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Abstract

This is the first paper on consumer search where the cost of going back to stores already searched is explicitly taken into account. We show that the optimal sequential search rule under costly second visits is very different from the traditional reservation price rule in that it is nonstationary and not independent of previously sampled prices. We explore the implications of costly second visits on market equilibrium in two celebrated search models. In the Wolinsky model some consumers search beyond the first firm and in this class of models costly second visits do make a substantive difference: equilibrium prices under costly second visits can both be higher and lower than their perfect recall analogues. In the oligopoly search model of Stahl where consumers do not search beyond the first firm, there remains a *unique* symmetric equilibrium that has firms use pricing strategies that are identical to the perfect recall case.

JEL-codes: D11, D40, D83, L13

Key Words: Search, Costly Second Visits, Oligopoly, Competition

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1 Introduction

The main focus of consumer search theory is to analyze how market outcomes are affected if the cost consumers have to make to get information about the prices and/or qualities firms offer is explicitly taken into account. One of the basic results of the extensive literature is that firms have some market power that they can exploit even if there are many firms in the market and that price dispersion emerges as a consequence of the fact that some firms aim at selling to many consumers at low prices, while others make higher margins over fewer customers (see, e.g., Stigler (1961) and Reinganum (1979)).

Most, if not all, of the consumer literature implicitly or explicitly makes the assumption of perfect or free recall: consumers can always come back to previously sampled firms without making a cost.[1] One of the important consequences of this assumption is that consumer search behavior is characterized by one reservation price that is constant over time (Kohn and Shavell (1974)): for any observed price sequence, consumers stop searching and buy at the firm from which they received a price quote if that price is not larger than this reservation price; otherwise they continue searching.[2]

The assumption of perfect recall is, so we argue in this paper, at odds with the general philosophy of the consumer search literature which has search frictions at its core. If consumers have to make a cost to go to a shop in the first place, then in almost any natural environment it is also costly (in terms of time, effort, or money) to go back to that shop. Even while searching on the internet, where the costs of search are arguably lower than in nonelectronic markets, it takes some mouse clicks and time to go back to previously visited websites. In other words, in consumer search it is not only important to remember the offers previously received, but one also has to make a cost to activate these offers again.

In this paper we replace the perfect recall assumption by the more natural assumption of costly second visits, where the cost of going back to stores previously sampled is explicitly modeled. In doing so we concentrate on the implications for both the optimal consumer sequential search strategy and

[1]See, e.g., Reinganum (1979), Morgan and Manning (1985) , Stahl (1989) and Stahl (1996) for early papers and Janssen et al. (2005), Tse (2006) and Waldeck (2008) for more recent papers explicitly using the perfect recall assumption.

[2]An alternative setting is studied by Weitzman (1979). He considers the interesting case where alternatives differ in the cost of inspection as well as in the distribution of revenues and he asks the question in which order the alternatives should be explored.

the equilibrium pricing strategies of firms. Under costly second visits, we show that consumer search is no longer characterized by a reservation price that is constant over time. Instead, the reservation price at any moment in time depends on (i) the number of firms that are not yet sampled and (ii) the lowest price sampled so far. In particular, for a given lowest price in the sample the reservation price is (weakly) decreasing in the number of firms that are not yet sampled (increasing over time) and increasing in the minimum price in the sample if this minimum price is not too large. Of course, if no prices are sampled yet, the reservation price is just a constant (depending on the number of firms that quote prices). Only when there are infinitely many prices to sample (in a perfectly competitive market), stationarity reappears and the reservation price in that case coincides with the reservation price under perfect recall. Thus, one conclusion is that competitive search models are robust to introducing costly second visits.

These two differences in the characterization of reservation prices have important consequences for the actual search behaviour of consumers. Under costly second visits it may very well happen that if consumers observe as part of a price sequence two prices p_t and p_{t+1}, with $p_t < p_{t+1}$, they will rationally decide to accept to buy at p_{t+1} and not at p_t. This behaviour is not possible under perfect recall and rational consumer behaviour. The main reason for the fact that different behaviours are possible is that under costly second visits, no matter how small the cost of retrieving previously sampled information, the search process is no longer stationary. In addition, the fewer the number of firms not yet sampled, the worse the chance of observing a low price if one continues searching. Together, this implies that the class of search behaviours that are consistent with rational behaviour on the part of consumers becomes much richer.

In contrast to the assumption of perfect recall commonly employed in the literature on consumer search, many papers in the literature on job search assume that only current offers can be accepted as previous offers that are not accepted are foregone. Karni and Schwartz (1977) have interpreted these two applications of search theory as making specific assumptions on the probability with which past observations can be successfully retrieved: in consumer search, the probability of successful retrieval is one, in job market search, this probability is zero. They then go on to study situations with "uncertain recall", where the probability that past observations can be successfully retrieved is less than one but greater than zero (see also Landsberger and Peleg (1977)). We interpret the difference between consumer search and job market search differently, namely in terms of the cost one has to make to retrieve information. This cost is either zero or prohibitively high. We study the intermediate case where the cost is positive, but not too high to make it uninteresting to consider the option of going back to previously sampled firms.[3]

Our next (main) question is how these changes in the optimal search strategy impact on the optimal pricing behaviour of firms.[4] Is the assumption of perfect recall crucial for the analysis of search markets? The answer to this question may depend on the particular industry setup considered. To answer this question, we divide the class of sequential search models in two subclasses: (i) models "with true search", i.e. models where in equilibrium some consumers search beyond the first firms, and (ii) models "without true search", i.e. models where in equilibrium consumers stop at the first alternative they observe. We first analyze the implications of costly second visits in the Wolinsky model as a celebrated example of the first class of search models distinguished. We show that in this model, the equilibrium pricing behaviour of firms is affected when we go beyond perfect recall. Since consumers do search in these models, there are some options which are worse than the reservation option value. Costly second visits do matter here as they affect the reservation prices and thus the expected demand from searching consumers.

As an important example of the second class of search models, we use a conventional model of oligopolistic competition with homogeneous goods and sequential consumer search, which was pioneered by Stahl (1989). The distinguishing feature of the Stahl model is that there are two types of consumers, informed and uninformed consumers. Informed consumers have zero search cost and always buy the product at the lowest price in the market. Uninformed consumers have positive search cost and engage in optimal sequential search. In the Stahl model N firms set prices simultaneously to maximize

[3] As far as we are aware, there is no paper studying this most relevant case. Kohn and Shavell (1974) say that some of their results continue to hold if there is no possibility of recall, but they also do not analyze the situation of costly recall. Some of the results of Landsberger and Peleg (1977) are similar in nature to ours. Most notably that for every search there is a time-dependent reservation price and that this price is constant in case of perfect recall and infinitely many firms. In the operations research literature Kang (1999) studies an optimal stopping problem where the cost of a second visit is a percentage of the utility derived from previous observations and arrives at a technical analysis that resembles our analysis of the optimal search rule. See Section 2 for a more detailed comparison.

[4] An extensive overview of this literature has recently been given by Baye et al. (2006).

profits, where demand potentially comes from both types of consumers.

The surprising result we obtain for the Stahl (1989) model is that even though the consumer's search strategy is different and more complicated, the market equilibrium does not involve firms choosing different pricing strategies. We have two types of results that underline this general conclusion. First, the symmetric equilibrium that is found by Stahl (1989) remains an equilibrium. In this equilibrium firms choose a price from a price distribution that is such that consumers with a positive search cost buy immediately in the first store they visit. Even the definition of the reservation price does not need to be adjusted. This first result is quite intuitive: at the reservation price (which is the upper bound of the price distribution) consumers are indifferent between buying immediately and continuing to search and buy (with probability one) at the next store and thus consumers never consider seriously to go back to previously visited stores. The second result is less intuitive: we show there are no other types of symmetric equilibria. Thus, the Stahl equilibrium remains the unique symmetric equilibrium if we allow for costly second visits. With costly second visits in principle firms may benefit from setting prices above the reservation price of the first search round. The standard argument why firms will not set such prices is that a firm that charges a price equal to the upper bound of the price distribution will not sell to any consumer as even the uninformed consumers will continue to search after observing such a price and have then at least two prices to compare where the other price(s) are strictly smaller with probability one. This argument does not hold with costly second visits as competitors that are visited first may have prices that are lower than the upper bound, but not so much lower that it pays for consumers to pay the cost of going back to these previously visited stores. At first look one might think that if the valuation of the good is sufficiently high firms can always compensate the low probability of such event with sufficiently high prices. We show, however, that the structure of the profit function is such that if firms charge prices above first round reservation prices, they can never compensate the loss of demand with higher revenue per consumer.

Armstrong and Zhou (2010) give a particular interpretation of costly second visits. They show that costly second visits can be re-interpreted as buy-now discounts, i.e. as discounts consumers only get when they visit a firm for the first time: as soon as they walk out of the store without buying the possibility to receive the discount disappears. The main difference between their paper and ours is that the buy-now discount in Armstrong and

Zhou (2010) is a strategic variable chosen by firms, whereas in our model the cost of a second visit to a firm is an exogenous feature of the search technology. This means that our analysis may have various other applications, such as in a search theoretic explanation for the existence of shopping malls (see Non (2010)).

