Universität St.Gallen

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June 2009 Discussion Paper no. 2009-12

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Electronic Publication: http://www.vwa.unisg.ch

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Abstract

This paper extends the model with narrow framing suggested by Barberis and Huang (2009) to also account for probability weighting and a convex-concave value function in the specification of cumulative prospect theory preferences on narrowly framed assets. We show that probability weighting is needed in order that investors reduce their holding of narrowly framed risky assets in the presence of negative skewness and high Sharpe ratios, which are typical characteristics of stock index returns. The model with framing and probability weighting can thus explain the stock participation puzzle under realistic assumptions on stock market returns. We also show that a convex-concave value function generates wealth effects that are consistent with empirical observations on stock market participation. Finally, we address the asset pricing implications of probability weighting in the model with narrow framing and show that in the case of negative skewness the equity premium of narrowly framed assets is much higher than when probability weighting is not taken into account.

Keywords:
Narrow framing, cumulative prospect theory, probability weighting function, negative skewness, simulation methods.

JEL Classification

D1, D8, G11, G12.

1 Financial support from the National Center of Competence in Research "Financial Valuation and Risk Management" (NCCR-FINRISK) and from the Foundation for Research and Development of the University of Lugano is gratefully acknowledged.

1 Introduction

Experimental studies on decision making under risk have described several departures of individual choice behavior from the principles defining expected utility theory (EUT). There is growing consensus that cumulative prospect theory by Tversky and Kahneman (1992) is superior to EUT as a descriptive model of preferences. Cumulative prospect theory (CPT) distinguishes two phases: framing and valuation. Framing is the way the decision problem is structured and organized, while valuation is the process by which framed prospects are ranked. Valuation occurs using a reference-dependent, kinked and convex-concave value function and probability weighting functions.

Recently, Barberis and Huang (2009) proposed a model with cumulative prospect theory preferences that includes framing into the utility specification.[1] For the sake of analytically tractability, Barberis and Huang (2009) assume no probability weighting and a piecewise-linear value function, which is everywhere concave. The model is then applied to the consumption-portfolio problem. In this context, when adding a new stock to the portfolio, the investor's utility is not only given by how the stock's return impacts the distribution of the investor's overall wealth, but also by the distribution of the stock alone, which, for some reason, might be narrowly framed. Barberis and Huang (2009) find that "for a wide range of preference parameterizations, the narrow framer is less likely to buy a given stock than is an investor who maximizes a standard utility function defined

[1]In this paper we refer to Barberis and Huang (2009) since it is in the paper that portfolio selection and asset pricing are formally analyzed, with an explicit derivation of optimal portfolios and the characterization of equilibrium prices. Applications of narrow framing already appear in Barberis, Huang, and Thaler (2006) and Barberis and Huang (2008a).

over wealth or consumption." It seems that narrow framing can help explaining several observed features of observed real portfolios, more importantly, the stock participation puzzle.

Piecewise-linear value functions and no probability weighting are common assumptions in many applications of CPT in finance.[2] Besides the fact that considering the "full" CPT is quite challenging, ignoring probability weighting and assuming piecewise-linear value functions is often motivated by the fact that loss aversion has the highest impact on portfolio choices. However, when no probability weighting is used and the value function is piecewise-linear, CPT emerges as a special case of EUT, with, additionally, the assumption that investors are risk neutral on gains and losses, and display loss aversion. Being a special case of EUT under these specifications, CPT cannot address several of the observed violations of EUT and thus loses most of its descriptive validity. Moreover, probability weighting in CPT describes decision makers' behavior with respect to low-probability payoffs. In the context of portfolio choice, these payoffs usually refer to the tails of the assets' returns distribution. We thus expect that probability weighting in CPT plays a crucial role for portfolio selection when returns displays positive or negative skewness, as is the case for real returns.[3]

This paper extends the model with framing suggested by Barberis and Huang (2009) to also

[2]Exceptions are De Giorgi, Hens, and Levy (2003), Levy and Levy (2004), Barberis and Huang (2008b) and Jin and Zhou (2008).

[3]Barberis and Huang (2008b) present a representative agent, one-period model where some assets display positive skewness. They show that CPT investors are skewness seeking and hold positively skewed assets, which at equilibrium display a negative excess return.

include probability weighting and a convex-concave value function in the specification of CPT preferences. As far as we known, this is the first paper that also includes probability weighting in a dynamic consumption-portfolio model with CPT preferences. We show that adding probability weighting does not increase the complexity of the model and its analytical tractability remains intact. Moreover, while it is true that in general with a convex-concave value function first-order conditions are not sufficient for a global optimum and non-standard optimization algorithms are required (see De Giorgi, Hens, and Mayer 2007), this is not an issue in the model with framing studied in this paper, since CPT is only applied to framed stocks and utility is derived from the return of single stocks not from the portfolio's return. Therefore, no portfolio optimization involves probability weighting and a convex-concave value function. Nevertheless, under a convex-concave value function we lose an important property of the model, namely the homogeneity property which plays a crucial role when deriving analytical solutions. As a consequence, a convex-concave value function still significantly complicates the theoretical analysis and we thus rely on simulations. When a convex-concave value function is used, the model also delivers interesting wealth effects that will be discuss below.

To study the impact of probability weighting and a convex-concave value function on narrow framing, we perform a number of numerical exercises. First, we show that when probability weighting is added to the model with framing, investors reduce their holding of narrowly framed assets as the degree of framing increases. This finding is robust with respect to different specifications of the return distribution and of investors' preferences, as long as framed assets do not display a very high

Sharpe ratio *and* high positive skewness. Second, even if a convex-concave value function is added, which implies a risk seeking behavior on losses with moderate or high probability, still investors reduce their holding of narrowly framed assets as the degree of framing increases. Third, when narrowly framed risky assets have a high Sharpe ratio and negative skewness, their holding is strongly reduced as the degree of narrow framing increases. In contrast, without probability weighting, even with negative skewness, the investor's allocation to narrowly framed assets increases as a function of the degree of framing when the Sharpe ratio is high. It follows that probability weighting is needed in the model with framing to explain why investors only hold a small proportion of their wealth in stocks, the so called stock participation puzzle. Finally, we show that with a convex-concave value function, important wealth effects emerge. As the initial wealth increases, the impact of narrow framing is reduced. Consequently, wealthy investors appear to be less affected by narrow framing and participate more in stock markets, while the opposite is true for investor with low initial wealth. This is consistent with the empirical observation on stock market participation reported by Hong, Kubik, and Stein (2004), showing that stock market participation grows fast as a function of wealth.

We also studied asset pricing implications of narrow framing with probability weighting. First, we show that with probability weighting the impact of narrow framing on equilibrium prices is qualitatively similar to when no probability weighting is added to the model: as the degree of narrow framing is increased, the equilibrium risk-free rate decreases and the equity premium of the narrowly framed asset increases. However, probability weighting has a significant impact on the value of the equilibrium risk-free rate and the equity premium. We observe that when the framed asset displays positive skewness (as it is the case with log-normal returns) and loss aversion is not too high, then the equity premium in the presence of probability weighting is slightly lower (and the risk-free rate slightly higher) compared to the case without probability weighting. This can be easily explained by the fact that investors with probability weighting like positive skewness more than investors without probability weighting when loss aversion is not too high (see also Barberis and Huang 2008b). In contrast, when loss aversion is high or the framed asset displays negative skewness, the equity premium with probability weighting is much higher (and the risk-free rate much lower) than in case of no probability weighting. The difference can be very high and reach 3-3.5%.

The remainder of the paper is structured as follows. In Section 2 we present the model with framing suggested by Barberis and Huang (2009) and its extensions to allow for probability weighting and convex-concave value functions. In Section 3 we briefly describe the algorithm used to solve the consumption-portfolio problem when a convex-concave value function is added. In Section 4 we present a numerical exercise and discuss the effect of framing in the general models studied in this paper. Section 5 concludes our proposal.

2 Model

In this section we present a generalization of the model with framing suggested by Barberis and Huang (2009) that also includes probability weighting and a convex-concave value function, which are important features of cumulative prospect theory. The description of the model closely follows

Barberis and Huang (2009).

2.1 Investors' Preferences

At time t the investor chooses a consumption level C_t and decides how to invest her remaining wealth $W_t - C_t$ to n assets with gross returns $R_{1,t+1}, \ldots, R_{n,t+1}$ between time t and $t+1$. Time $t+1$ wealth is therefore given by

$$W_{t+1} = (W_t - C_t) \sum_{i=1}^{n} \theta_{i,t} \, R_{i,t+1}$$

where $\theta_{i,t} \geq 0$ is the proportion of post-consumption wealth invested in asset i. We assume that short-sale is not allowed.

The investor frames assets $m+1, \ldots, n$ narrowly. Investor's utility at time t is then given as follows:

(1) $$V_t = H\left(C_t, \mu(V_{t+1}|I_t) + b_0 \sum_{i=m+1}^{n} U_t(G_{i,t+1}) \right)$$

where

(2) $$H(C,x) = ((1-\beta)\, C^\rho + \beta\, x^\rho)^{\frac{1}{\rho}}, \quad 0 < \beta < 1, 0 \neq \rho < 1,$$

(3) $$\mu(k\, x) = k\, \mu(x), \quad k > 0,$$

(4) $$G_{i,t+1} = \theta_{i,t}\, (W_t - C_t)\, (R_{i,t+1} - R_{i,z}), \quad i = m+1, \ldots, n.$$

In Equation (1), $\mu(V_{t+1}|I_t)$ is the certainty equivalent at time t of the investor's (random) utility at time $t+1$. Equation 1 is thus a recursive utility specification that allows for narrow framing, i.e., the investor can also get utility directly from assets $i > m$. The utility function U_t is defined as follows. For a random variable G_{t+1} with cumulative distribution F_t at time t we have

$$U_t(G_{t+1}) = \int_{-\infty}^{0} \bar{v}(x)\frac{d}{dx}[w^-(F_t(x))]\,dx + \int_{0}^{\infty} \bar{v}(x)\frac{d}{dx}[-w^+(1-F_t(x))]\,dx. \tag{5}$$

with

$$\bar{v}(x) = \begin{cases} x^\zeta & x \geq 0 \\ -\lambda\,(-x)^\zeta & x < 0 \end{cases}, \quad \zeta \in (0,1] \tag{6}$$

$$w^+(p) = \frac{p^{\delta^+}}{(p^{\delta^+} + (1-p)^{\delta^+})^{1/\delta^+}}, \quad \delta^+ \in (0.3, 1] \tag{7}$$

$$w^-(p) = \frac{p^{\delta^-}}{(p^{\delta^-} + (1-p)^{\delta^-})^{1/\delta^-}}, \quad \delta^- \in (0.3, 1]. \tag{8}$$

The function U_t corresponds to the cumulative prospect theory (CPT) value function, consistent with the motivation given by Barberis and Huang (2009) suggesting that (cumulative) prospect theory is the natural choice to be coupled with narrow framing. Note that when $\zeta = 1$ ($\bar{v}$ is a piecewise-linear function) and $\delta^+ = \delta^- = 1$ (no probability weighting, i.e., $w^+(p) = w^-(p) = p$ for all $p \in [0,1]$), then

$$U_t(G_{i,t+1}) = \mathbb{E}_t\left[\bar{v}(G_{i,t+1})\right]$$

which is the case considered by Barberis and Huang (2009).

