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On the Impact of Independence of Irrelevant Alternatives*

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Abstract

On several classes of n-person NTU games that have at least one Shapley NTU value, Aumann characterized this solution by six axioms: Non-emptiness, efficiency, unanimity, scale covariance, conditional additivity, and independence of irrelevant alternatives (IIA). Each of the first five axioms is logically independent of the remaining axioms, and the logical independence of IIA is an open problem. We show that for $n = 2$ the first five axioms already characterize the Shapley NTU value, provided that the class of games is not further restricted. Moreover, we present an example of a solution that satisfies the first 5 axioms and violates IIA for 2-person NTU games (N, V) with uniformly p-smooth $V(N)$.

Keywords: NTU game, Shapley NTU value, positive smoothness

Journal of Economic Literature Classification: C71

1 Introduction

Several versions of the axiom of *independence of irrelevant alternatives* (IIA) have been employed and discussed in the literature in various fields of social sciences. In the context of NTU games, IIA (see Axiom 2 in Section 2 for a formal definition) requires that, quoting Aumann (1985), "a value y of a game W remains a value when one removes outcomes other than y ('irrelevant alternatives') from the set $W(N)$ of all feasible outcomes, without changing $W(S)$ for coalitions other than the all player coalition." IIA is a natural generalization of one of the four properties – *weak Pareto efficiency*, *equal treatment of equals*, and *scale covariance* are the three others – in Nash's (1950) definition of the "Nash" solution for bargaining problems. The NTU value introduced by Shapley (1969), called "Shapley" NTU value, generalizes, on the one hand, the TU Shapley (1953) value and, on the other hand, the Nash solution for bargaining problems. According to Aumann, the Shapley NTU value is characterized by IIA and five further axioms whose TU versions characterize the TU Shapley value. Thus, the open question whether IIA is really needed when NTU games are considered, is of particular interest. For the case of 2-person games, we present a complete answer to the foregoing question.

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The paper is organized as follows. In Section 2 the basic notation is provided and those definitions and results due to Aumann (1985) that are relevant for our presentation are recalled, including his characterizations of the Shapley NTU value by 6 axioms, i.e., Theorem A and Theorem B.

Section 3 formulates our main results: In the 2-person case, IIA may or may not be logically independent of the remaining axioms employed in Theorems A and B, depending on the considered set of NTU games. Sections 4 and 5 are devoted to the proofs of the two main results and in Section 6 we discuss and show the logical independence of the remaining 5 axioms.

2 Some Notation and Preliminaries

Let N be a finite nonempty set. We denote by $\mathbb{R}^N$ the set of all real functions on N. So $\mathbb{R}^N$ is the $|N|$-dimensional Euclidean space. (Here and in the sequel, if D is a finite set, then $|D|$ denotes the cardinality of D.) If $x, y \in \mathbb{R}^N$, then we write $x \geqslant y$ if $x_i \geqslant y_i$ for all $i \in N$. Moreover, we write $x > y$ if $x \geqslant y$ and $x \neq y$ and we write $x \gg y$ if $x_i > y_i$ for all $i \in N$. We denote $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \mid x \geqslant 0\}$ and $\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N \mid x \gg 0\}$. A *coalition* (in N) is a nonempty subset of N and 2^N denotes the set of all subsets of N. For every $S \in 2^N$ and any $x, \lambda \in \mathbb{R}^N$, the *indicator function* on S is denoted by $\chi^S \in \mathbb{R}^N$, i.e.,

$$\chi^S_j = \begin{cases} 1, \text{ if } j \in S, \\ 0, \text{ if } j \in N \setminus S, \end{cases}$$

the scalar product $\sum_{i \in N} \lambda_i x_i$ is denoted by $\lambda \cdot x$, $\lambda * x = (\lambda_i x_i)_{i \in N}$, λ_S is the restriction of λ to S, and 0^S denotes the zero of $\mathbb{R}^S$, i.e., $0^S = 0\lambda_S$. For $A, B \subseteq \mathbb{R}^N, t \in \mathbb{R}$, we write $A + B = \{a + b \mid a \in A, b \in B\}$, $tA = \{ta \mid a \in A\}$, $\lambda * A = \{\lambda * a \mid a \in A\}$, and the boundary of A, $cl(A) \cap cl(\mathbb{R}^N \setminus A)$, is denoted by ∂A, where "*cl*" means "closure". If A is convex and closed, then we say that A is *smooth* if it has a unique supporting hyperplane at each $z \in \partial A$. We call A *comprehensive* if $A = A - \mathbb{R}^N_+$.

A *TU game* on N is a mapping $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. An *NTU game* on N is a mapping V that assigns to each coalition S in N a nonempty comprehensive closed proper subset of $\mathbb{R}^S$ such that

(1) $V(N)$ is convex and smooth;

(2) $V(N)$ is non-leveled, i.e., if $x, y \in V(N)$ and $x > y$, then $y \notin \partial V(N)$;

(3) for each $S \in 2^N \setminus \{\emptyset, N\}$ there exits $x^S \in \mathbb{R}^N$ such that $V(S) \times \{0^{N \setminus S}\} \subseteq V(N) + \{x^S\}$.

