INITIAL AND BOUNDARY VALUE PROBLEMS FOR DIFFERENCE EQUATIONS

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Abstract: We consider initial and boundary value problems for linear nonhomogeneous difference equations with constant coefficients. For such problems we compute the numerical values of the solutions using the discrete deconvolution. The method can be easily used in applications and implemented on computer.

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1. Discrete convolution and deconvolution

Being given two finite sequences of numbers, of the same length, $a=(a_0;a_1;\ldots;a_k)$ and $b=(b_0;b_1;...;b_k)$, one consider the (*Cauchy*) *discrete convolution* (see [2]) $c=a*b=(c_0;c_1;\ldots;c_k)$, of components $c_0=a_0b_0$, $c_1=a_1b_0+a_0b_1,\ldots,c_k=\sum_{j=0}^{k}a_{k-j}b_j$ .It is commutative, associative, distributive with respect to the addition of the sequences and has the unit $\delta=(1;0;0;...;0)$.

If a and c are known and $a_0\neq 0$, we can determine b, such that $c=a*b$. This sequence is called the *deconvolution* of c by a, is denoted $b=c/a$ and has the components

$$b_0=\frac{c_0}{a_0},b_1=\frac{1}{a_0}(c_1-a_1b_0),\ldots,b_k=\frac{1}{a_0}\left(c_k-\sum_{j=0}^{k-1}a_{k-j}b_j\right)$$

hence it can be calculated by the algorithm

$$\begin{array}{cccc|cccc}
 & c_0 & c_1 & \cdots\ c_k & a_0 & a_1 & \cdots & a_k \\
 & c_0 & a_1b_0 & \cdots\ a_kb_0 & & & & \\
\hline
b_0=\frac{c_0}{a_0} & b_1=\frac{c_1-a_1b_0}{a_0} & \cdots & b_k/ & c_1-a_1b_0\cdots c_k-a_kb_0 & c_1-a_1b_0 & \cdots & a_kb_1
\end{array}$$

Particularly, one denote $a^{-1}=\delta/a$, hence $c/a=c*a^{-1}$. If a is a finite sequence and b a finite or infinite sequence, we denote by $(a;b)$ the sequence obtained by adjoining the two sequences`.

The notions of convolution and deconvolution can also be considered, with the same definitions and notations, in the case of the infinite sequences.

The convolution and the deconvolution operations considered above for finite sequences (of the same length) represent a truncate version of the usual operations for finite sequences, considered for example in MATLAB, by the instructions "conv" and "deconv" and which represent the multiplication and the division of the polynomials.

2. The main result.

We consider the nonhomogeneous linear difference equation

$$\sum_{j=0}^{n} a_{n-j} u_{j+k} = b_k ,\, k = 0,1,...$$

with the coefficients $a_0 \neq 0,\ a_1,...,a_n$ and the right hand terms $b_0, b_1,..., b_k,...$.

We denote $a = (a_0\,; a_1\,; \ldots ; a_n\,; 0; 0; \ldots)$ and $b = (b_0\,; b_1\,; \ldots ; b_k\,; \ldots)$.

The unique solution $u = (u_0\,; u_1\,; \ldots ; u_k\,; \ldots)$ *of the equation with known initial values* $u_0, u_1, \ldots, u_{n-1}$*, is given by the deconvolution formula* (see [1])

$$u = \left(a_0 u_0\,; a_1 u_0 + a_0 u_1\,; \sum_{j=0}^{n-1} a_{n-1-j} u_j\,; b_0\,; b_1\,; \ldots\right) \Big/ (a_0\,; \ldots ; a_n\,; 0; 0; \ldots) =$$

$$= ((a_0\,; a_1\,; \ldots ; a_{n-1}) * (u_0\,; u_1\,; \ldots ; u_{n-1}); b) * a^{-1} =$$

$$= (\underbrace{0; \ldots ; 0}_{n}; b * a^{-1}) + u_0\ (a_0\,; a_1\,; \ldots ; a_{n-1}\,; 0; 0; \ldots) * a^{-1} +$$

$$+ u_1\ (0; a_0\,; \ldots ; a_{n-2}\,; 0; 0; \ldots) * a^{-1} + \cdots + u_{n-1}\ (\underbrace{0; \ldots ; 0}_{n-1}; a_0\,; 0\,; 0; \ldots) * a^{-1}.$$

Remark. In the last form of u the sequence $(\underbrace{0; \cdots ; 0}_{n}; b * a^{-1})$ is a particular solution of the nonhomogeneous difference equation is the sum between its particular solution $(\underbrace{0; \cdots ; 0}_{n}; b * a^{-1})$ while the sum of the other terms represents the general solution, with arbitrary coefficient $u_0, \ldots, u_{n-1}$, of the homogeneous associated equation

$$\sum_{j=0}^{n} a_{n-j} u_{j+k} = 0\ ,\ \ k = 0,1,\ldots$$

2.2. Initial value problem

If we know the first n components $u_0, u_1, \ldots, u_{n-1}$, called *initial values*, of the solution u of the equation, the deconvolution formula gives the possibility to calculate as many components of u as required.

