On the Borda Method for Multicriterial Decision-Making

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Abstract

The present paper discusses two issues with multicriterial decision-making methods of Borda type (when scores such as *n, n-1,..., 2, 1* are given to the objects to be ranked and the hierarchy is obtained based on the totals of these scores). The first issue is related to the influence on the result of various transformations of the scores. We show that a linear transformation of the scores does not change the final ranking and that (almost) any polynomial of second degree or more, with positive coefficients, can alter the solution (ranking). The same happens if one changes the scores by employing the logarithm, exponential, or square root functions. In the second part of the paper we consider an iterated version of the Borda method. We show that this method is not robust: there are cases when different solutions are returned at different iterations.

1. Introduction

It is well-known that most decisions (either personal, social, or economic) are multicriterial: several possibilities of action (objects) are taken into account given several criteria, or are considered by several decision-makers. The goal is to select a particular course of action (a unique object), which is often times the one at the top of a synthesis ranking of objects. There are numerous methods to address this problem – the reader is guided towards Andraşiu et al. (1986) and the references it provides.

In the 18th century, the French mathematician Jean-Charles de Borda has proposed a method which is still widely used. In short, the method requires that, using a pre-defined scale, points be allocated to objects that are to be ranked, and that these points be summed; the synthesis hierarchy is obtained by sorting in decreasing order these "scores". The points that are allocated can decrease in constant steps (for instance, the object on the ith place in a departing hierarchy may receive $m - i + 1$ points, where m is the number of objects to be ranked), or they can belong to a specific set (for example, Onicescu (1970) proposes that *1, ½, ¼,...,½^i,...,½$^{m-1}$, ½^m* be the points allocated to objects on the *1st, 2nd,...,ith,...,mth* place in a departing hierarchy).

It has been shown that such ranking methods contradict one of the main rationality conditions of the Arrow theorem[1]: *independence*. In short, a multicriterial decision-making method (a method which aggregates the departing hierarchies) satisfies the independence property if the final ordering of any two objects does not depend on the presence (or absence) of a third object. For instance, let us employ the example from Păun (1987, p. 30), who considers five decision-makers who need to rank three objects. The departing hierarchies are

$(a_1, a_2, a_3), (a_1, a_2, a_3), (a_3, a_1, a_2), (a_2, a_3, a_1), (a_2, a_3, a_1).$

If we allocate three points to the first object in a hierarchy, two points to the second object and one point to the third, we obtain the following scores:

$a_1 \rightarrow 10, a_2 \rightarrow 11, a_3 \rightarrow 9,$

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[1] For an exhaustive view, the reader is guided towards the seminal work of Arrow (1963).

leading to synthesis hierarchy *(a₂, a₁, a₃)*. However, if we eliminate object a_2 from all departing hierarchies, and thus have the five decision-makes indicate

$$(a_1, a_3), (a_1, a_3), (a_3, a_1), (a_3, a_1), (a_3, a_1),$$

then, in a similar manner as above, a_1 obtains 7 points while a_3 obtains 8 points, and thus the synthesis hierarchy is *(a₃, a₁)*. Clearly, the relationship between a_1 and a_3 is opposite now to the one obtained when a_2 was present.

Another weakness of many methods for hierarchy aggregation is the possibility to manipulate the final result by adjusting some individual hierarchies (votes). This point has been emphasized in the early '70s – see Gibbard (1973) and Satterthwaite (1975). One way to counter this possibility is to make the method more complicated, so that forecasting and influencing the result are less likely. A classic example is the Copeland method, thoroughly presented by Nurmi (1989). The first degree Copeland method is in fact a Borda method: the object on the ith place is allocated $m - i$ points (and not $m - i + 1$ as above). The method can be manipulated in polynomial time. The second degree Copeland method is more evolved. After the first step, in which objects receive points as described above, in the second step the total score for each object is obtained by summing the first step points of those objects dominated by this particular object. Manipulation of this method is significantly more difficult; this has been shown to be a NP-complete problem.

The idea to iterate a Borda-like method further appears naturally. However, this raises two important issues:

- First, total scores at each step increase rapidly, which may lead to difficulties when implementing large, real-life applications on computer. A usual solution in such cases is data "normalization", obtained by applying a specific transformation on the initial figures. We will prove that such transformations may alter the final result, and so this should be avoided. Precisely, we show that a linear transformation does not alter the result, but some polynomial transformations do change the hierarchy. A similar situation is found when one uses the exponential, logarithmic, or square root transformations.
- Second, a desirable feature of iterated methods is that hierarchies at different iterations remain unchanged. A numerical example will show that this goal is sometimes missed.

