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A Note on Equity Premia in Markets with Heterogeneous Agents

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A Note on Equity Premia in Markets with Heterogeneous Agents

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Abstract

We analyze a static partial equilibrium model where the agents are not only heterogeneous in their beliefs about the return on risky assets but also in their attitude to it. While some agents in the economy are subjective utility maximizers others behave ambiguity averse in the sense of Knight (1921). If ambiguity averse agents meet overly optimistic subjective utility maximizers in the market lower equity premia can arise in the equilibrium than in a purely subjective utility framework.

Key words and phrases: Ambiguity, Partial Equilibrium, Heterogeneous Agents, No-Trade Interval,

1 Introduction

The main goal of the paper is to highlight the impact of ambiguity on market prices in a static market model with heterogeneous agents. We show that the presence of ambiguity averse agents can lead to higher prices of the asset.

Classical asset pricing literature relies heavily on the assumption that agents have homogeneous correct expectations about future returns. Lintner (1969) first analyzes a partial equilibrium model where agents have heterogeneous beliefs about the profitability of the asset. It turns out that the equilibrium price corresponds to the weighted average of opinions of market participants. While bullish (optimistic) investors demand security, bearish (pessimistic) investors supply it by shortselling. In equilibrium the price reflects the average opinion on the market. In this setting, Miller (1977) and Jarrow (1980) analyze the effect of short selling constraints on the equilibrium price and show that short selling constraint together with heterogeneous expectations may lead to overpricing. Since pessimistic agents cannot express their beliefs by selling the asset short, the price is biased upward and reflects the opinion of more optimistic agents. If this trend is not corrected over several periods a bubble can arise.

This idea gave rise to a series of papers analyzing bubbles and speculative overpricing caused by heterogeneous beliefs. Harrison and Kreps (1978) constructed a speculative bubble model in discrete time, Scheinkman and Xiong (2003) modeled overpricing as a consequence of overconfidence in continuous time. However, all papers explaining bubbles by heterogeneity of agents rely on the impossibility of short selling. This restriction is set exogenously and is justified by the market structure, high costs of short selling or regulation. The assumption which seems reasonable in some situations is hard to support in general. Most of developed financial markets explicitly allow short selling, a vast majority of stocks traded on exchanges is shortable at low cost[1]. At the same time short supply, i.e. the supply generated through short selling constitutes only a small fraction of the market. Based on an empirical analysis Lamont and Stein (2004) come to the conclusion that "..the problem is not too much short selling in falling markets ... but rather, too little in rising markets". To explain this phenomena authors refer to internal restrictions set by companies' chartas or reluctance to sell short. However, there is no model rationalizing this reluctance to sell short.

In this paper we relax the assumption of impossibility of short selling and establish

[1] A review of short selling constraints over the world can be found in Saffi and Sigurdsson (2008).

the reluctance to sell the asset short through preferences. The reluctance to short sell is not an exogenously given attitude as before but comes as optimal behavior of agents that have certain preferences. To model this we use ambiguity averse preference in the sense of Knight, axiomatized by Gilboa and Schmeidler (1989).

Already Miller (1977) in his paper referred to Knightian uncertainty and pointed out that "[i]n practice, uncertainty, divergence of opinion about a security's return, and risk go together". Thus, in markets where agents have different views on the asset at least some market participants are likely to experience uncertainty and behave like ambiguity averse agents. We formalize this idea and analyze a market where heterogeneous agents face risk and uncertainty.

