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Abstract

Scarf's economy has been a vehicle in understanding stability properties in exchange economies. The full set of market equilibria and Pareto optimal allocations for this economy has not been analysed. This paper aims to do that. Firstly, we examine the Pareto optima and we find three different classes. Only Class I exhausts the aggregate endowments of all the goods. Class II and III involve throwing away partially or totally one good in order to achieve Pareto efficiency. Secondly, we explore the price and endowment distribution combinations which sustain the different Pareto Optima as market equilibria. A Pareto optimum which involves throwing away the whole endowment of one of the goods is globally stable.

Keywords: Exchange economy; Complements; Stability.

JEL classification: D50; D61.

1 Introduction

In 1960 Scarf introduced an exchange economy with cyclic preferences to highlight the possibility of global instability of general equilibrium. Scarf showed that for special initial endowment distributions between individuals in an exchange economy with particular preferences exhibiting complementarity, there is a unique market equilibrium with equal prices of the goods that is globally unstable. His example has perfect complementarities in tastes between pairs of goods in a three-good and three-individual economy in which there are equal aggregate endowments of each good. Each good enters the preferences of two individuals, but no pair of individuals care about exactly the same two goods. This was referred to as an economy with cyclical preferences. His work had a strong impact on the development of general equilibrium theory, since it was the first clear example of global instability of the tatonnement process.

We would argue that Scarf's example is also interesting from an empirical and normative point of view. In terms of preferences, all individuals have perfect complementarity in those goods that they wish to consume. But the goods they desire overlap just partially. It is as if any two of the individuals have something in common but not everything. Several examples can be found in the household environment, or in the international trade scenario when countries specialise in consumption in different set of products that overlap.

Market equilibrium in this exchange economy has been previously investigated with particular initial endowment distributions. In Scarf's seminal paper, it is assumed that each individual has only the total endowment of one of the goods that he wants to consume. Later research contributions are still characterised by special endowment restrictions. For example, Hirota (1981) analyses the equilibrium assuming that the sum of the initial endowment across goods is equal for each individual and coincides with the aggregate endowment. Anderson *et al* (2004) develop an experimental double auction and allowed prices to adjust under a nontatonnement rule, based on the same endowment restrictions as those imposed by Hirota. A consequence of focussing attention just on these endowments is that there is a unique market equilibrium which gives equal utility to all individuals and the prices of all goods are equal.

The full set of Pareto optima and market equilibria with a general individual endowment distribution for this economy are still unknown. In particular, an interesting issue concerns the characterisation of the efficient allocations in this general environment and how it is possible to decentralise them as market equilibria. Scarf's preferences are not strictly convex and also not strictly nonsatiated in all goods. Therefore,

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the second fundamental theorem of welfare economics cannot be easily invoked. Finding the prices and initial endowment distributions that support the different types of efficient allocations is still an open research task.

In this paper, firstly we fully characterise the efficient allocations allowing for general endowment distributions. We show that only three classes of Pareto optima arise. There is a single Pareto optimum in which the efficient allocation exhausts the endowment of all the goods. In all the other cases, the endowment of one good is totally or partially wasted. We then define the set of prices and initial endowment distributions which will decentralise each type of Pareto optimum. Finally we conduct stability analysis.

Specifically, the unique efficient allocation which exhausts the supply of all goods and gives equal "utility" to all individuals can be decentralised using many different combinations of prices and endowment sets. Firstly, this efficient allocation can be supported by unequal positive prices for all goods if the initial endowment distributions are different for each individual and satisfy a mild set of inequality restrictions. Particular special subclasses of endowment distributions within this group are of interest. In one special case, if and only if the endowment distribution of individuals 1 and 2 satisfy a single restriction, this allocation is decentralised by good x costing twice as much as good y. Secondly we can have equal prices for two of the goods if and only if just two of the individuals have a "similar aggregate" initial endowment distribution in a sense we make precise below. This case tends to exhibit local stability and it is certainly stable if just two individuals have exactly the same initial endowment of each good. Thirdly, it is possible to support the equal utility allocation with three equal prices if and only if the endowment distribution satisfies the restrictions introduced by Hirota; heuristically all three individuals have a "similar aggregate" initial endowment distribution. His restrictions imply that in equilibrium since prices of all goods are equal, individuals have equal wealth individuals and they can trade goods on a one-for-one basis. This allows individuals to specialise in consumption on the goods they want through trade. The stability properties of equilibrium with Hirota endowment distributions have been discovered by Scarf and Hirota. We give stability results for the more general endowment distributions.

We next show that the other efficient allocations in which there is one good which is not totally consumed or is completely wasted can be decentralised if the price of this good is zero, and in terms of the initial endowments there is a top dog citizen who is relatively wealthy in the endowment of the goods he likes. The particular efficient allocation in which the good is totally wasted emerges only if one individual is in such a favoured endowment position that he has the total endowments of the goods that he likes and no trade occurs in the market. These Pareto corner allocations in which there is one good totally wasted are globally stable.

The paper is organised as follows. After stating Scarf's preferences, we find the three classes of Pareto optima of this economy. In Section 3 we analyse the feasible types of market equilibria. We next define the set of prices and initial endowment distributions that can decentralise the different Pareto optima (Section 4). The stability analysis concludes this paper (Section 5).

2 Perfect Complements: the Scarf Economy

Scarf's economy has 3 individuals and 3 goods. Individual preferences are given by

$$\begin{aligned} u_1(x_1, y_1, z_1) &= \min\{y_1, z_1\} \\ u_2(x_2, y_2, z_2) &= \min\{x_2, z_2\} \\ u_3(x_3, y_3, z_3) &= \min\{x_3, y_3\} \end{aligned}$$

There is an interlocking set of perfect complementarities in preferences between the three goods. As an example think of a father, mother and daughter with the three consumption activities being attending football matches, using perfume, participating in the bridge club. The father may want football match spectating and bridge club participation in equal proportions but care not at all about perfume. The mother wants perfume and bridge club participation but has no use for football matches. The daughter hates bridge but likes football matches and perfume in equal proportions. Any two of the three people have one common and one distinct desire.

The aggregate endowments of the goods are $X = Y = Z = 1$, we call this a square economy since there are equal aggregate endowments of all goods. For convenience we normalise the scale of the economy at 1 unit of each good. Obviously, changing the size of the economy does not affect the nature of the results.

2.1 Pareto Optima

With the strong complementarities we would expect efficient allocations to involve specialisation in consumption on those goods which individuals wish to consume. For example allocating any of good 3 to individual 1 yields no Pareto improvement. Moreover we find that there can be two efficient allocations in which one allocation involves more consumption of one good by one individual (which is of no utility value to him). But both allocations are efficient.

The set of feasible allocations is given by

$$F = \{x, y, z | \Sigma x_h \leq 1, \Sigma x_h \leq 1, \Sigma x_h \leq 1, x \geq 0, y \geq 0, z \geq 0\}$$

where $= (x_1, x_2, x_3)$, etc. The set of efficient allocations are most easily shown in terms of the efficient utility distributions. Define

$$\begin{aligned}
P_1 &= \{x, y, z | (x, y, z) \varepsilon F, u_1(x_1, y_1, z_1) = 1 - a, u_2(x_2, y_2, z_2) = a, u_3(x_3, y_3, z_3) = a, 0 \leq a \leq 1/2\} \\
P_2 &= \{x, y, z | (x, y, z) \varepsilon F, u_1(x_1, y_1, z_1) = a, u_2(x_2, y_2, z_2) = 1 - a, u_3(x_3, y_3, z_3) = a, 0 \leq a \leq 1/2\} \\
P_3 &= \{x, y, z | (x, y, z) \varepsilon F, u_1(x_1, y_1, z_1) = a, u_2(x_2, y_2, z_2) = a, u_3(x_3, y_3, z_3) = 1 - a, 0 \leq a \leq 1/2\}
\end{aligned}$$

Thus P_h is a set of feasible utility distributions which favour individual 1 in the sense that as a varies, u_1 varies in the interval $(1/2, 1)$ while $u_2 = u_3$ vary in $(0, 1/2)$. In this situation we refer to the most favoured individual as the top dog. Similarly in P_2, P_3 a different individual is favoured. The set of efficient utility distributions is given by

$$P = P_1 \cup P_2 \cup P_3$$

The set of efficient allocations is characterised by three types of Pareto optima. Only the first type exhausts the aggregate feasibility constraint. The other cases imply throwing out totally or partially the endowment of one of the goods

(a) *Class I: total exhaustion.* There is a Pareto optimum in which the individuals get equal utility $u_1 = u_2 = u_3 = 1/2$

$$\begin{aligned}
y_1 &= z_1 = 1/2 = 1/2 \\
x_2 &= z_2 = 1/2 = 1/2 \\
x_3 &= y_3 = 1/2 = 1/2
\end{aligned}$$

and none of the goods is wasted.