The structure of the rest of the paper is as follows. Section 2 analyzes the optimal sequential search behavior of consumers in a setting where they have a finite number of objects to discover. Section 3 then investigates the implications of this optimal search rule for the Wolinsky (1986) model and Section 4 presents the results for the model of Stahl (1989). Section 5 concludes. All technical proofs can be found in the appendix.

2 Optimal Sequential Consumer Search

The environment we discuss in this section and that will be relevant in the market setting discussed in the next two Sections is one where consumers have a choice whether or not to buy one alternative out of a finite number of alternatives. The utility each alternative delivers is unknown before consumers investigates the properties of the alternative. Before inspection all alternatives look the same, but ex post they are likely to be different. The notation we use in this Section is based on the idea that the alternatives only differ in price p, but this is not in anyway essential. Thus, we concentrate on an environment where the alternative i has a price p that is distributed according to the distribution function $F_i(p)$ and $F_i(p) = F(p)$ for all i. We assume that $F(p)$ is a continuous function and has a finite support. We define $\underline{p}$ to be the lower bound of the support of the distribution and $\overline{p}$ be the upper bound. Consumers engage in sequential search and get their first price quotation for free (following most of the literature),[5] but any subsequent price quotation comes at a search cost c. Consumers have unit demand and an identical valuation for the good which we denote by v and $v > c$. If the consumer decides to go back to the store she already visited before she incurs costs b where $0 \leq b \leq c$.

The main issue we are interested in in this section is how the presence of costly second visits ($b > 0$) affects the optimal search rule when $F(p)$ is known. Since the expected value of continuing to search depends on future

[5] Alternatively, we can easily incorporate the case where the first search comes at a cost as well.

period expected values we use backward induction to analyze the optimal stopping rule. To this end, define p^s_{k-1} as the smallest price in a sample of $k-1$ prices previously sampled. We will argue that for each value of p^s_{k-1} there is a unique value of p_k such that an individual consumer is indifferent between buying at p_k and either going back to one of the previously sampled firms and buying there or continue searching. We denote this price by $\rho_k(p^s_{k-1})$. If $p_k \leq \rho_k(p^s_{k-1})$, the consumer decides to buy at p_k. Otherwise, he either buys at p^s_{k-1} (if this price is relatively small) or continues to search.

The proof is by induction starting at the last firm. The following lemma introduces the base of induction.

Lemma 2.1. *Let $F(p)$ be a distribution of prices. Then for $k = N-1$ the reservation price ρ_{N-1} is uniquely defined as a function of $p^s_{N-2} \in [\underline{p}, \overline{p}]$ by*

$$\rho_{N-1}(p^s_{N-2}) = \min\left(p^s_{N-2} + b,\ c + p^s_{N-2} + b - \int_{\underline{p}}^{p^s_{N-2}+b} F(p)dp,\ p^*_{N-1}\right)$$

where p^*_{N-1} satisfies the equation

$$p^*_{N-1} = c + E(p_N | p_N < p^*_{N-1} + b)F(p^*_{N-1} + b) + (1 - F(p^*_{N-1} + b))(p^*_{N-1} + b).$$

Moreover, if the consumer decides to continue searching, the continuation cost of search, defined as the additional net expected cost of continuing to search conditional on optimal behaviour after the search is made, is given by

$$C_{N-1}(p^s_{N-1}) = c + p^s_{N-1} + b - \int_{\underline{p}}^{p^s_{N-1}+b} F(p)dp.$$

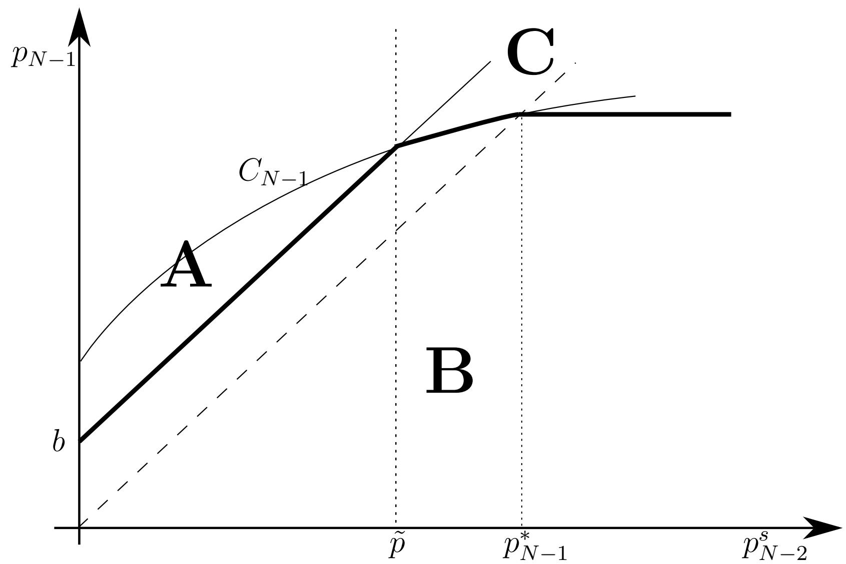
The following picture illustrates the lemma.

The reservation price as a function of p^s_{N-2} is presented by the bold curves. It is easy to see that this line consists of three parts:

(i) for $p^s_{N-2} < \tilde{p}$[6] the best alternative to buying at p_{N-1} is to go back to the lowest-priced firm in the sample so far. Thus, the reservation price is determined by $\rho_{N-1} = p^s_{N-2} + b$.

[6] We give a formal definition of $\tilde{p}$ later.

Figure 1: Reservation Price ρ_{N-1} as a function of p^s_{N-2}



(ii) for $\tilde{p} \leq p^s_{N-2} < p^*_{N-1}$ the option to continue searching is always preferred to the option of going back to the lowest-priced firm in the sample so far. Thus, the consumer's optimal choice is based on a comparison between the current price and the expected continuation costs of continuing to search;

(iii) for the region $p^s_{N-2} \geq p^*_{N-1}$ the situation is similar to the previous case, except that the current price is always the lowest price in the sample so far, implying that the continuation cost does not depend on p^s_{N-2}. Therefore, the reservation price is independent of p^s_{N-2} in this case.

Along the bold curve the consumer is indifferent between buying now at the shop he is currently visiting or either continuing to search or to go back to the lowest-priced firm in the sample so far.

Since optimal search behaviour is completely determined by the pair (p_{N-1}, p^s_{N-2}) we can characterize it in the same figure. Indeed, in region A, which is bounded from below by ρ_{N-1} and from the right by $\tilde{p}$, the consumer always goes back and buys at the lowest-priced firm in the sample so far. In region B, which is bounded from above by the reservation price, the consumer always buys at the current shop. Finally in region C, which is bounded from below by the reservation price and for which $p^s_{N-2} > \tilde{p}$, the consumer always continues to search.

Next we show that on any step $1 < k < N-1$ the reservation price as a function of the lowest price in the sample is uniquely defined and has essentially the same shape as in Figure 3.1. The proof is by induction. Before we give the formal statement of the result and the proof, we have to provide a technical result that turns out to be useful in making the induction step. To this end, assume that y is a random variable with a continuous distribution function $F(y)$. Let for a given search and return cost c and b, the following function be defined

$$\begin{aligned} C^*(x) = & \mathbb{P}(y < \min(x+b, C(\min(x,y))))\cdot \\ & \cdot \mathbb{E}(y|y < \min(x+b, C(\min(x,y))))+ \\ & + \mathbb{P}(y \geq \min(x+b, C(\min(x,y))))\cdot \\ & \cdot \mathbb{E}(\min(x+b, C(\min(x,y)))|y > \min(x+b, C(\min(x,y)))) + c. \end{aligned} \tag{2.1}$$

The function $C^*(x)$ can be interpreted as a generalized continuation cost of additional search given continuation cost on the next step.

For any function $f(x)$ let us define

$$f^-(x) = \liminf_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f^+(x) = \limsup_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Then the following lemma holds.

Lemma 2.2. *If $C(z)$ is a continuous function and for any z in he support of $F(\cdot)$ $0 \leq C^-(z) \leq C^+(z) < 1$ and $C(\underline{y}) > b$, where $\underline{y}$ is the lower bound of the support of $F(y)$, then $C^*(x)$ is a continuous function and for any x in the support of $F(\cdot)$ except the lower bound, $0 \leq C^{*-}(x) \leq C^{*+}(x) < 1$.*[7]

Given these two lemmas, we are now ready to state and prove the main result of the chapter. The result says that the reservation price as a function of p^s_{k-1} is well-defined and unique and a monotone function of p^s_{k-1}. In later

[7]The result of Kang (1999) for the case where the costs of going back are a percentage of utility observed is similar in nature to this lemma. His proof relies on convexity of the value function, while we focus on the slope.

results, we prove that the time- and history-dependency of these reservation prices cannot be neglected, unlike the case of costless recall.