As shown in Barberis and Huang (2008b), if $\zeta < 2\min(\delta^+, \delta^-)$, which is the case in many calibrations of CPT (see Abdellaoui 2000), then the function U_t can be written as

$$U_t(G_{t+1}) = -\int_{-\infty}^{0} w^-(F(x))\,d\bar{v}(x) + \int_{0}^{\infty} w^+(1-F(x))\,d\bar{v}(x)$$

When $G_{i,t+1}$ is given by Equation (4) with a fixed reference point $R_{i,z}$, $\theta_{i,t} > 0$ and $W_t > C_t$, then the following holds:

$$F_{i,t}(x) = \mathbb{P}\left[G_{i,t+1} \leq x | I_t\right] = \mathbb{P}\left[R_{i,t+1} \leq \frac{x}{\theta_{i,t}\left(W_t - C_t\right)} + R_{i,z} | I_t\right] = F^R_{i,t+1}\left(\frac{x}{\theta_{i,t}\left(W_t - C_t\right)} + R_{i,z}\right)$$

and thus:

$$\begin{aligned} U_t(G_{i,t+1}) &= -\int_{-\infty}^{0} w^-\left(F^R_{i,t}\left(\frac{x}{\theta_{i,t}\left(W_t - C_t\right)} + R_{i,z}\right)\right) d\bar{v}(x) \\ &\quad + \int_0^{\infty} w^+\left(1 - F^R_{i,t}\left(\frac{x}{\theta_{i,t}\left(W_t - C_t\right)} + R_{i,z}\right)\right) d\bar{v}(x) \\ &= \theta^{\zeta}_{i,t}\left(W_t - C_t\right)^{\zeta}\left[-\int_{-\infty}^{0} w^-\left(F^R_{i,t}\left(y + R_{i,z}\right)\right) d\bar{v}(y) \right. \\ &\quad \left. + \int_0^{\infty} w^+\left(1 - F^R_{i,t}\left(y + R_{i,z}\right)\right) d\bar{v}(y)\right] \\ &= \theta^{\zeta}_{i,t}\left(W_t - C_t\right)^{\zeta} U_t(R_{i,t+1} - R_{i,z}). \end{aligned}$$

Finally,

$$V_t = H\left(C_t, \mu(V_{t+1}|I_t) + b_0\left(W_t - C_t\right)^{\zeta} \sum_{i=m+1}^{n} \theta^{\zeta}_{i,t} U_t(R_{i,t+1} - R_{i,z})\right). \tag{9}$$

This equation has important implications for our analysis. First, while in general a convex-concave value function and probability weighting in CPT might cause the optimization problem to possess several local optima (see De Giorgi, Hens, and Mayer 2007), this is not the case in this model. Indeed, as long as $\theta_{i,t} \geq 0$, the shape of $\bar{v}$ and probability weighting functions w^+ and w^- only determine the value that the investor attributes to each framed stock, but do not apply to portfolio returns and thus do not impact the portfolio selection problem. Second, wealth at time t enters into the investor's utility as an additional weighting factor for framed stocks. Namely, suppose that an investor with wealth W_t at time t has the *optimal* consumption plan $(C_\tau)_{\tau=t,t+1,\ldots}$, *optimal* strategies $(\theta_{i,\tau})_{i=1,\ldots,n,\tau=t,t+1,\ldots}$ and a framing parameter that depends on his wealth and is given by $b_0\,W_t^{1-\zeta}$. Let V_t be his utility at time t. Since $(C_\tau/W_t)_{\tau=t,t+1,\ldots}$ is a feasible consumption plan for an investor with wealth 1, and $(\theta_{i,\tau})_{i=1,\ldots,n,\tau=t,t+1,\ldots}$ are feasible investment strategies we derive the utility for this second investor at time t assuming the framing parameter $b_0\,1^{1-\zeta} = b_0$ and we obtain

$$\begin{aligned} & H\left(\frac{C_t}{W_t}, \mu\left(\frac{V_{t+1}}{W_t}|I_t\right) + b_0\left(1 - \frac{C_t}{W_t}\right)^{\zeta} \sum_{i=m+1}^{n} \theta_{i,t}^{\zeta}\, U_t(R_{i,t+1} - R_{i,z})\right) \\ = \;& \frac{1}{W_t}\, H\left(C_t, \mu(V_{t+1}|I_t) + b_0\, W_t^{1-\zeta}\,(W_t - C_t)^{\zeta} \sum_{i=m+1}^{n} \theta_{i,t}^{\zeta}\, U_t(R_{i,t+1} - R_{i,z})\right) = \frac{V_t}{W_t}. \end{aligned}$$

It follows from this last equation that the consumption plan $(C_\tau/W_t)_{\tau=t,t+1,\ldots}$ and the investment strategies $(\theta_{i,\tau})_{i=1,\ldots,n,\tau=t,t+1,\ldots}$ are optimal for the investor with wealth 1 at time t. Therefore, as long as $\zeta \in (0,1)$, wealth at time t determines the impact of the framed stock on the investor's utility. In other words, an investor with lower wealth will have the same investment strategies as an investor with higher wealth who weights framed stocks more. This is an interesting observation which implies that in our model with a convex-concave function $\bar{v}$, wealthy investors are less concerned by the effect of narrow framing.

2.2 Consumption and Portfolio Selection Problems

We now address the consumption-portfolio problem. From Equation (9) we derive the Bellman equation

$$V_t = J(W_t, I_t) = \max_{\theta_t, C_t} H\left(C_t, \mu(J(W_{t+1}, I_{t+1})|I_t) + b_0\,(W_t - C_t)^{\zeta} \sum_{i=m+1}^{n} \theta_{i,t}^{\zeta}\, U_t(R_{i,t+1} - R_{i,z})\right).$$

where H and μ are given by Equations (2) and (3), respectively. Unfortunately, when a convex-concave function $\bar{v}$ is used, H is not homogeneous, which implies that the consumption and portfolio decisions problems cannot be separated as in Barberis and Huang (2009) where $\zeta = 1$. Consequently, analytical tractability of the model is lost. We will discuss in Section 3 how we deal with the case $\zeta \in (0, 1)$ using simulations, while in this section we focus on the case $\zeta = 1$ and probability weighting.

When $\zeta = 1$, H is homogenous of degree one and the consumption and portfolio decision problems can be separated. This works exactly as in Barberis and Huang (2009) and we thus refer to their paper, while here we only report the main steps. First, it can be shown that $J(W_t, I_t) = A_t(I_t)\, W_t$, where $A_t(I_t) = J(1, I_t)$ and

$$A_t(I_t)\, W_t = \max_{\theta_t, C_t} \left[(1-\beta)\, C_t^{\rho} + \beta\, (W_t - C_t)^{\rho} \left[\mu(A_{t+1}\, \theta_t'\, \mathbf{R}_{t+1}) | I_t) + b_0 \sum_{i=m+1}^{n} \theta_{i,t}\, U_t(R_{i,t+1} - R_{i,z}) \right]^{\rho} \right]^{1/\rho}$$

where $\theta_t = (\theta_t^1, \ldots, \theta_t^n)'$ and $\mathbf{R}_{t+1} = (R_{1,t+1}, \ldots, R_{n,t+1})'$. Second, consumption and the portfolio problems can be separated, given that the last terms in the bracket of the last equation only depend on θ_t, while the first term only depends on C_t. Therefore, the portfolio selection problem is

$$B_t^{\star} = \max_{\theta_t} \left[\mu(A_{t+1}\, \theta_t'\, \mathbf{R}_{t+1}) | I_t) + b_0 \sum_{i=m+1}^{n} \theta_{i,t}\, U_t(R_{i,t+1} - R_{i,z}) \right]$$

while the consumption problem is

$$A_t(I_t) = \max_{\alpha_t} \left[(1-\beta)\, \alpha_t^{\rho} + \beta\, (1-\alpha_t)^{\rho}\, (B_t^{\star})^{\rho} \right]^{1/\rho}$$

where

$$\alpha_t = \frac{C_t}{W_t}$$

is the consumption-to-wealth ratio. Third, we solve the first-order condition

$$(1-\beta)\,\alpha_t^{\rho-1} = \beta\,(1-\alpha_t)^{\rho-1}\,(B_t^\star)^\rho \tag{10}$$

for the consumption problem to find the optimal consumption-to-wealth ratio $\alpha_t^\star$ and the optimal value $A_t(I_t) = (1-\beta)^{1/\rho}\,(\alpha_t^\star)^{1-1/\rho}$ as a function of $\alpha_t^\star$. Finally, since the same applies at time $t+1$ we put $(1-\beta)^{1/\rho}\,(\alpha_{t+1}^\star)^{1-1/\rho}$ instead of $A_{t+1}(I_{t+1})$ into the portfolio selection problem and we obtain

$$B_t^\star = \max_{\theta_t}\left[\mu((1-\beta)^{1/\rho}\,\alpha_{t+1}^{1-1/\rho}\,\theta_t'\,\mathbf{R}_{t+1}|I_t) + b_0\sum_{i=m+1}^{n}\theta_{i,t}\,U_t(R_{i,t+1}-R_{i,z})\right]. \tag{11}$$

The only difference that appears in Equations (10) and (11) relative to the case where no probability weighting applies (i.e., $\delta^+ = \delta^- = 1$) is that the expected utility $\mathbb{E}_t\left[\bar{v}(R_{i,t+1}-R_{i,z})\right]$ is replaced by $U_t(R_{i,t+1}-R_{i,z})$, which contains the probability weighting functions w^+ and w^-. It is therefore clear that while probability weighting obviously impacts the optimal portfolio strategy θ_t, the complexity of the portfolio selection problem remains unchanged.

2.3 Equilibrium Analysis

In this section we briefly address asset pricing implications of narrow framing when $\zeta = 1$ and probability weighting applies. Our analysis is a straightforward extension of that presented by Barberis and Huang (2009). The only reason why we decided to report it here, is that we will refer to the equilibrium conditions below when we present the numerical examples in Section 4.

Suppose that a representative agent exists with preferences as described in Equations (1)–(8) and μ has the functional form

$$\mu(x) = \mathbb{E}\left[x^{1-\gamma}\right]^{1/(1-\gamma)} \tag{12}$$

with $\rho = 1-\gamma$. We additionally assume that the reference returns for the narrowly framed asset is the risk-free gross return $R_{f,t}$, i.e., $R_{i,z} = R_{f,t}$ in Equation (4). Then the following holds:

Lemma 2.1.

$$\beta^{1/(1-\gamma)}\, R_{f,t}\, \mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right] \mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R_{W,t+1}\right]^{\gamma/(1-\gamma)} = 1$$

and for $i = 2, \ldots, n$ we have

$$\frac{\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(R_{i,t+1}-R_{f,t})\right]}{\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]} + b_0\, 1_{\{i>m\}}\, R_{f,t}\left(\frac{\beta}{1-\beta}\right)^{1/(1-\gamma)}\left(\frac{1-\alpha_t}{\alpha}\right)^{-\gamma/(1-\gamma)} U_t(R_{i,t+1}-R_{i,z}) = 0.$$

where $R_W = W_{t+1}/(W_t - C_t)$ is the gross return of the total wealth portfolio.

Again, the only difference between the model without probability weighting and the model with probability weighting is that $\mathbb{E}_t\left[\bar{v}(R_{i,t+1}-R_{i,z})\right]$ is replaced by $U_t(R_{i,t+1}-R_{i,z})$. In other words, since probability weighting impacts the value that investors attribute to framed stocks, it also affects equilibrium prices.