(b) *Class II: the aggregate endowment of one good is partially wasted.* There is an infinite number of other efficient utility distributions which can be reached without consuming the total endowment of one of the goods. For example set $u_1 = u_2 = a, u_3 = 1 - a$. This is attained by consumptions

	x_h	y_h	z_h	u_h
$h = 1$	0	a	a	a
$h = 2$	a	0	a	a
$h = 3$	$1 - a$	$1 - a$	0	$1 - a$
Total	1	1	$2a$	

So long as $0 \leq a \leq 1/2$ these allocations are feasible and they cannot be bettered. There is a surplus of good z available but it cannot usefully be consumed by either individual 3 (he does not want it) nor by individuals $1, 2$ (since there is no matching remaining amount of their complementary good available).For example if $a = 1/4$ efficient utility distributions and consumptions are:

	x_h	y_h	z_h	u_h
$h = 1$	0	$1/4$	$1/4$	$1/4$
$h = 2$	$1/4$	0	$1/4$	$1/4$
$h = 3$	$3/4$	$3/4$	0	$3/4$
Total	1	1	$1/2$	

In such a case, 50% of one of the goods is wasted. Similarly there are two alternative Pareto optima in which only half of one good is not fully consumed but in which there is a different top dog individual**:**

u_1	u_2	u_3
1/4	3/4	1/4
3/4	1/4	1/4

(c) *Class III: the aggregate endowment of one good is totally wasted.* This class is characterised by three Pareto optima in which one individual gets the total endowment of two of the goods and the third good is just wasted

$$u_1 = 1 \text{ with } y_1 = z_1 = 1; u_2 = u_3 = 0.$$

Here 1 uses all of Y, Z which since these are essential goods for $2, 3$ means that $2, 3$ are restricted to the utility associated with zero consumption of the goods they care about.

(a)-(c) above define the only types of Pareto optima. In any Pareto optimum two of the goods must be fully allocated for consumption, at most one good may have no useful consumption purpose. If two of the goods were not fully allocated, we could raise the utility of the person who wants those two goods by giving them the lower amount of whatever is leftover so that worthwhile consumption increases.

In Figure 1 we represent the full set of Pareto optima. The apex shows the Pareto optimum in which all individuals get equal utility. The upper boundary of the pyramid shows the other two classes in which one good is totally or partially wasted.

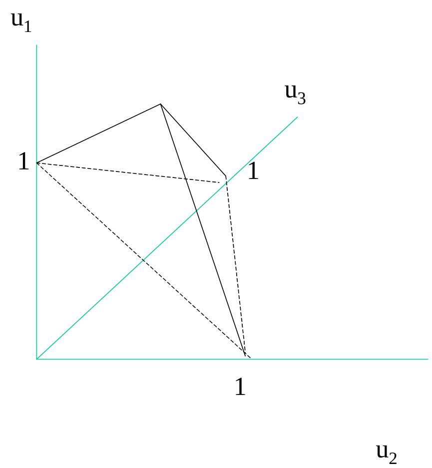

Fig 1 A graphic representation of
the different types of Pareto Optima

3 Market Equilibria

Initial endowments for h are given by X_h, Y_h, Z_h. Prices are p_x, p_y, p_z. Note also that homogeneity of degree zero in prices implies that we can impose a price normalisation. The two most common are either to set one price equal to unity (but this assumes that any equilibrium will have a positive price in that particular market i.e. the numeraire good is not in excess supply in equilibrium) or $\Sigma p_i = 1$[1]. Here we use the latter normalisation.

Initially suppose that all goods are owned by some individual so that, as the aggregate endowment of each good is unity,

$$\Sigma_h X_h = \Sigma_h Y_h = \Sigma_h Z_h = 1$$

[1] By contrast, Scarf used an unusual price normalisation: $\Sigma p_i^2 = 1$ which, combined with the non-negativity of prices, means that prices are restricted to the surface of a non-negative quartersphere.

Demands are given by

$$\begin{aligned} f_{x1} &= 0, f_{y1} = f_{z1} = \frac{p_x X_1 + p_y Y_1 + p_z Z_1}{p_y + p_z} \\ f_{y2} &= 0, f_{x2} = f_{z2} = \frac{p_x X_2 + p_y Y_2 + p_z Z_2}{p_x + p_z} \\ f_{z3} &= 0, f_{x3} = f_{y3} = \frac{p_x X_3 + p_y Y_3 + p_z Z_3}{p_x + p_y} \end{aligned}$$

These are continuous in prices for $p_x, p_y, p_z > 0$, satisfy the individual budget constraints with equality and are homogeneous of degree zero in p. Note that they are also continuous at a point at which just one price is zero and the other two prices are positive. However they are discontinuous at a point at which any two prices are zero since some of the demands are not defined at such a point.

Since the aggregate endowments of each good are equal to unity, the excess demands are:

$$\begin{aligned} E_x &= f_{x2} + f_{x3} - 1 \\ E_y &= f_{y1} + f_{y3} - 1 \\ E_z &= f_{z1} + f_{z2} - 1 \end{aligned}$$

As an algebraic identity we have Walras Law[2]. So the three excess demand equations are dependent.

An equilibrium, for a fixed initial endowment distribution between individuals is a price vector p such that there is no aggregate excess demand, and for any good i if there is excess supply at p of good i then $p_i = 0$. That is goods which in equilibrium are in excess supply are priced at zero. Formally for a given initial endowment distribution between individuals, an equilibrium is a set of prices p_i such that

$$E_i \leq 0, p_i \geq 0, p_i E_i = 0 \quad i = x, y, z$$

Note that an equilibrium of this economy can never have two prices zero, if for example $p_x = p_y = 0$ then individual 3 will have an infinite demand for goods x, y. Since excess demands are continuous (except where two prices are equal) and satisfy Walras Law, a competitive equilibrium exists (Arrow-Hahn (1971) for example). We do not know if it is unique or (under tatonnement) stable.

4 The Decentralisation of Pareto Optima

4.1 The Equal Utility Pareto Optimum

Here we have $u_h = 1/2$ and all goods are consumed. To represent this as a market equilibrium it must be the case that we can find an initial endowment distribution and prices such that all excess demands are zero and prices are all positive. This follows because in this Pareto optimum we know that

$$\Sigma x_{ih} - 1 = \Sigma y_{ih} - 1 = \Sigma z_{ih} - 1 = 0$$

or equivalently

$$E_x = E_y = E_z = 0$$

and hence the prices must be positive.

[2] Walras Law:

$$\begin{gathered} p_x E_x + p_y E_y + p_z E_z = \\ p_x(f_{x2} + f_{x3} - 1) + p_y(f_{y1} + f_{x3} - 1) + \\ p_z(f_{y1} + f_{x2} - 1) = \\ (p_y + p_z) f_{y1} + (p_x + p_z) f_{x2} + (p_x + p_y) f_{x3} - (p_x + p_y + p_z) = \\ p_x X_1 + p_y Y_1 + p_z Z_1 + p_x X_2 + p_y Y_2 + p_z Z_2 + p_x X_3 + p_y Y_3 + p_z Z_3 \\ -(p_x + p_y + p_z) = 0 \end{gathered}$$

From Walras law we can focus on just two excess demands:

$$\begin{aligned} E_x &= f_{x2} + f_{x3} - 1 \\ E_y &= f_{y1} + f_{x3} - 1 \end{aligned}$$

In fact since the market equilibrium demands must equal the Pareto optimal allocation, we must have $f_{x2} = f_{x3} = f_{y1} = 1/2$. The analysis of the equilibrium will be based on these equations.

Again these equations are not all independent and we can impose any price normalisation we like. We take the sum of the prices as equal to unity. We also select the two equations $f_{x2} \leq 1/2, f_{y1} \leq 1/2$ with which to work.

The market equilibrium allocation requires just two equations to be satisfied, whilst there are two normalised prices and six free initial endowment variables that can be selected. So there will be an infinity of ways of decentralising the equal utility efficient allocation.

Using the price normalisation $\Sigma p_i = 1$ and Walras Law, an equilibrium giving the allocation corresponding to the equal utility Pareto optimum requires prices and an endowment distribution such that:

$$\begin{aligned} f_{y1} &= \frac{p_x X_1 + p_y Y_1 + p_z Z_1}{(p_y + p_z)} = 1/2 \\ f_{x2} &= \frac{p_x X_2 + p_y Y_2 + p_z Z_2}{(p_x + p_z)} = 1/2 \end{aligned} \tag{1}$$

where $p_z = 1 - p_y - p_x$.

We can interpret these equations in terms of the net trades individuals make in equilibrium. We can rewrite the first equation as

$$p_x X_1 = p_y(1/2 - Y_1) + (1 - p_y - p_x)(1/2 - Z_1)$$

individual 1 sells x to buy the shortfall of y, z below his Pareto optimal consumption level of 1/2. Similarly for individual 2

$$p_y Y_2 = p_x(1/2 - X_2) + (1 - p_y - p_x)(1/2 - Z_2)$$

Individual 2 sells y and buys the difference between x and z below his Pareto optimal consumption level.

4.2 Supporting the equal utility Pareto Optimum with unequal prices

In the next sections we show the different combinations of prices and quantities that will support this Pareto optimum. We start presenting the more general case (equal utility and unequal prices), showing that any initial endowment distribution which meets some weak inequality conditions will satisfy a general Hirota-type condition, which we define below and from which we can infer the equilibrium prices. One particularly interesting example of this case arises when the endowment distribution is proportionally distributed among all the individuals, in which case the equilibrium prices are unequal between all goods but are in a fixed proportional relationship. Then we show that the equal utility Pareto optimum is also decentralised by just two goods having equal prices if the endowment of just two individuals is similarly allocated. Further this Pareto optimum can be attained as a market equilibrium with equal prices for the goods if and only if the initial endowments are similarly distributed among individuals in the sense of Hirota.