Theorem 2.3. *The reservation price $\rho_k(p^s_{k-1})$ is uniquely defined for any k and any p^s_{k-1} from the support of $F(p)$. Moreover, the time- and history-dependent reservation prices $\rho_k(p^s_{k-1})$ are nondecreasing in p^s_{k-1}.*

Proof. The proof is by induction using Lemma 2.1 as induction base and Lemma 2.2 for the induction step. See the Appendix for details. □

The proof of the theorem basically shows that the function $\rho_{k+1}(p^s_k)$ is defined over three separate intervals and essentially looks like the reservation price for the last step (see Figure 3.1). When p^s_{k-1} is relatively small $\rho_k(p^s_{k-1}) = p^s_{k-1}+b$. Then for intermediate values of p^s_{k-1}, $\rho_k(p^s_{k-1}) = C_k(p^s_{k-1})$ and for higher values $\rho_k(p^s_{k-1})$ is independent of p^s_{k-1}. One can thus, define a price $\tilde{p}_k$ as the price such that on step k the consumer is indifferent between going back to the shop charging this price and continuing to search, i.e., $\tilde{p}_k + b = C_k(\tilde{p}_k)$.

We are now in the position to prove some special properties of the reservation price function that turn out useful in the next section. To this end, define ρ^{pr} as the reservation price under perfect recall, i.e., as noted, e.g., by Stahl (1989),

$$c = \int_{\underline{p}}^{\rho^{pr}} F(p)dp.$$

By considering the limiting case where the cost of recall is zero we provide more insight into the reason why the cases of perfect recall and costly second visits are so different from one another. Moreover, the reservation price under perfect recall turns out to play an important role in further characterizing the optimal search behaviour under costly second visits.

Proposition 2.4. [8] *Let $b = 0$. Then for any k the reservation price is defined by:*

$$\rho_k = \min(p^s_{k-1}, \rho^{pr}).$$

[8] As this fact is intuitively obvious the proof is available upon request.

Under perfect recall, the search rule is stationary, but (interestingly) slightly different from what is commonly thought as in any period the reservation price is still dependent on the lowest of previously sampled prices. When the current price is smaller than any of the previously sampled prices, then the consumer simply compares the current price with ρ^{pr} and decides whether or not to buy. If the current price is larger, the consumer simply forgets about the current price. Because of stationarity, previously sampled prices are in a full model including price setting behaviour of the firms, irrelevant. Either these previously sampled prices are below ρ^{pr}, but then the consumer simply does not continue to search, or they are above ρ^{pr}, but then the consumer never considers buying there unless he has visited all the stores and knows for sure that there are no lower prices in the sample.[9]

To further characterize the optimal search rule, under costly second visits we show that the price $\tilde{p}_k$ is intimately related to the price ρ^{pr} under perfect recall.

Proposition 2.5. *For all k, $\tilde{p}_k \equiv \tilde{p} = \rho^{pr} - b$.*

Next, we show that rational consumers never use the option of going back to previously sampled stores, unless they have visited every store available. This result is especially useful in the context of the analysis of the Wolinsky model with costly second visits in the next section.

Corollary 2.6. *Assume the consumer behaved optimally on all steps $1 \leq k \leq K$. Then if $K < N$, it is never optimal for this consumer to go back.*

Next, we show that reservation prices are non-decreasing over time. In particular, if a price smaller than $\tilde{p} = \rho^{pr} - b$ is sampled before, then the reservation price is simply $\rho_k(p^s_{k-1}) = p^s_{k-1} + b$ and therefore if $p^s_k = p^s_{k-1}$, then $\rho_{k+1}(p^s_k) = \rho_k(p^s_{k-1})$. However, if a price strictly larger than $\tilde{p} = \rho^{pr} - b$ is the lowest price in the sample so far, then $\rho_{k+1}(p^s_k) > \rho_k(p^s_{k-1})$. Thus, under costly second visits reservation prices are essentially nonstationary.

[9] However, in equilibrium even this could not be the case with $b = 0$ as then the traditional argument kicks in that no firm wants to charge the highest price above ρ^{pr} as no consumer will ever buy at this price, implying that no firm will want to choose a price above ρ^{pr}.

Proposition 2.7. *If $p_k^s = p_{k-1}^s$, then $\rho_{k+1}(p_k^s) \geq \rho_k(p_{k-1}^s)$, i.e., reservation prices are non-decreasing over time. Moreover, $\rho_{k+1}(p_k^s) > \rho_k(p_{k-1}^s)$ for all p_k^s and p_{k-1}^s such that $p_k^s = p_{k-1}^s > \tilde{p} = \rho^{pr} - b$.*

Knowing that reservation prices are non-decreasing in the search round, we next establish a relationship between the highest and the lowest reservation prices implying that they cannot be more than a factor 2 apart.

Lemma 2.8. *For any p in the support of $F(p)$ $\frac{\rho_{N-1}(p)+b}{\rho^{pr}} < 2$.*

Since, using Proposition 2.7, $\rho_k(p) \leq \rho_{N-1}(p)$ and $\rho_1 \geq \rho^{pr}$ we have

$$\frac{\rho_k(p) + b}{\rho_1} < \frac{\rho_{N-1}(p) + b}{\rho^{pr}}$$

and thus

Corollary 2.9. $\forall k \in (2, N) : \frac{\rho_k(p)+b}{\rho_1} < 2$

We finally consider the limiting case (of perfect competition) where there are potentially infinitely many prices to sample. As the time dependency of the reservation prices disappears due to the fact that now the cost of continuing to search is independent of time, i.e., $\rho_k(p_{k-1}^s) = \rho_{k+1}(p_k^s)$. For prices below $\tilde{p}$, we knew already that this equality holds. Interestingly, with infinitely many firms and previously sampled prices above $\tilde{p}$, the reservation prices becomes independent of previously sampled prices and equal to the reservation price under perfect recall. Thus, the cost of going back to previously sampled firms does not play an important role under perfect competition.

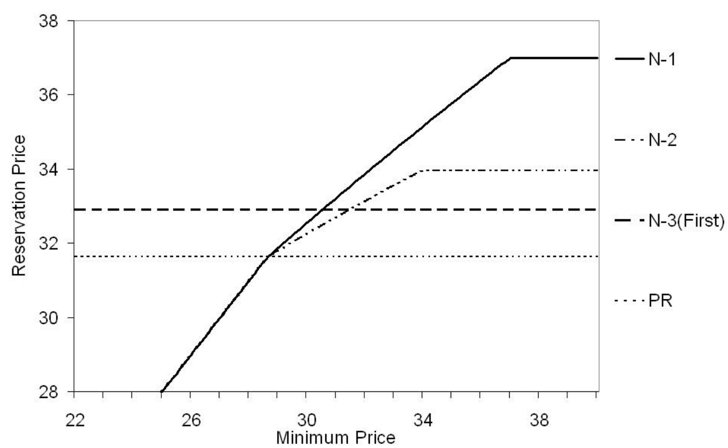
Proposition 2.10. *Let $K \in \mathbb{N}$. Then for any $p \geq \tilde{p}$ $\lim_{N\to\infty} \rho_K(p) = \rho^{pr}$.*

Thus, under perfect competition the reservation price under costly second visits is exactly identical to the case where consumers have perfect recall.

We end this Section by providing a numerical example to illustrate some features of the reservation prices. The example clearly shows that it can be rational to accept a price in a future period even if a lower price has been observed in the past.

Consider the uniform distribution of prices on $[0, 100]$. Assume there are 4 firms in the market, search costs c are equal to 5 and the costs of going back to a previously sampled firm b equals 3. The reservation prices after visiting no, one and two firms as well as the reservation price under perfect recall are presented in Figure 2. In this case, the reservation price under perfect recall equals approximately 31.62, while the reservation price before visiting any shop under costly second visits equals 32.90. Thus, if a consumer faces, say, a price of 33 in the first period he decides to continue searching. From Figure 2 it is clear, however, that if the third price the consumer encounters is say 34 it is optimal for him to stop.

Figure 2: Simulation Results for Uniform Distribution .



Parameters of simulation: $N = 4, b = 3, c = 5$.

The figure also illustrates most of the results we proved in the previous section. In particular, it is easy to observe that all reservation price functions are non-decreasing in p_k^s (Theorem 2.3) , and that the sequence of reservation prices is non-decreasing in the number of firms left, and strictly increasing for all prices above $\tilde{p}$ (Proposition 2.7).

3 A Wolinsky-type Model with Costly Second Visits

After we have defined the optimal search behaviour of consumers it is natural to look at the equilibrium implications of such a behaviour and ask whether costly second visits imply different firm behaviour in equilibrium. In this Section we study the Wolinsky (1986) model as a prominent example of a model with true search where some consumers search beyond the first option. the next Section deals with the Stahl model as an example of a search model where no consumer with positive search cost searches beyond the first firm.

