Risk Aversion as Attitude towards Probabilities: A Paradox*

James C. Cox[a] and Vjollca Sadiraj[b]

a, b. Department of Economics and Experimental Economics Center, Georgia State University, 14 Marietta St. NW, Atlanta, GA 30303

b. Corresponding author

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Abstract: Theories of decision under risk that challenge expected utility theory model risk attitudes at least partly with transformation of probabilities. We explain how attributing risk aversion (partly or wholly) to attitude towards probabilities, can produce extreme probability distortions that imply paradoxical risk aversion.

Keywords: risk aversion, probability transformation, calibration, reference dependence, loss aversion

1. Introduction

The first paradox to challenge expected utility theory was offered by Allais (1953). The Allais patterns challenge the independence axiom, which gives the expected utility functional its idiosyncratic feature of linearity in probabilities. In order to rationalize the Allais paradox, theories of decision under risk that relax the independence axiom were developed (see Machina, 1987 for an accessible presentation).

The idea of representing risk aversion with transformed probabilities originated in the psychology literature about mid-twentieth century (Preston and Baratta, 1948; Edwards, 1954) and entered the economics literature about 30 years later (Handa, 1977; Kahneman and Tversky, 1979; Quiggin, 1982). Some early models of probability weighting (Handa, 1977; Kahneman and Tversky, 1979) violated stochastic dominance. Subsequent models with rank dependence avoided that problem (Tversky and Kahneman, 1992; Quiggin, 1993). Further development and applications of rank dependent models, and other alternatives to expected utility, have continued to the present. A comprehensive presentation of the literature containing many references is offered by Wakker (2010).

The present paper, however, demonstrates that attributing risk aversion to attitude towards probabilities produces extreme probability distortions; hence paradoxical risk aversion emerges.

2. Risk Aversion as Probability Transformation: Examples of Paradoxes

We begin with two illustrative examples of risk-avoiding choices. One example includes choices between pairs of lotteries with non-negative outcomes whereas the other example involves lotteries with mixed outcomes, gains and losses.

2.1 An Example of Risk-Avoiding Choices over Lotteries with Non-Negative Payoffs

Consider two urns, A_r and B_r. Urn B_r contains 100 balls: 10 white balls, b black balls and r red balls. Numbers of black and red balls, b and r are known. If one replaces the 10 white balls with 5 red balls and 5 black balls then one has urn A_r; it contains 100 balls: b+5 black balls and r+5 red balls.[1] Let a red ball pay \$4 million, a black ball pay \$0, and a white ball pay \$1 million.[2]

Suppose that an individual is offered a choice between the two urns and we observe that he is indifferent or prefers urn B_r over urn A_r when the number of red balls r is contained in $\{r_*, r_*+5, ..., r^*-5, r^*\}$ for some $0 \le r_* < r^* \le 90$. Modeling these risk avoiding choices through attitudes towards probabilities produces extreme probability distortion for the individual who values \$4 million more than twice as much as he values \$1 million. Indeed, weak preference for urn B_r over urn A_r reveals that $f(\frac{r+10}{100}) - f(\frac{r+5}{100}) \ge (c-1)[f(\frac{r+5}{100}) - f(\frac{r}{100})]$, where $f(\cdot)$ is the transformation of decumulative probabilities and c is the ratio of values ("utilities") of 4 million and 1 million, that is $c = v(4M)/v(1M)$.[3] It follows from iteration of this inequality

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[1] For theories that include the reduction axiom one can, alternatively, think of urn A as also having 10 white balls, r red balls and b black balls. Again, each red ball pays \$4M and each black ball pays 0; however, in case of urn A, each white ball pays \$4M or 0 with equal probability whereas in case of urn B it still pays \$1 million for sure.

[2] Formally option A is a two outcome lottery, $\{4M_p, 0\}$ that pays \$4M with probability p (and 0 otherwise). Option B is a three outcome lottery, $\{\$4M_{p-1/2n}, \$1M_{1/n}, 0\}$ that pays \$4M with probability p-1/2n and \$1M with probability 1/n. In this example, the probability of receiving \$1M is 10/100 (there are10 white balls) hence n=10. All risky options A and options B considered in this paper are of these types.

[3] Without any loss of generality, we assume that v(0)=0.