Moreover, we use the convention that $V(\emptyset) = \emptyset$. Let γ_N and Γ_N denote the set of all TU games and NTU games on N, respectively. For any $v \in \gamma_N$ the *associated* NTU game $V_v \in \Gamma_N$ is defined by $V_v(S) = \{y \in \mathbb{R}^S \mid y \cdot \chi^S_S \leqslant v(S)\}$ for all coalitions S in N. Denote $\Gamma^{TU}_N = \{V_v \mid v \in \gamma_N\}$. For $T \in 2^N \setminus \{\emptyset\}$, the *unanimity* game on T, $u_T \in \gamma_N$, is defined by $u_T(S) = 1$ for all S such that $T \subseteq S \subseteq N$ and $u_T(S) = 0$ for all $S \subseteq N$ with $T \setminus S \neq \emptyset$. The NTU unanimity game U_T is the NTU game associated with u_T. The set γ_N with coalition-wise operations is the real vector space of dimension $2^{|N|}-1$ and the set of TU unanimity games forms a basis of γ_N. Moreover, Γ_N is closed under positive scalar multiplication, but, if $U, V \in \Gamma_N$, then $U + V$ may not be a member of Γ_N. However, for any $\lambda \in \mathbb{R}^N_{++}$ and $V \in \Gamma_N$,

$\lambda * V \in \Gamma_N$ (for any coalition S, $(\lambda * V)(S) = \lambda_S * V(S)$). One further notation is useful for the sequel. For any $V \in \Gamma_N$ let $d(V) \in \mathbb{R}^N$ be defined by

$$d_i(V) = \max V\{i\} \ \forall i \in N. \tag{2.1}$$

Let $V \in \Gamma_N$. By (1) and comprehensiveness of $V(N)$, for any $x \in \partial V(N)$, there exists a unique $\lambda^{V,x} \in \mathbb{R}^N_+$ such that

$$\chi^N \cdot \lambda^{V,x} = 1 \text{ and } V(N) \subseteq \{y \in \mathbb{R}^N \mid \lambda^{V,x} \cdot y \leqslant \lambda^{V,x} \cdot x\}. \tag{2.2}$$

Moreover, by (2), $\lambda^{V,x} \gg 0^N$ and, by (3), for any $S \in 2^N$,

$$v_x^V(S) = \sup\{\lambda_S^{V,x} \cdot y \mid y \in V(S)\} \in \mathbb{R}, \tag{2.3}$$

with the convention that $v_x^V(\emptyset) = 0$, so that $v_x^V \in \gamma_N$. Using this notation note that

$$\begin{aligned} &\text{if} \quad U, V, W = U + V \in \Gamma_N, x \in U(N), y \in V(N), \text{ and } z = x + y \in \partial W(N), \\ &\text{then} \quad x \in \partial U(N), y \in \partial V(N), \lambda^{U,x} = \lambda^{V,y} = \lambda^{W,z}, \text{ and } v_x^U + v_y^V = v_z^W. \end{aligned} \tag{2.4}$$

Now, the *Shapley NTU value* (the *NTU value* for short) of V introduced by Shapley (1969), denoted by $\Phi(V)$, is defined by

$$\Phi(V) = \{x \in \partial V(N) \mid \lambda^{V,x} * x = \phi(v_x^V)\},$$

where, for any $v \in \gamma_N$, the Shapley value (see Shapley (1953)) of v, denoted by $\phi(v) \in \mathbb{R}^N$ is defined by

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!}(v(S \cup \{i\}) - v(S)) \ \forall i \in N. \tag{2.5}$$

Let $\Gamma \subseteq \Gamma_N$. A *solution* on Γ is a mapping σ that assigns to each $V \in \Gamma$ a subset $\sigma(V)$ of $V(N)$. The following properties of a solution σ on $\Gamma \subseteq \Gamma_N$ are employed.

- Axiom 1 (Non-Emptiness, NE): $\sigma(V) \neq \emptyset$ for all $V \in \Gamma$.
- Axiom 2 (Efficiency, EFF): $\sigma(V) \subseteq \partial V(N)$ for all $V \in \Gamma$.
- Axiom 3 (Conditional Additivity, CADD): If $U, V, W = U + V \in \Gamma$, then $\sigma(W) \supseteq (\sigma(U) + \sigma(V)) \cap \partial W(N)$.
- Axiom 4 (Unanimity, UNA): If $U_T \in \Gamma$, then $\sigma(U_T) = \left\{\frac{\chi^T}{|T|}\right\}$ for $T \in 2^N \setminus \{\emptyset\}$.
- Axiom 5 (Scale Covariance, SCOV): If $V \in \Gamma$, $\lambda \in \mathbb{R}^N_{++}$, and $\lambda * V \in \Gamma$, then $\sigma(\lambda * V) = \lambda * \sigma(V)$.
- Axiom 6 (Independence of Irrelevant Alternatives, IIA): If $U, V \in \Gamma$, $U(N) \subseteq V(N)$, and $U(S) = V(S)$ for all $S \subsetneq N$, then $\sigma(U) \supseteq \sigma(V) \cap U(N)$.

In order to recall Aumann's characterization of Φ, the following definition is useful.

Definition 2.1 *Let N be a finite nonempty set and $\Gamma \subseteq \Gamma_N$. Then Γ is a* **feasible domain** *if*

(1) $\Phi(V) \neq \emptyset$ for all $V \in \Gamma$;

(2) $\Gamma_N^{TU} \subseteq \Gamma$*;*

(3) If $V \in \Gamma$ *and* $\lambda \in \mathbb{R}^N_{++}$*, then* $\lambda * V \in \Gamma$*;*

(4) If $V \in \Gamma$*, then the game that is obtained by replacing* $V(N)$ *by any of its supporting half-spaces is an element of* Γ*, i.e., if* $x \in \partial V(N)$*, and if* $W \in \Gamma_N$ *is the game that may differ from* V *only inasmuch as* $W(N) = \{y \in \mathbb{R}^N \mid \lambda^{V,x} \cdot y \leqslant \lambda^{V,x} \cdot x\}$*, then* $W \in \Gamma$*.*

Let N be a finite nonempty set. We remark that $\Gamma_N^{\Phi} = \{V \in \Gamma_N \mid \Phi(V) \neq \emptyset\}$ is a feasible domain.