We make a few innocuous simplifications to the Model of Wolinsky (1986) in order to focus on our main point – the influence of costly second visits on equilibrium outcomes. Each of N firms can produce a single distinct brand. All the firms have identical cost functions $C(x) = F + C \cdot x$, where x is the quantity of goods sold. There is a unit mass of consumers and each consumer is interested in buying one unit of the product and also derives utility from consuming the numeraire good x_0. The utility function of consumers is given by $u(x_0, i) = x_0 + v_i$, where v_i is a value attached by a consumer to brand i. The v_i's are realizations of independent and identically distributed variables with distribution function G with finite support $[\underline{v}, \overline{v}]$. Given the utility function it is clear that consumers are interested in maximizing the surplus $v_i - p_i$. Consumers are not informed about the prices and values v_i before they search a particular firm. The search process is costly with cost c. We supplement this assumption with the assumption of costly second visits: in order to return to a previously sampled firm consumers have to pay b. We look for the symmetric equilibrium of the model, where all firms charge price p^*.

Though our search results are formulated in terms of costs and prices, they can be easily interpreted in terms of values and utilities. From Theorem 2.3 it follows that the optimal stopping rule is characterized by a set of reservation utility functions $\omega_i(v_{i-1}^b - p^*)$, where v_i^b is the best utility so far. Each moment a consumer compares the current option $(v_i - p_i)$ with the reservation utility $\omega_i(v_{i-1}^b - p^*)$ and makes her decision. We denote the reservation utility in the first search round by ω_1 without arguments, since there is no history on which to condition the decision.

Now we construct the demand function for a firm and show that it differs from the demand function of the original model. Note, that Corollary 2.6 simplifies the solution a lot: we do not need to consider the possibility a consumer returns to previously visited firms until all firms are visited.

Demand for a firm that charges a price p given the equilibrium price p^* comes from four different sources. First, some consumers (randomly) come to the firm in their first search round and immediately buy (if the match value is below their reservation utility). The demand from this source is given by

$$I_1 = \frac{1}{N}\left[1 - G(w_1 + p)\right].$$

Second, other consumers come to the firm for the first time after the first search round but before the last search round and then buy immediately when they have first visited this particular firm:

$$I_2 = \frac{1}{N}\sum_{j=2}^{N-1}\int_{\omega_j(v^b_{j-1}-p^*)+p}^{\overline{v}}\int_{\underline{v}}^{\omega_{j-1}(v^b_{j-2}-p^*)}\dots\int_{\underline{v}}^{\omega_1} dG(v_1)\dots dG(v_j).$$

This expression represents the fact that along the search path each of the utilities of consuming the good provided by a firm before visiting the particular firm in question was smaller than the corresponding (step and history dependent) reservation utility, while the current utility level is larger than the appropriate optimal stopping level.

Third, some consumers come to the firm in the last search round. Here the following conditions have to be satisfied. First, the offer is acceptable and, second, not more than b worse than any other offer. These two conditions determine the lower limit of the first integral as, first, v_i must be larger than p and, second, the current offer must be more attractive than going back to any previous offer, i.e., $v^b_{N-1} - p^* - b < v_i - p$, or $v_i > v^b_{N-1} - p^* - b + p$. Thirdly, all other offers along the search path have to be rejected yielding the remaining $N-1$ integrals.

$$I_3 = \frac{1}{N}\int_{\max\{v^b_{N-1}-p^*-b+p,p\}}^{\overline{v}}\int_{\underline{v}}^{\omega_{N-1}(v^b_{N-2}-p^*)}\dots\int_{\underline{v}}^{\omega_1} dG(v_1)\dots dG(v_N).$$

Fourth, some consumers were at the firm but left it and decided to return back later. Let us denote $s_i = \max_{j<i}(v_j - p_j)$, $s_1 = 0$. Assuming that the firm we are interested in was visited on search round i this implies that the following three conditions are satisfied. Firstly, all other firms except the last one provide worse alternatives and were rejected on the previous search rounds: $v_j < \min(\omega_j(s_j), v_i - p_i + p^*)$ for all $j \neq i$ and $j \neq N$. Secondly, firm i was rejected on round i ($v_i < \omega_i(s_i)$), but the offer is in principle acceptable ($v_i \geq p$). Thirdly, the last firm gives an offer which is worse than the offer of firm i by more than b: $v_N < v_i - p + p^* - b$. Thus, summing over all search paths:

$$I_4 = \frac{1}{N}\sum_{i=1}^{N-1}\int_p^{\omega_i(s_i)}\int_{\underline{v}}^{v_i-p+p^*-b}\int_{\underline{v}}^{\min(\omega_{N-1}(s_{N-1}),v_i-p_i+p^*)}\dots$$

$$\dots\int_{\underline{v}}^{\min(\omega_1(s_1),v_i-p_i+p^*)} dG(v_1)\dots dG_{i-1}dG_{i+1}\dots dG(v_N)dG_i.$$

The resulting expression for a firm's demand is

$$D(p,p^*) = I_1 + I_2 + I_3 + I_4.$$

Obviously the resulting demand function is quite different from the result obtained by Wolinsky (1986) (see formula 5 in that paper): the result by Wolinsky can be obtained by substituting $b = 0$ and a stationary reservation price in the demand function. In particular the demand is higher for any given pair (p, p^*), however the equilibrium price is not necessarily higher. With or without costly second visits, the equilibrium price is determined by

$$p^* = C - \frac{D(p^*,p^*)}{D'_p(p^*,p^*)}$$

and so the slope of the demand function together with the demand itself determine the equilibrium prices.[10] Inspection of the above expressions reveals that it is extremely difficult to get analytical results for the Wolinsky model with costly second visits. To show that the equilibrium prices under costly second visits are indeed different from the Wolinsky paper (i.e. it is not just that the formula looks different) we have performed a numerical analysis for the case where the utilities are uniformly distributed over $[a0, 1]$ and $N = 2$.[11] The table below shows for different values of the search cost parameter c the equilibrium prices for the Wolinsky model (where $b = 0$) and our model for different values of b.

The table reveals a few interesting facts. First, it confirms the overall result that equilibrium prices are increasing in search cost, i.e., even at positive cost of second visits, equilibrium prices are clearly increasing in search cost (whether it is measured keeping the ratio of b to c fixed, or whether only b is fixed and the impact of the cost of the first visit is investigated).

[10]We thank Jidong Zhou for pointing out and illustrating this fact.

[11]The model can be solved numerically for higher values of N, but for each higher value of N even the numerical solution becomes harder to calculate and requires to set up a different algorithm for each N.

Table 1: The equilibrium price as a function of c, b

c/b	0	0.2c	0.4c	0.6c	0.8c	1.0c
0.05	0.429900	0.431155	0.432282	0.433290	0.434186	0.434978
0.10	0.443128	0.444777	0.446062	0.447014	0.447655	0.448005
0.15	0.454977	0.456583	0.457524	0.457852	0.457609	0.456828
0.20	0.465919	0.467188	0.467426	0.466709	0.465098	0.462639
0.25	0.476211	0.476915	0.476157	0.474039	0.470640	0.466018

Second, and more surprising, the impact of costly second visits the equilibrium price as a function of b can be both increasing (for small values of c), and non-monotone (for larger values of c) with equilibrium prices becoming decreasing in b when b becomes relatively large.

We conclude that the introduction of costly second visits changes the equilibrium outcomes in the model with true search and these models are therefore not robust to the introduction of costly second visits.

4 The Stahl model with costly second visits

In this section we analyse the question whether costly second visits imply different equilibrium behaviour of firms in models where under perfect recall consumers do not search beyond the first firm. To do so, we focus on the celebrated model by Stahl (1989).

Recall that Stahl (1989) considers a market where N firms produce a homogenous good and have identical production costs, which we normalize to zero. Each firm decides upon the price at which it is going to sell the good in the market. There are two types of consumers in the market. A fraction λ of all consumers are "shoppers", i.e. these consumers like shopping or have zero search costs for other reasons. We assume that these consumers know all prices in the market and buy at the firm with the lowest price. The remaining fraction $1-\lambda$ of consumers is uninformed. These consumers engage in sequential search and get their first price quotation for free. They will search optimally in the way analyzed in Section 2. The timing in the model is simple: first, firms simultaneously decide on their prices, where the strategy of firm i is described by $F_i(p)$. Stahl (and we) concentrate on symmetric equilibria where $F_i(p) = F(p)$ for all i.

We start with the question whether a "Stahl-type" of pricing strategy, i.e.

where all firms play mixed strategies with the support up to first round reservation price is indeed an equilibrium in the model with costly second visits. Then we proceed with the investigation whether other types of equilibria are possible.

Our first result states that the "Stahl-type" of equilibrium is also an equilibrium in the model with costly second visits.

Proposition 4.1. *There is a mixed strategy equilibrium where all firms charge prices below the first-round reservation price, which equals the reservation price under perfect recall* ρ^{pr}*.*

We can explain this result as follows. Since nobody searched with perfect recall, the upper bound of the price distribution (the worst option for consumers) was not worse than the reservation price (value) under perfect recall. Thus, provided that firms stick to the same strategy, all the reservation prices under costly second visits are equal to the reservation price with perfect recall. Therefore, none of the firms individually has a profitable deviation.