We assume that some market participants are subjective utility maximizer (SEU) that differ in their expectation on the security's return while others have minimax preferences axiomatized by Gilboa and Schmeidler (1989). An ambiguity averse decision maker (AA) uses a class of models instead of one to assess utilities to future payoff streams and commits to a position only if the expected utility of this position is positive for all models she considers. In this setting both types of agents determine their demand for the risky/uncertain security by solving their utility maximization problem. Subjective utility maximizers then demand the security if the price is below their subjective expected return and supply it otherwise. In any case they hold a position of the security except for the knife edge case. The picture is different for ambiguity averse investors: there exists an interval of prices within which it is optimal for them to hold zero position in the security. This so called no trade interval first studied by Dow and Werlang (1992) arises because the expected utility of a short and long position is positive for some but not for all models the investors takes into account. As a result the ambiguity averse investor refuses to participate in the market at all. This has two implications: First, the agents do not demand the asset and the risk has to be taken by fewer investors. This leads to higher risk premia required by investors to hold the asset. This effect has been studied extensively in the non-participation literature. On the other hand, the ambiguity averse agents also refuse to short the asset and fail to generate short supply. Thus the supply is lower compared to a market with SEU agents only. This can lead to higher equilibrium prices. Even though short selling is not forbidden, an upward biased price (compared to the average risk adjusted valuation) can arise if subjective utility maximizers are overly optimistic and bid the price up. As a result an increase in ambiguity may force ambiguity averse decision maker to stop short selling and thus increase the equilibrium price. This effect does not arise in the previous equilibrium models with ambiguity where an increase in ambiguity decreases demand and leads to lower equilibrium prices.

The result is an extension of the series of papers following the ideas of Miller (1977). Unlike previous papers on overpricing we do not impose the short selling assumption exogenously. Here, it is a result of a rational utility maximization of agents having minimax preferences.

Several papers investigate the impact of ambiguity on the portfolio and investment choice and its consequences for markets. Epstein and Wang (1994) use ambiguity aversion and no trade interval to explain non participation in the markets and portfolio inertia. It is showed to be a reason for underdiversification in Uppal and Wang (2003) and market incompleteness in Mukerji and Tallon (2001). Caballero and Krishnamurthy (2007) highlights the role of ambiguity in flights to quality. In the latter model an increase in ambiguity that is unrelated to fundamental value causes a sell out in the security pressing the price down and resulting in a flight to quality. Ui (2009) studies a model of non-participation in a financial market with finitely many different potential investors who exhibit heterogeneous levels of ambiguity and obtain private signals. As Cao, Wang, and Zhang (2005) and Easley and O'Hara (2009) they also note that equity premium can decrease if non-participation arises. This happens because investors that exhibit higher levels of ambiguity aversion and therefore demand a higher premium for holding the asset leave the market. This decreases the average premium required to hold the security. The heterogeneity in this models refers to different degrees of uncertainty aversion of the investors and not to their point estimate of the returns of the asset.

Our paper differs from this literature in two aspects. First, non-participation literature concentrates on the interaction of ambiguity averse and perfectly rational agents having correct beliefs for the asset return. In contrast, our paper studies the interaction of ambiguity averse decision makers with heterogeneous risk averse agents. As a result we do not assume anyone being rational and having correct beliefs. The prices on the market reflect solely average beliefs and not necessarily fundamental values. All our predictions about the changes of the equilibrium price are made with respect to the average opinion and not with respect to the fundamental value.

Second, previous models on ambiguity aversion in equilibrium models suggest that the presence of ambiguity averse decision makers results in lower prices since ambiguity aversion increases the premium required by the investor to hold the asset. In our model ambiguity can cause a price increase. As more agents become ambiguity averse or ambiguity about the return of investment increases, more investors become reluctant to short the asset, thus lowering the supply of the asset. If SEU agents are optimistic enough the effect of lower supply is stronger than the effect of lower demand. In equilibrium we get higher prices.

Our paper can also be seen in the spirit of limits of arbitrage studied by Shleifer and Vishny (1997). There, rational arbitrageurs refuse to correct a bubble and to take advantage of an arbitrage because this arbitrage is risky. They only step in if the mispricing is high enough to be rewarded for the risk they take. Our story is similar: the arbitrage here is not only risky but also ambiguous. Moreover, given the price is in the no trade interval, a short position in the security is arbitrage for some models an ambiguity averse decision maker takes into account but not in all. Similar to Shleifer and Vishny (1997) overpricing can persist due to its ambiguous nature.