Theorem 2.2 (Aumann (1985, Theorem A)) *Let* $\Gamma \subseteq \Gamma_N$ *be a feasible domain. Then the Shapley NTU value is the unique solution on* Γ *that satisfies Axioms 1 through 6.*

Axiom 6, the IIA axiom, in the foregoing theorem may be replaced by "maximality":

Theorem 2.3 (Aumann (1985, Theorem B)) *Let* $\Gamma \subseteq \Gamma_N$ *be a feasible domain. Then the Shapley NTU value is the maximum solution on* Γ *that satisfies Axioms 1 through 5; i,e.,* Φ *satisfies Axioms 1 through 5 on* Γ*, and if the solution* σ *on* Γ *satisfies Axioms 1 through 5, then* $\sigma(V) \subseteq \Phi(V)$ *for all* $V \in \Gamma$*.*

3 The Main Results

The main results of our investigation are the following two theorems that provide new insight into the role of IIA in Aumann's characterization of the Shapley NTU value for the 2-person case.

Theorem 3.1 *If* $|N| = 2$*, then the Shapley NTU value on* Γ_N^{Φ} *is characterized by Axioms 1 through 5.*

For the proof of Theorem 3.1 see Section 4. In order to state the other theorem, the following definition is needed. Let N be a finite nonempty set and let $V \in \Gamma_N$. Then V is called *uniformly p-smooth* if there exists $\varepsilon > 0$ such that $\lambda^{V,x} \geqslant \varepsilon \chi^N$ for all $x \in \partial V(N)$ (for the definition of $\lambda^{V,x}$ see (2.2)).

Theorem 3.2 *If* $|N| = 2$ *and* $\Gamma \subseteq \Gamma_N$ *is the set of uniformly p-smooth NTU games, then* Γ *is a feasible domain and Axiom 6 (IIA) is logically independent of the remaining axioms in Theorem A.*

Section 5 is devoted to the proof of Theorem 3.2 by means of an example of an appropriate subsolution of the Shapley NTU value.

4 Proof of Theorem 3.1

Throughout this section, let $|N| = 2$, say $N = \{1, 2\}$. We postpone the proof and present several preparatory remarks and lemmas.

Remark 4.1 Let $V \in \Gamma_N$. If $d(V) \in V(N)$, then $|\Phi(V)| = 1$. If $d(V) \in \partial V(N)$, then $\Phi(V) = \{d(V)\}$.

For any function $g : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ let $\text{dom}(g)$ denote the *effective domain* of g, i.e., $\text{dom}(g) = \{x \in \mathbb{R} \mid g(x) \in \mathbb{R}\}$. We say that $g : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is *differentiable* if $g'(x)$ exists for any $x \in \text{dom}(g)$. Let

$$\begin{aligned} \Gamma^0 &= \{V \in \Gamma_N \mid d(V) = 0\} \text{ and} \\ \mathcal{G} &= \{g : \mathbb{R} \to \mathbb{R} \cup \{-\infty\} \mid g \text{ is concave and differentiable, } \text{dom}(g) \neq \emptyset, g'(x) < 0 \ \forall x \in \text{dom}(g)\}. \end{aligned}$$

Note that, for any $g \in \mathcal{G}$, by concavity of g, the derivative of g on $\text{dom}(g)$ is continuous.

The mapping that assigns to each $V \in \Gamma^0$ the function $g_V : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ defined by

$$g_V(x) = \sup\{y \in \mathbb{R} \mid (x, y) \in V(N)\},$$

where $\sup \emptyset = -\infty$, is a bijection from Γ^0 to $\mathcal{G}$. Hence, for each $V \in \Gamma^0$,

$$\{(x, g_V(x)) \mid x \in \text{dom}(g_V)\} = \partial V(N).$$

Let $V \in \Gamma^0$ and $g = g_V$. It is well-known that

$$(x, y) \in \Phi(V) \Leftrightarrow x \in \text{dom}(g), y = g(x), g'(x)x = -g(x). \tag{4.1}$$

It is useful to use another parametrization of $\partial V(N)$. Substituting any $(x, g(x))$, $x \in \text{dom}(g)$, by $(t - f(t), -t - f(t))$ yields $g'(x)(1 - f'(t)) = -1 - f'(t)$ so that

$$f'(t) = \frac{-1 - g'(x)}{1 - g'(x)} \tag{4.2}$$

and hence $-1 < f'(t) < 1$ and $f : \mathbb{R} \to \mathbb{R}$ is convex. We have deduced that the mapping that assigns to each $V \in \Gamma^0$ the convex differentiable function $f_V := f$ is a bijection from Γ^0 to $\mathcal{F}$, where

$$\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a convex } C^1 \text{ function, } -1 < f'(t) < 1 \ \forall t \in \mathbb{R}\}.$$

Lemma 4.2 *Let $V \in \Gamma^0$. For all $t \in \mathbb{R}$, $(t - f_V(t), -t - f_V(t)) \in \Phi(V)$ iff $t = f_V(t){f_V}'(t)$.*

Proof: Let $f = f_V$, $g = g_V$, $x \in \text{dom}(g)$, and $t = \frac{x - g(x)}{2}$. By (4.2),

$$f(t)f'(t) - t = \left(\frac{-x - g(x)}{2}\right)\left(\frac{-1 - g'(x)}{1 - g'(x)}\right) - \frac{x - g(x)}{2} = \frac{g'(x)x + g(x)}{1 - g'(x)}.$$

We conclude that $f(t)f'(t) = t$ if and only if $g'(x)x = -g(x)$. The proof is complete by (4.1). **q.e.d.**

Corollary 4.3 *Let $U^0 \in \Gamma^0$ satisfy*

$$\begin{aligned} f_{U^0}(0) &> 0, & (4.3) \\ f_{U^0}(t){f_{U^0}}'(t) &> t \ \forall t > 0, & (4.4) \\ f_{U^0}(t){f_{U^0}}'(t) &< t \ \forall t < 0. & (4.5) \end{aligned}$$