2. Notation

In what follows, n will always represent the number of decision-makers (or the number of criteria), which we denote $d_1, d_2,...,d_n$, and m will represent the number of objects to be ranked, denoted $o_1, o_2,...,o_m$. Each decision-maker conveys a hierarchy of objects (a total, linear order) and objects in these departing hierarchies are allocated points according to their position in the hierarchy (more points being allocated to objects ranked higher). Let $p(d_i, o_j)$, where $1 \leq i \leq n$, $1 \leq j \leq m$, denote the number of points received by object o_j in decision-maker's d_i hierarchy.

We can now define the quality of an object:

$$Q(o_j) = \sum_{i=1}^{n} p(d_i, o_j).$$

Let us consider a function $f : \boldsymbol{R} \rightarrow \boldsymbol{R}$. If the objects are allocated points such as $f(p(d_i, o_j))$, hence transformed by applying function f, then the corresponding quality of an object will be denoted $Q_f(o_j)$.

3. The sensitivity of the Borda method to point transformations

Lemma 1. (The additivity lemma) Let $f_1, f_2 : \boldsymbol{R} \rightarrow \boldsymbol{R}$ be two functions. Let us assume we have a multicriterial decision-making problem such that, for two objects o_j, o_k, we have

$$Q_{f_1}(o_j) > Q_{f_1}(o_k), Q_{f_2}(o_j) > Q_{f_2}(o_k).$$

Let function $f(x) = a_1 f_1(x) + a_2 f_2(x) + a_3$, where a_1 and a_2 are two positive numbers. Then,

$$Q_f(o_j) > Q_f(o_k).$$

Proof.

$$
\begin{aligned}
Q_f(o_j) &= \sum_{i=1}^{n}(a_1 f_1(p(d_i, o_j)) + a_2 f_2(p(d_i, o_j)) + a_3) \\
&= a_1 \sum_{i=1}^{n} f_1(p(d_i, o_j)) + a_2 \sum_{i=1}^{n} f_2(p(d_i, o_j)) + a_3 n \\
&= a_1 Q_{f_1}(o_j) + a_2 Q_{f_2}(o_j) + a_3 n, \\
Q_f(o_k) &= a_1 Q_{f_1}(o_k) + a_2 Q_{f_2}(o_k) + a_3 n.
\end{aligned}
$$

Clearly, $Q_f(o_j) > Q_f(o_k)$. □

The following example will be used several times below. We consider four decision-makers who have to decide on five objects, and let the departing hierarchies be:

	d_1	d_2	d_3	d_4
1st place	o_3	o_2	o_3	o_2
2nd place	o_1	o_4	o_1	o_1
3rd place	o_5	o_1	o_2	o_4
4th place	o_4	o_5	o_4	o_3
5th place	o_2	o_3	o_5	o_5

If we allocate 5 points to the object on the first place, 4 to the object on the second, and so on, 1 point to the fifth placed object, then the total scores the five objects obtain are 15, 14, 13, 11, 7 and thus the Borda hierarchy is $(o_1, o_2, o_3, o_4, o_5)$.

Lemma 2. Given the previous example, for any function $f(x) = x^k$, with $k \geq 2$, we obtain $Q_f(o_2) > Q_f(o_1)$.

Proof. It is obvious that

$$
\begin{aligned}
Q_f(o_1) &= 3 \cdot 4^k + 3^k, \\
Q_f(o_2) &= 2 \cdot 5^k + 3^k + 1.
\end{aligned}
$$

It suffices to show that $2 \cdot 5^k > 3 \cdot 4^k$. We can write $5^k = (4 + 1)^k$ and use the Newton binomial series. Then,

$$
\begin{aligned}
2 \cdot (4 + 1)^k &= 2 \cdot (C_k^0 4^k + C_k^1 4^{k-1} + \ldots + C_k^{k-1} 4^1 + C_k^k 4^0) \\
&= 2 \cdot 4^k + 2 \cdot k \cdot 4^{k-1} + \alpha,
\end{aligned}
$$

where α is a strictly positive number. Since $k \geq 2$, we obtain $2 \cdot 4^k + 2 \cdot k \cdot 4^{k-1} \geq 3 \cdot 4^k$, which means that $2 \cdot 5^k > 3 \cdot 4^k$.

Observation 1. In the previous setup, it can be shown that for $k \geq 3$ an even stronger result emerges: $Q_f(o_3) > Q_f(o_1)$. The inequality is clearly true for $k = 3$ while, for k greater than 3, it can be

proved by induction and by relying once again on the Newton binomial series. We can now demonstrate the main result of this section.

Theorem 1. (i) A linear transformation, with positive coefficients, of the allocated points does not alter the ranking produced by the Borda method. (ii) Let us consider a polynomial of second degree or more, with positive coefficients, such that the coefficient of the first degree term is either zero or equal to the coefficient of the second degree term. For any such polynomial $f(x)$, there are multicriterial decision-making problems for which the solution obtained by the Borda method is altered if one applies the $f(x)$ transformation.