Now, using the results from section 2 we can formally prove the idea that there are no other symmetric equilibria than the Stahl equilibrium. For such an equilibrium to exist it must be the case that the upper bound of the price distribution is strictly larger ρ^{pr}. To simplify notation we introduce the following definition.

Definition 4.2. *Let's denote* r_k *to be the maximum possible reservation price in the k-th search round, i.e.,* $r_k = \max_p \rho_k(p)$*.*

The claim that there are no other symmetric equilibria is now proved in three consecutive steps. Lemma 4.3 shows that there are no equilibria where the upper bound of the support is smaller than to r_{N-1}. Lemma 4.4 shows that there are no equilibria where the upper bound of the support is in between r_{N-1} and $r_{N-1} + b$). Finally, Lemma 4.5 shows that there are no equilibria where the upper bound of the support is above $r_{N-1} + b$.

Lemma 4.3. *There is no equilibrium price distribution with* $r_1 < \overline{p} < r_{N-1}$*.*

Now we analyze the "intermediate" case where the upper bound would be $\overline{p} \in [r_{N-1}, r_{N-1} + b]$. The proof of this Lemma is based on the fact that in order to compensate firms for the loss in demand resulting from charging above r_{N-1}, the upper bound of the distribution has to be above $2r_1 - b$, which contradicts the relationship between reservation prices that is consistent with the search perspective as established Lemma 2.8.

Lemma 4.4. *There is no equilibrium price distribution with* $\overline{p} \in [r_{N-1}, r_{N-1}+b]$.

Finally, we analyze the case where the upper bound is quite well above the highest reservation price. This part of the overall proof is the most complicated part. The idea of the proof is that if the upper bound of the support is larger that the highest reservation price, it is anyway bounded from above due to the structure of the upper part of the support of an equilibrium price distribution. This gives an upper bound on the profits firms receive from choosing a price equal to the upper bound. On the other hand, we argue that a price equal to the first-round reservation price should also be charged in equilibrium. Moreover, we show that this first-round reservation price should be larger than some lower bound, creating some lower bound on equilibrium profits. The last part of the proof shows that the upper bound we construct is smaller than the constructed lower bound yielding an inconsistency.

Lemma 4.5. *There is no equilibrium price distribution with* $\overline{p} > r_{N-1} + b$.

These three Lemmas together allow us to state that the "Stahl" equilibrium is the unique symmetric equilibrium in the model.

Theorem 4.6. *The unique symmetric equilibrium in the model with costly second visits is the equilibrium characterized in Proposition 4.1.*

5 Conclusions

Consumer search models have assumed that consumers have costless access to prices in stores they already visited, but have to pay a search cost to visit the store in the first place. We have argued that this assumption is often not justified and that when there are search cost for visiting a store in the first place, there are also (smaller) costs of going back to a store (second visits). We have shown that without the assumption of costless second visits, the optimal sequential search rule is no longer characterized by a unique, stationary reservation price. Instead, the reservation price in a particular search round is a function of the number of firms that are still not-visited and the lowest price sampled so far.

We have studied the implications of costly second visits for two strands of literature, one where -under perfect recall- in the market equilibrium firms price in such a way that some consumers do search beyond the first firm and another class where no consumer does so. In the first class of search models, inspired by Wolinsky (1983), costly second visits imply a change in the equilibrium behavior of firms where costly second visits may imply both higher and lower equilibrium prices.

In the class of models "without true search" we have shown for the celebrated paper by Stahl (1989) that the equilibrium analysis is robust to the assumption of costly second visits. Our analysis shows that the equilibrium analyzed by Stahl remains an equilibrium under the alternative assumption of costly second visits and that, in addition, there do not exist other possible symmetric equilibrium outcomes in the oligopolistic competition setup. Even though the optimal search behaviour of the consumers can be very complicated, firms behave in such a way that they do not charge prices above the first search round reservation price. The main reason for this finding is that if a firm charges a price above this first search round reservation price, it loses relatively so many consumers that this loss in demand can never be sufficiently compensated by the increase in price.

Appendix: Proofs

Lemma 2.1. *Let $F(p)$ be a distribution of prices. Then for $k = N-1$ the reservation price ρ_{N-1} is uniquely defined as a function of $p^s_{N-2} \in [\underline{p}, \overline{p}]$ by*

$$\rho_{N-1}(p^s_{N-2}) = \min\left(p^s_{N-2} + b,\ c + p^s_{N-2} + b - \int_{\underline{p}}^{p^s_{N-2}+b} F(p)dp,\ p^*_{N-1}\right)$$

where p^*_{N-1} satisfies the equation

$$p^*_{N-1} = c + E(p_N | p_N < p^*_{N-1} + b)F(p^*_{N-1} + b) + (1 - F(p^*_{N-1} + b))(p^*_{N-1} + b).$$

Moreover, if the consumer decides to continue searching, the continuation cost of search, defined as the additional net expected cost of continuing to search conditional on optimal behaviour after the search is made, is given by

$$C_{N-1}(p^s_{N-1}) = c + p^s_{N-1} + b - \int_{\underline{p}}^{p^s_{N-1}+b} F(p)dp.$$

Proof. We consider the situation where $N-2$ firms have been sampled and the consumer has decided to make one more search. In this case, the consumer has three options: to buy now at the newly observed price p_{N-1}, to buy now at lowest price among the previously sampled prices p^s_{N-2}, or to continue searching. Knowing the value of $\min(p_{N-1}, p^s_{N-2})$, the last option gives an expected value of

$$\begin{aligned} v - c - E(p_N | p_N &< \min((p_{N-1}, p^s_{N-2}) + b)F(\min(p_{N-1}, p^s_{N-2}) + b) - \\ &\quad (1 - F(\min(p_{N-1}, p^s_{N-2}) + b))(\min(p_{N-1}, p^s_{N-2}) + b). \end{aligned}$$

Let us first concentrate on the case where $p_{N-1} \geq p^s_{N-2}$. In this case the payoff of continuing to search does not depend on p_{N-1} so that the reservation price is given by the point where the consumer is either (i) indifferent between buying now at p_{N-1} or buying at p^s_{N-2} (and paying the additional cost of going back b) or (ii) indifferent between buying now at p_{N-1} and continue searching. In the first case $\rho_{N-1}(p^s_{N-2}) = p^s_{N-2} + b$; in the second case

$$\begin{aligned}
\rho_{N-1}(p^s_{N-2}) &= c + E(p_N | p_N < p^s_{N-2} + b)F(p^s_{N-2} + b) + \\
&+ (1 - F(p^s_{N-2} + b))(p^s_{N-2} + b) = \\
&= c + \int_{\underline{p}}^{p^s_{N-2}+b} p dF(p) + (1 - F(p^s_{N-2} + b))(p^s_{N-2} + b) = \\
&= c + p^s_{N-2} + b - \int_{\underline{p}}^{p^s_{N-2}+b} F(p)dp.
\end{aligned}$$

It is easily seen that the first-order derivative of this expression w.r.t. p^s_{N-2} is positive and strictly smaller than 1. Moreover, it is easily seen that at $p^s_{N-2} = \underline{p}$, this expression equals $p^s_{N-2}+c > p^s_{N-2}+b$. Hence, by continuity, for small values of p^s_{N-2} the reservation price is given by $\rho_{N-1}(p^s_{N-2}) = p^s_{N-2}+b$. For larger values of p^s_{N-2} it is $\rho_{N-1}(p^s_{N-2}) = c + p^s_{N-2} + b - \int_{\underline{p}}^{p^s_{N-2}+b} F(p)dp$, at least when $\rho_{N-1}(p^s_{N-2})$ is still larger than p^s_{N-2}.

Let us next concentrate on the case where $p_{N-1} \leq p^s_{N-2}$. In this case the consumer will never go back to previously sampled prices and thus the reservation price is implicitly characterized by the price that solves

$$p_{N-1} = c + E(p_N | p_N < p_{N-1} + b)F(p_{N-1} + b) + (1 - F(p_{N-1} + b))(p_{N-1} + b).$$

Because of continuity at $p_{N-1} = p^s_{N-2}$, the fact that when $p^s_{N-2} < \rho_{N-1}(p^s_{N-2}) < p^s_{N-2}+b$, the derivative of the reservation price is strictly smaller than 1, and the fact that left differentiability holds at $p_{N-1} = p^s_{N-2}$, we should have that there is exactly one p_{N-1} that solves the above equation. This implies that in the region where $p_{N-1} \leq p^s_{N-2}$, $\rho_{N-1}(p^s_{N-2})$ is independent of p^s_{N-2}. Thus, also in this case $\rho_{N-1}(p^s_{N-2})$ is uniquely defined and non-decreasing in p^s_{N-2}.

