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Non-Committed Procurement under Intricate Uncertainty

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Abstract

This paper studies non-committed procurement in which (i) it is not economically viable for a buyer (e.g. government) to commit herself to scoring rules due to the complex nature of the good and (ii) asymmetric sellers have affiliated signals on production costs with interdependent values. In non-committed procurement, a buyer advertises open invitations to potential sellers without committing to scoring rules, sellers submit menus of alternative contracts, and finally the buyer selects a winning seller by choosing a contract from the winning seller's menu. This paper establishes the existence of a continuum of monotone equilibria given the multiplicity of continuation equilibrium that the buyer would choose to follow for her contract choice from the winning seller's menu. Monotone equilibrium is bounded above by joint ex-post efficiency and below by joint interim efficiency. Among multiple equilibria, the jointly ex-post efficient equilibrium is not only jointly ex-post renegotiation-proof but also ex-ante robust to the possibility that the buyer might choose to follow any alternative continuation equilibrium upon any seller's deviation. The results also suggest the practical importance of the buyer's reputation for a jointly ex-post efficient contract choice under interdependence values without her commitment.

1 Introduction

Procurement of goods or services is an important part of the economy. For example, public procurement by governments accounts for 10 to 15% of GDP in

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developed countries and up to 20% of GPD in developing countries. The items acquired through procurement vary from simple stationary items to highly complex goods and services such as infrastructure projects, nuclear power plants, and military weapons. Recently, US infrastructure projects financed under the American Recovery and Reinvestment Act have been awarded to numerous private construction companies in order to stimulate the US economy.[1] In 2009, the United Arab Emirates awarded $25 billion construction and nuclear power projects to a team led by Korea Electric Power Corporation (KEPCO). In 2010, Canada announced a $9 billion plan to purchase sixty five F-35 joint strike fighter jets from Lockheed Martin.

In order to model procurement, various scoring auctions where a buyer (e.g., government) can commit to a scoring rule are proposed in the literature. In scoring auctions, each seller submits a single bid (or equivalently a single contract), i.e., a pair of characteristics of the good and an amount of monetary payment that the buyer pays to the seller. The scoring rule calculates each seller's score given his bid and the seller with highest score wins procurement.[2] Examples of scoring auctions include the first scoring auction, the second scoring auction, and the handicap auction.[3] Che (1993) and Asker and Cantillon (2008) study scoring auctions in situations where sellers' signals on production costs have only private values. Branco (1997) considers scoring auctions with symmetric sellers, independent signals and common values.

While scoring auctions generate competitive bidding in an intuitive way, scoring rules must specify scores for all possible bids that sellers may submit. This may be quite complex, especially when characteristics of the good are highly multidimensional. For example, when a government is considering awarding a contract for the construction of a tunnel in a mountainous area, the specification of the tunnel to be built would be highly multidimensional. The characteristics may include the possible route, length, and radius of the tunnel, the construction method to be utilized, the air ventilation system, the construction time, and the operating issues after the construction: the list of the specifications goes on and on. In this case, it may not be economically viable for the government to commit itself to a scoring rule that specifies a score for every possible bid. The difficulty of procurement of tunnel construction is compounded because the construction cost may not be fully known to the construction companies. The

[1] The US government is by far the biggest buyer in the world. Every 20 seconds of each working day, the federal government awards a contract with an average value of $465,000.

[2] We use feminine pronouns for the buyer and masculine pronouns for sellers.

[3] Given the scoring rule in each auction, the seller with the highest score wins procurement. In the first scoring auction, the winning seller executes the contract he submits. In the second scoring auction, the winning seller can execute any contract that matches the highest rejected score. The handicap auction can give different additional scores to different sellers on top of scores based on the contracts that they submit.

construction cost will depend on the geological characteristics of the mountain, the composition and distribution of minerals in the area in which the tunnel is to be constructed. Different construction companies may receive different signals on construction costs. Those signals have interdependent values in the sense that each company's estimate of its construction cost depends on all companies' signals and its estimate would be more precise if other companies' signals were known to the company.