$k\ (\le (r^* - r_*)/5)$ times that weak preferences for urn B_r over urn A_r, when the number of red balls is from $\{r_*, ..., r^*\}$, reveal that

$$(1)\quad f(\frac{r^*}{100}+\frac{2}{20}) - f(\frac{r^*}{100}+\frac{1}{20}) \ge q\left[f(\frac{r^*}{100}+\frac{1}{20}) - f(\frac{r^*}{100})\right] \ge .... \ge q^{k+1}\left[f(\frac{r^*}{100}-\frac{k-1}{20}) - f(\frac{r^*}{100}-\frac{k}{20})\right].$$

where $q = c-1$. Thus the slope of the transformation of decumulative probabilities increases geometrically at a rate $q(>1)$ and therefore severe underweighting of probabilities is implied. The larger is $r^* - r_*$, the more severe is the distortion of probabilities, and the more absurd is the implied risk aversion. For example, let $v(\$4M)/v(\$1M) \ge 2.7$ and the agent be indifferent or prefer urn B over urn A when there are {0, 5, 10,…,r*} red balls in urn B. Then he has revealed the following distortion of probabilities: $f(0.1) < 0.00943 f(0.5)$ if $r^* = 40$; and $f(0.1) \le 0.000047$ and $f(0.5) < 0.004952$ if $r^* = 90$.

These extreme probability distortions imply implausible risk aversion. For subadditive value functions over large positive payoffs, implied risk aversion includes: (a) \$1 million with one-half probability is preferred to \$100 million with 0.1 probability if rejection of option A holds for r from {0,5,…,40}; and (b) \$100 for sure is preferred to \$2.1 million with 0.1 probability if rejection of option A holds for r^* from {0,5,…,90}.[4]

2.2 An Example of Risk-Avoiding Choices over Lotteries with Positive and Negative Payoffs

Alternatively, consider the following options a_r and b_r. Compositions of balls in the urns are the same as above but payoffs are different: each red ball now pays \$3M and each black ball imposes a loss of \$1M; the white balls pay 0.[5] Let υ be the value function for gains and μ be

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[4] $v(100M)f(0.1) \le 100v(1M) \times 0.00943 f(0.5) < v(1M) f(0.5)$ follows from subadditivity of $v(\cdot)$ and $f(0.1) \le 0.00943 f(0.5)$.

[5] Formally option a is a two outcome lottery, $\{3M_p, -1M\}$ whereas option b is a three outcome lottery, $\{3M_{p-1/2n}, 0_{1/n}, -1M\}$, where $p \in \{1/20, 2/20, ..., 19/20\}$.

the value function for losses. Let f^+ and f^- be the probability transformation functions for the gain and loss domains. Indifference or preference for option b_r over option a_r when there are r red balls in urn a_r reveals

$$(2) \qquad \upsilon(3M)\, f^+(\frac{r-5}{100}) + \mu(-1M)\, f^-(1-\frac{r+5}{100}) \geq \upsilon(3M)\, f^+(\frac{r}{100}) + \mu(-1M)\, f^-(1-\frac{r}{100}) \,.$$

Using $f^-(1-\frac{r}{100}) + f^+(\frac{r}{100}) = 1$, and rearranging terms, one has $f^+(\frac{r+5}{100}) - f^+(\frac{r}{100}) \geq R[\, f^+(\frac{r}{100}) - f^+(\frac{r-5}{100})]$ where $R = \upsilon(3M)/(-\mu(-1M))$. If option b_r is weakly preferred to option a_r when the number of red balls, r (in urn a_r) is from $\{r_*, r_*+1, ..., r^*\}$, then iteration of the last inequality $k\ (\leq r^* - r_*)$ times implies

$$(3)\ f^+(\frac{r^*}{100}+\frac{1}{20}) - f^+(\frac{r^*}{100}) \geq R\left[f^+(\frac{r^*}{100}) - f^+(\frac{r^*}{100}-\frac{1}{20})\right] \geq .... \geq R^k \left[f^+(\frac{r^*}{100}-\frac{k-1}{20}) - f^+(\frac{r^*}{100}-\frac{k}{20})\right].$$

Thus the slope of the transformation of decumulative probabilities over gains increases, again geometrically, at a rate R and therefore severe underweighting of probabilities is expected when the agent values a gain of \$3M more than a loss of \$1M. For example, if $R \geq 1.7$ then implications with respect to probability distortions are similar to the ones reported in section 2.1; so is the paradoxical risk aversion that emerges. For subadditive value functions over large positive payoffs one has: \$100 for sure is preferred to \$2.1 million with 0.1 probability if indifference or preference for option b_r holds for r (in urn b_r) from {0,5…,90}; or \$10 with probability ¾ is preferred to \$15,000 with probability 0.1 if indifference or preference for option b_r holds for r (in urn b_r) from {0,5,…,65}.