Once price p_{N-1} is observed the continuation costs of search are defined by

$$\begin{aligned}
C_{N-1}(p^s_{N-1}) &= c + E(p_N | p_N < p^s_{N-1} + b)F(p^s_{N-1} + b) + \\
&+ (1 - F(p^s_{N-1} + b))(p^s_{N-1} + b) = \\
&= c + \int_{\underline{p}}^{p^s_{N-1}+b} p dF(p) + (1 - F(p^s_{N-1} + b))(p^s_{N-1} + b) = \\
&= c + p^s_{N-1} + b - \int_{\underline{p}}^{p^s_{N-1}+b} F(p)dp.
\end{aligned}$$

$\square$

Lemma 2.2. *If $C(z)$ is a continuous function and for any z in he support of $F(\cdot)$ $0 \le C^-(z) \le C^+(z) < 1$ and $C(\underline{y}) > b$, where $\underline{y}$ is the lower bound of the support of $F(y)$, then $C^*(x)$ is a continuous function and for any x in the support of $F(\cdot)$ except the lower bound, $0 \le C^{*-}(x) \le C^{*+}(x) < 1$.*

Proof. Continuity follows immediately from the definition of C^*.

Consider the following inequality: $y < \min(x + b, C(\min(x, y)))$. Since $C^+(z) < 1$, this inequality can be rewritten in the form $y < g(x) = \min(x + b, C(x), a)$, where a satisfies equation $a = C(a)$. It is clear that $g^+(x) \le 1$.

Thus, we can rewrite C^* in the following form:

$$\begin{aligned} C^*(x) = \mathbb{P}(y < g(x))\mathbb{E}[y|y < g(x)]+ \\ + \mathbb{P}(y \ge g(x))\mathbb{E}[\min(x + b, C(\min(x, y)))|y \ge g(x)] + c \end{aligned}$$

Now note, that if $x \le a$ then given that $y \ge g(x)$ we get $\min(x + b, C(\min(x, y)) = \min(x + b, C(x))$ which is just $g(x)$ for $x < a$. Then we get

$$C^*(x) = \mathbb{P}(y < g(x))\mathbb{E}[y|y < g(x)] + \mathbb{P}(y \ge g(x))\mathbb{E}[g(x)|y \ge g(x)] + c$$

and therefore

$$C^{*+}(x) = \left(\frac{F(g(x))}{F(g(x))} \int_{\underline{y}}^{g(x)} y f(y) dy + (1 - F(g(x)))g(x) \right)^+ =$$
$$= [g(x)f(g(x)) + (1 - F(g(x))) - g(x)f(g(x))]\, g^+(x) = [1 - F(g(x))]g^+(x) < 1.$$

with the second equality coming from the continuity of $g(x)$. It is also clear that $C^{*-} \ge 0$.

Another case is if $x > a$. Here, given $y \ge g(x)$we get $\min(x+b, C(\min(x, y)) = C(\min(x, y))$. Then we get

$$C^*(x) = \mathbb{P}(y < g(x))\mathbb{E}[y|y < g(x)] + \mathbb{P}(y \ge g(x))\mathbb{E}[C(\min(x, y))|y \ge g(x)] + c$$

Or

$$C^*(x) = \int_{\underline{y}}^{g(x)} yf(y)dy + \int_{g(x)}^{x} C(y)f(y)d(y) + \int_{x}^{\infty} C(x)f(y)d(y) + c$$

Now, because of the continuity of $g(x)$ and $C(x)$ again we get

$$C^{*+}(x) = [g(x)f(g(x)) - C(g(x))f(g(x))]g^+(x) + C^+(x)(1 - F(x))$$

Now note, that for $x > a$ we have $g(x) = a$ and therefore $C(g(x)) = a$. Thus,

$$C^{*+} = C^+(x)(1 - F(x)) < 1$$

In the same way

$$C^{*-} = C^-(x)(1 - F(x)) < 1$$

which completes the proof since $C^-(x) \geq 0, 1 - F(x) \geq 0$.

□

Theorem 2.3. *The reservation price $\rho_k(p^s_{k-1})$ is uniquely defined for any k and any p^s_{k-1} from the support of $F(p)$. Moreover, the time- and history-dependent reservation prices $\rho_k(p^s_{k-1})$ are nondecreasing in p^s_{k-1}.*

Proof. Let $C_k(p^s_k)$ be a continuation cost of additional search on the k-th step given realizations of (p^s_{k-1}, p_k) (recall that $p^s_k = \min(p^s_{k-1}, p_k)$). Then, given the optimal search behaviour of the consumer, $C_k(p^s_k)$ is the expected payoff of two events: either the consumer buys at the next firm to be searched (first event) or he continues to search onwards or goes back (second event). Thus, we get that

$$\begin{aligned} C_k(p^s_k) = c + {} & \mathbb{P}(p_{k+1} < \min(p^s_k + b, C_{k+1}(p^s_{k+1})))\cdot \\ & \cdot \mathbb{E}(p_{k+1} | p_{k+1} < \min(p^s_k + b, C_{k+1}(p^s_{k+1}))) + \\ & + \mathbb{P}(p_{k+1} \geq \min(p^s_k + b, C_{k+1}(p^s_{k+1})))\cdot \\ & \cdot \mathbb{E}(\min(p^s_k + b, C_{k+1}(p^s_{k+1})) | p_{k+1} \geq \min(p^s_k + b, C_{k+1}(p^s_{k+1}))) \end{aligned} .$$

We prove that $0 \leq C^-_k(p^s_k) \leq C^+_k(p^s_k) < 1$. The proof is by backward induction. From lemma 2.1 it is easy to see that $0 \leq C^-_{N-1}(p^s_{N-1}) \leq$

$C^+_{N-1}(p^s_{N-1}) < 1$, thus the base of induction is proven. We will now argue that this property also holds for any other period. For proving the induction step we can apply lemma 2.2 by substituting in the equation (2.1) $x = p^s_k$, $y = p_{k+1}$, $C^*(x) = C_k(p^s_k)$, $C(\min(x,y)) = C_{k+1}(p^s_{k+1})$. Therefore, from $0 \leq C^-_{k+1}(p^s_{k+1}) \leq C^+_{k+1}(p^s_{k+1}) < 1$ it follows that $0 \leq C^-_k(p^s_k) \leq C^+_k(p^s_k) < 1$ and thus, by induction it follows that for any k it holds that $0 \leq C^-_k(p^s_k) \leq C^+_k(p^s_k) < 1$.

The rest of the proof is straightforward. If $p_k \geq p^s_{k-1}$, then $\rho_k(p^s_{k-1}) = \min(p^s_{k-1} + b, C_k(p^s_{k-1}))$, which is well-defined and unique. Moreover, it is non-decreasing in p^s_{k-1} since both $p^s_{k-1} + b$ and $C_k(p^s_k)$ are non-decreasing in p^s_{k-1}. If, on the other hand, $p_k < p^s_{k-1}$, then the reservation price is a solution to the equation $p_k = C_k(p_k)$, which is unique since $C_k(p_k)$ has a slope strictly smaller than 1. In this case, the reservation price does not depend on p^s_{k-1} and is thus nondecreasing in p^s_{k-1}.

□

Proposition 2.5. *For all k, $\tilde{p}_k = \tilde{p} = \rho^{pr} - b$.*

Proof. Note that the price $\tilde{p}_k$ is defined such that after visiting k stores, the consumer is indifferent between continuing searching and going back to the lowest-priced store in the sample so far. Therefore, at $\tilde{p}_k$ the reservation price $\rho_k(\tilde{p}_k) = \tilde{p}_k + b$. The expected costs of continuing to search are:

$$c + F(\tilde{p}_k + b)\mathbb{E}(p_{k+1}|p_{k+1} < \tilde{p}_k + b) + (1 - F(\tilde{p}_k + b))(\tilde{p}_k + b)$$

By equating it to the best current option $(\tilde{p}_k + b)$ and some simplifications we have also used in previous proofs, we get

$$c = \int_{\underline{p}}^{\tilde{p}_k + b} F(p)dp.$$

It follows therefore that $\tilde{p}_k$ does not depend on k and that (by comparing this equation to the definition of ρ^{pr}) it is actually just equal to $\rho^{pr} - b$.

□

Corollary 2.6. *Assume the consumer behaved optimally on all steps $1 \leq k \leq K$. Then if $K < N$, it is never optimal for this consumer to go back.*

Proof. Note, that the option of going back is preferred to continue searching or stopping only if $p_K^s < \tilde{p}$. On the first step any price $p_1 \leq \rho^{pr}$ would be accepted immediately. So, if the consumer continued his search it must be the case that $p_1 > \rho^{pr}$. Given $p_1^s > \rho^{pr}$ on the second step any price $p_2 \leq \rho^{pr}$ also would be accepted immediately. Thus, if consumer continued his search it must be the case that $p_2 > \rho^{pr}$. Then by induction if customer reached step K it must be the case that for any $1 \leq k \leq K$ it was the case that $p_k > \rho^{pr}$. Therefore $p_K^s > \rho^{pr} > \tilde{p}$ and it is never optimal to go back, except possibly at the last step.