4.2.1 The General Case

Suppose we take an arbitrary initial endowment distribution $X_h, Y_h, Z_h, = 1, 2$ with $X_3 = 1 - X_1 - X_2, Y_3 = 1 - Y_1 - Y_2, Z_3 = 1 - Z_1 - Z$ such that:

$$\begin{aligned} \alpha X_1 + \beta Y_1 + (1 - \alpha - \beta) Z_1 &= (1 - \alpha)/2 \\ \alpha X_2 + \beta Y_2 + (1 - \alpha - \beta) Z_2 &= (1 - \beta)/2 \end{aligned} \tag{2}$$

where α, β are exogenously fixed numbers in the unit interval.

From the aggregate availability of each good, (2) implies that a similar relation holds for individual 3.

Lemma 1 *If (2) holds for some numbers α, β then*

$$\alpha X_3 + \beta Y_3 + (1 - \alpha - \beta) Z_3 = (\alpha + \beta)/2$$

This endowment distribution leads to a market equilibrium with prices fixed at the exogenous values given by α, β. This is of interest since it relates easily to the Hirota conditions and directly generalises those. Indeed, (2) have the form of a generalised Hirota condition.

Proposition 2 *The equal utility Pareto optimum is supported by unequal prices if*

$$\begin{aligned} \alpha X_1 + \beta Y_1 + \gamma Z_1 &= \kappa \\ \alpha X_2 + \beta Y_2 + \gamma Z_2 &= \kappa_1 \end{aligned} \tag{3}$$

where α, β, κ are constants with $\kappa = 1/2(1-\alpha)$, $\kappa_1 = 1/2(1-\beta)$ and $\gamma = 1 - \alpha - \beta > 0$, and $0 < \alpha \neq \beta \neq \gamma < 1$.

Any pair of endowment distributions with the same value of α, β will generate the same price equilibrium with the same equal equilibrium utility distribution. For example the endowment distribution

$$Z_1 = 0.3; Y_1 = 0.7; Z_2 = 0.35; Y_2 = 0.1; X_1 = .043; X_2 = .591$$

yields $p_x = \alpha = 0.28, p_y = \beta = 0.33$. But the endowment distribution

$$Z_1 = 0.3; Y_1 = 0.4; Z_2 = 0.35; Y_2 = 0.1; X_1 = .396, X_2 = .59$$

yields exactly the same equilibrium prices and utility distribution.

In fact there is an alternative way of thinking of (2). Suppose that $\alpha(...), \beta(...)$ are functions of the any given individual endowment distribution such that identically (2) are satisfied[3]. The functions are just a convenient algebraic summary of combinations of terms in initial endowments. They are particularly useful because if we use (2) to eliminate X_1, X_2 from the excess demand functions then we can write the equilibrium prices very simply in terms of the functions α, β :

$$\begin{aligned} p_x &= \alpha \\ p_y &= \beta \end{aligned}$$

These prices solve the equilibrium equations (1). This also tells us that variations in the initial endowment distribution that keep α, β constant lead to the same equilibrium prices. This means that any and all endowment distributions at which the computed values of α, β satisfy $0 < \alpha + \beta < 1, \alpha > 0, \beta > 0$ will decentralise the equal utility Paret optimum. Any endowment distribution satisfying inequalities (4) below will do the job.

The requisite inequalities have the form

$$\begin{aligned} 2(Y_1X_2 - Y_2X_1) - (Y_1 + X_2) + 1/2 &< 0 \\ 2(Y_1Z_2 - Y_2Z_1) + (Y_2 - Y_1 + Z_2) + 1/2 &> 0 \\ 2(Z_1X_2 - Z_2X_1) + (X_1 - X_2 - Z_1) + 1/2 &> 0 \\ 2(Y_1(Z_2 - X_2) - Y_2(Z_1 - X_1) + Z_1X_2 - Z_2X_1) + (Y_2 + X_1 - Z_2 - Z_1) + 1/2 &> 0 \end{aligned} \tag{4}$$

[3] Indeed values

$$\begin{aligned} \alpha &= \frac{2(Y_1Z_2 - Y_2Z_1) + Y_2 - Y_1 - Z_2 + 1/2}{2(Y_1Z_2 - Y_1X_2 + Z_1X_2 - Y_2Z_1 + Y_2X_1 - Z_2X_1) + (Y_2 + X_1 - Z_2 - Z_1) + 1/2} \\ \beta &= \frac{2(Z_1X_2 - Z_2X_1) + (X_1 - Z_1 - X_2) + 1/2}{2(Y_1Z_2 - Y_1X_2 + Z_1X_2 - Y_2Z_1 + Y_2X_1 - Z_2X_1) + (Y_2 + X_1 - Z_2 - Z_1) + 1/2} \end{aligned}$$

will satisfy these conditions so long as the denominator does not vanish.

There are an infinity of endowment distributions which satisfy these inequalities. For example if individuals have a zero endowment of the good which they do not wish to consume ($X_1 = 0, Y_2 = 0, Z_3 = 0$ which implies that $Z_2 = 1 - Z_1$) the inequalities assume the form

$$\begin{aligned} 2Y_1X_2 - (Y_1 + X_2) + 1/2 &< 0 \\ 1 - Z_1(1 + 2Y_1) + Y_1 + 1/2 &> 0 \\ 2Z_1X_2 - (X_2 + Z_1) + 1/2 &> 0 \\ 2(Y_1(1 - Z_1 - X_2) + Z_1X_2) - (1 - X_1) + 1/2 &> 0 \end{aligned} \tag{5}$$

These will be satisfied for similarly low or similarly high values of Y_1, X_2, Z_1.

Proposition 3 *The equal utility Pareto optimum can be reached as a market equilibrium iff the inequalities (5) hold.*

4.2.2 Proportional Prices

Any endowment distribution satisfying the inequalities will lead to prices that generate a market equilibrium with equal utility of 1/2. Within this, special classes of endowment distributions will result in special relations between the equilibrium prices. For example one interesting case might be that in which in equilibrium good x is say twice as expensive as good y. Suppose that in (2) $\beta = 2\alpha$ and $\gamma = (1 - 3\alpha)$ so that

$$\begin{aligned} \alpha X_1 + 2\alpha Y_1 + (1 - 3\alpha)Z_1 &= 1/2(1 - \alpha) \\ \alpha X_2 + 2\alpha Y_2 + (1 - 3\alpha)Z_2 &= 1/2(1 - 2\alpha) \end{aligned} \tag{6}$$

Then any endowment distribution satisfying these two restrictions can decentralise the equal utility Pareto optimum with prices

$$p_x = \alpha, p_y = 2\alpha, p_z = 1 - 3\alpha \tag{7}$$

Eliminating α between the two equations in (6) gives the required endowment restriction

$$\frac{(1 - 2Z_1)}{(Y_1 + 2X_1 + 1 - 3Z_1)} = \frac{(1 - 2Z_2)}{(Y_2 + 2X_2 + 1/2 - 3Z_2)}$$

For example if we choose a=1/6, then $p_x = 1/6$; $p_y = 1/3$ and $p_z = 1/2$ and $X_1 = 2.5 - 2Y_1 - 3Z_1$, and $X_2 = 2 - 2Y_2 - 3Z_2$. Then to be sure that $X_1, X_2 \geq 0$ we need $2Y_1 + 3Z_1 \leq 2.5$ and $2Y_2 + 3Z_2 \leq 2$. This implies $Y_1 \leq 1.25 - 1.5Z_1$, and $Y_2 \leq 1 - 1.5Z_2$. This gives us a four parameter family of endowment distributions. But we need $Y_1, Y_2 \geq 0$ which means we must have $Z_1 \leq 1.25/1.5, Z_2 \leq 1/1.5$. Suppose we fix $Z_1 = Z_2 = 1/3$ for example. Then we can choose any values $Y_1 \leq .75$ and $Y_2 \leq .5$ which make $X_1 = 1.5 - 2Y_1 \leq 1$ (for example $Y_1 \geq .25$) and $X_2 = 1 - 2Y_2 \leq 1$ (for example $Y_2 \geq 0$). The upshot is an infinite number of endowments satisfying $.25 \leq Y_1 \leq .75, 0 \leq Y_2 \leq .5$ all of which will give equilibrium with prices satisfying (6) and equal utility for the individuals. Obviously this could be replicated for any $0 < a < 1/3$ and for other values of Z_1, Z_2

Proposition 4 *The equal utility Pareto optimum is supported by proportional prices iff*

$$\begin{aligned} \alpha X_1 + 2\alpha Y_1 + (1 - 3\alpha)Z_1 &= \kappa \\ \alpha X_2 + 2\alpha Y_2 + (1 - 3\alpha)Z_2 &= \kappa_1 \end{aligned} \tag{8}$$

where $\kappa = 1/2(1 - \alpha)$, $\kappa_1 = 1/2(1 - 2\alpha)$ *and* $\gamma = 1 - 3\alpha > 0$, *and* $0 < \alpha < 1/3$.