2.3 Scale Invariance

The example in section 2.1 used payoffs of \$4 million, \$1 million, and 0 and reported probability transformation implications of a pattern of risk preferences. Inspection of statement (1) (or statement (3)) reveals that the only way in which the prize values enter the inequalities is through $q = \upsilon(4M)/\upsilon(1M) - 1$ (or through $R = \upsilon(3M)/(-\mu(-1M))$. Hence, irrespective of the size of payoffs (whether they are very large or very small or moderate in size), rejection of urn B_r over a range of red balls implies the same paradoxical risk aversion for all payoffs with the same ratio of valuation q (or R). For example, the prizes can be \$40 (for a red ball), \$10 (for a white ball) and 0 (for a black ball) and calibration implications are the same as for prizes in section 2.1 that involve millions as long as the valuation of \$40 is at least twice as much as valuation of \$10. Similarly for example 2.2, the prizes can be \$30 (for a red ball), \$0 (for a white ball) and -\$10 (for a black ball) and implications are still the same as for prizes in section 2.2 that involve millions as long as gaining \$30 (in absolute value) is valued more than losing \$10.

The following section provides general results.

3. Implausibility of Modeling Risk Aversion with Probability Attitudes

We report several propositions and corollaries that state implications of attributing risk-aversion to attitudes towards probabilities. All proofs are collected in the appendix.

Let $L = \{x_{p_j}, x\}, j = 1, \ldots, n$ denote a prospect with n+1 possible non-negative outcomes; it pays x_{p_j} with probability $p_j, j = 1,...,n$, and x with probability $1 - \sum_{j=1}^{n} p_j$. As usual, let the outcomes be ordered from the largest to the smallest: $x_{p_k} \geq x_{p_j} \geq x$, for all $k > j$.

Consider a decision theory D that represents preferences over lotteries L with functional

(*) $$U(L) = \sum_{j \geq 1} v(x_{p_j})\left(f(P_j) - f(P_{j+1}) \right)$$

where: $P_j = \Pr(y : y \geq x_{p_j}\}$; $f(\cdot)$ is the transformation of decumulative probabilities, P_j; and $v(\cdot)$ is the outcome valuation (utility) function.

3.1 Risk-Avoiding Choices over Risky Gains

For any given integer *n*, consider pairs of lotteries $B_i = \{h_{p_i - 1/2n}, m_{1/n}, \ell\}$ and $A_i = \{h_{p_i}, \ell\}$, where $p_i = i / 2n$, and $i = 1, 2, \cdots, 2n-1$. We use the following notations: $\succsim$ for weak preference and $\succ$ for strong preference; $C = (v(h) - v(m)) / (v(m) - v(\ell))$; $K(t, M, N) = 1 + \sum_{j=0}^{M} t^{j+1} \Big/ \sum_{i=0}^{N} t^{-i}$; and $\Psi(p_*, p^*) = \{p_*, p_* + 1/2n, ..., p^*\}$, for some $1/2n \leq p_* < p^* \leq 1 - 1/2n$.

Proposition 1. Let non-negative $h > m > \ell$, probabilities $p^* > p_*$ and some integer *n* such that $n \geq \max\{1/2p_*, 1/2(1-p^*)\}$ be given. If

(P.1) $\{h_{p-1/2n}, m_{1/n}, \ell\} \succeq \{h_p, \ell\}$, for all $p \in \Psi(p_*, p^*)$

then for any $q \in \Psi(p_*, p^*)$

(Q.1) $$f(q) \leq \frac{1}{\mathrm{K}} f(p^* + \frac{1}{2n}) + (1 - \frac{1}{K}) f(p_* - \frac{1}{2n})$$

where $\mathrm{K} = K(C, 2n(p^* - q), 2n(q - p_*))$.