Then, for any $U \in \Gamma^0$ that satisfies

$$\begin{aligned} f_U(0) &= f_{U^0}(0), & (4.6) \\ {f_U}'(t) &\geqslant {f_{U^0}}'(t) \ \forall t > 0, & (4.7) \\ {f_U}'(t) &\leqslant {f_{U^0}}'(t) \ \forall t < 0, & (4.8) \end{aligned}$$

the following two properties are satisfied:

$$f_U{}'(\mathbb{R}) \quad = \quad]-1,1[, \tag{4.9}$$

$$\Phi(U) \quad = \quad \{(-f_U(0), -f_U(0))\}. \tag{4.10}$$

Proof: In order to show (4.9), by (4.7) and (4.8), it suffices to verify that $\sup_{q\in\mathbb{R}} f_{U^0}{}'(q) = 1$ and that $\inf_{q\in\mathbb{R}} f_{U^0}{}'(q) = -1$. However, by (4.4) and (4.5), $f_{U^0}(t) > t$ for all $t > 0$ and $f_{U^0}(t) < t$ for all $t < 0$ so that the foregoing equations are implied by

$$f_{U^0}(t) \leqslant t \sup_{q\in\mathbb{R}} f_{U^0}{}'(q) + f_{U^0}(0) \text{ and } f_{U^0}(t) \geqslant t \inf_{q\in\mathbb{R}} f_{U^0}{}'(q) + f_{U^0}(0)\ \forall t \in \mathbb{R}.$$

By (4.3) – (4.8), $t = f_U(t) f_U{}'(t)$ iff $t = 0$. Thus, (4.10) follows from Lemma 4.2. **q.e.d.**

We now construct, for any $\alpha > 0$, a game $U^0 \in \Gamma^0$ that satisfies (4.4), (4.5), and $f_{U^0}(0) = \alpha$. Secondly, a useful technical Lemma is proved.

For $\varepsilon > 0$ and $c \in \mathbb{R}$, let $V^{\varepsilon,c} \in \Gamma^0$ be defined by

$$V^{\varepsilon,c}(N) = \left\{x \in \mathbb{R}^N \,\middle|\, x_1 < 0, x_1 x_2 \geqslant \varepsilon^2\right\} - \left\{c\chi^N\right\}. \tag{4.11}$$

Remark 4.4 It is straightforward to verify that, for any $t \in \mathbb{R}$,

$$f_{V^{\varepsilon,0}}(t) = \sqrt{t^2 + \varepsilon^2} \tag{4.12}$$

so that, by Lemma 4.2, $\Phi(V^{\varepsilon,0}) = \partial V^{\varepsilon,0}(N)$. By Definition of $V^{\varepsilon,c}$, for any $c \in \mathbb{R}$, $f_{V^{\varepsilon,c}}(t) = f_{V^{\varepsilon,0}}(t) + c$. Again by Lemma 4.2, $\{(c,c)\} = \Phi(V^{\varepsilon,c})$ for all $c \in \mathbb{R} \setminus \{0\}$. Furthermore, for any $c > 0$, $U^0 = V^{\varepsilon,c}$ satisfies (4.3) – (4.5) and $f_{U^0}(0) = \varepsilon + c$.

Lemma 4.5 *Let $g, h : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and nondecreasing functions such that $g(0) = h(0) = 0$ and $g(t) \leqslant h(t)$ for all $t \in \mathbb{R}_+$. Then there exist continuous and nondecreasing functions $\widetilde{h}, s : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\widetilde{h}(0) = 0, \widetilde{h}(t) \geqslant h(t) \qquad \forall t \in \mathbb{R}_+, \tag{4.13}$$

$$\widetilde{h}(\mathbb{R}_+) = h(\mathbb{R}_+), \tag{4.14}$$

$$\widetilde{h}(s(t)) = g(t) \qquad \forall t \in \mathbb{R}_+, \tag{4.15}$$

$$s(0) = 0 \leqslant s(t) - s(t') \leqslant t - t' \quad \forall t, t' \in \mathbb{R}_+, t' \leqslant t. \tag{4.16}$$

Proof: In order to construct $\widetilde{h} : \mathbb{R}_+ \to \mathbb{R}$, we introduce, for any $q \in \mathbb{R}_+$, the auxiliary function $g_q : \mathbb{R}_+ \to \mathbb{R}$ defined by $g_q(t) = g(t+q)$ for all $t \geqslant 0$. Moreover, let $f : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ be defined by $f(q) = \inf\{t \in \mathbb{R}_+ \mid g_q(t) = h(t)\}$ for all $q \geqslant 0$ (with the convention that $\inf \emptyset = \infty$). Note that "inf" is in fact "min", because g and h are continuous. Now, define

$$\widetilde{h}(t) = \sup\left(\{h(t)\} \cup \{g_q(t) \mid q \geqslant 0, f(q) \leqslant t\}\right)\ \forall t \in \mathbb{R}_+.$$

By construction, $\widetilde{h}$ is nondecreasing and satisfies (4.13).

Let $t \in \mathbb{R}_+$. If there exists q with $f(q) \leqslant t$ and $g_q(t) > h(t)$, then $\{q \in \mathbb{R}_+ \mid f(q) \leqslant t\}$ is a compact interval so that "sup" is, in fact, "max" in any case. Consequently, the continuities of h and g imply the continuity of $\widetilde{h}$ and, hence, (4.14).

