td>
<td>0</td>
</tr>
<tr>
<td>$\nu &lt; W_0$</td>
<td>$\infty$</td>
<td>0 and $\infty$</td>
</tr>
<tr>
<td>$\eta_1 &lt; 0$</td>
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<td>0</td>
</tr>
<tr>
<td>$\eta_1 = 0$</td>
<td>$\infty$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: The table reports the time horizon $T_j$ for which the ratio $w_{j,0}^*/b_j$ is maximal, as function of $\eta_1 = r - \log(\delta) - (1/2)s^2$ and the wealth ratio $WR_0 = W_0/\sum_{j=1}^{J} W_j \exp(-r T_j)$. We have $\hat{T}_j = -\log(\nu/(\xi_0 W_0))/\eta_1$. 

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\( WR_0 \)

<table>
<thead>
<tr>
<th></th>
<th>Small ((\nu &gt; W_0))</th>
<th>High ((\nu &lt; W_0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_2 &lt; 0 )</td>
<td>( T_j )</td>
<td>0</td>
</tr>
<tr>
<td>( \eta_2 &gt; 0 )</td>
<td>( \infty )</td>
<td>0 and ( \infty )</td>
</tr>
<tr>
<td>( \eta_2 = 0 )</td>
<td>( \infty )</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2: The table reports the time horizon \( T_j \) for which the probability to reach the investment goal is maximal, as function of \( \eta_2 = r - \log(\delta) + (1/2) s^2 \) and the wealth ratio \( WR_0 = W_0 / \sum_{j=1}^J W_j \exp(-rT_j) \). We have \( T_j = -\log(\nu/(\xi_0 W_0)) / \eta_2 \).
Figure 1: Optimal wealth shares at time $t = 0$ as function of the of $\beta_j^+$, when the investor has three investment goals with same discounted value at time $t = 0$ (i.e., $W_j \exp(-r T_j) = $50,000 for all $j$) at horizons 1 year (full line), 5 years (dashed line) and 20 years (dotted line). Investor’s preferences are characterized by $\beta_j^- = 2.25$ (infinite loss aversion) and $\alpha_j = 0.88$ for $j = 1, 2, 3$. We have $r = 0.03$, $\sigma = 0.2$ and $\kappa = 0.2$, which imply an equity premium of 4%. The wealth ratio $WR = W_0/\sum_{j=1}^3 W_j \exp(-r T_j)$ corresponds to 0.4 (top panel) and 0.75 (bottom panel).
Figure 2: Optimal proportion of risky assets at time $t = 0$ as function $\beta_j^+$, when the investor has three investment goals with same discounted value at time $t = 0$ (i.e., $W_j \exp(-r T_j) = 50,000$ for all $j$) at horizons 1 year (full line), 5 years (dashed line) and 20 years (dotted line). Investor’s preferences are characterized by $\beta_j^- = 2.25$ (infinite loss aversion) and $\alpha_j = 0.88$ for $j = 1, 2, 3$. We have $r = 0.03$, $\sigma = 0.2$ and $\kappa = 0.2$, which imply an equity premium of 4%. The wealth ratio $WR = W_0 / \sum_{j=1}^{3} W_j \exp(-r T_j)$ corresponds to 0.4 (top panel) and 0.75 (bottom panel).
Figure 3: Optimal wealth shares at time $t = 0$ as function of the wealth ratio $WR = W_0 / \sum_{j=1}^{3} W_j \exp(-r T_j)$, when the investor has three investment goals with same discounted value at time $t = 0$ (i.e., $W_j \exp(-r T_j) = 50,000$ for all $j$) at horizons 1 year (full line), 5 years (dashed line) and 20 years (dotted line). Investor’s preferences are characterized by $\beta^+_j = 0, \beta^-_j = 2.25$ (infinite loss aversion) and $\alpha_j = 0.88$ for $j = 1, 2, 3$. We have $r = 0.03, \sigma = 0.2$. We have $\kappa = 0.2$ (top panel) and $\kappa = 0.4$ (bottom panel), which imply an equity premium of 4% and 8%, respectively.