It can be easily verified (see Lemma A.1 in the Appendix) that if $v(h) - v(m) > v(m) - v(\ell)$ then K increases exponentially which generates implausible risk aversion through underweighting probabilities. For example take n=50 and outcomes *h>m>0* such that $v(h)/v(m) \geq 2.5$. If statement P.1 is satisfied for $p \in \Psi(0.01, 0.5)$, then $f(0.25) < 0.0000265 f(0.51)$. If statement P.1 is satisfied for $p \in \Psi(0.5, 0.99)$, then $f(0.75) - f(0.49) \leq 0.00004$.

We say that non-negative payoffs h, m, ℓ are *v*-favorable if $v(h) + v(\ell) > 2v(m)$. For *v*-favorable payoffs one has the following straightforward corollary.

Corollary 1.1. For any small $\varepsilon > 0$ and $q \in (p_*, p^*)$ there exists $\mathrm{n}^* \in \mathrm{N}$ such that if statement (P.1) is satisfied for some *v*-favorable payoffs then $f(q) - f(p_*) < \varepsilon$.

Corollary 1.2 applies Proposition 1 for the special case of $p_* = 1/2n$.[6]

Corollary 1.2. Let $p_* = 1/2n$. If statement (P.1) is satisfied then for any $q \in \Psi(1/2n, p^*)$, $f(q) \leq \frac{1}{\mathrm{K}} f(p^* + \frac{1}{2n})$ where $\mathrm{K} = K(C, 2n(p^* - q), 2nq - 1)$.

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[6] Cox, Sadiraj, Vogt and Dasgupta (2010) reports six different experiments designed to test empirical validity of pattern P.1. The percentages of subjects whose choices revealed $p_*=1/2n$ varied from 61% to 83% across the six experiments.

Some examples help one to appreciate these results. Take n=25 and outcomes *h>m>0* such that $v(h)/v(m) \geq 2.5$. If statement P.1 is satisfied for p from {0.02,0.04,…,0.98} then $f(0.5) < 0.00004$. If statement P.1 is satisfied for p from {0.02, 0.04,…,0.48} then $f(0.1) < 0.00027 f(0.5)$. Such probability distortions generate paradoxical risk aversion. For example, for any subadditive value function over (large) positive payoffs, $f(0.5) < 0.00004$ implies preference for a certain payoff of $1,000 over getting $25 million or 0 with equal probabilities.[7] Similarly, $f(0.1) < 0.00027 f(0.5)$ implies preference for $1000 with probability 0.5 over getting $3.7 million with probability 0.1. Table 1 reports severe probability distortions for different values of n and for $(v(h)-v(m)/(v(m)-v(\ell)) \geq 2$.

Proposition 2 states risk aversion implications of Corollary 1.2 for subadditive value functions over positive payoffs.[8]

Proposition 2. Suppose that statement (P.1) is satisfied. Then for any given $q \in \Psi(1/2n, p^*)$ and any positive z

a. $\{z_{p^*+1/2n}, 0\} \succeq \{zK\left(C, 2n(p^*-q), 2nq-1\right)_q, 0\}$

b. If payoffs h, m, ℓ are v-favorable then for any given integer G, for all p^*, n such that $2n(p^*-q) \geq \ln G/\ln C - 1$, $\{z_{p^*+1/2n}, 0\} \succeq \{zG_q, 0\}$

Part (a) of Proposition 2 says that pattern (P.1) implies preference for getting z with probability p*+1/2n against getting $zK\left(C, 2n(p^*-q), 2nq-1\right)$ with probability q. To get a feeling for these expressions consider a special case of *p**=1-1/2*n* and (a) *q*=1/2 or (b) *q*=1/4; expressions for K

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[7] Indeed, $v(25million) f(0.5) \leq 25000 v(1000) \times 0.0004 = v(1000)$

[8] General risk aversion implications for Proposition 1 (for $p_* > 1/2n$) are similar to the ones reported in Proposition 2 but the expressions look cumbersome. Since these general results are not more enlightening we do not include them in the paper; they are available upon request to the authors.