For $\gamma \in g(\mathbb{R}_+)$ denote

$$\begin{aligned} \alpha_g(\gamma) &= \min\{t \in \mathbb{R}_+ \mid g(t) = \gamma\}, & \alpha_{\widetilde{h}}(\gamma) &= \min\{t \in \mathbb{R}_+ \mid \widetilde{h}(t) = \gamma\}, \\ \beta_g(\gamma) &= \sup\{t \in \mathbb{R}_+ \mid g(t) = \gamma\}, & \beta_{\widetilde{h}}(\gamma) &= \sup\{t \in \mathbb{R}_+ \mid \widetilde{h}(t) = \gamma\}. \end{aligned}$$

We may now define $s : \mathbb{R}_+ \to \mathbb{R}_+$ as follows. For $t \geqslant 0$ let $s(t) = \min\{\alpha_{\widetilde{h}}(\gamma) + t - \alpha_g(\gamma), \beta_{\widetilde{h}}(\gamma)\}$, where $\gamma = g(t)$. By construction, s is nondecreasing and satisfies (4.15). As $\beta_{\widetilde{h}}(\gamma) - \alpha_{\widetilde{h}}(\gamma) \leqslant \beta_g(\gamma) - \alpha_g(\gamma)$ (note that $\beta_g(\gamma) = \infty$ is just possible if $\max_t g(t)$ exists and $\gamma = \max_t g(t)$), s is continuous, and it satisfies (4.16). **q.e.d.**

Now, we are prepared for the proof.

Proof of Theorem 3.1: By Aumann's Theorem B we only have to show uniqueness. Let σ be a solution on Γ_N^Φ that satisfies NE, PO, CADD, UNA, and SCOV, let $V \in \Gamma_N^\Phi$. Again by Theorem B, $\sigma(V) \subseteq \Phi(V)$ so that it suffices to prove that $\Phi(V) \subseteq \sigma(V)$. If $\Phi(V)$ is a singleton, then the proof is finished by NE. Hence, by Remark 4.1 we may assume that $d \notin V(N)$, where $d = d(V)$. Let $\widehat{x} \in \Phi(V)$. It remains to show that $\widehat{x} \in \sigma(V)$. By SCOV we may assume that $\widehat{x} = d - 2\chi^N$.

By CADD and Remark 4.1 it suffices to construct $U, W \in \Gamma_N^\Phi$ such that $\Phi(U) = \{-2\chi^N\}$, $d = d(W) \in \partial W(N)$, and $V = U + W$.

In order to construct U, an auxiliary game $U^1 \in \Gamma_N^\Phi$ is constructed. Let U^1 be the NTU game defined by $U^1(N) = \frac{1}{2}(V(N) - \{d\}) - \{\chi^N\}$ and $d(U^1) = 0$. Then $U^1 \in \Gamma_N^\Phi$ and $-2\chi^N \in \Phi(U^1)$. By Remark 4.4 there exists $U^0 \in \Gamma^0$ that satisfies (4.3) – (4.5) and $f_{U^0}(0) = 2$. Let $f_i = f_{U^i}$ for $i = 0, 1$. Recall that $f_0'(\mathbb{R}) =]-1, 1[$. Let $\widetilde{F} : \mathbb{R} \to]-1, 1[$ be any continuous and strictly increasing function that satisfies

$$\widetilde{F}(t) \begin{cases} \geqslant \max_{i \in \{0,1\}} f_i'(t) & \text{, if } t \geqslant 0, \\ \leqslant \min_{i \in \{0,1\}} f_i'(t) & \text{, if } t < 0. \end{cases}$$

By the aforementioned properties of the functions f_i', $\widetilde{F}(\mathbb{R}) =]-1, 1[$ and $\widetilde{F}(0) = 0$.

Applying Lemma 4.5 to $g, h : \mathbb{R}_+$ given by $g(t) = f_1'(t)$ and $h(t) = \widetilde{F}(t)$ (or given by $g(t) = -f_1'(-t)$ and $h(t) = -\widetilde{F}(-t)$, respectively), for all $t \geqslant 0$, guarantees the existence of continuous nondecreasing functions $F : \mathbb{R} \to]-1, 1[$ and $s : \mathbb{R} \to \mathbb{R}$ that satisfy

$$F(t) \geqslant \widetilde{F}(t), F(-t) \leqslant \widetilde{F}(-t) \qquad \forall t \in \mathbb{R}_+, \tag{4.17}$$

$$F(s(t)) = f_1'(t) \qquad \forall t \in \mathbb{R}_+, \tag{4.18}$$

$$s(0) = 0 \leqslant s(t) - s(t') \leqslant t - t' \qquad \forall t, t' \in \mathbb{R}, t' \leqslant t. \tag{4.19}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be the unique function defined by $f' = F$ and $f(0) = 2$. Then f a convex C^1 function. Let U be the 0-normalized NTU game defined by

$$U(N) = \{x \in \mathbb{R}^N \mid \exists t \in \mathbb{R} : x \leqslant (t - f(t), -t - f(t))\}.$$

As $f'(t) \geqslant f_0'(t)$ for all $t > 0$ and $f'(t) \leqslant f_0'(t)$ for all $t < 0$, $\Phi(U) = \{-2\chi^N\}$ by Corollary 4.3 so that $U \in \Gamma^0$.

By (4.19), the real function $\widehat{s} : \mathbb{R} \to \mathbb{R}$ defined by $\widehat{s}(t) = 2t - s(t)$ for all $t \in \mathbb{R}$ is a monotonic continuous bijection that satisfies $\widehat{s}(0) = 0$. Hence there exists a unique C^1 function g that satisfies $g(0) = 2$ and $g'(t) = f_1'(\widehat{s}^{-1}(t))$. Then g is convex, $g'(0) = 0$, and $g'(t) \in]-1, 1[$ so that the NTU game W defined by $W(\{i\}) = V(\{i\})$ for $i \in N$ and

$$W(N) = \{x \in \mathbb{R}^N \mid \exists t \in \mathbb{R} : x \leqslant (t - g(t), -t - g(t))\} + \{2\chi^N + d\}$$

satisfies (1) and (2) of Section 2. As $d = d(W) \in \partial W(N)$, $\Phi(W) = \{d\}$ by Remark 4.1.