3 Computational Aspects

As discussed in the previous section, when probability weighting is added to the narrow framing model of Barberis and Huang (2009) it remains possible to separate the consumption and portfolio problems. This allows us to use their method to efficiently compute the investor's optimal strategy. Specifically, the computation begins with initial values for θ_{t+1} and α_{t+1} and then applies a function minimization algorithm to Equation (9) to solve for θ_t, and thus compute B_t^*. Using these values, we then obtain the consumption-to-wealth ratio α_t. To compute the optimal consumption and portfolio, simply repeat the above procedure updating the values of θ_t and α_t each time, until convergence. We have found that this procedure converges with high precision within a few minutes on a computer using unoptimized code.

The addition of a convex-concave value function makes the computational problem significantly more challenging as the investor's consumption and portfolio decisions are no longer separable. As such, we can no longer apply the above procedure. Instead, we use a significantly slower but much more flexible simulation method which computes the optimal consumption and portfolio decisions over time directly from the investor's Bellman equation. Although this technique is conceptually simple, performing the necessary computations in a manner that is sufficiently fast and accurate requires careful engineering.

We now describe our computational approach to deal with a convex-concave value function. Here we consider a general consumption-portfolio problem and we do not directly refer to the model with framing presented in the previous section, given that our approach is very flexible and can be applied to any model that satisfies minimal consistency requirements. We consider an investor at time t who is in state S_t. Typically this state is just the investor's wealth W_t at time t, but it can also include other state information about the investor or the market. During period t the investor consumes C_t and invests the remaining wealth across some risky assets, the proportions of which are specified by the vector θ_t. At the end of the period the investor is in state S_{t+1}. This is a random variable due to the stochastic nature of the asset returns. The investor's optimal utility V_t, optimal consumption C_t and optimal portfolio θ_t are then given by the Bellman equation which can be written,

$$V_t(S_t) = \max_{C_t,\theta_t} H\left(C_t, \mathcal{U}_t(V_{t+1}(S_{t+1}), \mathcal{V}_{t+1}(C_t, \theta_t, S_t))\right) = \max_{C_t,\theta_t} J(C_t, \theta_t, \mathcal{U}_t(V_{t+1}(S_{t+1}), \mathcal{V}_{t+1}(C_t, \theta_t, S_t)), S_t)$$

where the function H defines how we combine the utility derived from the consumption in the current time period t with the current value of optimal utility $V_{t+1}(S_{t+1})$ and of other sources of utility $\mathcal{V}_{t+1}(C_t, \theta_t, S_t)$ at time $t + 1$. For example, in the model with framing discussed in the previous section, $\mathcal{V}_{t+1}(C_t, \theta_t, S_t)$ is the sum of gains and losses from framed assets. The function $\mathcal{U}_t$ defines how the current value of next period's utility is computed. In general, $\mathcal{U}_t$ is an expectation operator. In the model with framing of the previous section, $\mathcal{U}_t$ sums up the certainty equivalent at time t of $V_{t+1}(S_{t+1})$ and the CPT value of gains and losses $\mathcal{V}_{t+1}(C_t, \theta_t, S_t)$ from framed assets; see Equation (1). For convenience we define $J(C_t, \theta_t, \mathcal{U}_t(V_{t+1}(S_{t+1}), \mathcal{V}_{t+1}(C_t, \theta_t, S_t)), S_t)$ to be the function that is to be maximized in each step. In theory, if we know how to compute $\mathcal{U}_t(V_{t+1}(S_{t+1}), \mathcal{V}_{t+1}(C_t, \theta_t, S_t))$, and from this we can obtain the value of J, we can then determine the optimal behavior and utility V_t of the investor by using a function maximization algorithm.

To do this in practice requires the following steps. Firstly, we must be able to estimate the value of $V_{t+1}(S_{t+1})$ for any given state S_{t+1}. If the V_{t+1} is a fairly smooth functions over the state space, which is typically the case, then we can estimate these values by using a function approximator that fits a smooth surface over a finite set of values. Next, given a distribution for S_{t+1}, which can depend on θ_t, C_t and other variables such as the assets' drift rates and volatility, we can use numerical integration to compute $\mathcal{U}_t(V_{t+1}(S_{t+1}), \mathcal{V}_{t+1}(C_t, \theta_t, S_t))$. Indeed, as discussed above, $\mathcal{U}_t$ is usually an expectation operator, or, as for CPT, can be written as a Lebesque integral. Finally, using H to compute the value of J is straightforward.

In this way we can compute the optimal behavior of the investor at time t so long as we know the optimal utility in a sufficiently dense set of points in state space at time $t + 1$ for our function approximations of V_{t+1} to be accurate. Thus, if we start at the end of the investor's life with some default action, such as to consume the entire wealth, we can step by step compute the optimal strategy for this investor backwards over time.

In the above algorithm the selection of states to sample, the estimations of V_{t+1} based on a surface fitted to these points, the maximization over C_t and θ_t, and the backwards recursion of this process over time, are all essentially the same for every model, except perhaps for some model specific parametrization. Thus we have created a general computation engine that performs the above steps in a standardized way. To test some model, all the user has to do is provide the engine with an appropriate function to compute $J(C_t, \theta_t, \mathcal{U}_t(V_{t+1}(S_{t+1}), \mathcal{V}_{t+1}(C_t, \theta_t, S_t))$. As the estimates of $V_{t+1}(S_{t+1})$ is provided by the simulation engine, the computation of J is usually fairly straightforward. Indeed, it often requires just a few lines of code to express the core equations of the model being studied, perhaps using some mathematical libraries for the required probability distributions and numerical integration to work out the expectations. Besides J, all the model needs to specify is the size of the state space, the number of assets, and any constraints that we would like to place on these, for example to limit short selling.

The main advantage of this approach is that it allows us to freely experiment with a wide range of models: we can compute an optimal strategy for essentially any dynamically consistent model which has a smooth utility function over some state space. Furthermore, the simulation reveals not just a steady state solution, but the full dynamics of the investor's optimal utility, consumption and asset allocations over the investor's life. The main disadvantage is clearly computational cost and thus some effort has gone into ensuring that the required computations are performed as quickly and accurately as possible. For the Merton model (Merton 1973) we can calculate an investor's optimal portfolio strategy over a 70 year period in a few minutes. For a complex model, such our narrow framing model with probability weighting and a convex-concave value function, this computation takes about an hour. To approximate investors with an unlimited lifetime we continue the computation until convergence, which typically takes two or three hours for a complex model.

We have applied our simulation engine to solve the Merton model with up to three assets, the Wachter model with mean reverting Sharpe ratio (Wachter 2002), and the model with framing discussed in the previous section, without and with probability weighting and a piecewise-linear value function. For these models analytic solutions exist, so that we can easily verify the accuracy of the solutions obtained. Under a range of parameter settings the simulator produced results that were within a fraction of a percent of the analytic solutions.

4 Numerical Examples

4.1 Consumption and Portfolio Selection Problems

In this section we study the impact of probability weighting and a convex-concave value function on a simple consumption-portfolio problem taken from Barberis and Huang (2009). This problem consists of an investor who makes consumption and portfolio decisions on a yearly basis. The investor has an unlimited lifespan and a temporal discount rate of $\beta = 0.98$. Each year the investor allocates her post consumption wealth across three assets. The first asset is risk-free and has a net return of 2%. The second and third assets are risky and their gross returns $R_{2,t+1}$ and $R_{3,t+1}$ are given by

$$\log R_{i,t+1} = g_i + \sigma_i \, \epsilon_{i,t+1}, \quad i = 2, 3$$

where

$$\begin{pmatrix} \epsilon_{2,t} \\ \epsilon_{3,t} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \omega \\ \omega & 1 \end{pmatrix}\right), \quad \text{i.i.d. over time.}$$

The investor's wealth then evolves according to

$$W_{t+1} = (W_t - C_t)((1 - \theta_{2,t} - \theta_{3,t})R_f + \theta_{2,t}R_{2,t+1} + \theta_{3,t}R_{3,t+1}.$$

To simplify the problem, we fix $\theta_{2,t} = 0.5$, so that the investor just has to split the remaining post-consumption wealth $(1 - \theta_{2,t})\,(W_t - C_t)$ between the risk-free asset and the third asset. The second asset might represent some non-financial asset, such as housing wealth or human capital, while asset 3 might represent the domestic stock market. The third asset is the only asset that is narrowly framed, i.e., the investor possesses preferences according to Equations (1)-(4), where $n = 3$ and $m = 2$. We additionally assume that μ has the functional form given in Equation (12) and $\rho = 1 - \gamma$, where γ is the parameter of risk aversion. Moreover, the reference gross-return $R_{3,z}$ used to specify U_t in Equation (1) is set equal to 1.02 and corresponds to the risk-free gross return. Our discussion will focus on the proportion $\theta_3/(1 - \theta_2)$ of the remaining post-consumption wealth that is allocated to the narrowly framed asset.

The parameters' specification of our model is reported in Table 1. We deviate from the numerical example reported by Barberis and Huang (2009) in two ways. Firstly, we simply set $\omega = 0$, rather than the value of 0.01. Although this does not significantly change the investor's behavior, having independent returns will make the computations easier when we will extend our numerical example to also address the impact of narrow framing when assets' returns display negative skewness. The other change we make is to use higher volatility and drift rates based on annual returns for the S&P 500 index from January 1946 to January 2009: $g_1 = g_2 = 6.15\%$ and $\sigma_1 = \sigma_2 = 15.49\%$. These numbers are significantly higher than the values used by Barberis and Huang (2009): $g_1 = g_2 = 4\%$ and $\sigma_1 = \sigma_2 = 10\%$. Our calibrations imply a mean of 7.6% and a standard deviation of 17.0% for the annual returns of the S&P 500 index. Note that from January 2008 to January 2009, the S&P 500 index has lost 37.2% of its value. If we take out from our dataset this last observation, we obtain the estimations $g_1 = g_2 = 8.31\%$ and $\sigma_1 = \sigma_2 = 15.53\%$.

[Table 1 about here.]

Our first results are reported in Figure 1 and in Table 2 (column 2) and refers to the case where CPT is specified using a piecewise-linear value function ($\zeta = 1$ in Equation (4)) and no probability weighting ($\delta^+ = \delta^- = 1$ in Equations (7) and (8)). We compute how the remaining post-consumption wealth (after that 50% has already been invested in asset 2) is allocated to the third asset as a function of the parameter of risk aversion γ and the degree of narrow framing b_0. Here, and throughout our numerical exercises, we fix the parameter of loss aversion to be $\lambda = 2.25$, as calibrated by Tversky and Kahneman (1992) from their experiments.

[Figure 1 about here.]

[Table 2 about here.]

With no narrow framing, that is, $b_0 = 0$, we see that it is only with a risk aversion of 4 or greater that the investor no longer allocates 100% of the remaining post-consumption wealth to asset 3. Even with $\gamma = 8.0$ the investor still allocates slightly less than 55% of the remaining post-consumption wealth to asset 3, while 50% of post-consumption wealth is already allocated to the risky asset 2. It is well-known that investors with expected utility preferences invest a high proportion of their wealth in risky assets, even if their degree of risk aversion is high, in contrast to the empirical observation that individual investors only hold a small proportion of their wealth in stocks (Mankiw and Zeldes 1991).

If the degree of narrow framing b_0 is increased, we observe that the investor further increases her holding of asset 3. This result is different from that reported by Barberis and Huang (2009).