2 The Model

2.1 Setup

We consider a two period exchange economy with two assets: one ambiguous and one riskfree. The risky asset is traded at $t = 0$ and pays a liquidating[2] dividend x at time $t = 1$. The supply of the asset is fixed at Q while demand is determined by the maximization problem of investors. The riskfree asset is traded in infinite supply at a given riskfree rate [3].

There is a continuum of risk averse agents in the economy that share the same v NM index defining the CARA utility with risk aversion coefficient γ:

$$u(x) = -e^{-\gamma x}. \tag{1}$$

Investors that take prices as given differ in their beliefs about returns on stock and their attitude towards uncertainty. There are two types of investors: subjective utility maximizer (SEU agents) that maximize their expected utility under their subjective belief and ambiguity averse decision maker (AA agents) that take a class of models into account since they do not trust the validity of one particular model.

While all market participants agree that the dividend in the next period is normally distributed with volatility σ they disagree about the expected return μ of the asset. This

[2]To set up a meaningful static model we assume that the risky asset is withdrawn from the market and has zero value after paying the dividend at time $t = 1$.

[3]We consider a partial equilibrium model and set the riskfree rate as exogenous.

disagreement might be a result of overconfidence in own ability to evaluate signals as the model of Scheinkman and Xiong (2003) suggests or be the result of the use of different models.

The range of possible drifts is given by $[\underline{\mu}, \overline{\mu}]$, investors are distributed across this interval according to a distribution M having a density m.

2.2 Maximization Problem

2.2.1 Subjective expected utility investors

Heterogeneous risk averse agents agree to disagree about the asset's return. Each risk averse investor has a point estimate μ_i for the expected return which he uses for his evaluations. For expositional simplicity we assume that risk averse investors are optimists, having a belief above some threshold $\hat{\mu} \in [\underline{\mu}, \overline{\mu}]$. From the modeling point of view this assumption seems reasonable since agents that are optimistic about the returns on a new security are also likely to be confident about the choice of their model. An example for this type of behavior was the overoptimism combined with a high level of confidence during the Internet bubble. The assumption made here is not crucial for the validity of the result but simplifies greatly the analysis. We will discuss how to relax it later on.

Given the belief μ_i the agent maximizes her expected utility. Due to the shape of the utility function endowments of agents do not affect their demand for the risky asset. The problem of the SEU investor with belief μ_i then reads

$$\text{Maximize } \mathbb{E}^i(-\exp(-\gamma d_i^s(x-p))) \text{ over } d_i \in \mathbb{R} \tag{2}$$

where p denotes the equilibrium price of the asset and d_i^s the number of risky asset in the portfolio. The expectation is taken with respect to her personal belief μ_i. Note that unlike the seminal paper of Miller (1977) we do allow for short selling, i.e. negative values of d_i. Thus, depending on the equilibrium price, the agent can be either supplier or demander of the risky asset.

Standard techniques show that the demand of the investor i is given by

$$d_i^s = \frac{\mu_i - p}{\gamma\sigma^2} \tag{3}$$

An SEU agents is a net demander of the asset if the price is below the mean return and a net supplier if the price is above. In any case the optimal position in the asset is nonzero except for the knife edge case $\mu_i = p$. Thus, SEU agents are always active on the market trading in one or other direction and generating both demand and short sell supply depending on the individual belief μ_i.

Using the mean value theorem the aggregate demand of SEU investors can be calculated as

$$D^s = \int_{\hat{\mu}}^{\overline{\mu}} d_i m(d\mu_i) \tag{4}$$

$$= \int_{\hat{\mu}}^{\overline{\mu}} \frac{\mu_i - p}{\gamma\sigma^2} m(d\mu_i) \tag{5}$$

$$= \frac{\underline{\mu} + k^* - p}{\gamma\sigma^2} (1 - M(\hat{\mu})) \tag{6}$$

where $k^* \in [\hat{\mu} - \underline{\mu}, \overline{\mu} - \hat{\mu}]$. The demand of SEU agents is determined by the weighted average of opinions of SEU agents $\underline{\mu} + k^*$ and the mass of the SEU agents in the economy $(1 - M(\hat{\mu}))$.