Proof. The Borda method obtains the synthesis hierarchy by ordering $Q(o_j) = \sum_{i=1}^{n} p(d_i, o_j)$, $1 \leq j \leq m$. A linear transformation $f(x) = ax + b$, with $a \geq 0$, leads to $Q_f(o_j)$ which, according to Lemma 1, will keep the hierarchy unchanged[2].

Let us consider a polynomial $f(x) = a_1 x^k + a_2 x^{k-1} + \ldots + a_k x + a_{k+1}$, with $a_i \geq 0$, $1 \leq i \leq k + 1$, and either $a_k = 0$ or $a_k = a_{k-1}$. If we denote $f_i(x) = a_i x^{k-i+1}$, $1 \leq i \leq k + 1$, we have $f(x) = f_1(x) + f_2(x) + \ldots + f_{k-2}(x) + f_{k+1}(x) + (f_{k-1}(x) + f_k(x))$. According to Lemma 2, for any $i = 1, 2, \ldots, k - 2$, we have that $Q_{f_i}(o_2) > Q_{f_i}(o_1)$. Using Lemma 1 repeatedly, we get that $Q_g(o_2) > Q_g(o_1)$, where $g(x) = f_1(x) + f_2(x) + \ldots + f_{k-2}(x) + f_{k+1}(x)$.

Let us analyze separately the case of transformation $h(x) = x^2 + x$. Using this function, 30 points are allocated to the object on the first place, 20 to the object on the second place, 12 to the third placed object, 6 to the object on the fourth place and 2 points to the last object in a hierarchy. Using the previous example of departing hierarchies, $Q_h(o_1) = 72 < Q_h(o_2) = 74$. According to Lemma 1, the same is the case if we apply transformation $g'(x) = a_{k-1} \cdot g(x)$. We employ Lemma 1 once again for functions $h(x)$ and $g'(x)$, which, by addition, lead precisely to polynomial $f(x)$. $\square$

Observation 2. The condition we imposed on the coefficients of the first and second degree terms of the polynomial is needed in the above proof, given the particular example we were working with. Indeed, let us consider the polynomial $f(x) = x_2 + 4x$. The points to be allocated to objects are now 45, 32, 21, 12, and 5. Accordingly, $Q_f(o_1) = 117 > Q_f(o_2) = 116$, so the order of the two objects in reversed. As expected, other nonlinear transformations also change the hierarchy.

Theorem 2. Let us consider transformations $f_1(x) = 2^x$, $f_2(x) = \ln x$, and $f_3(x) = x^{1/2}$. There are multicriterial decision-making problems for which the solution obtained by the Borda method is altered if one applies these transformations.

Proof. If we consider the previous example and transformation $f_1(x) = 2^x$, then the points to be allocated are 32, 16, 8, 4, and 2, which leads to

$$Q_{f_1}(o_1) = 56, Q_{f_1}(o_2) = 74, Q_{f_1}(o_3) = 70, Q_{f_1}(o_4) = 28, Q_{f_1}(o_5) = 16,$$

and thus to hierarchy $(o_2, o_3, o_1, o_4, o_5)$. This is completely different from the hierarchy obtained when 5, 4, 3, 2, 1 points were allocated.

For transformation $f_2(x) = \ln x$ we consider the example below:

	d_1	d_2	d_3	d_4	points
1st place	o_1	o_3	o_5	o_4	1.61
2nd place	o_2	o_1	o_1	o_3	1.38
3rd place	o_5	o_2	o_2	o_2	1.09
4th place	o_3	o_4	o_4	o_5	0.69
5th place	o_4	o_5	o_3	o_1	0

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[2] This is a particular case of Lemma 1, in which $f_1(x) = x$, $f_2(x) = x$, $a_1 = a_2 = a/2$, and $a_3 = b$.

For this example, the Borda method, using the "standard" points 5, 4, 3, 2, 1, leads to hierarchy $(o_1, o_2, o_3, o_5, o_4)$. We now allocate ln 5, ln 4, ln 3, ln 2, and ln 1 points (using the approximate values we have listed in the above table), which leads to total scores of 4.37, 4.65, 3.68, 2.99, and 3.39 and thus to hierarchy $(o_2, o_1, o_3, o_5, o_4)$, different from the previous one.

Lastly, for transformation $f_3(x) = x^{1/2}$ we consider the following example:

	d_1	d_2	d_3	d_4	$d5$	points
1st place	o_1	o_3	o_3	o_4	o_4	2.236
2nd place	o_2	o_1	o_1	o_5	o_5	2
3rd place	o_5	o_2	o_2	o_3	o_3	1.732
4th place	o_4	o_4	o_4	o_2	o_2	1.414
5th place	o_3	o_5	o_5	o_1	o_1	1

The Borda method using the "standard" point system leads to hierarchy $(o_3, o_4, o_1, o_2, o_5)$. By transforming the points to be allocated using function f_3 (see the approximate values in the table above), the five objects get the scores 8.236, 8.292, 8.936, 8.714, and 7.732, which leads to hierarchy $(o_2, o_1, o_3, o_5, o_4)$, different from the previous one.