4.2.3 Two prices equal

Another case of some interest is that in which in equilibrium goods x and y are equally expensive. Then two individuals trading these goods between themselves would be in a similar position of relative advantage. We can define the class of endowment distributions which will lead to such equilibrium prices with equal utilities by setting $\alpha = \beta$: the endowment distribution must satisfy

$$\begin{aligned} \alpha(X_1 + Y_1) + (1 - 2\alpha)Z_1 &= 1/2(1 - \alpha) \\ \alpha(X_2 + Y_2) + (1 - 2\alpha)Z_2 &= 1/2(1 - \alpha) \end{aligned} \tag{9}$$

Solving the first equation of (9) for $\alpha = p_x = p_y = p$

$$p = \alpha = \frac{(1/2 - Z_1)}{(X_1 + Y_1 - 2Z_1 + 1/2)} \tag{10}$$

Using this price p in the second equation of (9), the endowment distribution must satisfy

$$\frac{(1/2 - Z_1)}{(X_1 + Y_1 - 2Z_1 + 1/2)} = \frac{(1/2 - Z_2)}{(X_2 + Y_2 - 2Z_2 + 1/2)} \tag{11}$$

It is also true that if we start from (11), and define α from (10) we get exactly conditions (9).

The condition (11) is certainly satisfied if individuals 1 and 2 have exactly the same amount of each good ($X_1 = X_2$; $Y_1 = Y_2$; $Z_1 = Z_2$). More generally it is also satisfied when the sum of the endowments of two goods of individual 1 and 2 is equal and also they have identical endowments of the third good in the sense that[4]

$$X_1 + Y_1 + Z_1 = X_2 + Y_2 + Z_2 = k \text{ and } Z_1 = Z_2 \tag{12}$$

When (9) holds, we can also write[5]

$$\begin{aligned} p &= \frac{Z_3}{(2Z_3 + 1 - X_3 - Y_3)} = \frac{Z_3}{k + 3Z_3} \\ p_z &= 1 - 2p = \frac{k + Z_3}{k + 3Z_3} \end{aligned} \tag{13}$$

where $k = 1 - X_3 - Y_3 - Z_3$. This gives a whole family of values of the initial endowment distributions all of which generate positive prices $p_y = p_x \neq p_z$ and which generate the market equilibrium quantities

$$y_1 = z_1 = 1/2; x_2 = z_2 = 1/2; x_3 = y_3 = 1/2$$

corresponding to the Pareto optimum with equal utilities for all individuals. We can plot these alternative equilibrium prices as a function of $k = 1 - X_3 - Y_3 - Z_3$ and Z_3.

---

[4] If $Z_1 = Z_2$ (11) can be written

$$\frac{X_2}{2} + \frac{Y_2}{2} - Z_2(X_2 + Y_2) = \frac{X_1}{2} + \frac{Y_1}{2} - Z_1(X_1 + Y_1)$$

[5] This is easiest to see if $X_1 + Y_1 + Z_1 = X_2 + Y_2 + Z_2 = k$ and $Z_1 = Z_2$. From $Z_1 = Z_2$ we have $Z_3 = 1 - 2Z_1$.so

$$p = \frac{Z_3/2}{(X_1 + Y_1 - 2Z_1 + 1/2)}$$

Also from

$$\begin{aligned} X_1 + Y_1 &= X_2 + Y_2 \\ X_3 + Y_3 &= 2(1 - X_1 - Y_1) \\ X_1 + Y_1 &= 1 - (X_3 + Y_3)/2 \end{aligned}$$

so

$$\begin{aligned} p &= \frac{Z_3/2}{1 - (X_3 + Y_3)/2 - (1 - Z_3) + 1/2} \\ &= \frac{Z_3/2}{1/2 - (X_3 + Y_3)/2 + Z_3} \end{aligned}$$

It is also true under just (11): use the aggregate endowment constraints to eliminate the endowments of individual 1, then use (11) to eliminate X_2 in terms of the endowments of individual 3 and Z_2. Replace these expressions in (10) to get (13).

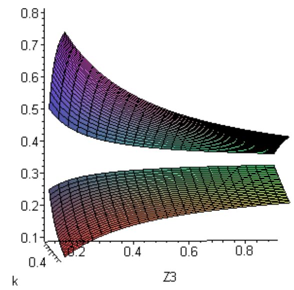

Fig 2 Equilibrium Prices p, p_z as a function of k and Z_3

For example if $k = 0.2, Z_3 = 0.25$ then $p_x = .263, p_z = .474$. And so on for other combinations.

Proposition 5 *The equal utility Pareto optimum is supported by $p_x = p_y \neq p_z$ for all goods iff*

$$\begin{aligned} \alpha(X_1 + Y_1) + \gamma Z_1 &= \kappa \\ \alpha(X_2 + Y_2) + \gamma Z_2 &= \kappa \end{aligned} \tag{14}$$

with $\kappa = 1/2(1-\alpha)$ and $\gamma = (1-2a)$.

This is an extended version of the Hirota condition applied just to two individuals.

4.2.4 Supporting the equal utility Pareto Optimum with equal positive prices

Scarf and Hirota use particular distributions of initial endowments and show that with these $p_x = p_y = p_z = 1/3$ gives an equilibrium with equal utilities of 1/2. Hirota's class is defined by

$$X_h + Y_h + Z_h = 1 \quad \text{for all } h$$

In fact Scarf's endowments, $Y_1 = Z_2 = X_3 = 1$ and all others zero, are a special case of Hirota's class of endowments. Hirota's endowments have the strong interpretation that when they hold all individuals have equal wealth if prices are equal for all goods. We can derive this class of endowments from (2) by setting $\alpha = \beta = 1/3$

$$\begin{aligned} X_1 + Y_1 + Z_1 &= 1 \\ X_2 + Y_2 + Z_2 &= 1 \end{aligned}$$

We can then ask what is the full set of initial endowment distributions which make $p_x = p_y = p_z = 1/3$ a market equilibrium and which leads to the equal utility Pareto optimum.

Proposition 6 *The equal utility Pareto optimum is supported by equal positive prices for all goods iff the Hirota conditions hold*

Thus we have shown that a market equilibrium with equal prices for all three goods supports the equal utility Pareto optimum iff the initial endowments satisfy the Hirota endowment conditions. The equilibrium with equal quantity and prices is obtained when the total endowment, $X + Y + Z = 3$, is equally distributed among individuals. On average, every individual has the same power in contracting since every individual has got a third of the total initial endowment. Each individual endowment is $X_h + Y_h + Z_h = 1$ for each $h = 1, ..., 3$). Setting the prices equal allows one unit of any good to exchange for one unit of any other good so eg individual 1 can sell say 1/3 of a unit of X (which he does not want) and buy 1/6 of a unit of each of Y, Z which he does want.

5 Decentralisation of Corner Pareto Optima

In the corner Pareto optima by definition one individual has higher utility than the other two who have equal utility. We refer to the individual who is better off in the Pareto optimum as the top dog. Markets can ensure that this utility distribution is reached by finding prices and a suitable endowment to ensure that the top dog has higher equilibrium wealth than the other individuals. Below we characterise the prices and the exact endowment distribution restriction for each type of corner Pareto optimum. One aspect of the endowment restriction is that the top dog must have a sufficiently large endowment of at least one of the goods which he wishes to consume.

5.1 Unequal Utility Pareto Optima (Class II and III)

Pareto optimum Class II have the form $u_h = 1 - a, u_k = a = u_l$ for $h, k, l = 1, 2, 3$. If we analyse one case say $u_1 = 1 - a, u_2 = a = u_3$ the others will follow.

In this case we know that $y_1 = z_1 = 1 - a; x_2 = z_2 = a; x_3 = y_3 = a$ with other consumptions being zero. Generally we think of 1 as being the favoured individual so that $a < 1/2$ in which case less than the total endowment of x is consumed at the Pareto optimum. In market terms prices must be such that x is in excess supply so for this to be reached as a market equilibrium it must be that $p_x = 0$. We know that the total endowment of goods y, z is consumed so in market equilibrium they must exhibit zero excess demand. So we can take $p_y, p_z > 0$ and for example normalise the prices so that $p_x + p_y + p_z = p_y + p_z = 1$. It follows that $p_z = 1 - p_y$. This leaves p_y as the only price to be determined, and we have two equations that must hold: the demand for goods consumed by individual 1 must equal $1 - a$ and those by individuals 2 must equal a. It follows by Walras law that also the demand for individual 3 must equal a.

Individual 1 wants to sell good x and buy a units of good z and y. But good x does not have any value ($p_x = 0$). The net trade condition equivalent to his demands is

$$0 = p_y((1 - a) - Y_1) + (1 - p_y)((1 - a) - Z_1) \tag{15}$$

Note that if $Y_1 = Z_1$ the individual will not trade at all but will just consume his initial endowment. Also 1 must own initially at least $1 - a$ of one of the goods that he wishes to consume since otherwise (15) cannot hold.