are: (a) $\mathrm{K}\left(C, n-1, n-1\right)$ and (b) $\mathrm{K}\left(C, 3n/2-1, n/2-1\right)$. For $C \geq 2$ and n=10, one has (a) $\mathrm{K}\left(C,9,9\right) \geq 1025$ and (b) $\mathrm{K}\left(C,14,4\right) \geq 33825$. The implications of pattern (P.1) of risk aversion are then paradoxical preferences for a sure \$100 against a binary lottery that pays: (a) 102,000 or 0 with even odds; or (b) \$3.3 million or 0 with odds 1 to 4. What are implications of pattern (P.1) when *p**(=1/2-1/2n) < 0.5? For (c) *q*=0.25 and (d) *q*=0.1 expressions for K become (c) $\mathrm{K}\left(C, n/2-1, n/2-1\right)$ and (d) $\mathrm{K}\left(C, 4n/5-1, n/5-1\right)$. So, for $C \geq 2$ and n=10 one has (c) $\mathrm{K}\left(C,4,4\right) \geq 33$ and (d) $\mathrm{K}\left(C,7,1\right) > 341$; paradoxical risk aversion implications are preference for \$1,000 or 0 with even odds against: (c) \$33,000 or 0 with odds 1 to 3; or (d) \$341,000 or 0 with odds 1 to 9. Table 2 reports different levels of paradoxical risk aversion that follow from probability distortions that are reported in Table 1.

It can be verified (see Lemma A.1) that $K(C, 2n(p^* - q), 2nq - 1)) \geq C^{2n(p^* - q)}$, which for C >12 becomes as large as one wants it for sufficiently large n. Hence, the larger the value of n, the more extreme are the implications of the (P.1) pattern of risk aversion.[9] Also, for any given n, the larger the value of C the more extreme is the risk aversion implied by (P.1).

Part (b) of Proposition 1 states exactly this result: when the value of the high outcome h is larger than twice the value of the intermediate outcome *m* then for any given G as large as one wants it to be, and for any given positive z, pattern (P.1) with $\ell = 0$ implies getting z with probability *p**+1/2*n* is preferred to getting zG with probability *q* for any *n* and *q* such that $2n(p^* - q) \geq \ln G / \ln C - 1$. To illustrate part (b), suppose one is interested in identifying a pattern (P.1) that implies that (*) \$100 or 0 with odds 8:2 is preferred to \$12,500 or 0 with odds 1:9. Plugging in G=125, C=2, *p**=0.7 and *q*=0.1 in the last inequality one finds that the threshold for n is as low as 5. That is, for $\ell = 0$ and any *h* and *m* such that the individual values *h* at least 3 times as much as *m*, rejection of option A in favor of option B for p from

[9] Note that this proposition makes no explicit assumption on the curvature of the probability transformation; all valuations of $f(\cdot)$ follow from agent's revealed preferences over options A_i and B_i.

{0.1,0.2,…,0.7} implies statement (*).[10] Similarly, \$1,000 or 0 with even odds is preferred to \$125,000 or 0 with odds 1:9 by an individual who values *h* at least 2.5 times as much as *m* and rejects option A in favor of option B for *p* from {0.03, 0.20,…,0.47}, that is for n=15.

3.2 Risk-Avoiding Choices over Risky Gains and Losses

Next, let's consider slightly different pairs of lotteries that involve both gains and losses. For any given integer n, consider pairs of lotteries $S_i = \{g_{p_i - 1/2n}, 0_{1/n}, -\ell\}$ and $R_i = \{g_{p_i}, -\ell\}$, where $p_i = i/2n$, and $i = 1, 2, \cdots, 2n-1$. Let $\mu(\cdot) < 0$ denote the value function for negative payoffs and define $L = -\upsilon(g)/\mu(-\ell)$.

Proposition 3. Let positive $g > \ell$, integer n and probabilities $p^* > p_*$ be given. If

(P.3) $\{g_{p-1/2n}, 0_{1/n}, -\ell\} \succeq \{g_p, -\ell\}$, for all $p \in \Psi(p_*, p^*)$

then for any $q \in \Psi(p_*, p^*)$

(Q.3) $$f^+(q) \le \left[\frac{1}{\kappa} f^+(p^* + \frac{1}{2n}) + (1 - \frac{1}{\kappa}) f^+(p_* - \frac{1}{2n}) \right]$$

where $\kappa = K(L, 2n(p^* - q), 2n(q - p^*))$.