□

4. Iterating the Borda method

As mentioned in the introduction, the Copeland method of second degree is a two-step procedure: in the first step points are allocated to objects in a similar fashion to the Borda method, while in the second step the total for each object is obtained by summing the first step points of the objects that particular object dominates. We now proceed to iterate this procedure further.

Formally, for a multicriterial decision-making problem of the type we have analyzed so far, where objects' quality is evaluated by $Q(o_j)$, with $1 \leq j \leq m$, we define $Q_i(o_j)$, with $i \geq 0$, in this manner:

$$Q_1(o_j) = Q(o_j),$$

$$Q_{i+1}(o_j) = \sum_{k=1}^{n} \sum_{l \in D_k(j)} Q_i(o_l), i \geq 1,$$

where $D_k(j)$ is the set of objects in decision-maker's d_k hierarchy which are dominated by object o_j. Then, at each step i, the decreasing order of $Q_i(o_j)$, with $1 \leq j \leq m$, indicates the hierarchy at that particular iteration.

Naturally, the question is whether, given a particular problem, hierarchies remain unchanged at various steps. If so, this would mean the procedure is robust, and thus full confidence should be placed on that (constant) hierarchy. Unfortunately, this is not always the case, and we will demonstrate this by employing a numerical example. Four decision-makers need to rank five objects, and their departing hierarchies are provided in the table below:

	d_1	d_2	d_3	d_4
1st place	o_1	o_2	o_1	o_4
2nd place	o_3	o_4	o_5	o_2
3rd place	o_2	o_1	o_2	o_1
4th place	o_4	o_3	o_4	o_3
5th place	o_5	o_5	o_3	o_5

Given the above, the results of the first ten iterations (returned by a simple computer program) are the following:

Scores at iteration 1: 12, 11, 5, 9, 3
Hierarchy at iteration 1: $(o_1, o_2, o_4, o_3, o_5)$

Scores at iteration 2: 72, 75, 29, 59, 25
Hierarchy at iteration 2: $(o_2, o_1, o_4, o_3, o_5)$

Scores at iteration 3: 484, 483, 209, 381, 163
Hierarchy at iteration 3: $(o_1, o_2, o_4, o_3, o_5)$

Scores at iteration 4: 3216, 3227, 1353, 2567, 1073
Hierarchy at iteration 4: $(o_2, o_1, o_4, o_3, o_5)$

Scores at iteration 5: 21292, 21411, 9013, 16937, 7147
Hierarchy at iteration 5: $(o_2, o_1, o_4, o_3, o_5)$

Scores at iteration 6: 141336, 141875, 59789, 112475, 47361
Hierarchy at iteration 6: $(o_2, o_1, o_4, o_3, o_5)$

Scores at iteration 7: 937300, 941547, 396433, 745997, 314139
Hierarchy at iteration 7: $(o_2, o_1, o_4, o_3, o_5)$

Scores at iteration 8: 6217376, 6244307, 2629961, 4947836, 2083977
Hierarchy at iteration 8: $(o_2, o_1, o_4, o_3, o_5)$

Scores at iteration 9: 41240092, 41420155, 17444101, 32820873, 13822131
Hierarchy at iteration 9: $(o_2, o_1, o_4, o_3, o_5)$

Scores at iteration 10: 273546984, 274741499, 115707421, 217699035, 91685129
Hierarchy at iteration 10: $(o_2, o_1, o_4, o_3, o_5)$

As indicated above, hierarchies at the first four steps alternate (o_1 and o_2 switch their places). From the fourth step on the hierarchy remains constant (however, we have not explored past the 10th iteration).

5. Concluding remarks
The two main results of the present paper are:
(1) Transforming the initial points allocated to objects in a multicriterial decision-making problem solved by the Borda method may alter the final result. This is the case for many of the usual nonlinear functions one may employ: polynomial of second degree or more (with some restrictions placed on coefficients), logarithm, exponential, square root. The hierarchy is not modified if one employs a linear transformation of the initial points.
(2) Iterating the Borda method may lead to different hierarchies at different iterations.

We need to emphasize once again that Borda-like methods are the most commonly used for multicriterial decision problems. Then, the significance of the above results is obvious. First, caution should be exerted when modifying the initial data (by normalization or other operations imposed, for instance, by limitations of computing capacity). Also, one should always bear in mind that the final result may depend on the actual method employed, or even on the version of that particular method.

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