The difference can be easily explained. In our numerical exercise calibrated on historical returns, the framed asset displays a Sharpe ratio of 0.329, while in Barberis and Huang (2009) it is lower at 0.248. Obviously, the characteristics of the framed assets, e.g., their Sharpe ratio, play an important role in the way framing impacts the asset allocation. To get a deeper understanding of this, we compute how the asset allocation to the framed asset changes when the Sharpe ratio is between 0.27 and 0.38. Figure 2 reports the allocation to the narrowly framed asset as a function of the Sharpe ratio, with fixed risk aversion $\gamma = 5$ and three different values for the degree of narrow framing. We see that up to a Sharpe ratio of 0.31 (slightly below our estimated value of 0.329), the effect of narrow framing is to reduce the allocation to the framed asset (as reported by Barberis and Huang (2009)), while the opposite holds when the Sharpe ratio is higher than 0.31. In this case, the framed asset is attractive for the investor who increases her holding as the degree of framing increases. It follows that framing does not explain why investors only hold a small proportion of stocks when the Sharpe ratio is high. Note that the crossing point in Figure 2 corresponds to the point where the framed asset has zero CPT value. When the CPT value of the framed asset is zero, i.e., $U_t = 0$ in Equation (1), then obviously framing does not have any impact on the investor's asset allocation.

[Figure 2 about here.]

We now add probability weighting into the specification of CPT. The parameters of the probability weighting functions given in Equation (7) and (8) are $\delta^+ = 0.61$ and $\delta^- = 0.69$, respectively, as calibrated by Tversky and Kahneman (1992) from their experiments. The allocation to the narrowly framed risky asset for the case with a piecewise-linear value function and probability weighting is reported in Figure 3 and Table 2 (column 3).

[Figure 3 about here.]

Obviously, with no narrow framing ($b_0 = 0$) the results are identical to the case without probability weighting, as probability weighting only enters into the valuation of narrowly framed assets. However, as the investor narrowly frames the third asset, she now reduces her allocation to that asset. We point out that even with probability weighting the effect of framing is weak, and even in this case we still expect that for a high enough Sharpe ratio investors with probability weighting will increase their allocation to the framed asset as the degree of narrow framing increases. Indeed, it is well-know that CPT investors with no short-sale constraints engage in a infinite leverage of the market portfolio when the Sharpe ratio is high enough and the distribution of returns presents positive skewness (see De Giorgi, Hens, and Levy 2003), as it is the case with a log-normal distributions. To see how the Sharpe ratio impacts the effect of narrow framing when probability weighting is added to the model, Figure 4 reports the allocation to the framed asset as a function of the Sharpe ratio, with a fixed parameter of risk aversion $\gamma = 5$ and three different values for the degree of narrow framing. We see that narrow framing reduces the allocation to the framed asset as the Sharpe ratio is smaller than 0.35 (slightly above our estimated Sharpe ratio), while the opposite occurs when the Sharpe ratio is above 0.35. This number is slightly higher than that found in the case of no probability weighting.

[Figure 4 about here.]

The assumption of a log-normal distribution implies that returns display positive skewness: it is 0.47 in our numerical example. In the presence of probability weighting, negative skewness is expected to have an impact on framing, so a model that excludes negative skewness a priori is not well-suited for our analysis. Indeed, probability weighting describes people's preferences with respect to payoffs that occur with a small probability, i.e., the observation that decision makers dislike extreme losses and like high gains. Together with loss aversion, probability weighting implies that investors dislike payoffs which do not offer a high upside potential relative to the downside risk. In the context of assets' returns, this means that CPT investors like positive skewness and dislike negative skewness (see Barberis and Huang 2008b). Finally, it is well-known that negative skewness is a characteristic of the distribution of stocks' returns in many financial markets, so allowing for negative skewness in the return's distribution of asset 3 seems important.[4]

We thus extend our analysis by considering the case where asset returns' follows a skew-normal distribution. We opt for the skew-normal distribution for the following reasons: a) is very flexible and can account for positive and negative skewness, (b) can be easily estimated, (c) the normal distribution is a special case of it. The density function of the skew-normal distribution with mean μ, standard deviation σ and skewness ν is given by

$$\frac{2}{\kappa_2}\,\phi\left(\frac{x-\kappa_3}{\kappa_2}\right)\,\Phi\left(\kappa_1\,\left(\frac{x-\kappa_3}{\kappa_2}\right)\right)$$

where ϕ is the standard normal probability density function, Φ is the standard normal probability

[4]There is a growing empirical evidence showing that investors ask a premium for systematic skewness; see Harvey and Siddique (2000) and the references therein.

cumulative function, and

$$\kappa_1 = \frac{\kappa}{\sqrt{1-\kappa^2}}, \qquad \kappa_2 = \sqrt{\frac{\sigma^2}{1-\frac{2\,\kappa^2}{\pi}}}, \quad \text{and} \quad \kappa_3 = \mu - \kappa\,\kappa_2\sqrt{\frac{2}{\pi}}, \tag{13}$$

are the shape, scale and location parameters, respectively. The parameter

$$\kappa = \frac{\operatorname{sign}(\nu)\sqrt{\frac{\pi}{2}}|\nu|^{\frac{1}{3}}}{\sqrt{|\nu|^{\frac{2}{3}} + \left(\frac{4-\pi}{2}\right)^{\frac{2}{3}}}},$$

which only depends on skewness ν, is zero when $\nu = 0$. In this case, the skew-normal distribution corresponds to the normal distribution with mean μ and standard deviation σ.

We calibrate the skew-normal distribution on yearly returns of the S&P 500 index from January 1946 to January 2009. The calibrated parameters are reported in Table 1. These imply a mean return of 7.6%, volatility of 15.8% and a skewness of −0.339. We see that allowing for negative skewness, the volatility decreases from 17% to 15.8%, while the mean remains almost unchanged. In this way, the Sharpe ratio increases from 0.329 to 0.354 and it is not clear whether investors will perceive the framed asset as being less or more attractive than in the case of log-normal returns. We will come back to this point below. Figure 5 reports the calibrated density functions under the skew-normal distribution assumption as well as under the log-normal distribution assumption considered above. We see that the skew-normal distribution attaches higher probability to extreme negative returns.

[Figure 5 about here.]

We then replace the log-normal distribution for returns $R_{2,t+1}$ and $R_{3,t+1}$ with the skew-normal distribution, i.e.,

$$R_{i,t+1} - 1 \sim SN(\kappa_1, \kappa_2, \kappa_3), \quad \text{for } i = 2, 3$$

while we keep the correlation between returns equal to zero. Unlike the log-normal distribution, the lower tail of the skew-normal does not terminate at the origin. This means that there is a strictly non-zero probability for risky assets having more than 100% negative return, which can can excluded. Fortunately, under our fitted skew-normal distribution the estimated probability of the risky assets losing more than 75% in value in any given year is almost zero. Thus we simply clip the lower tail of the returns distribution at −80%.

For our first test of skew-normal returns we consider again the narrow framing model with piecewise-linear value function and no probability weighting. The results for this case are reported in Figure 6 and in Table 2 (column 4). Similarly to the case with log-normal returns, we see that the allocation to the framed asset increases as the degree of narrow framing increases. The effect is now weaker than in the case of log-normal returns, because the investor is loss averse and dislikes negative skewness. However, with no probability weighting, it seems that the Sharpe ratio is the main driver of her preference for the framed asset.

[Figure 6 about here.]

Next we consider the behavior of the probability weighting investor with skew-normal returns. These results appear in Figure 7 and Table 2 (column 5). As expected, the probability weighting investor is far more sensitive to the negative skewness of the returns. While the Sharpe is higher, negative skewness causes the the investor with probability weighting to behave in the same way as we discussed above for the case with log-normal returns. However, in this case very little narrow framing is required before the investor decides to completely move out of holding the third asset. Since high Sharpe ratio and negative skewness are typical characteristics of stock index returns, we conclude that in the model with framing probability weighting is crucial in order to explain why investors only hold a small proportion of their wealth in stocks.

[Figure 7 about here.]

In all our numerical exercises presented above we assume that the value function $\bar{v}$ given in Equation (6) is piecewise-linear, i.e., $\zeta = 1$. As we discussed in Section 2, this assumption is very convenient, since consumption and portfolio decisions can be separated and the model is analytically tractable. However, CPT preferences are characterized by a convex-concave value function $\bar{v}$, where ζ is usually taken equal to 0.88, as calibrated by Tversky and Kahneman (1992) from their experiments. With a convex-concave value function, consumption and portfolio decisions are not separable anymore, but we can solve them numerically using the approach discussed in Section 3. Figure 8 and Table 2 (column 6) report the results when we additionally assume probability weighting and skew-normal distributed returns, as defined above. With a convex-concave value function, we also have to specify the initial wealth, since the allocation now depends on it. We take $W_0 = \$100,000$. We see that a convex-concave value function slightly reduces the effect of narrow framing. This does not come as a surprise, given that a convex value function on losses (additionally to probability weighting) implies a risk-seeking attitude on losses with moderate to high probability. However, investors' attitude on extreme losses is mainly determined by the probability weighting function and in the presence of negative skewness, investors with convex-concave value function behave in a similar manner as investors with a piecewise-linear value function.

[Figure 8 about here.]

As we discuss in Section 2, a convex-concave value function in the specification of CPT preferences on framed assets, introduces an interesting wealth effect. As wealth increases, a higher degree of narrow framing is needed in order that framing impacts investors' decisions. In our numerical example above, we have $W_0 = \$100,000$. An investor with initial wealth $W_0 = \$1,000,000$ would behave as the investor of our numerical example when her degree of narrow framing were $10^{1-0.88} = 1.32$ times higher. If we combine this effect with the results above about the impact of narrow framing, we see that in the model with probability weighting and a convex-concave value function, wealthy investors allocate more to framed assets than investors with low wealth level do. As an example, the investor with initial wealth $\$1,000,000$, if she displays risk aversion $\gamma = 5$ and degree of narrow framing $b_0 = 0.1$, she would invest 51.6% is the remaining post-consumption wealth to the narrowly framed asset, while an investor with initial wealth $\$100,000$ with same parameters, would invest 35.6% of remaining post-consumption wealth to the framed asset. If we take the framed asset in our numerical example to be the stock market, this implies that wealthy investors will have a higher participation in stock markets. This is consistent with the observation reported by Hong, Kubik, and Stein (2004) that stock market participation grows very quickly as a function of wealth.

4.2 Equilibrium Analysis

We now address asset pricing implications of narrow framing with probability weighting. We consider the numerical example of the previous subsection, where investors narrowly frame asset 3, while asset 2 can be seen as a non-financial asset. We assume a representative investor with preferences according to Equations (1)-(4), where $n = 3$ and $m = 2$. Additionally, μ has the functional form given in Equation (12) and $\rho = 1 - \gamma$, where γ is the parameter of risk aversion. Moreover, the reference gross-return $R_{3,z}$ used to specify U_t in Equation (1) is set equal to the risk-free gross return $R_{f,t}$.

We also impose the following conditions, taken from Barberis and Huang (2009): (i) the risk-free rate is constant, $R_{f,t} = R_f$ for all t; (ii) consumption growth C_{t+1}/C_t is log-normal with parameters g_C and σ_C, i.e.,

$$\log \frac{C_{t+1}}{C_t} = g_C + \sigma_C \, \epsilon_{C,t+1}$$

where $\epsilon_{C,t} \sim N(0, 1)$; (iii) the consumption-to-wealth ratio $\alpha_t = C_t/W_t = \alpha$ is constant over time, (iv) the fraction of post-consumption wealth $\theta_{3,t} = \theta_3$ invested in stock 3 is also constant over time. Under these conditions, we can easily solve the characterization of equilibrium prices reported in Lemma 2.1.