2.2.2 Ambiguity averse investors

An ambiguity averse decision maker is uncertain about the right model and takes a set of models into account. Instead of using their own model ambiguity averse decision maker use all models that they see on the market. More precisely, they build a belief about the return and if the belief is below the ambiguity threshold $\hat{\mu}$ they use all estimates they see on the market to assess the profitability. Thus, the set of priors used by ambiguity averse agents is given by

$$\mathcal{P} = \{P : x_P \sim \mathcal{N}(\mu, \sigma^2), \mu \in [\underline{\mu}, \overline{\mu}]\} \tag{7}$$

Being ambiguity averse she maximizes her minimal expected payoff, i.e.

$$\text{Maximize } \inf_{P \in \mathcal{P}} \mathbb{E}^P(-\exp(-\gamma d_i(x - p))) \text{ over } d_i \in \mathbb{R} \tag{8}$$

where $\mathcal{P}$ is defined by (7). It is known from Dow and Werlang (1992) that the demand function of the ambiguity averse investor is continuous and has kinks at $p = \underline{\mu}$ and $p = \overline{\mu}$. The ambiguity generates a so called no trade interval, in which the agents refuse to trade the risky asset. The exact expression for the demand function of an ambiguity averse agent in our setting is given by

$$d^a = \begin{cases} \frac{\underline{\mu} - p}{\gamma\sigma^2} & \text{if } p < \underline{\mu} \\ 0 & \text{if } \underline{\mu} < p < \overline{\mu} \\ \frac{\overline{\mu} - p}{\gamma\sigma^2} & \text{if } p > \overline{\mu} \end{cases} . \tag{9}$$

The ambiguity averse investor demands the asset if the equilibrium price is low enough. However, if the price is sufficiently high, i.e. $p > \underline{\mu}$ the ambiguity averse agents refuses to invest in risky asset. Unlike the SEU agent who starts short selling as soon as she stops buying, ambiguity averse agent is also reluctant to short sell the asset at a price $p \in [\underline{\mu}, \overline{\mu}]$. Thus, ambiguity has two effects: first, if the price is above $\underline{\mu}$ ambiguity averse agents stop investing in the asset, decreasing aggregate demand, on the other hand, they do not start short selling thus decreasing supply of the asset.

Clearly, the aggregate demand of ambiguity averse agents is then given by

$$D^a = \int_{\underline{\mu}}^{\hat{\mu}} d^a m(d\mu_i) \tag{10}$$

2.3 Equilibrium Analysis

Before we perform equilibrium analysis we note that the demand is only positive if $p < \overline{\mu}$ since all agent aim to sell the asset if $p > \overline{\mu}$. Thus, in equilibrium ambiguity averse agents will never short the asset since the price for the risky asset will never exceed the most optimistic valuation $\overline{\mu}$ in equilibrium. Therefore, the demand of an ambiguity averse investor in equilibrium is

$$d^a = \max\left\{\frac{\underline{\mu} - p}{\gamma\sigma^2}, 0\right\} \tag{11}$$

Then, the aggregate demand of ambiguity averse decision makers amounts to

$$D^a = \int_{\underline{\mu}}^{\hat{\mu}} \max\left\{\frac{\underline{\mu} - p}{\gamma\sigma^2}, 0\right\} m(d\mu_i) \tag{12}$$

$$= M(\hat{\mu}) \max\left\{\frac{\underline{\mu} - p}{\gamma\sigma^2}, 0\right\} \tag{13}$$