As above, it can be shown that $\kappa > L^{1+2n(p^*-q)}$; hence for $L > 1$, κ increases exponentially, which generates implausible risk aversion through underweighting probabilities over gains. Figures reported in Table 1 can be used to illustrate Proposition 3 as follows. Let gain and loss figures be such that $-\upsilon(g)/\mu(-\ell) \ge 2$. Then $L + 1 \ge 3$ and pattern (P.3) with $p_* = 1/2n$ implies exactly the same probability distortions as the ones reported in Table 1. For example, if n=25 then rejection of $\{g_p, -\ell\}$ for $p^* = 0.48$ implies $f^+(0.1) \le 0.93 \times 10^{-6} f^+(0.5)$.

For $\upsilon(\cdot)$ subadditive on positive payoffs one has:

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[10] Cox et al (2010) report that the estimated percentage of subjects that revealed p* ≥ 0.7 is 73, 40, 60 and 88 in Calcutta 400/80/0, Magdeburg 40/10/0, Atlanta 40/10/0 and Atlanta 14/4/0, respectively.

Proposition 4. Suppose that statement (P.3) is satisfied. Then for any given $q \in \Psi(1/2n, p^*)$ and any positive z

a. $\{z_{p^*+1/2n}, 0\} \succ \{z\kappa\left(L, 2n(p^* - q), 2nq - 1\right)_q, 0\}$

b. If $\upsilon(g) > -\mu(-\ell)$ then for any integer G, for all p^*, n such that $2n(p^* - q) \geq \ln G / \ln(L) - 1$, $\{z_{p^*+1/2n}, 0\} \succeq \{zG_q, 0\}$.

Figures reported in Table 2 can also be used to illustrate Proposition 4. As stated above, if gain and loss figures are such that $\upsilon(g)/[-\mu(-\ell)] \geq 2$ then $L \geq 2$ and pattern (P.3) with $p_* = 1/2n$ implies exactly the same implausible risk aversion as the ones reported in Table 2. For example, if n=25 then rejection of $\{g_p, -\ell\}$ for *p**=0.48 implies that getting 100 with probability 0.5 is preferred to getting 100 million with probability 0.1.

Both results stated in Proposition 4 extend straightforwardly to a reference-dependent model that incorporates variable reference amounts of money payoff. Suppose that the reference point is not zero but the intermediate outcome *m* when a choice between Options A and B need to be made. For $\upsilon(\cdot)$ sub-additive on positive payoffs one has:

Corollary 4.1 (endogenous reference point). Suppose that statement (P.1) is satisfied. Let the variable reference point be the intermediate payoff *m*. Then both statements a and b in Proposition 4 are satisfied with $L = \upsilon(h - m)/[-\mu(\ell - m)]$.

4. Implausibility of Modeling Risk Aversion as Attitude towards Probabilities

Previous literature has focused on the inability of expected utility theory to rationalize some supposed patterns of risk averse preferences. Allais (1953) introduces patterns of choices under risk that (if observed) refute expected utility.[11] Allais's critique was directed at the linearity in probabilities property of the EU functional. Patterns of risk aversion that originated with Hansson (1988), and were made famous by Rabin (2000), question the ability of expected utility of terminal wealth to rationalize risk aversion at large stakes and at small stakes. This critique attacks concavity of the utility of money as an explanation of risk aversion.

This paper shows how attributing risk aversion partly or wholly to attitude towards probabilities produces extreme probability distortions that have paradoxical implications. Patterns of risk aversion call into question the plausibility of theories of decision under risk that, to accommodate the Allais paradox, relax linearity in probabilities.

[11] Empirical validity of the Allais patterns is still disputed in the literature (for direct tests of these patterns (one-task experimental studies) see: Conlisk, 1989; Cubbit, Starmer and Sugden, 1998; Cox, Sadiraj and Schmidt, 2011).