Turning to the other individuals, individual 2 wants to sell good y and buy good x and z:

$$p_y Y_2 = (1 - p_y)(a - Z_2)$$

whilst individual 3 wants to sell good z and buy good x and y):

$$(1 - p_y)Z_3 = p_y(a - Y_3)$$

We can take an arbitrary price $p_y = k$; giving $p_z = (1 - k)$ for any $0 < k < 1$ and look for endowment distributions which will lead to the Pareto efficient consumptions yielding $u_1 = 1 - a, u_2 = u_3 = a$ for an arbitrary $a \leq 1/2$. The endowment distribution must satisfy the net trade conditions

$$\begin{aligned} 0 &= k((1 - a) - Y_1) + (1 - k)((1 - a) - Z_1) \\ kY_2 &= (1 - k)(a - Z_2) \\ (1 - k)Z_3 &= k(a - Y_3) \end{aligned} \tag{16}$$

Any endowment distribution satisfying

$$\frac{Y_2}{a - Z_2} = \frac{(a - Y_3)}{Z_3} \Rightarrow \frac{(a - Y_3)}{Z_3} = \frac{a - Y_2 - Y_3}{Z_2 + Z_3 - a} = \frac{(1 - a) - Y_1}{(Z_1 - (1 - a))} \tag{17}$$

will satisfy these conditions by setting $(1 - k)/k$ to be equal to $(a - Y_3)/Z_3$. In fact there is a two parameter set of distributions of Y_2, Y_3 defined by arbitrary values of $Y_2 \leq a, Y_3 \leq a$ (which then determine all other endowments in Y, Z as in the above equation) all of which will lead to the prices $p_y = k; p_z = (1 - k), p_x = 0$ which support the Pareto optimum with the fixed value a. Since good x has a zero value at the equilibrium, its endowment distribution is irrelevant.

Proposition 7 *Pareto optima with utility distributions* $u_1 = 1-a, u_2 = u_3 = a$ *can be supported with prices* $p_x = 0, 0 < p_y = k \neq p_z < 1$ *and any endowment distribution satisfying*

$$\begin{aligned} kY_1 + (1-k)Z_1 &= 1-a \\ kY_2 + (1-k)Z_2 &= (1-k)a \\ kY_3 + (1-k)Z_3 &= ka \end{aligned}$$

with $k \neq (1-k)$.

In this case we have $p_y = k; p_z = 1-k$. Note that the conditions in the proposition are like the Hirota linear endowment restrictions but involving only two goods y, z. Of course this is because the distribution of x is immaterial since its price is zero.

To support the corner Pareto optima what matters is the endowment/wealth distribution. In the examples above individual 1 is like a top dog with most of the endowment. The wealth of individuals 2 and 3 valued at the equilibrium prices is lower than the wealth of individual 1 valued at the equilibrium prices, since $a \leq 1/2$ and $0 < k < 1$. Note that although the bottom dogs 2 and 3 have equal equilibrium utility, in general their wealths valued at equilibrium prices differ. If $k = 1/2$ they have equal wealth but if $p_y = k < 1/2$ (and so $p_z > 1/2$) individual 3 who wants to consume x, y has lower wealth than individual 2 who wants to consume x, z.

A special case of (17) is of particular interest. Suppose that we select the endowments so that

$$Y_2 + Z_2 = Y_3 + Z_3 = a \Rightarrow Y_1 + Z_1 = 2(1-a)$$

then the ratio of endowments in (17) are equal to unity. But since these common ratios are equal to the price ratio between goods y and z this then means that we can take $p_y = k = 1/2 = p_z$ so that the two goods that have positive value in equilibrium are equally valued. Then this initial endowment distribution gives equal wealth to individuals 2 and 3 when valued at the equilibrium prices. But each of these individuals is worse off in wealth than the top dog individual 1.

Proposition 8 *Let* $a \leq 1/2$. *Pareto optima with utility distributions* $u_1 = 1-a, u_2 = u_3 = a$ *can be supported with prices* $p_x = 0, p_y = p_z = 1/2$ *and any endowment distribution satisfying*

$$Y_1 + Z_1 = 2(1-a), Y_2 + Z_2 = Y_3 + Z_3 = a$$

In equilibrium individuals 2 *and* 3 *are equally wealthy but both are clearly less wealthy than individual* 1.

For example setting $a = 1/4; k = .5; Z2 = 1/6$ gives $Y_2 = .167, Y_3 = .183, Z_3 = .033$. The point is that for any a there is an infinity of positive but unequal prices $p_y \neq p_z$ with associated individual initial endowment distributions which lead to the market equilibrium with $u_1 = a, u_2 = u_3 = 1-a$.

Another special case is that in which $a = 0$ in which case the Pareto optimum displays extreme inequity: $u_1 = 1, u_2 = u_3 = 0$. This can be supported as a market equilibrium only if individual 1 has got all the endowment of the two goods that he likes, whatever the distribution of the good that he does not want among the other individuals. Setting $a = 0$ in (16) gives $kY_2 + (1-k)Z_2 = 0$; $kY_3 + (1-k)Z_3 = 0$ which implies that $Y_2 = Z_2 = Y_3 = 0$ (since $0 < k < 1$ and $Y_h \geq 0, Z_h \geq 0$) and so from the aggregate endowment availability $Y_1 = Z_1 = 1$. For example if $X_1 = 0.3, Y_1 = 1, Z_1 = 1, X_2 = 0,5, X_3 = 0,2; Y_2 = Z_2 = Y_3 = Z_3 = 0$ individual 1 has the total endowment of the two goods y, z that he wishes to consume. Then $u_1 = 1$, $u_2 = u_3 = 0$ and no trade occurs. Each individual just keeps his original endowment although for both individuals 2 and 3 they have no use for one of the goods with which they may be endowed. The prices p_y, p_z are then irrelevant and can be set at arbitrary levels within the price normalisation.

Proposition 9 *The Pareto optimum with utility distribution* $u_1 = 1, u_2 = u_3 = 0$ *can be supported with prices* $p_x = 0$,*and* $p_y > 0$, $p_z > 0$ *is and only if*

$$Y_1 = Z_1 = 1, Y_2 = Z_2 = Y_3 = Z_3 = 0$$

In this case the top dog interpretation is extremely inequitable: individuals 2 and 3 have zero wealth valued at any prices whilst individual 1 has wealth 1 again valued at any prices.

6 Stability Of Market Equilibria Under Tatonnement

The original interest in the economy put forward by Scarf was in the stability properties of the equal price equilibrium under a tatonnement rule for price adjustment. Scarf showed that with his particular initial endowment distribution the unique market equilibrium $p_i = 1/3$ corresponding to the Pareto optimum with equal utilities was globally unstable under the price normalisation that he used. Hirota showed that other initial endowment distributions also lead to the equal price equilibrium and that for these other distributions (within the Hirota class but excluding the Scarf case) there was a tendency to local and global stability.

In general for local stability the excess demand functions must be downward sloping in their own price and the feedback cross effects between markets should be "small" in comparison with the own price effects. Generally we can write the Jacobian of the excess demand functions for x, y as

$$J = \begin{bmatrix} \partial E_x/\partial p_x & \partial E_x/\partial p_y \\ \partial E_y/\partial p_x & \partial E_y/\partial p_y \end{bmatrix} \tag{18}$$

so that $det(J) =$

$$\partial E_x/\partial p_x \partial E_y/\partial p_y - \partial E_x/\partial p_y \partial E_y/\partial p_x$$

and $trace(J) = \partial E_x/\partial p_x + \partial E_y/\partial p_y$. If the excess demand functions are downard sloping in their own price then the trace is always negative. The condition for the determinant to be positive (and hence for two eigenvalues whose real parts are negative and local stability) is that

$$\partial E_x/\partial p_x \partial E_y/\partial p_y > \partial E_x/\partial p_y \partial E_y/\partial p_x$$

We can think of this as saying that the aggregate of cross market effects (the LHS) should be small in absolute value relative to the own price effects.

6.1 Stability of Equilibrium with Equal Utility

6.1.1 The General Case

To explore local stability with an arbitrary initial endowment distribution satisfying (2) we can linearise the excess demand functions around the equilibrium prices $p_x = \alpha, p_y = \beta$ and compute the trace and the determinant (see Appendix A).

In section 4.2.1 we have shown that in general there are alternative initial endowment distributions which generate equilibrium with the same unequal prices. Some of these endowment distributions which yield the equal utility equilibrium outcome are locally stable, others are locally unstable even though the equilibrium prices are the same (see the Appendix). For example, if $Z_1 = 0.3$; $Y_1 = 0.7$; $Z_2 = 0.35$; $Y_2 = 0.1$ and we take $X_1 = .043$, $X_2 = .591$ then $p_x = a = 0.28, p_y = b = 0.33$. With these values the determinant of the Jacobian in a neighbourhood of equilibrium is $-.095$ and the trace is $-.943$. The equilibrium is locally unstable since the determinant is negative-locally it is a saddlepoint. On the other hand if we take $Z_1 = 0.3; Y_1 = 0.4$; $Z_2 = 0.35$; $Y_2 = 0.1$; and $X_1 = .396$, $X_2 = .591$ then again $p_x = \alpha = 0.28, p_y = \beta = 0.33$ but now the determinant has a value of .331 while the trace is equal to -1.448. In this case the equilibrium is locally stable. In the two examples we have given what matters a lot is the relative ownership by individual 1 of goods y and z. This is interesting since they are both goods he wishes to consume.