Let $h = f \circ s + g \circ \widehat{s}$. We claim that

$$h'(t) = 2f_1'(t) \;\forall t \in \mathbb{R}. \tag{4.20}$$

In order to show (4.20) define $D_f, D_g : \mathbb{R}^2 \to \mathbb{R}$ by

$$D_f(t,t') = \begin{cases} \frac{f(t)-f(t')}{t-t'} & \text{, if } t \neq t', \\ f'(t) & \text{, if } t = t', \end{cases} \quad \text{and } D_g(t,t') = \begin{cases} \frac{g(t)-g(t')}{t-t'} & \text{, if } t \neq t', \\ g'(t) & \text{, if } t = t', \end{cases}$$

and note that D_f, D_g are continuous. Hence, for any $t \in \mathbb{R}$,

$$\begin{aligned} h'(t) &= \lim_{t' \to t} \tfrac{f(s(t)) - f(s(t')) + g(\widehat{s}(t)) - g(\widehat{s}(t'))}{t-t'} \\ &= \lim_{t' \to t} \tfrac{D_f(s(t),s(t'))\left(s(t)-s(t')\right) + D_g(\widehat{s}(t),\widehat{s}(t'))\left(\widehat{s}(t)-\widehat{s}(t')\right)}{t-t'} \\ &= \lim_{t' \to t} \tfrac{\left(D_f(s(t),s(t')) - D_g(\widehat{s}(t),\widehat{s}(t'))\right)\left(s(t)-s(t')\right) + D_g(\widehat{s}(t),\widehat{s}(t'))(2t-2t')}{t-t'} \end{aligned}$$

As $f'(s(t)) = f_1'(t) = g'(\widehat{s}(t))$ for all $t \in \mathbb{R}$, we may conclude from (4.19) and the continuities of s and $\widehat{s}$ that

$$\lim_{t' \to t} \frac{\left(D_f(s(t), s(t')) - D_g(\widehat{s}(t), \widehat{s}(t'))\right)(s(t) - s(t'))}{t - t'} = 0, \lim_{t' \to t} \frac{D_g(\widehat{s}(t), \widehat{s}(t'))(2t - 2t')}{t - t'} = 2g'(\widehat{s}(t))$$

so that our claim follows.

Now, $h(0) = 4 = 2f_1(0)$ so that $h = 2f_1$. By definition of f_1,

$$U^1(N) = \{x \in \mathbb{R}^N \mid \exists t \in \mathbb{R} : x \leqslant (t - f_1(t), -t - f_1(t))\}$$

so that

$$\begin{aligned} & \partial V(N) - \{d + 2\chi^N\} \\ = \; & 2\partial U^1(N) \\ = \; & \{(2t - 2f_1(t), -2t - 2f_1(t)) \mid t \in \mathbb{R}\} \\ = \; & \{(2t - h(t), -2t - h(t)) \mid t \in \mathbb{R}\} \\ = \; & \{(s(t) - f(s(t)) + \widehat{s}(t) - g(\widehat{s}(t)), -s(t) - f(s(t)) - \widehat{s}(t) - g(\widehat{s}(t))) \mid t \in \mathbb{R}\} \end{aligned}$$

so that $V(N) \subseteq U(N) + W(N)$ is shown. In order to show that $U(N) + W(N) \subseteq V(N)$, as $U(N) + W(N) \subseteq \{d\} + \{x \in \mathbb{R}^N \mid x(N) \leqslant -4\}$, it suffices to show that any element of $\partial(U(N) + W(N))$ belongs to $V(N)$. Let $x \in \partial(U(N) + W(N))$. Then there exist $y \in \partial U(N)$ and $z \in \partial W(N)$ such that $x = y + z$.

Let $t, \alpha \in \mathbb{R}$ such that $x - d - 2\chi^N = (t - \alpha, -t - \alpha)$. By the definition of U and W there exist $t', t'' \in \mathbb{R}$ such that $y = (t' - f(t'), -t' - f(t'))$ and $z - d - 2\chi^N = (t'' - g(t''), -t'' - g(t''))$. As the supporting hyperplane to $U(N)$ at y is parallel to the supporting hyperplane to $W(N)$ at z (see (2.4)), $f'(t') = g'(t'')$. As $s(t/2) + \widehat{s}(t/2) = t$, there exists $\beta \in \mathbb{R}$ such that $t' = s(t/2) + \beta$ and $t'' = \widehat{s}(t/2) - \beta$. As $f'(s(t/2)) = g'(\widehat{s}(t/2))$ and f' and g' are nondecreasing functions, $f'(t') = f'(s(t/2)) = g'(t'')$ and $\alpha = 2f_1(t/2)$. **q.e.d.**

5 Proof of Theorem 3.2

Throughout this section, let $|N| = 2$, say $N = \{1, 2\}$, and let Γ^{ups} denote the set of uniformly p-smooth games (see Section 3 for the definition of this property) in Γ_N. Clearly, Γ^{ups} satisfies (2) – (4) of Definition 2.1. In order to show that Γ^{ups} is a feasible domain in Γ_N, it suffices to construct, for any $V \in \Gamma^{ups}$, a nonempty subset of $\Phi(V)$. To this end let $V \in \Gamma^{ups}$ and define

$$\sigma_0(V) = \begin{cases} \Phi(V) & \text{, if } d(V) \in V(N), \\ \arg\max\{(d_1(V) - x_1)(d_2(V) - x_2) \mid x \in \partial V(N)\} & \text{, if } d(V) \notin V(N). \end{cases}$$

Note that σ_0 is well-defined. Indeed, if $d(V) \notin V(N)$, then $\partial V(N) \cap \left(\{d(V)\} - \mathbb{R}^N_+\right)$ is a nonempty compact set by uniform p-smoothness of $V(N)$ so that $\sup\{(d_1(V) - x_1)(d_2(V) - x_2) \mid x \in V(N)\}$ is attained by some $x \in \partial V(N), x \ll d(V)$.