Figure 4: Probability of reaching the investment goals as function of the wealth ratio $WR = W_0 / \sum_{j=1}^{3} W_j \exp(-r T_j)$, when the investor has three investment goals with same discounted value at time $t = 0$ (i.e., $W_j \exp(-r T_j) = \$50,000$ for all $j$) at horizons 1 year (full line), 5 years (dashed line) and 20 years (dotted line). Investor’s preferences are characterized by $\beta^+ = 0$, $\beta^- = 2.25$ (infinite loss aversion) and $\alpha_j = 0.88$ for $j = 1, 2, 3$. We have $r = 0.03$, $\sigma = 0.2$. We have $\kappa = 0.2$ (top panel) and $\kappa = 0.4$ (bottom panel), which imply an equity premium of 4% and 8%, respectively.
Figure 5: Optimal proportion of risky assets at time $t = 0$ as function of the wealth ratio $WR = W_0 / \sum_{j=1}^{3} W_j \exp(-r T_j)$, when the investor has three investment goals with same discounted value at time $t = 0$ (i.e., $W_j \exp(-r T_j) = $50,000 for all $j$) at horizons 1 year (full line), 5 years (dashed line) and 20 years (dotted line). Investor’s preferences are characterized by $\beta_j^+ = 0$, $\beta_j^- = 2.25$ (infinite loss aversion) and $\alpha_j = 0.88$ for $j = 1, 2, 3$. We have $r = 0.03$, $\sigma = 0.2$. We have $\kappa = 0.2$ (top panel) and $\kappa = 0.4$ (bottom panel), which imply an equity premium of 4% and 8%, respectively.
Figure 6: Optimal wealth shares at time $t = 0$ as function of the equity premium $\mu - r = \sigma \kappa$, when the investor has three investment goals with same discounted value at time $t = 0$ (i.e., $W_j \exp(-r T_j) = 50,000$ for all $j$) at horizons 1 year (full line), 5 years (dashed line) and 20 years (dotted line). Investor’s preferences are characterized by $\beta_j^+ = 0, \beta_j^- = 2.25$ (satisficing behavior) and $\alpha_j = 0.88$ for $j = 1, 2, 3$. We have $r = 0.03, \sigma = 0.2$. The wealth ratio $WR = W_0 / \sum_{j=1}^3 W_j \exp(-r T_j)$ corresponds to 0.40 (top panel) and 0.75 (bottom panel).
Figure 7: Probability of reaching the investment goals as function of the equity premium \( \mu - r = \sigma \kappa \), when the investor has three investment goals with same discounted value at time \( t = 0 \) (i.e., \( W_j \exp(-r T_j) = $50,000 \) for all \( j \)) at horizons 1 year (full line), 5 years (dashed line) and 20 years (dotted line). Investor’s preferences are characterized by \( \beta_j^+ = 0, \beta_j^- = 2.25 \) (satisficing behavior) and \( \alpha_j = 0.88 \) for \( j = 1, 2, 3 \). We have \( r = 0.03 \), \( \sigma = 0.2 \). The wealth ratio \( WR = W_0 / \sum_{j=1}^{3} W_j \exp(-r T_j) \) corresponds to 0.40 (top panel) and 0.75 (bottom panel).
Figure 8: Optimal proportion of risky assets at time 0 as function of the equity premium $\mu - r = \sigma \kappa$, when the investor has three investment goals with same discounted value at time $t = 0$ (i.e., $W_j \exp(-r T_j) = $50,000 for all $j$) at horizons 1 year (full line), 5 years (dashed line) and 20 years (dotted line). The total allocation to the risky asset is given by the red line. Investor’s preferences are characterized by $\beta_j^+ = 0$, $\beta_j^- = 2.25$ (satisficing behavior) and $\alpha_j = 0.88$ for $j = 1, 2, 3$. We have $r = 0.03$, $\sigma = 0.2$. The wealth ratio $WR = W_0 / \sum_{j=1}^3 W_j \exp(-r T_j)$ corresponds to 0.40 (top panel) and 0.75 (bottom panel).