6.1.2 Proportional Prices

The ambiguity of local stability extends to the case in which good x is twice as expensive in equilibrium as good y ($\alpha = 2\beta$). For example if we set $\alpha = 1/6$ the trace is equal to

$$-8 - Y_2 + 4Z_2 + 3.2Y_1 + 8.8Z_1$$

and the determinant is equal to

$$5.4 + 21.6Z_2Y_1 + 10.8(Y_2 - Z_2 - Y_1) - 21.6Z_1Y_2$$

Choosing $Z_1 = Z_2 = 1/3$ the trace becomes

$$-3.73 - Y_2 + 3.2Y_1$$

and the determinant

$$1.8(1 - 2(Y_1 - Y_2))$$

The determinant is positive if $Y_1 - Y_2 < 1/2$ but otherwise is negative. The trace is negative if $-3.73 - Y_2 + 3.2Y_1 < 0$ which is certainly satisfied if $Y_1 - Y_2 < 1/2$.

Thus there are examples of the case in which proportional prices can lead to a Pareto optimal outcome in which goods are fully consumed and all individuals receive utility 1/2 which are locally stable. But in other cases we have local instability. However there are relatively simple conditions like the relative ownership of y by individuals 1 and 2 which yield simple stability conditions.

6.1.3 Stability of Equilibrium with two equal prices

If the endowment restriction satisfies (9) or (11) local stability is uncertain. The appendix shows that we will tend to have local stability if individual 3 does not have a heavy concentration of the goods he wishes to consume (x, y) in his initial endowment.

For example if we fix $Z_3 = .25, Y_3 = .5, X_3 = .3$ we have $p_x = p_y = .357$ and $p_z = 0.286$ whilst the trace t and determinant d are respectively

$$\begin{aligned} t &= -2.769 + 3.111X_2 + 1.444Z_2 \\ d &= 1.219 - 1.084Z_2 - 1.355X_2 \end{aligned}$$

and we can plot these as functions of X_2, Z_2.

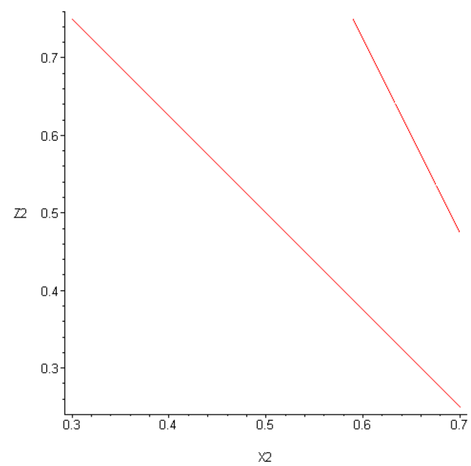

Fig 3. Trace and determinant as functions of X_2, Z_2

Here on the line further from the origin the trace is zero whilst on the line closer to the origin the determinant is zero. Below both lines we have the trace negative and the determinant positive so the equilibrium is locally stable; between the lines we have the determinant and trace both negative whilst above both lines the determinant is negative whilst the trace is positive. Thus in either of these second cases we have local instability.

However adding similarity to the potential endowment wealth of individuals 1, 2 gives more definite results.

Individuals $1, 2$ **Have Equal Endowments of** $z : Z_1 = Z_2$ We know that if $Z_1 = Z_2$ (9) reduces to a restricted form of the Hirota conditions applied just to individuals 1 and 2: the two individuals have an equal summed amount of good x and y ($X_1 + Y_1 = X_2 + Y_2 = k$) and $Z_1 = Z_2$. In this case the equilibrium is always locally stable if $Y_2 > Y_1$ which implies that $X_2 < X_1$. This means that the endowment of the good that they want to sell is greater than the endowment of the good that they want to buy. However the equilibrium is also locally stable if $Y_1 < 1/2$.

Two Individuals Have Identical Endowments If individuals 1 and 2 have exactly the same amount of each good, the equilibrium is always locally stable(see appendix).

6.1.4 Stability of Equilibrium with Equal Prices and Equal Utilities

When the endowment distribution satisfies the Hirota restrictions we ned a single endowment restriction to ensure that the equal price, equal utility equilibrium is locally stable. The endowments must satisfy

$$X_2 > \max\{1 - 2Y_1 - Y_2 - 2X_1, \frac{(2Y_1 - 1/2)}{2X_1 + 1. - 2Y_1}\}$$

which combines a condition which Hirota initially found (under a different price normalisation for the determinant to be positive) with a condition for the trace to be negative.

6.2 Stability of Equilibrium with Unequal Utilities

If the endowment distributions are such that a top dog individual is in the system, the equilibrium is always stable for initial conditions which start with one price at its equilibrium value of zero (see Appendix A 2.1). In fact if one individual is in such a favoured position that he has the total endowment of the two goods he wishes to consume and no-trade occurs in the market then the equilibrium is globally stable. There is an infinite set of prices that can support this no-trade equilibrium, in which the other two bottom class citizens are permanently rationed to their initial endowments without achieving any utility. In the other corner Pareto optima, still there is a top dog citizen but the difference in terms of wealth with regard to the other citizens is not so remarkable as in the no trade case. The equilibrium that emerges in this case is stable for any initial conditions starting with a zero price for the good which is in excess supply (see Appendix A 2.1). Starting with arbitrary initial conditions we show that for equilibria with some trade which have $p_x = 0$ and individual 1 as the top dog, the sign of the determinant and the trace are ambiguous (see Appendix A 2.2).

7 Conclusion

Why does this matter? Generally in the literature we view Scarf's example as an important demonstration that under tatonnement there can be no presumption that competitive equilibrium is even locally stable. Hirota's work then shows that actually Scarf's result, whilst important, is quite fragile since it depends not just on a certain type of complementarity but also on a kind of potential equal power of consumers in their initial endowments. If you keep the complementarities but change the endowments then global instability no longer prevails. These contributions are important in themselves in adding to our understanding of what we expect to be true theoretically.

However, this economy is also interesting for empirical and normative issues. It analyse situations in which decision units have a few shared peculiar features.

The obvious Pareto optimum is equal utility for the individuals. But there are many others in which there is a single top dog and the other two individuals are second class citizens.

We show that markets are actually quite flexible in this setting. For many initial endowments we define three different types of configurations of prices that will implement the equal utility optimum. Similarly there are prices that will implement the top dog/second class citizen Pareto optima for many initial endowments. Moreover when there is a top dog in the sense we define the market equilibrium supporting the unequal Pareto optimum has strong stability properties. However the source of a top dog is essentially in the initial endowment distribution, and with Scarf preferences prices cannot overcome the basic inequality in the endowment distribution.

The key result is that if one individual is strictly better off than others then one of the goods at least partially is not fully consumed in aggregate and the market equilibrium entails a zero price for that good. For this to be possible we cannot have the case in which one individual has an initial endowment consisting only of the good with a zero price unless we are looking at the Pareto optimum in which two of the individuals receive zero utility. Thus if the usual unpleasant case of equilibrium occurs in which one individual is rationed out of markets by prices, then at least he has some company.

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A Appendix

Proof of Lemma 1

Suppose that

$$\begin{aligned}\alpha X_1 + \beta Y_1 + (1-\alpha-\beta)Z_1 &= (1-\alpha)/2 \\ \alpha X_2 + \beta Y_2 + (1-\alpha-\beta)Z_2 &= (1-\beta)/2\end{aligned}$$

Summing these

$$\alpha(X_1+X_2) + \beta(Y_1+Y_2) + (1-\alpha-\beta)(Z_1+Z_2) = 1-\alpha/2-\beta/2$$

But there is an aggregate endowment of unity of each good so this implies

$$\alpha(1-X_3) + \beta(1-Y_3) + (1-\alpha-\beta)(1-Z_3) = 1-\alpha/2-\beta/2$$

and rearranging this we derive

$$\alpha X_3 + \beta Y_3 + (1-\alpha-\beta)Z_3 = \alpha/2 + \beta/2$$

■

Proof of Proposition 2 (Case with different prices)

a) Suppose that the price are unequal and such that: $p_x = \alpha$, $p_y = \beta$ and $(1-\alpha-\beta) = p_z$, with $0 < 1-\alpha-\beta > 1$, and $0 < \alpha \neq \beta \neq \gamma < 1$. The equations (1) become:

$$\begin{aligned}f_{y1} &= \frac{\alpha X_1 + \beta Y_1 + (1-\alpha-\beta)Z_1}{(1-\alpha)} = 1/2 \\ f_{x2} &= \frac{\alpha X_2 + \beta Y_2 + (1-\alpha-\beta)Z_2}{(1-\beta)} = 1/2\end{aligned}$$

which imply:

$$\begin{aligned}\alpha X_1 + \beta Y_1 + \gamma Z_1 &= \kappa \\ \alpha X_2 + \beta Y_2 + \gamma Z_2 &= \kappa_1\end{aligned}$$

where $\kappa = 1/2(1-\alpha)$, $\kappa_1 = 1/2(1-\beta)$ and $\gamma = 1-\alpha-\beta > 0$.