Ambiguity aversion has the equilibrium effect of preventing short selling by ambiguity averse agents. Aggregate demand of ambiguity averse agents corresponds to the demand in Miller (1977) where short selling restrictions where imposed exogenously. While ambiguity averse agents are internally constrained and only act as demander of the asset, SEU agents sell short if the price is high enough. The aggregate demand for the risky asset in the economy is given by the demand of the two groups of investors

$$D = D^s + D^a \tag{14}$$

In equilibrium,

$$D = Q \tag{15}$$

and we have the following lemma:

Lemma 2.1 *Under above conditions there exists a unique equilibrium in the market for the risky/uncertain asset with equilibrium price given by*

$$p = \begin{cases} \underline{\mu} + k^*(1 - M(\hat{\mu})) - Q\gamma\sigma^2 & \text{if } k^*(1 - M(\hat{\mu})) < Q\gamma\sigma^2 \\ \underline{\mu} + k^* - \frac{Q\gamma\sigma^2}{1 - M(\hat{\mu})} & \text{else} \end{cases} . \quad (16)$$

The first value in the above equation corresponds to the price when the ambiguity averse agents demand positive amounts of the asset, i.e. $p < \underline{\mu}$ and no short selling takes place. This price would arise in an unconstrained economy with unconstrained SEU maximizers $M(\hat{\mu})$ of them having belief $\underline{\mu}$ and the price equals the average opinion of all market participants .

The second value is the constrained equilibrium price when ambiguity averse agents do not demand the risky asset. Since the risk adjusted return of a long position in the worst case scenario is negative ambiguity averse agents do not demand the security and stay away from the market leaving it to the overly optimistic investors. However, they are reluctant to short sell the asset since the worst case return of the short position is negative as well. The price reflects the valuation of SEU agents only leading to a higher equilibrium price than in the pure heterogeneous expectation case. The effect of the reluctance to short sell is twofold. On the one hand, ambiguity averse agents stop demanding the asset, decreasing the scarcity of the asset by shifting the demand downwards. This potentially decreases the price. On the other hand, the agents with optimistic beliefs demand higher amounts of the asset causing higher prices.

3 Comparison with Miller (1977)

In this section we compare our results to the findings of Miller (1977). Recall that Miller (1977) assumed that heterogeneous investors are uniformly distributed across

$$[\underline{\mu}, \overline{\mu}] = [\hat{\mu} - k, \hat{\mu} + k] \quad (17)$$

The riskfree rate is zero and short selling is not allowed in this market. The maximization problem of the investor with belief μ_i then becomes

$$\text{Maximize } \mathbb{E}^i(-\exp(-\gamma d_i(x - p))) \text{ over } d_i \in \mathbb{R}^+ \quad (18)$$

and the individual demand amounts to

$$d^s = \max\left\{\frac{\mu_i - p}{\gamma\sigma^2}, 0\right\} \tag{19}$$

Aggregating over all investors and solving for equilibrium yields

$$p = \begin{cases} \hat{\mu} - Q\gamma\sigma^2 & \text{if } k < \gamma\sigma^2 Q \\ \hat{\mu} + k - 2\sqrt{kQ\gamma\sigma^2} & \text{else} \end{cases} . \tag{20}$$

The first value in the above equation denotes the equilibrium price that arises if the short selling constraint does not bind. The second value is the price resulting in the constrained equilibrium. Here, some of investors aim to hold negative amounts of the security but are prevented from it by the short selling constraint. It can be easily checked that the constrained price is higher than the unconstrained.