As in our discussion above, also for the equilibrium analysis we consider two cases. The first case is similar to the one studied by Barberis and Huang (2009) and assumes that the framed asset 3 possesses log-normally distributed returns, i.e.,

$$\log R_{3,t+1} = g_3 + \sigma_3\, \epsilon_{2,t+1}$$

and the correlation with consumption growth is ρ, i.e.,

$$\text{corr}\left(\log \frac{C_{t+1}}{C_t}, \log R_{3,t+1}\right) = \rho.$$

The parameter specification for this case is reported in Table 3.

[Table 3 about here.]

For the case with log-normally distributed returns, the equilibrium risk-free rate R_f and the equity premium $\mathbb{E}\left[R_{3,t}\right] - R_f$ are reported in Table 4. The first two panels and three columns simply replicate Table 4 in Barberis and Huang (2009). First, we see that under all specifications of the investor's preferences (with and without probability weighing and all choices of the parameters of risk aversion and loss aversion) the equilibrium risk-free rate decreases and the equity premium increases when the degree of framing is increased. Second, we observe that when adding probability weighing to the specification of CPT, the equilibrium risk-free rate slightly increases (up to 0.13%) and the equity premium slightly decreases (up to 0.3%) when loss aversion is 2 or 2.25. In contrast, when loss aversion is 3, the equilibrium risk-free rate is significantly smaller (up to 0.55% less) and the equity premium significantly higher (up to 1.2% more) when probability weighting is added to the specification of CPT. These findings are intuitive: when loss aversion is 3, the investor with probability weighting is more sensitive to the tail of the distribution than the investor without probability weighting. Consequently, even if returns are positively skewed, when loss aversion and the degree of narrow framing are high, the investor with probability weighting ask a higher equity premium to hold the risky asset. In contrast, when loss aversion is 2 or 2.25 the investor with probability weighting likes positive skewness more than the investor without probability weighting and is willing to hold the risky asset at a lower premium (see Barberis and Huang 2008b). Note that skewness is 0.61 in the numerical example of this subsection, i.e., much higher than the 0.47 we had in the numerical example of previous subsection. Therefore, the results of this subsection do not contradict the results of the previous subsection, where a loss aversion of 2.25 was enough to observe the investor with probability weighting moving out from the framed stock.

[Table 4 about here.]

The second case we consider is when the framed asset 3 possesses skew-normally distributed returns with mean μ_3, standard deviation σ_3 and skewness $\nu_3 < 0$, i.e.,

$$R_{3,t+1} \sim SN(\kappa_1, \kappa_2, \kappa_3)$$

where κ_1, κ_2 and κ_3 depend on μ_3, σ_3 and ν_3 as reported in Equation (13). We also assume that $R_{3,t+1}$ is independent from the consumption growth. The assumption about independence between the returns of the framed asset and consumption growth in mainly for analytical convenience, but does not qualitatively impact the results. Table 5 reports the parameter specification for the skew-normal case.

[Table 5 about here.]

The equilibrium risk-free rate R_f and the equity premium $\mathbb{E}[R_{3,t+1}] - R_f$ for the case of skew-normal returns are reported in Table 6. Again, we see that under all specifications of investor's preferences, when the degree of framing is increased, the equilibrium risk-free rate decreases and the equity premium increases. However, with skew-normal returns and negative skewness for the framed asset, the effect is much stronger when probability weighting is added to the specification of CPT. This result is now clear. The investor with probability weighting dislikes negative skewness much more than the investor without probability weighting and thus asks for a higher premium in order to hold the framed asset. The difference between the equity premium when probability weighting is present and when it is not is huge and ranges from 0.5% (low degree of framing, loss aversion of 2) to more than 3.5% (high degree of framing and loss aversion of 3). If the degree of framing and loss aversion are high, addition of probability weighting can double the equity premium for the framed asset.

[Table 6 about here.]

5 Conclusion

In this paper we extended the model with framing suggested by Barberis and Huang (2009) to also include probability weighting and a convex-concave value function in the specification of cumulative prospect theory preferences on narrowly framed assets. We showed that adding probability weighting does not increase the complexity of the model and analytical tractability is preserved. In contrast, with a convex-concave value function, we lose the homogeneity property of the model and we thus have to rely on simulations.

We showed that in the model with framing, probability weighting is needed in order to observe investors moving out from framed assets, when these display high Sharpe ratios and negative skewness. This is the typical case of stock index returns in many markets, so probability weighting seems to be crucial to explain the stock market participation puzzle. We also showed that a convex-concave value function generates important wealth effects, i.e., narrow framing has a small effect on the portfolio decision of wealthy investors, while the opposite holds when investors possess a low level of wealth. This is consistent with empirical observations of stock market participation.

We also studied asset pricing implications of narrow framing with probability weighting. When skewness is positive and loss aversion is not too high, probability weighting slightly reduces the equity premium of the framed asset. Investors with probability weighting like positive skewness more than investors without probability weighting as long as loss aversion is not too high. Consequently, when skewness is positive and loss aversion is not too high, they are willing to hold framed assets at a lower premium compared to investors without probability weighting. In contrast, as the framed asset displays negative skewness or loss aversion is high, the investor with probability weighting demands a much higher equity premium compared to the investor without probability weighting.

References

ABDELLAOUI, M. (2000): "Parament-Free Elicitation of Utility and Probability Weighting Functions," *Management Science*, 46(11), 1497–1512.

BARBERIS, N., AND M. HUANG (2008a): "The Loss Aversion / Narrow Framing Approach to the

Equity Premium Puzzle," in *Handbook of Investments: The Handbook of the Equity Premium*, ed. by R. Mehra, pp. 199–228. North Holland, Amsterdam.

Barberis, N., and M. Huang (2008b): "Stocks as Lotteries: The Implications of Probability Weighting for Security Prices," *American Economic Review*, 95(8), 2066–2100.

Barberis, N., and M. Huang (2009): "Preferences with Frames: A New Utility Specification that Allows for Framing of Risks," *Journal of Economic Dynamics and Control*, Forthcoming.

Barberis, N., M. Huang, and R. H. Thaler (2006): "Individual Preferences, Monetary Gambles, and Stock Market Participation: A Case for Narrow Framing," *American Economic Review*, 96(4), 1069–1090.

De Giorgi, E., T. Hens, and H. Levy (2003): "Existence of CAPM Equilibria with Prospect Theory Preferences," NCCR-FINRISK Working Paper, no. 85, Available at SSRN: http://ssrn.com/abstract=420184.

De Giorgi, E., T. Hens, and J. Mayer (2007): "Computational Aspects of Prospect Theory with Asset Pricing Applications," *Computational Economics*, 29(3-4), 267–281.

Harvey, C. R., and A. Siddique (2000): "Conditional Skewness in Asset Pricing Tests," *Journal of Finance*, 55(3), 1263–1295.

Hong, H., J. D. Kubik, and J. C. Stein (2004): "Social Interection and Stock-Market Participation," *Journal of Finance*, 59(1), 137–163.

Jin, H., and X. Y. Zhou (2008): "Behavioral Portfolio Selection in Continuous Time," *Mathematical Finance*, 18(3), 385–426.

Levy, H., and M. Levy (2004): "Prospect Thoery and Mean-Variance Analyisis," *The Review of Financial Studies*, 17(4), 1015–1041.

Mankiw, N. G., and S. P. Zeldes (1991): "The Consumption of Stockholders And Nonstockholders," *Journal of Financial Economics*, 29(1), 97–112.

Merton, R. (1973): "An Intertemporal Capital Asset Pricing Model," *Econometrica*, 41, 867–888.

Tversky, A., and D. Kahneman (1992): "Advances in Prospect Theory: Cumulative Representation of Uncertainty," *Journal of Risk and Uncertainty*, 5, 297–323.

Wachter, J. (2002): "Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets," *Journal of Financial and Quantitative Analysis*, 37, 63–91.

β	0.98
λ	2.25
R_f	1.02
g_1, g_2	6.15%
σ_1, σ_2	15.49%
ω	0
κ_1	-1.6175
κ_2	0.2150
κ_3	0.2219

Table 1: Parameters for the model with narrow framing, and the assets' returns distributions.

	Log-normal		Skew-normal		
b_0	PL+NoPW	PL+PW	PL+NoPW	PL+PW	CC+PW
$\gamma = 2$					
0.000	100.0	100.0	100.0	100.0	100.0
0.025	100.0	100.0	100.0	100.0	100.0
0.050	100.0	100.0	100.0	100.0	100.0
0.075	100.0	100.0	100.0	100.0	100.0
0.100	100.0	100.0	100.0	0.0	100.0
0.125	100.0	100.0	100.0	0.0	100.0
0.150	100.0	100.0	100.0	0.0	100.0
0.175	100.0	100.0	100.0	0.0	100.0
0.200	100.0	100.0	100.0	0.0	86.2
$\gamma = 5$					
0.000	88.5	88.5	83.6	83.6	83.2
0.025	93.5	84.7	87.8	52.3	74.6
0.050	98.1	80.7	91.7	7.6	64.2
0.075	100.0	76.6	95.3	0.0	51.6
0.100	100.0	72.2	98.7	0.0	35.6
0.125	100.0	67.6	100.0	0.0	0.0
0.150	100.0	62.7	100.0	0.0	0.0
0.175	100.0	57.7	100.0	0.0	0.0
0.200	100.0	52.4	100.0	0.0	0.0
$\gamma = 8$					
0.000	54.2	54.2	51.3	51.3	51.0
0.025	58.8	50.8	55.5	17.9	41.4
0.050	63.0	47.1	59.2	0.0	28.2
0.075	66.9	43.2	62.5	0.0	9.4
0.100	70.6	39.0	65.5	0.0	0.0
0.125	74.0	34.7	68.3	0.0	0.0
0.150	77.2	30.2	70.8	0.0	0.0
0.175	80.2	25.6	73.1	0.0	0.0
0.200	83.1	20.9	75.3	0.0	0.0

Table 2: The table displays the percentage $\theta_3/(1-\theta_2)$ of remaining post-consumption wealth allocated to the narrowly framed asset 3, as a function of the degree of narrow framing b_0 for different values of risk aversion γ: $\gamma = 2$ (top-panel), $\gamma = 5$ (middle panel) and $\gamma = 8$ (bottom panel). We assume two different distributional assumptions for the returns of the risky assets: log-normal (columns 2 and 3) and skew-normal (columns 4, 5 and 6); three different specifications of the CPT value function: piecewise-linear $\bar{v}$ and no probability weighting (PL+NoPW, columns 2 and 4), piecewise-linear $\bar{v}$ and probability weighting (PL+PW, columns 3 and 5), convex-concave $\bar{v}$ and probability weighting (CC+PW, column 6).

g_C	1.84%
σ_C	3.79%
ρ	0.1
σ_3	20%

Table 3: Parameter specification for the equilibrium analysis under the assumption that the framed asset possesses log-normally distributed returns: g_C and σ_C are mean and standard deviation, respectively, for the log consumption growth, ρ is the correlation between log-consumption growth and log-returns of asset 3, and σ_3 is the standard deviation of asset 3's log-returns.