(b) Conversely suppose the conditions (2) hold. Then we have to show that this implies that $p_x = \alpha; p_y = \beta$. Again multiplying through (1) we get the linear system

$$\begin{aligned}p_x X_1 + p_y Y_1 + (1-p_x-p_y)Z_1 &= (1-p_x)/2 \\ p_x X_2 + p_y Y_2 + (1-p_x-p_y)Z_2 &= (1-p_y)/2\end{aligned}$$

Solving these linear equations we get

$$\begin{aligned} p_y &= -\frac{(-\frac{1}{2}X_2 + Z_1X_2 + \frac{1}{2}X_1 - Z_2X_1 - \frac{1}{2}Z_1 + \frac{1}{4})}{(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2)} \\ p_x &= \frac{(Y_2 - Z_2 + \frac{1}{4} - Z_2Z_1 - \frac{1}{2}Y_1 + Y_1Z_2)}{(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2)} \end{aligned} \tag{19}$$

This solution requires that the determinant condition

$$(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2) \neq 0$$

should hold.

Imposing the general Hirota conditions (2)

$$X_1 = (\frac{-\beta Y_1 - \gamma Z_1 + \kappa}{\alpha})$$

$$X_2 = (\frac{-\beta Y_2 - \gamma Z_2 + \kappa_1}{\alpha})$$

and substituting in (19) gives $p_x = \alpha; p_y = \beta$. ■

Proof of Proposition 4 (Case with proportional prices)

Suppose that the two prices are equal and such that: $p_x = \alpha$, $p_y = 2\alpha$and $(1-3\alpha) = p_z$, with $0 < \alpha < 1/3$. The equations (1) become:

$$\begin{aligned} f_{y1} &= \frac{\alpha(X_1 + 2Y_1) + (1-3\alpha)Z_1}{(1-\alpha)} = 1/2 \\ f_{x2} &= \frac{\alpha(X_2 + 2Y_2) + (1-3\alpha)Z_2}{(1-2\alpha)} = 1/2 \end{aligned}$$

which imply:

$$\begin{aligned} \alpha(X_1 + 2Y_1) + \gamma Z_1 &= \kappa \\ \alpha(X_2 + 2Y_2) + \gamma Z_2 &= \kappa_1 \end{aligned}$$

where $\kappa = 1/2(1-\alpha)$, $\kappa_1 = 1/2(1-2\alpha)$ and $\gamma = 1 - 3\alpha > 0$.

(b) Conversely suppose the endowment conditions (6) hold. Then we have to show that this implies that $p_x = \alpha; p_y = 2\alpha$. Again multiplying through (1) we get the linear system

$$\begin{aligned} p_xX_1 + p_yY_1 + (1 - p_x - p_y)Z_1 &= (1 - p_x)/2 \\ p_xX_2 + p_yY_2 + (1 - p_x - p_y)Z_2 &= (1 - p_y)/2 \end{aligned}$$

Solving these linear equations we get

$$\begin{aligned} p_y &= -\frac{(-\frac{1}{2}X_2 + Z_1X_2 + \frac{1}{2}X_1 - Z_2X_1 - \frac{1}{2}Z_1 + \frac{1}{4})}{(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2)} \\ p_x &= \frac{(Y_2 - Z_2 + \frac{1}{4} - Z_2Z_1 - \frac{1}{2}Y_1 + Y_1Z_2)}{(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2)} \end{aligned}$$

This solution requires that the determinant condition

$$(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2) \neq 0$$

should hold.

Imposing the conditions (8)

$$X_1 = (\frac{-2\alpha Y_1 - (1-3\alpha)Z_1 + \kappa}{\alpha})$$

$$X_2 = (\frac{-2\alpha Y_2 - (1-3\alpha)Z_2 + \kappa_1}{\alpha})$$

and substituting in (19) gives $p_x = \alpha$, $p_y = 2\alpha$ and p_z =$(1-3\alpha)$■

Proof of Proposition 5 (Case with two equal prices)

a) Suppose that the two prices are equal and such that: $p_x = p_y = \alpha$, and $(1-2\alpha) = p_z$, with $0 < \alpha < 1/2$. The equations (1) become:

$$\begin{aligned} f_{y1} &= \frac{\alpha(X_1+Y_1) + (1-2\alpha)Z_1}{(1-\alpha)} = 1/2 \\ f_{x2} &= \frac{\alpha(X_2+Y_2) + (1-2\alpha)Z_2}{(1-\alpha)} = 1/2 \end{aligned}$$

which imply:

$$\begin{aligned} \alpha(X_1+Y_1) + \gamma Z_1 &= \kappa \\ \alpha(X_2+Y_2) + \gamma Z_2 &= \kappa_1 \end{aligned} \tag{20}$$

where $\kappa = 1/2(1-\alpha)$, $\kappa_1 = 1/2(1-\alpha)$ and $\gamma = 1-2\alpha > 0$.

(b) Conversely suppose that (9) hold. Then we have to show that this implies that $p_x = \alpha; p_y = \alpha$. Again multiplying through (1) we get the linear system

$$\begin{aligned} p_x X_1 + p_y Y_1 + (1-p_x-p_y)Z_1 &= (1-p_x)/2 \\ p_x X_2 + p_y Y_2 + (1-p_x-p_y)Z_2 &= (1-p_y)/2 \end{aligned}$$

Solving these linear equations we get

$$\begin{aligned} p_y &= -\frac{(-\frac{1}{2}X_2 + Z_1X_2 + \frac{1}{2}X_1 - Z_2X_1 - \frac{1}{2}Z_1 + \frac{1}{4})}{(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2)} \\ p_x &= \frac{(Y_2 - Z_2 + \frac{1}{4} - Z_2Z_1 - \frac{1}{2}Y_1 + Y_1Z_2)}{(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2)} \end{aligned}$$

This solution requires that the determinant condition

$$(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2) \neq 0$$

should hold.

Imposing the conditions (9)

$$X_1 = (\frac{-\alpha Y_1 - (1-2\alpha)Z_1 + \kappa}{\alpha})$$

$$X_2 = (\frac{-aY_2 - (1-2\alpha)Z_2 + \kappa_1}{\alpha}),$$

and substituting in (19) gives $p_x = p_y = \alpha$ and p_z =$(1-2\alpha)$■

Proof of Proposition 6 (Case with equal prices)

(a) When we have equal prices the equations (1) become

$$\begin{aligned} f_{y1} &= \frac{X_1+Y_1+Z_1}{2} = 1/2 \\ f_{x2} &= \frac{X_2+Y_2+Z_2}{2} = 1/2 \end{aligned}$$

which imply

$$
\begin{aligned}
X_1 + Y_1 + Z_1 &= 1 \\
X_2 + Y_2 + Z_2 &= 1
\end{aligned}
$$

then using the Lemma we also have $X_3 + Y_3 + Z_3 = 1$ and so equilibrium with prices all equal imply that the Hirota conditions hold.

(b) Conversely suppose the Hirota conditions hold. Then we have to show that this implies that $p_x = p_y = 1/3$. Again multiplying through (1) we get the linear system

$$
\begin{aligned}
p_x X_1 + p_y Y_1 + (1 - p_x - p_y) Z_1 &= (1 - p_x)/2 \\
p_x X_2 + p_y Y_2 + (1 - p_x - p_y) Z_2 &= (1 - p_y)/2
\end{aligned}
$$

Solving these linear equations we get

$$
\begin{aligned}
p_y &= -\frac{(-\frac{1}{2}X_2 + Z_1X_2 + \frac{1}{2}X_1 - Z_2X_1 - \frac{1}{2}Z_1 + \frac{1}{4})}{(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2)} \\
p_x &= \frac{(Y_2 - Z_2 + \frac{1}{4} - Z_2Z_1 - \frac{1}{2}Y_1 + Y_1Z_2)}{(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2)}
\end{aligned}
$$

This solution requires that the determinant condition

$$
(Y_2X_1 - Y_2Z_1 + \frac{1}{2}Y_2 - Z_2X_1 - \frac{1}{2}Z_2 + \frac{1}{2}X_1 - \frac{1}{2}Z_1 + \frac{1}{4} + Y_1Z_2 - Y_1X_2 + Z_1X_2) \neq 0
$$

should hold.

Imposing the Hirota conditions

$$
X_1 = 1 - Y_1 - Z_1; Y_2 := 1 - X_2 - Z_2
$$

gives $p_x = 1/3; p_y = 1/3$. ■

A.1 Stability of Equilibrium with Unequal Prices and Equal Utilities

The determinant of (18) when $p_x = \alpha$ and $p_y = \beta$ is equal to

$$
d = \frac{((1 - 2Z_2)(1 - 2Y_1) + 2Y_2(1 - 2Z_1))(1 - \alpha - \beta)}{2\alpha(1 - \alpha)(\alpha + \beta)(1 - \beta)} \tag{21}
$$

whose sign is given by that of $(1 - 2Z_2)(1 - 2Y_1) + 2Y_2(1 - 2Z_1)$.