With no ex-ante commitment to a scoring rule, a government may, in practice, simply advertise open invitations for the procurement of a highly complex good or service with a few descriptive objectives. Sellers can then submit and present their proposals, which often include multiple possible bids, i.e., pairs of characteristics of the good and monetary payment. It may take the government a few months or years to evaluate the proposals and start negotiating with the winning seller on the characteristics of the good to be provided and the corresponding monetary payment.[4] When sellers' signals have interdependent values, each seller's decision on the initial proposal may depend on his belief on how much the government's perception of production costs which it developed from viewing other sellers' proposals spills over to its negotiation with the winning seller.

This paper studies the existence of monotone equilibria and the efficiency properties of equilibrium allocations in non-committed procurement when (i) it is not economically viable for a buyer (e.g. government) to commit herself to scoring rules due to the complex nature of the good and (ii) asymmetric sellers privately receive affiliated signals on production costs with interdependent values. Abstracting from reality, this paper formulates non-committed procurement in which the buyer advertises open invitations, sellers submit menus of alternative contracts (pairs of characteristics of the good and monetary payment), and the buyer evaluates menus by identifying the best contract in them and awards the procurement to the winning seller by choosing the best contract from the winning seller's menu.

Given a seller's signal and the maximum payoff that he is willing to give to the buyer, the ex-post joint surplus between the seller and the buyer depends on both the seller's signal and competing sellers' signals under interdependent values. When a seller believes that competing sellers reveal their signals by offering menus of alternative contracts contingent on their signals, he also has incentives to offer a menu of alternative contracts contingent on his own signal in a way that his menu includes each alternative contract that maximizes the ex-post joint surplus for each possible array of all sellers' signals given the maximum payoff that he is willing to give to the buyer. The buyer may well then choose the contract that

[4]Public procurement follows the principle set of rules (e.g., Federal Acquisition Regulation in the U.S.) and the government often has an oversight agency (e.g., Office of the Procurement Ombudsman in Canada).

maximizes the seller's ex-post payoff from the menu given her belief on all sellers' signals as long as it gives her the highest payoff among all alternative contracts in the menu. Such a choice made by the buyer maximizes the seller's ex-post payoff given the maximum payoff that he is willing to give to the buyer.[5]

At the same time, such a menu leaves multiple optimal alternatives available to the buyer because the buyer's payoff depends only on the characteristics of the good and the monetary payment to be provided but not on the production cost signals. Therefore, it is also possible for the buyer to choose an arbitrary optimal contract from the winning seller's menu regardless of her belief on all sellers' signals that she acquires from their menu offers. In this case, the contract that the buyer chooses from the seller's menu must maximize the interim joint surplus conditional on winning the procurement given only the seller's own signal because it is always feasible for the seller to offer a degenerate menu consisting of a single contract.

A seller's equilibrium menu offer therefore depends on his belief on how likely the buyer would choose a contract that would maximize the ex-post joint surplus when it is equally optimal with some of other alternative contracts in the menu. This paper studies a truthful monotone equilibrium in which a seller reveals his signal by making his equilibrium menu offer contingent on his signal. Let τ_i denote the probability that the buyer chooses a contract that maximizes the ex-post joint surplus in a continuation equilibrium when it is optimal in seller i's menu. In turn, $1 - \tau_i$ denotes the probability that the buyer chooses the same contract in seller i's menu, regardless of her belief on the other sellers' signals, which must be jointly interim efficient in a continuation equilibrium. A vector $\tau = [\tau_1, \ldots, \tau_N] \in [0, 1]^N$ then denotes sellers' beliefs on how likely the buyer chooses a contract that maximizes the ex-post joint surplus across sellers' menus.