	PL+NoPW		PL+PW	
b_0	$R_f - 1$	EP	$R_f - 1$	EP
$\gamma = 1.5, \lambda = 2$				
0.000	4.73	0.12	4.73	0.12
0.010	4.15	1.39	4.23	1.23
0.020	3.70	2.41	3.81	2.15
0.030	3.36	3.15	3.50	2.85
0.040	3.13	3.66	3.26	3.36
$\gamma = 1.5, \lambda = 3$				
0.000	4.73	0.12	4.73	0.12
0.005	4.09	1.54	4.05	1.62
0.010	3.43	2.99	3.29	3.31
0.015	2.82	4.35	2.48	5.10
0.020	2.32	5.45	1.77	6.65
$\gamma = 2, \lambda = 2.25$				
0.000	5.56	0.16	5.56	0.16
0.005	5.12	0.89	5.15	0.85
0.010	4.70	1.60	4.74	1.52
0.015	4.30	2.26	4.35	2.17
0.020	3.94	2.86	3.99	2.78
0.025	3.62	3.38	3.65	3.33
0.030	3.35	3.83	3.36	3.81
$\gamma = 4, \lambda = 2.25$				
0.000	8.58	0.33	8.58	0.33
0.005	7.96	0.84	8.00	0.81
0.010	7.35	1.35	7.42	1.29
0.015	6.74	1.85	6.84	1.77
0.020	6.15	2.34	6.25	2.25
0.025	5.57	2.81	5.67	2.73
0.030	5.01	3.27	5.09	3.20

Table 4: The table shows equilibrium risk-free rate $R_{f,} - 1$ and equity premium (EP) $\mathbb{E}\left[R_{3,t+1}\right] - R_f$ for the framed asset, as function of risk aversion γ, loss aversion λ and degree of narrow framing b_0. CPT is specified using a *piecewise-linear value function* ($\zeta = 1$ in Equation (4)) without probability weighting (columns 2 and 3) and with probability weighting (columns 4,5). Asset 3 possesses log-normally distributed returns with standard deviation $\sigma_2 = 0.3$, and correlation $\rho = 0.1$ with log consumption growth.

g_C	1.84%
σ_C	3.79%
σ_3	15.8%
ν_3	-0.339

Table 5: Parameter specification for the equilibrium analysis under the assumption that the framed asset possesses skew-normally distributed returns: g_C and σ_C are mean and standard deviation, respectively, for the log consumption growth, σ_3 and ν_3 are standard deviation and skewness, respectively, of asset 3's returns.

b_0	PL+NoPW $R_f - 1$	EP	PL+PW $R_f - 1$	EP
$\gamma = 1.5, \lambda = 2$				
0.000	4.73	0.00	4.73	0.00
0.010	4.30	0.94	4.08	1.44
0.020	3.98	1.66	3.50	2.71
0.030	3.74	2.18	3.06	3.71
0.040	3.57	2.56	2.73	4.42
$\gamma = 1.5, \lambda = 3$				
0.000	4.73	0.00	4.73	0.00
0.005	4.26	1.03	4.04	1.53
0.010	3.83	1.99	3.25	3.27
0.015	3.45	2.83	2.39	5.18
0.020	3.14	3.52	1.62	6.88
$\gamma = 2, \lambda = 2.25$				
0.000	5.56	0.00	5.56	0.00
0.005	5.23	0.55	5.08	0.79
0.010	4.93	1.05	4.60	1.61
0.015	4.66	1.50	4.10	2.42
0.020	4.42	1.91	3.62	3.22
0.025	4.20	2.26	3.16	3.98
0.030	4.02	2.56	2.75	4.65
0.035	3.87	2.82	2.41	5.21
$\gamma = 4, \lambda = 2.25$				
0.000	8.58	0.00	8.58	0.00
0.005	8.12	0.38	7.91	0.55
0.010	7.67	0.75	7.22	1.13
0.015	7.25	1.10	6.49	1.73
0.020	6.85	1.43	5.70	2.38
0.025	6.47	1.74	4.81	3.10
0.030	6.12	2.03	3.72	3.98
0.035	5.80	2.30	2.17	5.29

Table 6: The table shows equilibrium risk-free rate $R_{f,} - 1$ and equity premium (EP) $\mathbb{E}[R_{3,t+1}] - R_f$ for the framed asset, as function of risk aversion γ, loss aversion λ and degree of narrow framing b_0. CPT is specified using a piecewise-linear value function ($\zeta = 1$ in Equation (4)) without probability weighting (columns 2 and 3) and with probability weighting (columns 4,5). Asset 3 possesses skew-normally distributed returns with standard deviation $\sigma_2 = 15.8\%$, skewness $\nu = -0.339$, and correlation 0 with log consumption growth.

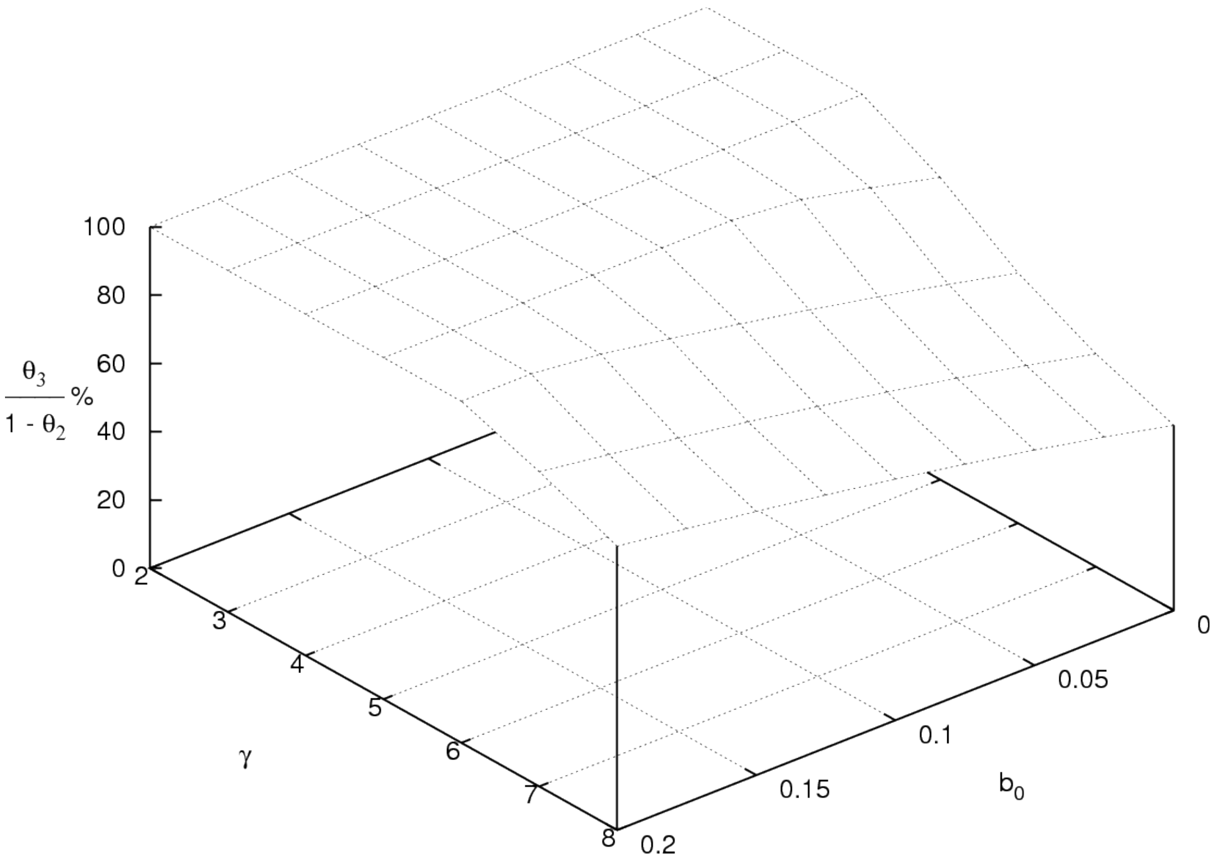

Figure 1: The figure shows the percentage $\theta_3/(1-\theta_2)$ of remaining post-consumption wealth allocated to the narrowly framed asset 3, as a function of risk aversion γ and of the degree of narrow framing b_0. CPT is specified using a *piecewise-linear value function* ($\zeta = 1$ in Equation (4)) and *no probability weighting*, i.e., $\delta^+ = \delta^- = 1$ in Equations (7) and (8), respectively. Assets' returns are assumed to be independent and identically distributed, with log-normal distribution with mean 7.6% and standard deviation 15.8%.

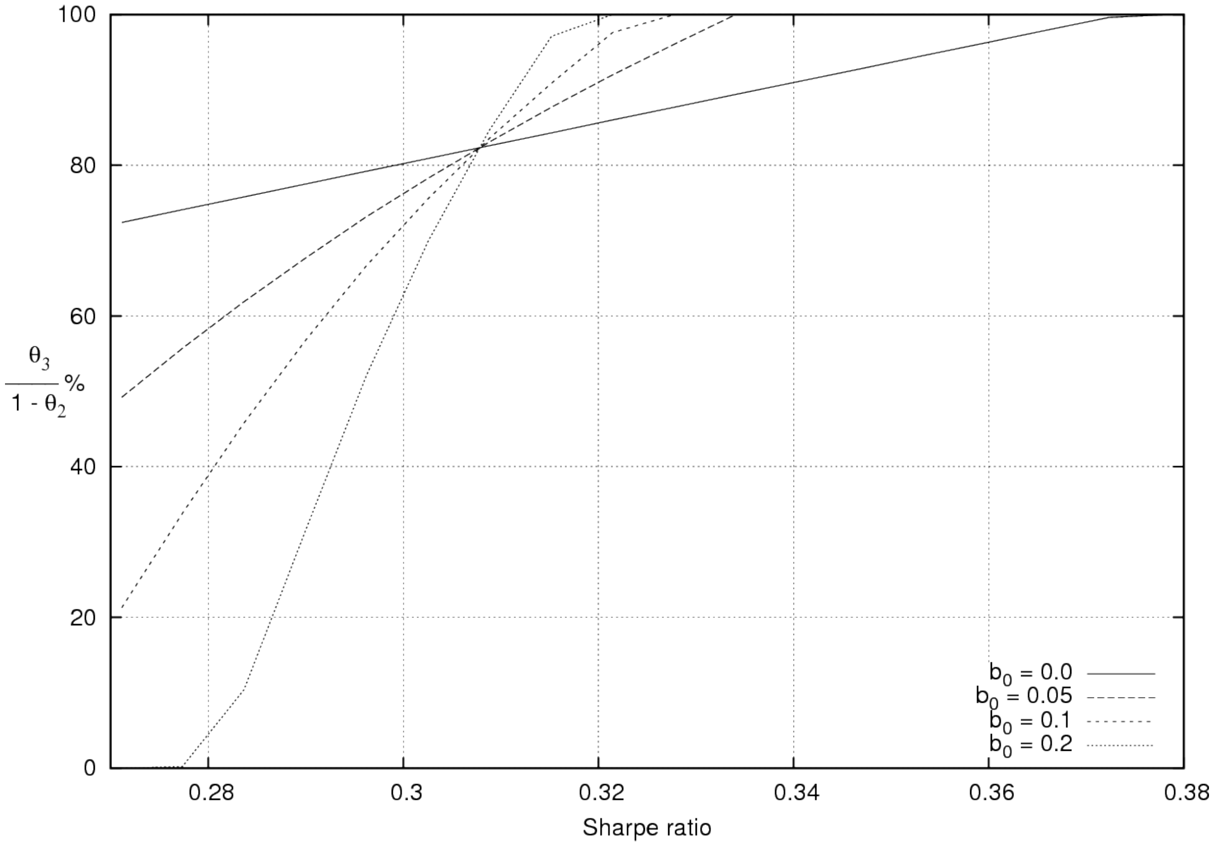

Figure 2: The figure shows the percentage $\theta_3/(1-\theta_2)$ of remaining post-consumption wealth allocated to the narrowly framed asset 3, as a function of the Sharpe ratio for $\gamma = 5$ and several degrees of narrow framing b_0. CPT is specified using a *piecewise-linear value function* ($\zeta = 1$ in Equation (4)) and *no probability weighting functions* ($\delta^+ = 1$ and $\delta^- = 1$ in Equations (7) and (8)), respectively. Assets' returns are assumed to be independent and identically distributed, with a log-normal distribution.