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Table 1. Distortion of Probabilities that Follows from Patterns of Risk Aversion

(P.1) for $\ell = 0$ and $v(h)/v(m) \geq 3$; ; (P.3) for $\upsilon(g)/(-\mu(-\ell)) \geq 2$

$p_* = 1/2n$	$p^* = 1 - 1/2n$			$p^* = 1/2 - 1/2n$	
n	p^*	$f(0.1) <$	$f(0.5) <$	p^*	$f(0.1) <$
5	0.9	0.00098	0.03031	0.4	0.03226 $f(0.5)$
10	0.95	0.29×10^{-5}	0.00098	0.45	0.00294 $f(0.5)$
25	0.98	0.28×10^{-5}	0.30×10^{-7}	0.48	0.93×10^{-6} $f(0.5)$
50	0.99	0.81×10^{-27}	0.89×10^{-15}	0.49	0.91×10^{-12} $f(0.5)$

Table 2. Paradoxical Risk Aversion that Follows from Distortion of Probabilities

	$100 \succ \{G_q, 0\}$			$\{100_{0.5}, 0\} \succ \{G_{0.1}, 0\}$	
n	p^*	$q = 0.1$	$q = 0.5$	p^*	**G**
5	0.9	102,300	3,300	0.4	3,100
10	0.95	$0.34 \text{x} 10^{8}$	102,500	0.45	34,100
25	0.98	$0.36 \text{x} 10^{16}$	$0.33 \text{x} 10^{10}$	0.48	$0.10 \text{x} 10^{9}$
50	0.99	$0.12 \text{x} 10^{30}$	$0.11 \text{x} 10^{18}$	0.49	$0.11 \text{x} 10^{15}$

Appendix: Proof of Propositions and Corollaries

Proof of Proposition 1. To simplify writing, let δ denote $1/2n$ and $\int_a^b df = f(b) - f(a)$.[12]

According to theory D, the supposition (P.1) writes as

(a.1) $$v(\ell)\int_{(i+1)\delta}^{2n\delta} df + v(m)\int_{(i-1)\delta}^{(i+1)\delta} df + v(h)\int_{0}^{(i-1)\delta} df \geq v(\ell)\int_{i\delta}^{2n\delta} df + v(h)\int_{0}^{i\delta} df,\ i = k_*,\ldots,k^*$$

where $k_* = 2np_*, k^* = 2np^*$.

Adding and subtracting $v(m)f(i\delta)$ on the left-hand-side of the above inequality and rearranging terms, one has

(a.2) $$\int_{i\delta}^{(i+1)\delta} df \geq C\int_{(i-1)\delta}^{i\delta} df,\ i = k_*,\ldots,k^*$$

where $C = \dfrac{v(h)-v(m)}{v(m)-v(\ell)}$. Write inequality (a.2) for $i + r(= k_*,\ldots,k^*)$ and apply it r times to get

(a.3) $$\int_{(i+r)\delta}^{(i+r+1)\delta} df \geq C\int_{(i+r-1)\delta}^{(i+r)\delta} df \geq \ldots \geq C^r\int_{i\delta}^{(i+1)\delta} df$$

To complete the proof it suffices to show that $\forall q \in \{(k_*+1)/2n,\ldots,(k^*-1)/2n\}$,

(a.4) $$f(q) - f(p_* - 1/2n) \leq \sum_{j=0}^{2nq-k_*} C^{-j}\int_{q-\delta}^{q} df \text{ and } f(p^*+1/2n) - f(q) \geq \sum_{j=0}^{k^*-2qn} C^{j+1}\int_{q-\delta}^{q} df$$

because two inequalities in (a.4) imply that $f(q) \leq \dfrac{1}{K}\left[f(p^*+1/2n) + (K-1)f(p_*-1/2n)\right]$;

To show the first inequality of (a.4) recall that $2n\delta = 1$ and $2np_* = k_*$ by notation and verify that

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[12] We do not require that f(.) is differentiable; to simplify writing we use $\int^b_a df$ as a symbol for the difference between the values of f(.) at b and a.

$$f(q)-f(p_*-1/2n)=f(2nq\delta)-f((k_*-1)\delta)=\sum_{i=k_*}^{2nq}\int_{(i-1)\delta}^{i\delta}df\le\sum_{j=0}^{2nq-k_*}C^{-j}\int_{q-\delta}^{q}df$$

(The inequality follows from inequality (a.3)). For the second inequality of (a.4) verify that

$$f(p^*+1/2n)-f(q)=f((k^*+1)\delta)-f(2nq\delta)=\sum_{j=2qn+1}^{k^*+1}\int_{(j-1)\delta}^{j\delta}df\ge\left(\sum_{j=0}^{k^*-2qn}C^{j+1}\right)\int_{q-\delta}^{q}df$$

Recall that $K(t,M,N)=1+\sum_{j=0}^{M}t^{j+1}\Big/\sum_{i=0}^{N}t^{-i}$.