This paper establishes a continuum of truthful monotone equilibria that spans the entire space of sellers' beliefs $[0, 1]^N$. Following Reny's existence result (2011), one could establish the existence of a truthful monotone equilibrium if each seller's interim payoff function were weakly quasisupermodular and weakly single crossing given the other sellers' non-decreasing strategies. However, a seller's interim payoff functions may fail to be weakly quasisupermodular or weakly single crossing at irrational bids or rational bids with a positive probability of ties just as Athey's single crossing condition may fail in first price auctions with single dimensional bids (Reny and Zamir 2004).[6] This paper extends Reny and Zamir's

[5] Joint ex-post efficiency is not reached in equilibrium under interdependent values if each seller is allowed to submit only a single contract.

[6] Reny and Zamir (2004) show that IRT-SCC is sufficient to show the existence of a monotone equilibrium in first price auctions with single dimensional bids in the general case involving asymmetric bidders, interdependent values, and affiliated signals. The existence of equilibrium in first price auctions can also be found in Athey (2001), Bresky (1999), Jackson and Swinkels (2003), Lebrun (1999), Maskin and Riley (2000). Those works restrict attention to either two

“individually rational tieless single crossing condition” (IRT-SCC) and introduces the “tieless supermodular condition” (TLS-SMC). This paper shows that IRT-SCC and TLS-SMC ensure both the weakly single crossing condition and the weakly quasisupermodular condition at individually rational tieless bids, which are sufficient to establish the existence of a truthful monotone equilibrium at any given $\tau = [\tau_1, \ldots, \tau_N]$. Subsequently the existence of a truthful monotone equilibrium is established at any given $\tau = [\tau_1, \ldots, \tau_N]$ by showing that IRT-SCC and TLS-SMC are satisfied in our model.

The continuum of truthful monotone equilibria is bounded above by the joint ex-post efficiency level and below by the level of joint interim efficiency. Given the continuum of truthful monotone equilibria, we examine which equilibrium is stable under two criteria. First of all, we adopt the notion of jointly ex-post renegotiation-proof equilibrium in which it is not mutually beneficial for the buyer and the winning seller to renegotiate their contract given a signal vector. If equilibrium is not jointly ex-post efficient, there is a positive probability that a mutually beneficial renegotiation between the buyer and the winning seller exists. While there is a continuum of equilibria, the jointly ex-post efficient equilibrium is the only one that is jointly ex-post renegotiation-proof.

Joint ex-post renegotiation is a notion that can be used to examine whether the buyer and the winning seller can improve upon the contract after the winning seller is chosen. Sellers may consider deviations from their menu offers (i.e., change their menus) even before the buyer chooses a winning seller. Given the multiplicity of continuation equilibrium that the buyer chooses to follow for her contract choice, sellers’ incentives for deviations depend on the continuation equilibrium that they believe the buyer would choose upon their deviations. For example, consider a truthful monotone equilibrium based on a continuation equilibrium in which the buyer always chooses a jointly interim efficient contract from seller 1’s menu upon accepting it, but always chooses a jointly ex-post efficient contract from the other sellers’ menus. If seller 1 believed that the buyer would in fact follow the continuation equilibrium where she always chooses a jointly ex-post efficient contract from his menu following his deviation, then seller 1 could deviate to a more aggressive menu, which provides a higher payoff to the buyer than his original menu does, in order to win procurement with a higher probability. Not only does such a deviation show that equilibrium menu offers (and consequently equilibrium contracts) in some equilibrium are not ex-ante robust to the possibility that the buyer would follow an alternative continuation equilibrium, but it also indicates that some equilibrium may not provide stable predictions on how likely each seller would win procurement, because seller 1’s deviation to a more aggressive menu makes it more likely for seller 1 to win and less likely for other sellers to win. In this light, we can examine how ex-ante robust an equilibrium is

bidders, symmetric bidders, independent signals, and private or common values.

to (sellers' beliefs on) the continuation equilibrium. We show that if the truthful monotone equilibrium is based on a continuation equilibrium in which the buyer chooses a jointly ex-post efficient contract from the winning seller's menu with positive probability, then it is ex-ante robust to a set of continuation equilibria in which the buyer chooses a jointly ex-post efficient contract with a lower probability. Therefore, if the truthful monotone equilibrium is based on a continuation equilibrium in which the buyer chooses a jointly ex-post efficient contract with probability one, then it is ex-ante robust to all continuation equilibria: no sellers have incentives to deviate regardless of the continuation equilibrium that they believe the buyer would choose to follow for her contract choice following their deviations.