Example 1. The initial value problem formed by the linear difference equation $u_{k+2} - 2u_{k+1} - 3u_k = k,\quad k = 0,1,2,...$ and the initial conditions $u_0 = u_1 = 1$, has $n = 2$, $a = (1; -2; -3; 0; 0; \ldots), b = (0; 1; 2; \ldots)$,

$$c = ((a_0\,; a_1) * (u_0\,; u_1); b) = ((1; -2) * (1; 1); b) = (1; -1; 0; 1; 2; 3; \ldots).$$

The deconvolution algorithm

```
1 -1  0   1   2 ... | 1 -2 -3  0  0 ...
1 -2 -3   0   0 ... | 1  1  5 14 45 ...
---------------------
   1  3   1   2 ...
   1 -2  -3   0 ...
   -----------------
      5   4   2 ...
      5 -10 -15 ...
      -------------
         14  17 ...
         14 -28 ...
         ----------
             45 ...
             45 ...
             ------
                ...
```

gives the sequence $u = c/a = (1;1;5;14;45;...)$ as numerical solution of the initial value problem.

3. Boundary value problem

We now consider the problem of finding the solution $u = (u_0;u_1;\ldots;u_k;\ldots)$ of the difference equation for which are known n components $u_{k_1}, u_{k_2}, \ldots, u_{k_n}$, called *boundary values*. By equalizing these values with the corresponding components of the solution of the equation, given by the last form of the deconvolution formula, we obtain a linear algebraic system of order n, from which it is eventually possible to calculate the initial values $u_0, u_1, \ldots, u_{n-1}$. Replacing these values in the deconvolution formula, we obtain the solution of the considered boundary value problem.

Example 2. We consider the same equation as in **example 1**, but now with the boundary conditions $u_4 = 45, u_6 = 409$. Because

$$a^{-1} = (1;-2;-3;0;0;\ldots)^{-1} = (1;0;0;\ldots)/(1;-2;-3;0;0;\ldots) = = (1;2;7;20;61;182;547;1640;\ldots),$$

(here and in the following examples, the deconvolution algorithms is not effectively presented), we have

$$u = (0;0;b * a^{-1}) + u_0\ (a_0;a_1;0;0;\ldots) * a^{-1} + u_1\ (0;a_0;0;0;\ldots) * a^{-1} =$$

$$= (0;0;b * a^{-1}) + u_0\ (1;-2;0;0;\ldots) * a^{-1} + u_1\ (0;1;0;0;\ldots) * a^{-1} =$$

$$= (0;0;0;1;4;14;44;135;\ldots) + u_0\ (1;0;3;6;21;60;183;546;\ldots) +$$
$$+ u_1\ (0;1;2;7;20;61;182;547;\ldots).$$

The boundary value conditions

$$u_4 = 4 + 21u_0 + 20u_1 = 45,$$
$$u_6 = 44 + 183u_0 + 182u_1 = 409$$

give $u_0 = u_1 = 1$. Replacing these values in the above formula for u, we obtain the same value for u as in **example 1**, namely

$u = (1;1;5;14;45;135;409;1228;\ldots)$.

4. Qualitative aspects of the boundary value problem

As will be seen in the example 3 below, a boundary value problem can have a unique solution, an infinity of solutions or it can have no solution.

Example 3. The difference equation $u_{k+2} + u_k = b_k$, $k = 0,1,2,\ldots$, with $b = (b_0;b_1;\ldots;b_k;\ldots) = (2;0;-2;0;2;0;-2;0;2;\ldots)$, has $a = (1;0;1;0;0;\ldots)$, and $a^{-1} = (1;0;-1;0;1;0;-1;0;\ldots)$, hence

$u = (0;0;b * a^{-1}) + u_0 \; (1;0;0;\ldots) * a^{-1} + u_1 \; (0;1;0;0;\ldots) * a^{-1} =$
$= (0;0;2;0;-4;0;6;0;\ldots) + u_0 \; (1;0;-1;0;1;0;-1;0;1;\ldots) + u_1 \; (0;1;0;-1;0;1;0;-1;\ldots)$

a) If $u_3 = 0, u_4 = -3$, one obtains $u_1 = -u_3 = 0$ and $u_4 = -4 + u_0 = -3$, hence $u_0 = 1$. The boundary value problem has the unique solution $u = (1;0;1;0;-3;0;5;0;\ldots)$.

b) If $u_2 = 1, u_4 = -3$, from $u_2 = 2 - u_0 = 1$ and $u_4 = -4 + u_0 = -3$,one obtains only $u_0 = 1$, hence the boundary value problem has an infinity of solutions given by the relation

$u = (1;0;1;0;-3;0;5;0;\ldots) + u_1 \; (0;1;0;-1;0;1;-1;0;\ldots)$, whe u_1 is an arbitrary parameter.

c) If $u_2 \neq 1, u_4 = -3$, one obtains both $u_0 \neq 1$ and $u_0 = 1$, contradiction, hence the considered boundary value problem has no solution.

References

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