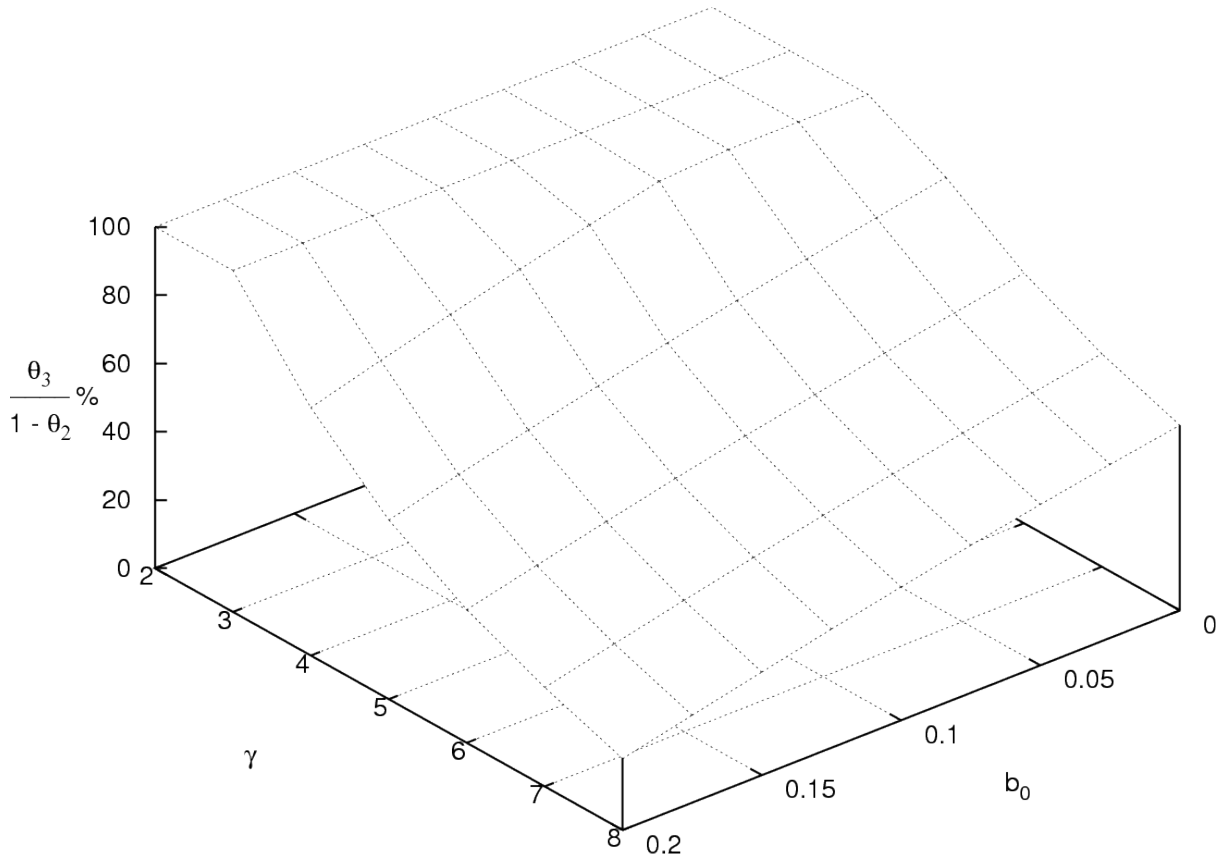

Figure 3: The figure shows the percentage $\theta_3/(1-\theta_2)$ of remaining post-consumption wealth allocated to the narrowly framed asset 3, as a function of risk aversion γ and of the degree of narrow framing b_0. CPT is specified using a *piecewise-linear value function* ($\zeta = 1$ in Equation (4)) and *probability weighting functions* with parameters $\delta^+ = 0.61$ and $\delta^- = 0.69$ in Equations (7) and (8), respectively. Assets' returns are assumed to be independent and identically distributed, with a log-normal distribution with mean 7.6% and standard deviation 17.0%.

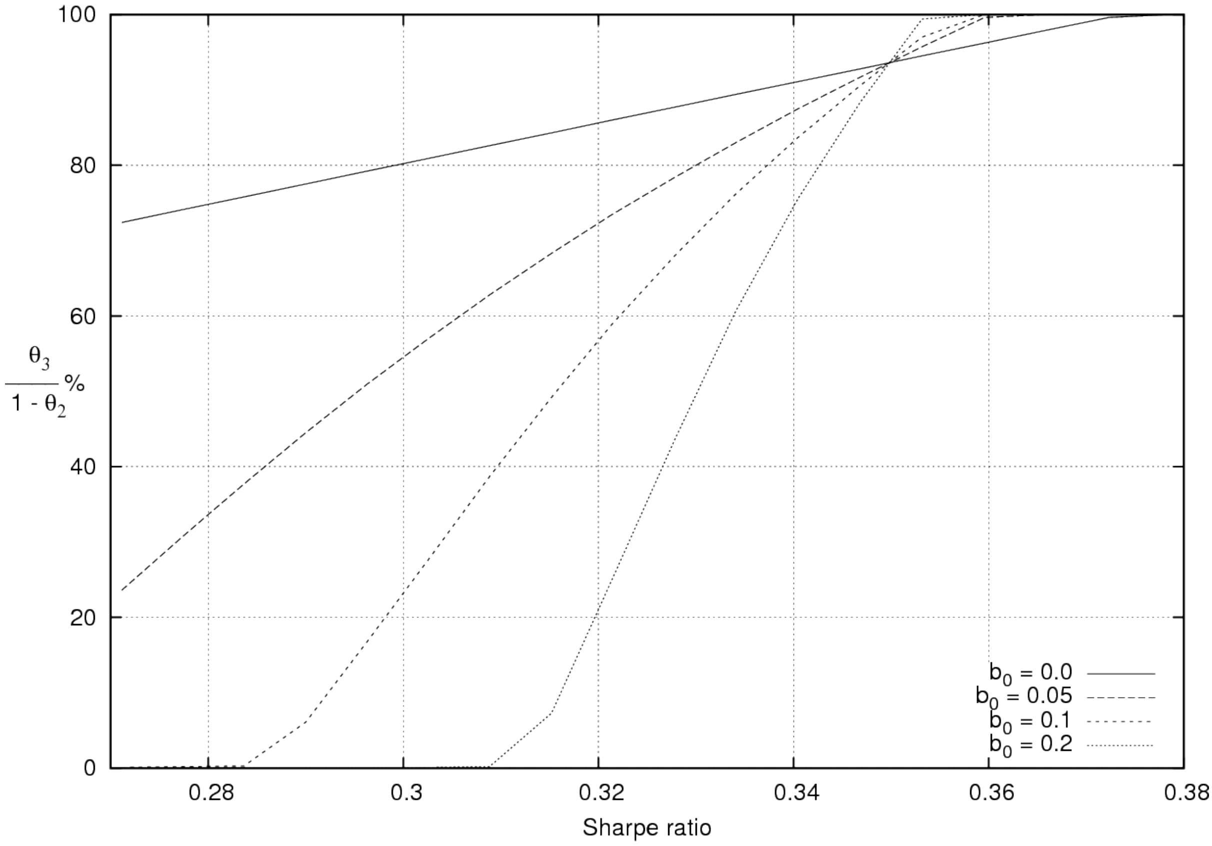

Figure 4: The figure shows the percentage $\theta_3/(1-\theta_2)$ of remaining post-consumption wealth allocated to the narrowly framed asset 3, as a function of the Sharpe ratio for $\gamma = 5$ and several degrees of narrow framing b_0. CPT is specified using a *piecewise-linear value function* ($\zeta = 1$ in Equation (4)) and *no probability weighting functions* ($\delta^+ = 1$ and $\delta^- = 1$ in Equations (7) and (8)), respectively. Assets' returns are assumed to be independent and identically distributed, with a log-normal distribution.

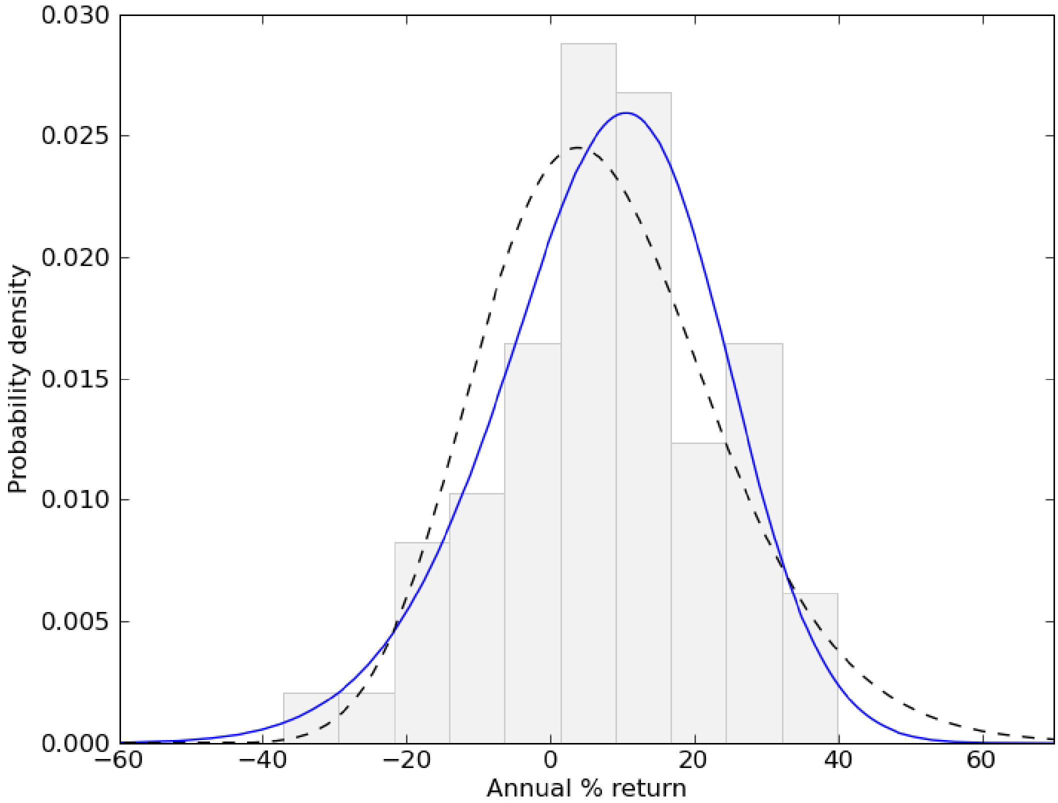

Figure 5: Calibrated probability density functions on historical yearly returns of S&P500 index from January 1946 to January 2009. The full line is the calibrated skew-normal distribution with mean 7.6%, standard deviation 15.8% and skewness -0.339. The dashed line is the calibrated log-normal distribution with mean 7.6% and standard deviation 17.0%.