The jointly ex-post efficient equilibrium is appealing because it is both jointly ex-post renegotiation proof and ex-ante robust to all continuation equilibria. With a lack of commitment on the buyer's side, it would be practically important for the buyer to build up a reputation for adhering to the jointly ex-post efficient equilibrium. In practice, the buyer would acquire sellers' information on production costs by reviewing and evaluating their proposals. It implies that the buyer plays the role of the collector of information on production costs by reviewing and evaluating sellers' proposals. Therefore, the buyer would have additional information on production costs that the winning seller does not have when she negotiates with him. If the buyer can establish her reputation in practice in a way that it leads sellers to believe her additional information on production costs would be used in negotiating a contract with the winning seller, sellers would submit their menus of contracts accordingly, and leading to the jointly ex-post efficient equilibrium.

2 Preliminaries

The buyer contracts with one of N sellers on the characteristics of the good to be provided and on the monetary payment to be given the seller. Let $\mathcal{N} = \{1, \ldots, N\}$ be the set of sellers. Let $t \in \mathbb{R}$ denote an amount of monetary payment from the buyer to a seller. Let $x \in \mathbb{X}$ denote the characteristics of the good. For any x, x' in $\mathbb{X}$, let $x \vee x'$ denote the least upper bound (join) of x and x', and $x \wedge x'$ denote the greatest lower bound (meet) of x and x'. If $\mathbb{X} \subseteq \mathbb{R}^n$, then the join of x and x' is the component-wise maximum and the meet is the component-wise minimum. A set $\mathbb{X}$ is a lattice if for any x and x' in $\mathbb{X}$, the joint and meet of x and x' exist as elements of $\mathbb{X}$.

Assumption 1. $\mathbb{X}$ is a compact metric space and a partially ordered lattice with a transitive, reflexive and antisymmetric order relation $\geq$.[7]

[7]An order relation is reflexive if $x \geq x$ for all $x \in \mathbb{X}$ and antisymmetric if $x \geq x'$ and $x' \geq x$

Each seller i receives a private signal $s_i \in [0,1]$ on the production costs of the good. Throughout the paper, the upper case letter S_i will denote seller i's signal as a random variable and the lower case letter s_i will denote its realization. The joint density of sellers' signals is denoted by $f : [0,1]^N \to \mathbb{R}_+$. When the buyer and seller i agree to execute a *contract* (x,t) given a vector of signals $s = [s_1, \ldots, s_N]$, seller i's payoff function is $t - c_i(x,s)$, the buyer's payoff function is $u(x) - t$, and the remaining sellers receive their reservation payoffs. If the buyer does not contract with any seller, the buyer and all sellers receive their reservation payoffs. All reservation payoffs are normalized to zero. Note that the payoff of the seller who contracts with the buyer depends on the other sellers' signals, so signals have interdependent values.

We will maintain the following assumptions on the buyer's and sellers' payoff functions, and on the joint density function f of sellers' signals.

Assumption 2 (i) $c_i(x,s)$ is bounded, measurable, and continuous in x at each s.
(ii) $u(x)$ is bounded, measurable and continuous.
(iv) $c_i(x,s)$ is strictly decreasing in s_i and non-increasing in s_{-i} at each x.
(v) For x and x' in $\mathbb{X}$ at each s,

$$u(x \wedge x') - c_i(x \wedge x', s) + u(x \vee x') - c_i(x \vee x', s) \geq \\ u(x') - c_i(x', s) + u(x) - c_i(x, s).$$

(vi) For any $x' \geq x$, $c_i(x',s) - c_i(x,s)$ is non-increasing in s.