To compare our results we assume that the set of all possible beliefs is given by (17), where $\hat{\mu}$ denotes the ambiguity threshold and M is the uniform. Using Lemma 2.3 we can compute the equilibrium price as

$$p = \begin{cases} \frac{4\mu - k}{4} - Q\gamma\sigma^2 & \text{if } k < \frac{4}{3}Q\gamma\sigma^2 \\ \frac{2\mu + k}{2} - 2Q\gamma\sigma^2 & \text{else} \end{cases} .$$

Note that the price in our setting is always below the price in the setting of Miller (1977). This happens for two reasons. First, the average expected return in our model is lower due to the presence of ambiguity averse investors. This leads to a lower price in the unconstrained equilibrium. Second, while the model of Miller completely excludes short selling, some short selling takes place in our model in the restricted equilibrium. The opinion of moderate investors that aim to go short is contained in the restricted price of our model. Since some short selling is executed by moderate investors with belief μ_i such that $p > \mu_i > \hat{\mu}$, the overpricing is not as severe as in the model of Miller.

4 Comparative Statics and Sensitivity Analysis

In the next section we analyze how a change of parameters changes the equilibrium price. Our main goal is to study the impact of ambiguity on the equilibrium price in the meaningful way.

4.1 Sensitivity with respect to ambiguity increase

While classical equilibrium models[4] with ambiguity predict that an increase in ambiguity lowers the equilibrium prices the situation differs here. The two main factors for the sensitivity analysis is the distribution of opinions and the ambiguity threshold. First we analyze the sensitivity of the price with respect to changes in ambiguity and distribution of opinions. We study how the change in ambiguity threshold affects the equilibrium price, assuming that the distribution remains the same. This can happen due to an exogenous shock on the market such as an unexpected market outcome causing more agents to doubt their models.

Lemma 4.1 *Denote by* $p^u = \underline{\mu} + k^*(1 - M(\hat{\mu})) - Q\gamma\sigma^2$ *the unrestricted equilibrium price and by* $p^c = \underline{\mu} + k^* - \frac{Q\gamma\sigma^2}{1-M(\hat{\mu})}$ *the restricted equilibrium price. Then* p^u *and* p^c *satisfy*

1. $\frac{dp^u}{d\hat{\mu}} < 0$ *and*
2. $\frac{dp^c}{d\hat{\mu}} > 0$

An increase in ambiguity caused by an increase of the ambiguity threshold lowers the prices in the unconstrained equilibrium. This result is intuitive, since all investor participate in the market and demand the asset. The decrease in $\hat{\mu}$ decreases the average expected return and thus the aggregate demand. This result is in line with the literature. In the constrained equilibrium however, the picture is different. Here, an increase in ambiguity leads to higher prices on the market. This happens for the following reason. On the one hand the increase of the ambiguity threshold forces some market participants into the no trade interval and the market becomes smaller. At the same time moderate agents who where willing to short sell before are now in the no trade interval and fail to generate short sale supply. If the concentration of optimists in the market is high enough they again can absorb the security available at the market and bid the price up. In extreme case where $\hat{\mu}$ is high enough there is no short selling at all and only the most optimistic investors determine the price.

Not only the ambiguity threshold is important for the equilibrium price but also the distribution of agents within intervals. In the following we analyze the impact of the change of the distribution of the agents along $[\underline{\mu}, \overline{\mu}]$. For example, due to a unfavorable outcome caused by a shock some agents that previously had the belief $\mu_i > \hat{\mu}$ may start to doubt their model and become ambiguity averse while others keep the belief μ_i and

[4] See Epstein and Wang (1994) and references therein.

remain insensitive to ambiguity. More ambiguity on the market thus means in this case, that although the ambiguity threshold remains constant, the mass of ambiguity averse investors changes.

In this case the direction of the price change depends heavily on the shape of the distribution of beliefs before and after the shock.

Lemma 4.2 *Let P and Q be distributions on $[\underline{\mu}, \overline{\mu}]$ with absolutely continuous densities f, g. Assume that P and Q satisfy the hazard rate condition, i.e.*

$$\frac{f(x)}{1-F(x)} \leq \frac{g(x)}{1-G(x)} \text{ for all } \underline{\mu} \leq x \leq \overline{\mu} \tag{21}$$

i.e. $P \succeq_{hr} Q$. Denote by p_P resp. p_Q the price on the market where the agents are distributed according to P, Q resp. Then

1. $p^c_P \geq p^c_Q$ *and*
2. $p^u_P \geq p^u_Q$

One could also think of an increase in ambiguity by means of an increase of $\overline{\mu}$ or decrease of $\underline{\mu}$. However this analysis is not meaningful without specific assumptions on the underlying distribution m. For this reason we omit this analysis here.