By Remark 4.1, σ_0 satisfies NE. Moreover, it satisfies SCOV and UNA. In order to show that $\sigma_0(V) \subseteq \Phi(V)$, we may assume that $d(V) \notin V(N)$. Let $x \in \sigma_0(V)$, $t = (d_1(V) - x_1)(d_2(V) - x_2)$, and $\lambda = \lambda^{V,x}$ (see (2.2)). Then the hyperplane $\{z \in \mathbb{R}^N \mid \lambda \cdot z = \lambda \cdot x\}$ is a tangent to the hyperbola

$$\{z \in \mathbb{R}^N \mid z \ll d(V), (d_1(V) - z_1)(d_2(V) - z_2) = t\}$$

so that $x \in \Phi(V)$ by (4.1) and the well-known *translation covariance* of Φ.

We now show that σ_0 satisfies CADD.

Lemma 5.1 *The solution σ_0 on Γ^{ups} satisfies* CADD*.*

Proof: For $i \in \{1, 2\}$, let $V^i \in \Gamma^{ups}$, $x^i \in \sigma_0(V^i)$ such that, with $V = V^1 + V^2$ and $x = x^1 + x^2$, $V \in \Gamma^{ups}$ and $x \in \partial V$. By CADD of Φ, $x \in \Phi(V)$. It remains to show that $x \in \sigma_0(V)$. If $d = d(V) \in V(N)$, then the proof is finished. Hence, we may assume that $d \notin V(N)$. As $x \in \partial V(N)$, $\lambda^{V^i,x^i} = \lambda^{V,x}$ for $i = 1, 2$, by (2.4). By (2.5), there exists $c \in \mathbb{R}$ such that $(d^2 - x^2) = c(d^1 - x^1)$, where $d^i = d(V^i)$ for $i = 1, 2$. As $d = d^1 + d^2$, Remark 4.1 implies that $x^1 \ll d^1$ or $x^2 \ll d^2$. Without loss of generality we may assume that $x^1 \ll d^1$. By definition of σ_0,

$$V^1(N) \supseteq \{z \in \mathbb{R}^N \mid z \ll d^1, (d^1_1 - z_1)(d^1_2 - z_2) \geqslant (d^1_1 - x^1_1)(d^1_2 - x^1_2)\} =: Z^1 \tag{5.1}$$

Let $Z = \left\{z \in \mathbb{R}^N \,\middle|\, z \ll d, \prod_{i\in N}(d_i - z_i) \geqslant \prod_{i\in N}(d_i - x_i)\right\}$. Two cases may occur:

(1) $x^2 \geqslant d^2$. By (5.1), $V(N) \supseteq \{x^2\} + Z^1$. Let $z \in Z$ and define $z^1 = z - x^2$. It suffices to show that $z^1 \in Z^1$. Now, $z^1 \ll d^1$, because $x^2 \geqslant d^2$ and $z \ll d$. The statement immediately follows from:

$$a, b \in \mathbb{R}^N_{++}, a_1a_2 \geqslant b_1b_2, \alpha \geqslant 0 \Longrightarrow (a_1 + \alpha b_1)(a_2 + \alpha b_2) \geqslant (b_1 + \alpha b_1)(b_2 + \alpha b_2) = (1 + \alpha)^2 b_1 b_2. \tag{5.2}$$

In order to show (5.2) we may assume that $a_1a_2 = b_1b_2$, i.e., $a_2 = b_1b_2/a_1$. Define $f(a_1) = (a_1 + \alpha b_1)\left(\frac{b_1b_1}{a_1} + \alpha b_2\right)$. Then f is a convex function and $f'(a_1) = 0$ iff $a_1 = b_1$.

(2) $x^2 \ll d^2$. Let

$$Z^2 = \{z \in \mathbb{R}^N \mid z \ll d^2, (d_1^2 - z_1)(d_2^2 - z_2) \geqslant (d_1^2 - x_1^2)(d_2^2 - x_2^2)\}.$$

By definition of σ_0, $V^2(N) \supseteq Z^2$. As $Z^1 + Z^2 \supseteq Z$, the proof is finished.

q.e.d.

Example 5.2 shows that $\sigma_0 \neq \Phi$.

Example 5.2 Let $X = \{x \in \mathbb{R}^N \mid x \ll 0, x_1x_2 = 1\}$ and $U \in \Gamma_N$ be defined by $U(N) = X - \mathbb{R}^N_+$ and $d(V) = 0^N$. If $Y = \{y \in X \mid x_i \geqslant -3\}$, then $Y \neq \emptyset$ so that $W(N) := \{z \in \mathbb{R}^N \mid \lambda^{U,y} \cdot z \leqslant \lambda^{U,y} \cdot y \; \forall y \in Y\}$ is uniformly p-smooth. Let $d(W) = 0^N$. We may easily deduce that $\Phi(W) = Y$. Let $d = \chi^N$ and $V \in \Gamma^{ups}$ be defined by $V(N) = W(N)$ and $d(V) = \chi^N$. By symmetry of V, $\Phi(V) \ni -d$. Define x by $x_1 = -3$ and $x_2 = -\frac{1}{3}$ and observe that $x \in \partial V(N)$. However, $(d_1 - x_1)(d_2 - x_2) = 16/3 > 4$ so that $-d \notin \sigma_0(V)$.