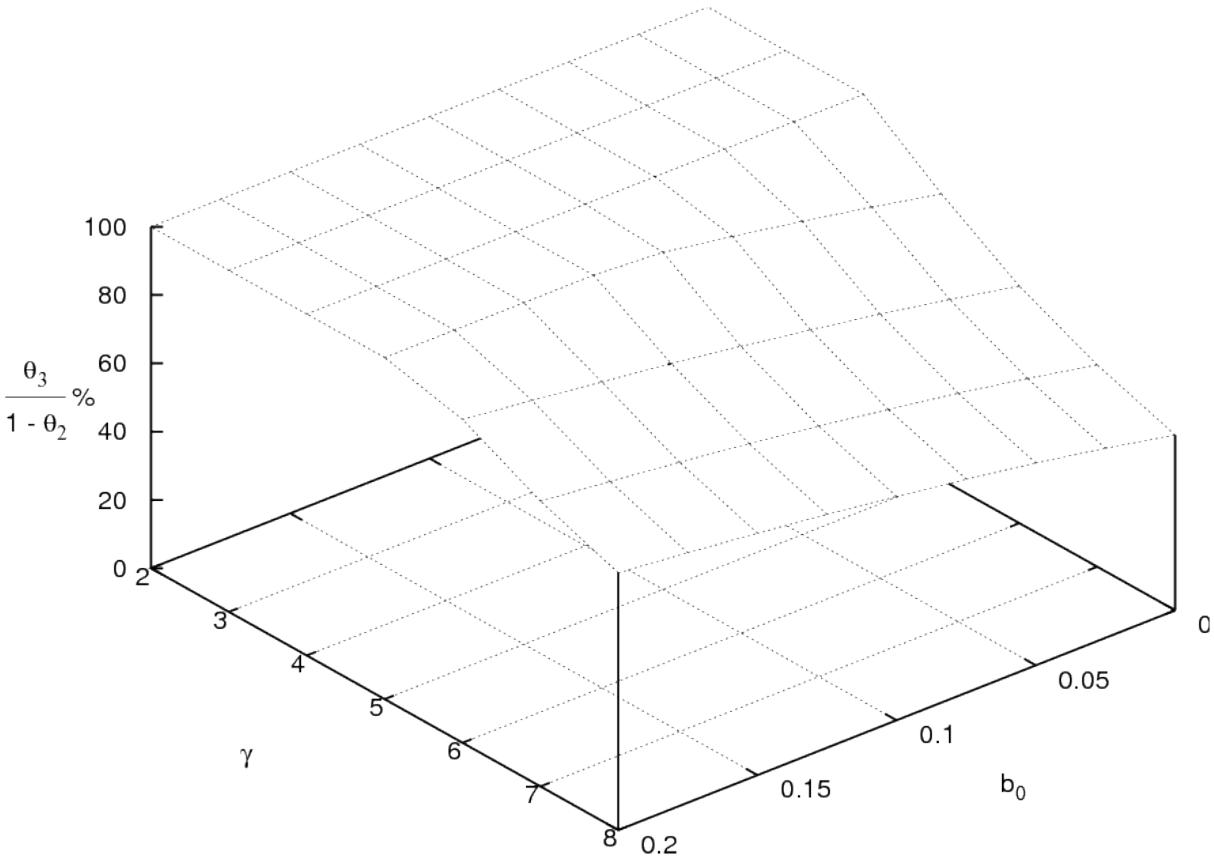

Figure 6: The figure shows the percentage $\theta_3/(1-\theta_2)$ of remaining post-consumption wealth allocated to the narrowly framed asset 3, as a function of risk aversion γ and of the degree of narrow framing b_0. CPT is specified using a *piecewise-linear value function* ($\zeta = 1$ in Equation (4)) and *no probability weighting functions*, i.e., $\delta^+ = \delta^- = 1$ in Equations (7) and (8), respectively. Assets' returns are assumed to be independent and identically distributed, with skew-normal distribution with mean 7.6%, standard deviation 15.8%, and skew -0.339.

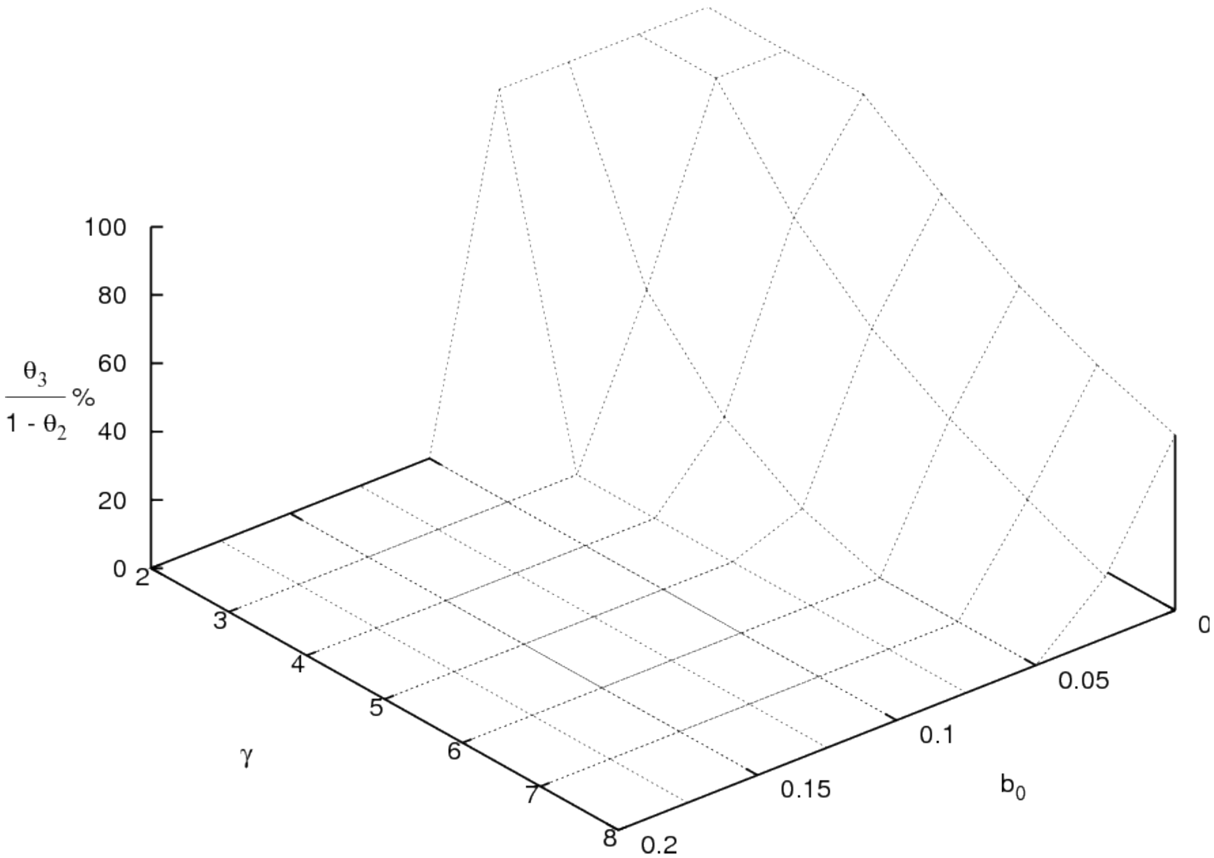

Figure 7: The figure shows the percentage $\theta_3/(1-\theta_2)$ of remaining post-consumption wealth allocated to the narrowly framed asset 3, as a function of risk aversion γ and of the degree of narrow framing b_0. CPT is specified using a *piecewise-linear value function* ($\zeta = 1$ in Equation (4)) and *probability weighting functions* with parameters $\delta^+ = 0.61$ and $\delta^- = 0.69$ in Equations (7) and (8), respectively. Assets' returns are assumed to be independent and identically distributed, with skew-normal distribution with mean 7.6%, standard deviation 15.8%, and skew -0.339.

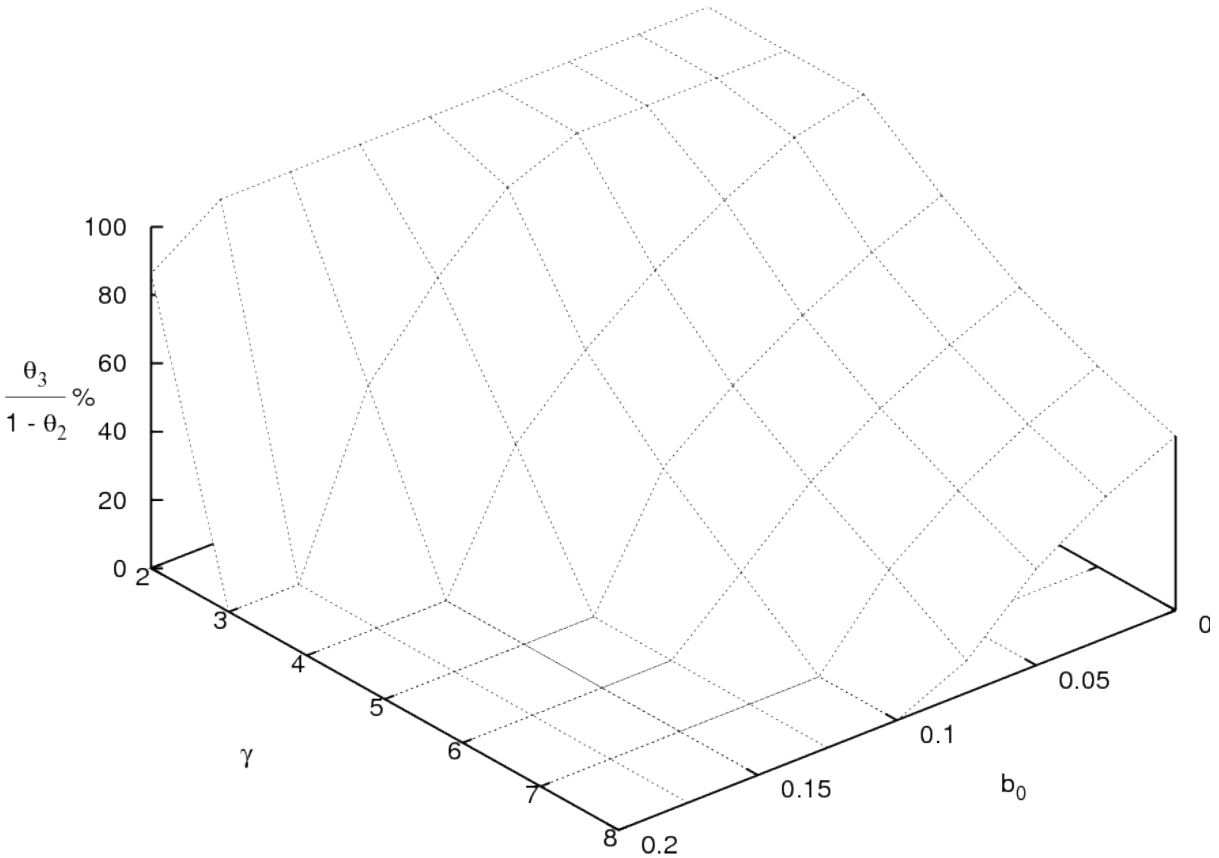

Figure 8: The figure shows the percentage $\theta_3/(1-\theta_2)$ of remaining post-consumption wealth allocated to the narrowly framed asset 3, as a function of risk aversion γ and of the degree of narrow framing b_0. CPT is specified using a *convex-concave value function* with parameter $\zeta = 0.88$ in Equation (4), and *probability weighting functions* with parameters $\delta^+ = 0.61$ and $\delta^- = 0.69$ in Equations (7) and (8), respectively. Initial wealth is $W_0 = \$100,000$. Assets' returns are assumed to be independent and identically distributed, with skew-normal distribution with mean 7.6%, standard deviation 15.8%, and skew -0.339.