The trace is equal to

$$
\begin{aligned}
t \quad = \quad & -\frac{(2\beta\alpha^2 + \beta^2 - 2\alpha^2 - \beta - \beta\alpha + \alpha)}{a(1 - a)(a + \beta)(1 - \beta)} Y_1 \\
& -\frac{(\beta\alpha + 2\beta^2 - \alpha^2 - 2\beta^2\alpha - \beta + \alpha)}{a(1 - a)(a + \beta)(1 - \beta)} Y_2 \\
& -\frac{(2\beta^2\alpha - 2\beta^2 - 1 + \alpha + 3\beta - 3\beta\alpha)}{a(1 - a)(a + \beta)(1 - \beta)} Z_2 \\
& -\frac{(-2\beta\alpha^2 - \beta^2 + 2\alpha^2 - 1 + 2\beta)}{a(1 - a)(a + b)(1 - b)} Z_1 \\
& -\frac{(1 - \alpha + 2\beta\alpha + \beta^2 - 2\beta - \beta^2\alpha)}{\alpha(1 - \alpha)(\alpha + \beta)(1 - \beta)}
\end{aligned} \tag{22}
$$

which we can write as

$$
\begin{aligned}
t = \quad & \frac{Y_1(\beta - \alpha + 2\alpha^2)}{(\alpha + \beta)(1 - \alpha)\alpha} + \frac{Y_2(\beta - \alpha - 2\beta^2)}{(1 - \beta)(\alpha + \beta)\alpha} \\
& -\frac{(2\beta - 1)Z_2}{(\alpha + \beta)\alpha} + \frac{Z_1(1 - \beta - 2\alpha^2)}{(\alpha + \beta)(1 - \alpha)\alpha} - \frac{(1 - \beta)}{(\alpha + \beta)\alpha}
\end{aligned}
$$

This is of ambiguous sign.

A.1.1 Proportional Prices

Substituting $\beta = 2\alpha$ in (21) and (22) we get:

$$d = \frac{(1 + 4Y_1 Z_2 - 2Y_1 - 4Y_2 Z_1 - 2Z_2 + 2Y_2)(1 - 3\alpha)}{6\alpha^2(1-\alpha)(1-2\alpha)}$$

and the trace:

$$\begin{aligned} t = & \frac{Y_1(\alpha + 2\alpha^2)}{(1-\alpha)3\alpha^2} + \frac{Y_2(\alpha - 8\alpha^2)}{(1-2\alpha)3\alpha^2} \\ & - \frac{(4\alpha - 1)Z_2}{3\alpha^2} + \frac{Z_1(1 - 2\alpha - 2a^2)}{(1-\alpha)3\alpha} - \frac{(1-2\alpha)}{3\alpha^2} \end{aligned}$$

Still the sign is ambiguous

A.1.2 Stability with two equal prices and equal utilities

1) General Case $\beta = a$. (21) and (22) become respectively:

$$d = \frac{[(1 - 2Z_2)(1 - 2Y_1) + 2Y_2(1 - 2Z_1)](1 - 2\alpha)}{4\alpha^2(\alpha - 1)^2}$$

$$t = \frac{(Y_1 - Y_2)}{(1-\alpha)} - \frac{Z_2(-1 + 2\alpha)}{2\alpha^2} + \frac{Z_1(1 - 2\alpha)(\alpha + 1)}{2\alpha^2(1 - a)} - \frac{(1-\alpha)}{2\alpha^2}$$

2) *Equal endowments of each good for individuals* 1, 2 ($X_1 = X_2$, $Y_1 = Y_2$, $Z_2 = Z_1$). This case is always stable.

Note that with equal endowments for the first two individuals, each endowment is at most equal to 1/2, i.e. $X_1 \leq 1/2$, $Y_1 \leq 1/2$, $Z_1 \leq 1/2$. The determinant is:

$$d = \frac{(2Z_2 - 1)(2\alpha - 1)}{4\alpha^2(\alpha - 1)^2}$$

which is always positive since $(2\alpha - 1) < 0$ and $(2Z_2 - 1) < 0$.

The trace becomes:

$$\begin{aligned} t &= \frac{4Z_2\alpha - 2\alpha + \alpha^2 - 2Z_2 + 1}{2\alpha^2(\alpha - 1)} \\ &= \frac{\alpha^2 + (1 - 2Z_2)(1 - 2\alpha)}{2\alpha^2(\alpha - 1)} \end{aligned}$$

that is always negative since $(\alpha - 1) < 0, 1 - 2Z_2 > 0$ and $1 - 2\alpha > 0$.

3) *Equal endowments of* Z *for individuals* 1, 2 $Z_1 = Z_2$:

$$d = \frac{-(1 - 2Z_2)(1 - 2(Y_1 - Y_2))(-1 + 2\alpha)}{4\alpha^2(-1 + \alpha)^2}$$

Since $(2\alpha - 1) < 1$ and $1 - 2Z_2 > 0$, the determinant is positive if $(1 - 2(Y_1 - Y_2) > 0$. This holds if $Y_2 > Y_1$ or if $Y_1 < 1/2$.

$$t = \frac{(Y_1 - Y_2)}{(1-\alpha)} + \frac{(2\alpha - 1)Z_2}{\alpha^2(\alpha - 1)} - \frac{(1-\alpha)}{2\alpha^2}$$

The trace is certainly negative when the conditions that make the determinant positive hold.

A.1.3 Stability with equal prices and equal utilities

In this case, $Z_1 = 1 - Y_1 - X_1; Z_2 = 1 - Y_2 - X_2$ and $a = b = 1/3$. The determinant is:

$$d = \frac{27}{14}(-\frac{1}{4} + \frac{1}{2}Y_1 + \frac{1}{2}X_2 - X_2Y_1 + Y_2X_1)$$

which is positive if:

$$X_1Y_2 > (\frac{1}{2} - X_2)(\frac{1}{2} - Y_1)$$

The trace is:

$$t = \frac{3}{2}(-Y_1 - 2Y_2 + 1 - X_2 - 2X_1)$$

The trace is negative if

$$-Y_1 - 2Y_2 + 1 - X_2 - 2X_1 < 0$$

We can combine (**??**) and (**??**) to derive an endowment restriction which is necessary and sufficient for local stability of this case: we require

$$X_2 > \max\{1 - 2Y_1 - Y_2 - 2X_1, \frac{(2Y_1 - 1/2)}{2X_1 + 1. - 2Y_1}\}$$

if this fails we can for example have the determinant and trace both positive. For example take $Y_1 = Y_2 = 1$ and $X_1 = .05, X_2 = .45$. Then (**??**) is .0025 and (**??**) is 0.15.

A.2 Stability with unequal utilities

A.2.1 Stability for any initial conditions starting with $p_x = 0$

In the more general case with equilibrium prices $p_y = k, p_z = 1 - k$, the endowment becomes

$$\begin{aligned} Y_2 &= ((1-k)/k)(a - Z_2); Z_3 = (k/(1-k))(a - Y_3); \\ Y_1 &= 1/k[1 - a - (1-k)Z_1] \\ Z_1 &= 1 - Z_2 - Z_3 \end{aligned}$$

$$E_y = \frac{(p_y(1-k)(a - Z_2))/k + (1 - p_y)Z_2)}{(1 - p_y)} + \frac{p_yY_3 + \frac{(1-p_y)(a-Y_3)k}{1-k}}{p_y} - 1$$

Computing its derivative and evaluating at $p_y = k$

$$\frac{\partial E_y}{\partial p_y} = \frac{-2a + Y_3 + Z_2}{k} < 0 \tag{23}$$

The equilibrium is always stable since $(Y_3 - a) < 0$ and $(Z_2 - a) < 0$.

As an example we know that a Pareto optimum with unequal utilities can be supported as an equilibrium with two equal prices $p_y = p_z = 1/2$ when the endowment distribution is:

$$Y_1 = 2 - 2a - Z1, Y_2 = a - Z_2; Y_3 = a - Z_3$$

With this endowment distribution, the excess demand function for y has the form

$$E_y = p_y(2 - 2a - Z_1) + (1 - p_y)Z1 + \frac{p_y(a - Z_3) + (1 - p_y)Z_3}{p_y} - 1$$

Then

$$\frac{\partial E_y}{\partial p_y} = 2 - 2a - 2Z_1 - \frac{Z_3}{p_y^2}$$

We should evaluate it at the equilibrium: $p_y = 1/2$

$$\frac{\partial E_y}{\partial p_y} = 2 - 2a - 2Z_1 - 4Z_3 < 0 \tag{24}$$

Note that $2(1 - Z_1 - Z_3) = 2Z_2$. Thus (24) becomes: $2(-a + Z_2 - Z_3) < 0$ since $Z_2 - a < 0$ (Y_2 cannot be negative).

A.2.2 Stability for arbitrary initial conditions

The determinant of (18) when $p_x = k$, $p_y = 1 - k$ and $Y_1 = (-(1-k)Z_1 + (1-a))/k$; $Y_2 = (-(1-k)Z_2 + (1-k)a)/k$ is

$$d = \frac{a[2(X_1 + X_2)(1-k) + k - 2(1 + Z_2)] + 2a^2 - 2(1-k)[(Z_2X_1 - Z_1X_2 + X_2)] + (1 - Z_1 + Z_2)])}{k^2(1-k)}$$

The trace is equal to

$$t = \frac{k^2(2a - 1 + 2X_2 - 2Z_2 + X_1) + k(2(1-a) - Z_1 - X_2 - X_1)) - 1 + Z_1 + Z_2}{k^2(1-k)}$$

The sign of the trace and of the determinant are ambiguous.