6 On the Logical Independence of the Remaining Axioms

Throughout this section, let N be a finite set such that $|N| \geqslant 2$. Let $\Gamma \subseteq \Gamma_N$ be a feasible domain. We are now going to define, for $i = 1, \ldots, 5$, a solution σ_i on Γ that exclusively violates Axiom i in Theorem A as well as in Theorem B, even if "maximum" is replaced by "unique maximal"[1].

In order to define σ_1, note that, as mentioned in Section 2, any TU game v on N is a linear combination of unanimity games, that is, there exist unique $c_T(v) = c_T, \emptyset \neq T \subseteq N$, such that $v = \sum_{\emptyset \neq T \subseteq N} c_T u_T$. As $|N| \geqslant 2$, there exist $2^{|N|} - 1 \geqslant 3$ coalitions. Select any two distinct coalitions T^1 and T^2 and define $\gamma_N^+ = \{v \in \gamma_N \mid c_{T^1}(v), c_{T^2}(v) \geqslant 0\}$ and $\gamma_N^{++} = \{v \in \gamma_N \mid c_{T^1}(v), c_{T^2}(v) > 0\}$. For any $V \in \Gamma$ define

$$\sigma_1(V) = \{x \in \partial(V) \mid v_x^V \in \gamma_N^{++}\} \cup \{x \in \Phi(V) \mid v_x^V \in \gamma_N^+\}. \tag{6.1}$$

Clearly, σ_1 satisfies EFF, SCOV, and IIA. As any unanimity TU game is an element of $\gamma_N^+ \setminus \gamma_N^{++}$, σ_1 satisfies UNA. CADD follows from (2.4). As $\sigma_1(V_{-u_N}) = \emptyset$, $\sigma_1 \neq \Phi$. Regarding the aforementioned modification of Theorem B, it remains to show that σ_1 is a maximal solution that satisfies the remaining axioms, i.e., Axioms 2 through 5. Assume, on the contrary, there exists a solution σ that satisfies EFF, CADD, UNA, SCOV, and contains σ_1 as a proper subsolution. Let $V \in \Gamma$ such that there exists $x \in \sigma(V) \setminus \sigma_1(V)$. By EFF, $x \in \partial V(N)$. Let $v = v_x^V$, $\lambda = \lambda^{V,x}$, $\widehat{\lambda} = (1/\lambda_i)_{i \in N}$, and $c_T = c_T(v)$ for $T \in 2^N \setminus \{\emptyset\}$. Let W be the NTU game associated with

$$w = \sum_{R \in 2^N \setminus \{\emptyset, T^2\}} (-c_R)u_R + (1 + |c_{T^2}|)u_{T^2}.$$

Two cases may occur:

[1] A solution σ is the *unique maximal* solution that satisfies certain axioms, if (a) σ satisfies the axioms, (b) σ is maximal under (a) (i.e., any solution that satisfies the axioms and contains σ coincides with σ), and (c) there exists no further maximal solution that satisfies the axioms.

(1) $c_{T^1} < 0$ or $c_{T^2} < 0$, say $c_{T^1} < 0$. Then $w \in \gamma_N^{++}$ so that $\partial(\widehat{\lambda} * W)(N) = \sigma_1(\widehat{\lambda} * W) \subseteq \sigma(\widehat{\lambda} * W)$. Now, $V + \widehat{\lambda} * W = \widehat{\lambda} * (1 + |c_{T^2}|)U_{T^2}$ so that, by SCOV, $\sigma(V + \widehat{\lambda} * W)$ is a singleton. On the other hand, by CADD, $\partial(V + \widehat{\lambda} * W)(N) \subseteq \sigma(V + \widehat{\lambda} * W)$ so that the desired contradiction has been obtained.

(2) $c_{T^1}, c_{T^2} \geqslant 0$, $c_{T^1}c_{T^2} = 0$, and $\lambda * x \neq \phi(v)$, say $c_{T^1} = 0$. Then $V + \widehat{\lambda} * W = \widehat{\lambda} * (1 + c_{T^2})U_{T^2}$ so that, By SCOV and UNA, $\sigma(V + \widehat{\lambda} * W) = \Phi(V + \widehat{\lambda} * W)$. As $w \in \gamma_N^{+} \setminus \gamma_N^{++}$ in this case, $\sigma(\widehat{\lambda} * W) = \Phi(\widehat{\lambda} * W)$ so that CADD, applied to x and the unique element of $\sigma(\widehat{\lambda} * W)$ yields the desired contradiction.

In order to define the solution σ_2 that exclusively violates EFF and contains Φ as a subsolution, we distinguish two cases: If $|N| > 2$, then let σ_2 be the solution defined by Peleg and Sudhölter (2007, Section 13.3, page 242), denoted by σ^2. If $|N| = 2$, then define

$$\sigma_2(V) = \begin{cases} \Phi(V) & \text{, if } d(V) \notin V(N) \text{ or } d(V) = 0, \\ \Phi(V) \cup \{d(V)\} & \text{, otherwise.} \end{cases} \tag{6.2}$$

Clearly, σ_2 satisfies NE and SCOV, and it violates EFF. By (2.4), σ_2 inherits CADD from Φ. Moreover, UNA and IIA are easily deduced using Remark 4.1.

The straightforward proofs that, for an arbitrary $|N| \geqslant 2$, the following solutions satisfy the desired properties, are left to the reader.

$$\begin{aligned} \sigma_3(V) &= \Phi(V) \cup \{x \in \partial V(N) \mid x \ll d(V)\}; \\ \sigma_4(V) &= \partial V(N); \\ \sigma_5(V) &= \Phi(V) \cup \left\{ x \in \partial V(N) \,\middle|\, \lambda^{V,x} \neq \frac{\chi^N}{|N|} \right\}. \end{aligned}$$

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