4.2 Sensitivity to changes of Q, γ, σ^2

The standard market factors work in the usual direction decreasing the equilibrium price.

Lemma 4.3 *Denote by $p^u = \underline{\mu} + k^*(1 - M(\hat{\mu})) - Q\gamma\sigma^2$ the unrestricted equilibrium price and by $p^c = \underline{\mu} + k^* - \frac{Q\gamma\sigma^2}{1-M(\hat{\mu})}$ the restricted equilibrium price. Then p^u and p^r satisfy*

1. $\frac{dp^u}{dQ} < 0$ *and* $\frac{dp^c}{dQ} < 0$
2. $\frac{dp^u}{d\gamma} < 0$ *and* $\frac{dp^c}{d\gamma} < 0$
3. $\frac{dp^u}{d\sigma^2} < 0$ *and* $\frac{dp^c}{d\sigma^2} < 0$

These sensitivity results are in line with the standard theory and the economics intuition carries over as well. Investors decrease their demands if risk aversion resp. volatility increase, thus, as a result prices decrease as well. Since this effects are well understood we keep the discussion short and omit the proof.

5 Extensions and Robustness Checks

5.1 Uncertainty about volatility

In the generic model we assumed that all agents agree on the volatility of the underlying asset. We can extend the model easily to the case with heterogeneous beliefs about volatility. The essence of result does not change much.

Again agents in the economy differ in their beliefs about the return of the asset. Assume that the asset is normally distributed according to (μ, σ) where beliefs about the actual value of (μ, σ) are given by

$$\mathcal{P} := \{P : x_P \sim \mathcal{N}(\mu, \sigma^2), \text{ s.t. } (\mu, \sigma^2) \in [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}^2, \overline{\sigma}^2]\}$$

Every agent is endowed with a belief (μ, σ) from the above interval. The set of all possible beliefs can be partitioned into two regions: an ambiguity averse region $A \subset [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}^2, \overline{\sigma}^2]$ and a subjective region S satisfying

$$\mathcal{A} + \mathcal{S} = [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}^2, \overline{\sigma}^2].$$

All agents that have a belief in $\mathcal{S}$ maximize their subjective utility given their belief while all agents in $\mathcal{A}$ maximize the minimal expected utility. The distribution M of agents across the interval is now two dimensional and we maintain the assumption that M has a density.

The analysis for the individual demand of an subjective utility investors carries over from the single dimensional case and we get

$$d_i^s = \frac{\mu_i - p}{\gamma \sigma_i^2} \tag{22}$$

resulting in aggregate demand

$$D^s = \int_{\mathcal{S}} d_i^s dm \tag{23}$$

$$= \int_{\mathcal{S}} \frac{\mu_i - p}{\gamma \sigma_i^2} dm \tag{24}$$

Using an appropriate version of the Mean Value Theorem we can show that

$$D^s = M(\mathcal{S}) \cdot \left(\frac{\mu^* - p}{\gamma (\sigma^*)^2} \right)$$

The effect of heterogeneity in the volatility may either decrease or increase the demand of SEU agents depending on the resulting average volatility σ^*. For the demand of ambiguity averse investors we only need to note that the highest variance minimizes their expected return. As in the standard literature on ambiguity averse portfolio choice we then get

$$d^a = \begin{cases} \frac{\underline{\mu}-p}{\gamma\overline{\sigma}^2} & \text{if } p < \underline{\mu} \\ 0 & \text{if } \underline{\mu} < p < \overline{\mu} \\ \frac{p-\overline{\mu}}{\gamma\overline{\sigma}^2} & \text{if } p > \overline{\mu} \end{cases} . \tag{25}$$