□

Proposition 2.7. *If $p_k^s = p_{k-1}^s$, then $\rho_{k+1}(p_k^s) \geq \rho_k(p_{k-1}^s)$, i.e., reservation prices are non-decreasing over time. Moreover, $\rho_{k+1}(p_k^s) > \rho_k(p_{k-1}^s)$ for all p_k^s and p_{k-1}^s such that $p_k^s = p_{k-1}^s > \tilde{p} = \rho^{pr} - b$.*

Proof. Note, that the reservation price essentially represents the cost of the next-best available alternative to buying now at the shop the consumer is currently visiting. If the next-best available alternative is to go back to the lowest-priced firm in the sample before visiting this shop, i.e., $p_{k-1}^s < \tilde{p}$ the reservation price is simply independent of the periods, i.e., $\rho_{k+1}(p_{k-1}^s) = \rho_k(p_{k-1}^s) = p_{k-1}^s + b$.

Now consider the case where the next-best available alternative is to continue searching. Let $\{\rho_k(p_{k-1}^s)\}_{k=1}^N$ be the sequence of the reservation price functions. Consider the following suboptimal strategy. If on step k the consumer makes a decision to visit one more firm he either buys at the firm he visits at step $k+1$ or continues his search but *forgets* about this firm later on (thus, he never comes back to that firm). Let us denote a reservation price under this suboptimal strategy by $\rho_k'(p_{k-1}^s)$. Then $\rho_k(p_{k-1}^s) \leq \rho_k'(p_{k-1}^s)$. On the other hand for any $p_{k-1}^s > \tilde{p}$ we get

$$\begin{aligned}\rho_k'(p_{k-1}^s) &= F(\rho_{k+1}(p_{k-1}^s))\mathbb{E}(p_{k+1}|p_{k+1} < \rho_{k+1}(p_{k-1}^s) + \\ &+ (1 - F(\rho_{k+1}(p_{k-1}^s)))\rho_{k+1}(p_{k-1}^s) < \rho_{k+1}(p_{k-1}^s)\end{aligned}$$

which completes the proof.

□

Lemma 2.8. *For any p in the support of $F(p)$ $\frac{\rho_{N-1}(p)+b}{\rho^{pr}} < 2$.*

Proof. Lemma 2.1 states that

$$\rho_{N-1}(p) = \min\left(p+b,\ c+p+b-\int_{\underline{p}}^{p+b} F(p)dp,\ p^*_{N-1}\right)$$

,

where p^*_{N-1} satisfies the equation

$$p^*_{N-1} = c + E(p_N | p_N < p^*_{N-1}+b)F(p^*_{N-1}+b) + (1-F(p^*_{N-1}+b))(p^*_{N-1}+b).$$

Note, that first, $\rho_{N-1}(p) \leq p^*_{N-1}$, and, second, p^*_{N-1} satisfies the equation

$$c+b = \int_{\underline{p}}^{p^*_{N-1}+b} F(p)dp. \tag{5.1}$$

The reservation price under perfect recall is defined by:

$$c = \int_{\underline{p}}^{\rho^{pr}} F(p)dp. \tag{5.2}$$

From (5.1) and (5.2) it follows that $p^*_{N-1}+b < 2\rho^{pr}$. Indeed, if this were not true, by subtracting one equation from the other we get:

$$b = \int_{\rho^{pr}}^{p^*_{N-1}+b} F(p)dp > \int_{\rho^{pr}}^{2\rho^{pr}} F(p)dp \geq \int_{0}^{\rho^{pr}} F(p)dp > c, \tag{5.3}$$

which contradicts the assumption $b < c$. The second inequality stems from the fact that $F(p)$ in a non-decreasing function, the last form the definition of ρ^{pr}. Therefore $p^*_{N-1}+b < 2\rho^{pr}$ and since $\rho_{N-1}(p) \leq p^*_{N-1}$ the lemma is proved.

□

Proposition 2.10. *Let $K \in \mathbb{N}$. Then for any $p \geq \tilde{p}$ $\lim_{N\to\infty} \rho_K(p) = \rho^{pr}$.*

Proof. Note, that for any $p \geq \tilde{p}$, $C_{N-1}(p)$ is fixed and does not depend on N. On the other hand for any $p \geq \tilde{p}$ we have

$$\begin{aligned}
C_k(p) &= F(\rho_{k+1}(p))\mathbb{E}(p_{k+1}|p_{k+1} < \rho_{k+1}(p)) + \\
&+ (1-F(\rho_{k+1}(p))\mathbb{E}(C_{k+1}(p_{k+1})|p_{k+1} \geq \rho_{k+1}(p)) \leq \\
&\leq C'_k(p) = F(\rho_{k+1}(p))\mathbb{E}(p_{k+1}|p_{k+1} < \rho_{k+1}(p)) + (1-F(\rho_{k+1}(p))\rho_{k+1}(p)
\end{aligned}$$

Note, that $C'_k(p)$ can be rewritten in the form:

$$C'_k(p) = \rho_{k+1}(p) + c - \int_{\underline{p}}^{\rho_{k+1}(p)} F(p)dp$$

Therefore, following our notation

$$C'^{+}_k(p) = \rho^{+}_{k+1}(p)(1 - F(\rho_{k+1}(p))) \leq C'^{+}_{k+1}(p)(1 - F(\rho_{k+1}(p)))$$

Then

$$C'^{+}_K(p) \leq \prod_{i=K}^{N-1} C'^{+}_{i+1}(p)(1 - F(\rho_{i+1}(p)))$$

As $1 - F(\rho_{i+1}(p)) < 1$ for any $p > \tilde{p}$ and $i > K$ (note, that $\rho_{i+1}(p) < \rho_{i+2}(p) \Rightarrow 1 - F(\rho_{i+1}(p)) > 1 - F(\rho_{i+2}(p)))$ we get

$$\lim_{N\to\infty} C'^{+}_K(p) = 0.$$

.

Now note that from proposition 3.5 it follows that $\rho_K(\tilde{p}) = \rho^{pr}$ and therefore $C_K(\tilde{p}) = \rho^{pr}$. Therefore, since $C'_K(p)$ is a continuous function we get that for any $p \geq \tilde{p}$,

$$\lim_{N\to\infty} C'_K(p) = \rho^{pr}.$$

Therefore

$$\lim_{N\to\infty} C_K(p) = \rho^{pr}.$$

□

Proposition 4.1. *There is a mixed strategy equilibrium where all firms charge prices below the first-round reservation price, which equals the reservation price under perfect recall ρ^{pr}.*

Proof. If the upper bound of the support $\overline{p} = \rho_1$, then $\max_p \rho_1(p) = \ldots = \max_p \rho_{N-1}(p) = \rho^{pr}$. Therefore, the equilibrium defined in Stahl (1989) is an equilibrium if none of the firms has a profitable deviation. The only (potentially profitable) way for firms to deviate is to charge prices above ρ_1. However, then this firm has a zero demand both from informed and uninformed consumers. Therefore, a profitable deviation does not exist, and the Stahl type of equilibrium is indeed an equilibrium. □

Lemma 4.3. *There is no equilibrium price distribution with* $r_1 < \overline{p} < r_{N-1}$.

Proof. It is easy to see that given the optimal search behavior all reservation prices are below or equal to the upper bound of the support of the distribution. Indeed, suppose $\overline{p} < r_{N-1}$. Recall, that

$$c + b = \int_{\underline{p}}^{r_{N-1}+b} F(p)dp$$

then

$$c = \int_{\underline{p}}^{r_{N-1}} F(p)dp$$

and therefore $r_{N-1} = \rho^{pr}$, which is not possible.

□

Lemma 4.4. *There is no equilibrium price distribution with* $\overline{p} \in [r_{N-1}, r_{N-1} + b]$.

Proof. First, consider profits at r_1 and at $\overline{p}$:

$$\pi(r_1) = \lambda(1 - F(r_1))^{N-1} r_1 + \frac{1-\lambda}{N} S r_1$$

and

$$\pi(\overline{p}) = Y\overline{p}$$

It is clear, that $S \geq 2 - F(r_1)$. If firm charges $\overline{p} > r_{N-1}$ it only sells something, if all other firms set prices at least above r_1 (otherwise all consumers stop on the first step). Therefore, $Y < (1 - F(r_1))^{N-1} < (1 - F(r_1))$. Then it should be that

$$\frac{1-\lambda}{N}(2 - F(r_1))r_1 < \frac{1-\lambda}{N}(1 - F(r_1))\overline{p} \leq \frac{1-\lambda}{N}(1 - F(r_1))(r_{N-1} + b)$$

and therefore $\frac{r_{N-1}+b}{r_1} > 2$ which contradicts Corollary 2.9.
Thus, the proposition is proved.

□

Lemma 4.5. *There is no equilibrium price distribution with* $\overline{p} > r_{N-1} + b$.