Lemma A.1. If $t>1$ then for any given $q\in(p_*,p^*)$,

a. $K(t,2n(p^*-q),2n(q-p_*))>t^{1+2n(p^*-q)}$

b. $\lim_{n\to\infty}\mathrm{K}\left(t,2n(p^*-q),2n(q-p_*)\right)=\infty$

Proof. Verify that

$$K(t,M,N)=1+\frac{\sum_{j=0}^{M}t^{j+1}}{\sum_{i=0}^{N}t^{-i}}=\frac{t^{2+M+N}-1}{t^{1+N}-1}>t^{1+M}$$

Hence, one has $K(t,2n(p^*-q),2n(q-p_*))>t^{1+2n(p^*-q)}\underset{n\to\infty}{\longrightarrow}\infty$

Proof of Corollary 1.1. Apply Proposition 1 and part (b) of Lemma A.1.

Proof of Corollary 1.2. It follows from plugging in $p_*=1/2n$ in (Q.1)

Proof of Proposition 2. To show part (a) first note that for $p_*=1/2n$ statement (Q.1) simplifies to $Kf(q)<f(p^*+1/2n)$; for any given z, the multiplication of both sides of the last inequality (a.ii) by v(z) and subadditivity of v(.) one has

$$v(zK)\,f(q) \le Kv(z)\,f(q) \le v(z)\,f(p^* + 1/2n)\,.$$

Part (b) follows from Lemma A.1: $2n(p^* - q) \le \ln G / \ln C - 1$ implies

$$K(C, 2n(p^* - 2q), 2nq - 1) \ge C^{1+2n(p^*-q)} \ge G$$

Proof of Proposition 3. Suppose that a loss averse agent satisfies statement (P.3). Then he has revealed

(a.5) $$\mu(-\ell)\,f^-(1-(i+1)\delta) + \upsilon((g)\,f^+((i-1)\delta) \ge \mu(-\ell)\,f^-(1-i\delta) + \upsilon(g)\,f^+(i\delta),$$

for all $i = 2np_*, \ldots, 2np^*$. This can be equivalently rewritten as

(a.6) $$\int_{i\delta}^{(i+1)\delta} df^+ = \int_{1-(i+1)\delta}^{1-i\delta} df^- \ge \frac{\upsilon(g)}{-\mu(-\ell)} \int_{(i-1)\delta}^{i\delta} df^+,\ i = 2np_*, \ldots, 2np^*.$$

(The first inequality follows from $f^+(p) = 1 - f^-(1-p)$.) Use notation L and apply the last inequality $j - i$ times to get

(a.7) $$\int_{j\delta}^{(j+1)\delta} df^+ \ge L^{j-i} \int_{i\delta}^{(i+1)\delta} df^+,\ \text{for all}\ j = i, \ldots, 2np^*$$

and then (repeat steps in the proof for Proposition 1 to) verify that the following inequality is true

$$f^+(q) \le \frac{1}{\kappa}\left[f^+(p^* + 1/2n) + (\kappa - 1)\,f^+(p_* - 1/2n) \right]$$

Proof of Proposition 4. (See the proof of Proposition 2) To show part (a) first note that for $p_* = 1/2n$ statement (Q.3) simplifies to $\kappa f^+(q) < f^+(p^* + 1/2n)$; for any given z, multiply both sides of the last inequality by v(z) and apply subadditivity of v(.) to complete the proof. Part (b) follows from Lemma A.1:

$2n(p^* - q) \le \ln(G)/\ln(L) - 1$ implies $K(C, 2n(p^* - 2q), 2nq - 1) \ge L^{1+2n(p^*-q)} \ge G$

Proof of Corollary 4.1 (endogenous reference point). This is a straightforward application of Proposition 4 with gain $h-m$ and loss $\ell - m$: that is $L = -\upsilon(h-m)/\mu(\ell - m)$.