Similarly for the aggregate demand:

$$D^a = \int_{\mathcal{A}} \max\left\{\frac{\underline{\mu}-p}{\gamma\overline{\sigma}^2}, 0\right\} m(d\mu_i) \tag{26}$$

$$= M(\mathcal{A}) \max\left\{\frac{\underline{\mu}-p}{\gamma\overline{\sigma}^2}, 0\right\} \tag{27}$$

Here, the uncertainty about the volatility reduces the demand of the ambiguity averse agents. However, only the mean return is essential for the decision to short sell or buy the asset. From this point on the model essentially reduces to the single dimensional case. In the same manner as above we can perform the equilibrium analysis.

5.2 N ambiguous independent assets

Already Jarrow (1980) in his paper investigated the effect of adding securities to Miller's model. It turns out that the answer depends on the distribution of assets. If the assets are correlated substitution effects influence demand and prices of securities and the effect of short selling constraint may go in both directions. However, if market participants agree on volatility of the assets, the results of Miller (1977) carry over to the multiple asset case.

We can easily extend our model to the multiple asset case. In case of independent assets the analysis does not change much. Due to independence of assets and the form of the utility function the demand for each asset is determined separately for SEU agents. The same holds true for ambiguity averse agents. Thus, the price for each asset is set independently and the equilibria can be analyzed one by one with the same technique as above. However, we cannot distinguish anymore between unconstrained and constrained equilibria since some assets might be in the constrained equilibrium while other in the unconstrained.

5.3 Different distribution of preferences

In our model we assumed that all agents with belief more pessimistic than the ambiguity threshold $\hat{\mu}$ are not only pessimistic about returns of the stock but also about the model they use. This assumption simplifies the analysis considerably but is not essential for the result. The main result still holds true if we relax the assumption that there are no SEU maximizer with a belief more pessimistic than $\hat{\mu}$. In this case although ambiguity averse agents still refuse to go short for prices within $[\underline{\mu}, \overline{\mu}]$ SEU agents with pessimistic belief can reflect their opinion by going short. However, the overpricing still can occur if the short sell supply of pessimistic agents is not big enough to offset the overoptimistic demand.

5.4 Market Crashes and Panics

The model we derived can also be used to explain panics and crashes. Those can happen if heterogeneous agents become pessimistic and want to sell the security causing a sell out of security. As in the optimistic case, ambiguity averse refuse to correct this overreaction of the price due to their ambiguity. In this way pessimistic agents can bring the prices to crash.

6 Discussion and Conclusion

In this paper we analyzed the impact of ambiguity on the equilibrium price on markets with heterogeneous agents. Agents' sensitivity to both risk and uncertainty may impose short selling constraint on their portfolio. This short selling constraint in turn affects the equilibrium price in an economy with heterogeneous agents by increasing the equilibrium price. While the effect of short selling constraint itself was already known, the paper rationalize the short selling constraint on markets through preferences. The model considered here has also interesting implications for the regulation. After the beginning of the financial crisis a lively discussion has started on how to regulate the markets better in order to prevent investors to take huge risks and to avoid bubbles. One of suggestions made by the theorists was to impose a minimax regulation. Within this kind of regulations agents on the market have to consider several models instead one when assessing riskiness to a future payoff. The claim is then acceptable if its return is nonnegative under all models the agents consider. This imposes a more conservative value assignment preventing investors from excessive risk taking. However, this kind of regulation can have side-effects highlighted in this paper. Agents regulated in the above sketched way behave as ambiguity averse investors in our model. If the regulation is imposed only locally they might be investors on the global market who behave like SEU agents in our model. As we have seen in the model such kind of interaction may lead to overpricing if beliefs are heterogeneous. Thus, a minimax regulation although conservative form of regulation may help to generate bubbles if it is not established in a careful way.

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