Proof. Let π_0 be the equilibrium profits. First, consider the profits of a firm that charges $\overline{p}$. As, by construction, $\overline{p}$ is in the support of the equilibrium price distribution, equilibrium profits are given by:

$$\pi_0 = \frac{1-\lambda}{N}(1 - F(\overline{p} - b))^{N-1}\overline{p} \tag{5.4}$$

As $\overline{p} > r_{N-1} + b$, a firm charging $\overline{p}$ does not get any informed consumers and only those uninformed consumers who have first visited all other firms, have observed these firms charge prices above r_{N-1} and then do not want to go back to these stores because of the cost of a second visit b. If a firm would charge $\overline{p} - b$ instead, its profits would be at least equal to

$$\left(\lambda(1 - F(\overline{p} - b))^{N-1} + \frac{1-\lambda}{N}(1 - F(\overline{p} - 2b))^{N-1}\right)(\overline{p} - b)$$

which is larger than or equal to

$$\left(\lambda(1 - F(\overline{p} - b))^{N-1} + \frac{1-\lambda}{N}(1 - F(\overline{p} - b))^{N-1}\right)(\overline{p} - b).$$

Whether or not $\overline{p}-b$ is in the support of the equilibrium price distribution, it should be the case that π_0 is larger than or equal to this expression, yielding

$$\overline{p} \leq \frac{1 - \lambda + \lambda N}{\lambda N} b \tag{5.5}$$

Therefore,

$$\pi_0 < \phi(\lambda, N) \equiv (1-\lambda)\frac{1 - \lambda + \lambda N}{\lambda N^2} b \tag{5.6}$$

this is the upper bound on the equilibrium profit. Next, we will construct a lower bound on the equilibrium profit. To this end, consider profits at r_1. It is easy to see that r_1 should be in the support of the equilibrium price distribution. Firstly, by definition of r_1 it cannot be the case that the whole price distribution lies above r_1. Secondly, if there is part (or whole) of probability distribution which lies below r_1, then there is proftable deviation from the largest price in this part of the support to r_1, since demand does not change between this to points. To simplify notation, let $F(r_1) = m$. We then have that

$$\pi_0 = \lambda(1-m)^{N-1} r_1 + \frac{1-\lambda}{N} S r_1,$$

where $S \geq 1$ is the total probability that a consumer buys from the firm, arising form all possible search paths of consumers. The firm charging r_1 gets at least $1/N$ consumers who randomly arrive at its store in the first search round and $\frac{N-1}{N}\frac{1}{N-1}(1-m)$ of consumers who first visit another store, observe a price strictly larger than r_1 and then randomly visits the store under consideration. thus, it follows that $S \geq 2-m$. Therefore, for any $p \leq r_0$ in the equilibrium support:

$$\pi_0 = \lambda(1-F(p))^{N-1}p + \frac{1-\lambda}{N}Sp,$$

which gives,

$$F(p) = 1 - \left(\frac{\pi_0}{p\lambda} - \frac{1-\lambda}{N\lambda}S\right)^{\frac{1}{N-1}}$$

and

$$\underline{p}(r_1) = \frac{N\lambda(1-m)^{N-1} + (1-\lambda)S}{N\lambda + (1-\lambda)S}r_1.$$

Now consider a family of probability distributions:

$$F(p;K) = 1 - \left(\frac{\pi_0}{p\lambda} - \frac{1-\lambda}{N\lambda}S\right)^{\frac{1}{K-1}}$$

Then for $M \geq K$ $F(p,M) \leq F(p,K)$ for every p. Moreover, if we define $r_1(K)$ as

$$\int_{\underline{p}(r_1(K))}^{r_1(K)+b} F(p;K)dp = c+b,$$

then we get that the solution of this equation $r_1(K)$ is an increasing function of K, because $\underline{p}(r_1(K))$ is linearly increasing in $r_1(K)$ with slope less than 1 and $F(p,K)$ is decreasing in K. Therefore, $r_1(2) \leq r_1(K)$ for any K. It is also clear that $r_1(K)$ is increasing in c, therefore, $r_1(2)|_{c=b} \leq r_1(2)|_{c>b}$. Let's denote $r^* = r_1(2)|_{c=b}$. It follows that r^* is implicitly defined by

$$\int_{\underline{p}(r^*)}^{r^*+b} F(p,2)dp = 2b$$

and therefore

$$\int_{\underline{p}(r^*)}^{r^*} F(p,2)dp \geq b$$

or

$$\int_{\underline{p}(r^*)}^{r^*} \left(1 - \frac{\pi_0}{p\lambda} + \frac{1-\lambda}{N\lambda}S\right) dp = \left(1 + \frac{1-\lambda}{N\lambda}S\right)(r^* - \underline{p}(r^*)) - \frac{\pi_0}{\lambda} \ln \frac{r^*}{\underline{p}(r^*)} \geq b.$$

As $r^* \leq r_1$ for any N, b, c and fixed S, m it follows that

$$\pi_0 \geq \lambda(1-m)^{N-1} r^* + \frac{1-\lambda}{N} S r^*. \tag{5.7}$$

By plugging in the expressions for $\underline{p}(r^*)$ and this lower bound on π_0 we get

$$\left(1 + \frac{1-\lambda}{N\lambda}S\right)(r^* - \underline{p}(r^*)) = \frac{\lambda N + (1-\lambda)S}{\lambda N} \frac{N\lambda(1-(1-m)^{N-1})}{\lambda N + (1-\lambda)S} r^* = (1-(1-m)^{N-1}) r^*$$

$$\frac{\pi_0}{\lambda} \ln \frac{r^*}{\underline{p}(r^*)} \geq \frac{r^*}{\lambda} \left(\lambda(1-m)^{N-1} + \frac{1-\lambda}{N}S\right) \ln \frac{N\lambda + (1-\lambda)S}{(1-m)^{N-1} N\lambda + (1-\lambda)S}$$

which gives a lower bound for r^*:

$$r^* \geq \frac{\lambda b}{\lambda(1-(1-m)^{N-1}) - \left(\lambda(1-m)^{N-1} + \frac{1-\lambda}{N}S\right) \ln \frac{N\lambda + (1-\lambda)S}{(1-m)^{N-1} N\lambda + (1-\lambda)S}}.$$

Therefore $\pi_0 \geq \psi_0(\lambda, m, N, S)$ where

$$\psi_0(\lambda, m, N, S) \equiv \frac{\lambda \left(\lambda(1-m)^{N-1} + \frac{1-\lambda}{N}S\right) b}{\lambda(1-(1-m)^{N-1}) - \left(\lambda(1-m)^{N-1} + \frac{1-\lambda}{N}S\right) \ln \frac{N\lambda + (1-\lambda)S}{(1-m)^{N-1} N\lambda + (1-\lambda)S}}.$$

This is the lower bound on equilibrium profits. It is straightforward to verify that $\frac{\partial}{\partial S}\psi_0(\lambda, m, N, S) > 0$ and because $S \geq 2 - m$ we get that

$$\pi_0 \geq \psi_0(\lambda, m, N, S) > \psi(\lambda, m, N) \equiv \psi_0(\lambda, m, N, 2-m).$$

Now, since $\pi_0 < \phi(\lambda, N)$ and $\pi_0 > \psi(\lambda, m, N)$ the equilibrium can only exist if the lower bound on profits is smaller than the upper bound, or $\xi(\lambda, m, N) \equiv \phi(\lambda, N) - \psi(\lambda, m, N) > 0$.

It is possible to verify that $\psi(\lambda, m, N)$ is decreasing function of m and that

$$\lim_{m \to 1} \frac{1}{(1-\lambda)b} \cdot \xi(\lambda, m, N) = \frac{1 - \lambda + \lambda N}{\lambda N^2} - \frac{\lambda}{N\lambda - (1-\lambda)\ln \frac{1-\lambda+N\lambda}{1-\lambda}}. \quad (5.8)$$

Therefore $\xi(\lambda, m, N) > 0$ only if the denominator of the second fraction in (5.8) is negative, which is equivalent to

$$\ln \frac{N\lambda + 1 - \lambda}{1-\lambda} > \frac{\lambda N}{1-\lambda}, \quad (5.9)$$

or, the denominator is positive, but the expression still holds, which is equivalent to

$$\ln \frac{N\lambda + 1 - \lambda}{1-\lambda} < \frac{\lambda N}{1 - \lambda + N\lambda}. \quad (5.10)$$

Let us start with (5.9). It is clear that at $\lambda = 0$ both the right hand side and the left hand side of (5.9) are equal to 0. However,

$$\frac{\partial}{\partial\lambda}\left(\ln \frac{N\lambda+1-\lambda}{1-\lambda} - \frac{\lambda N}{1-\lambda}\right) = -\frac{\lambda N^2}{(1-\lambda)^2(1-\lambda+N\lambda)} < 0$$

Thus, the left hand side of (5.9) increases slower than the right hand side, and thus (5.9) can never hold.

Now we proceed with (5.10). Again, at $\lambda = 0$ both the right hand side and the left hand side of (5.10) equal to 0. If we take the derivative of the difference again we get

$$\frac{\partial}{\partial\lambda}\left(\ln \frac{N\lambda+1-\lambda}{1-\lambda} - \frac{\lambda N}{1-\lambda+N\lambda}\right) = \frac{\lambda N^2}{(1-\lambda)(1-\lambda+N\lambda)^2} > 0.$$

Therefore, the left hand side of (5.10) increases faster than the right hand side, and so (5.10) cannot hold either. Therefore, there is no equilibrium with $\overline{p} > r_{N-1}$. $\square$

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