$R_i(x,s) = u(x) - c_i(x,s)$ is the joint ex-post surplus between seller i and the buyer when seller i sells the good with characteristics x to the buyer, given a signal vector s. Assumption 2.(v) implies that the joint ex-post surplus function is supermodular in x at each s. Assumption 2.(vi) implies that $c_i(x,s)$ is single crossing in s, and it follows that the ex-post surplus function $R(x,s)$ is also single crossing in s.

Assumption 3 (i) $f(s)$ is measurable and strictly positive on $[0,1]^N$.
(ii) $f(s \vee s')f(s \wedge s') \geq f(s)f(s')$ for all $s, s' \in [0,1]^N$.

Assumption 3.(i) implies that, given any s_i, the support of i's conditional distribution on the other signals is $[0,1]$. Assumption 2.(ii) implies that signals are affiliated.

For each s, let

$$X_i^*(s) \equiv \arg\max_{x \in \mathbb{X}} R_i(x,s)$$

implies that $x = x'$.

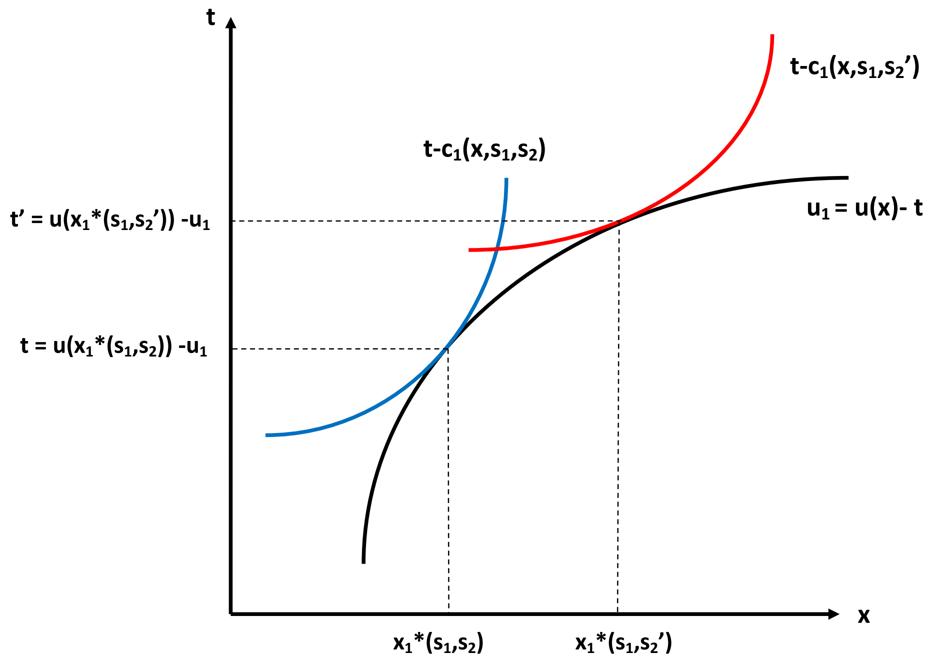

Figure 1: Preferences under interdependent values

be the set of jointly ex-post efficient characteristics of the good. $X_i^*(s)$ is non-empty because $R_i(x, s)$ is continuous in x at each s, and $\mathbb{X}$ is compact. Let $x_i^*(s)$ denote a typical element in $X_i^*(s)$. The following explains how a seller's preferences over contracts depend upon the vector of signals. Consider an example with two sellers and a one dimensional $\mathbb{X}$ as in figure 1. Suppose that seller 1 with signal s_1 is willing to give the buyer a payoff level u_1. In figure 1, the lower curve is the buyer's indifference curve associated with the utility level u_1 and the two curves above it represents the iso-profit curves for seller 1 with the signal s_1, one with seller 2's signal s_2 and the other with seller 2's signal s_2'.

Note that the buyer's payoff does not depend on the vector of production cost signals. While the buyer is indifferent between any contracts along her indifference curve, seller 1's payoff depends on both his signal and seller 2's signal. If seller 1 knew that seller 2's signal was s_2, seller 1, with the signal s_1, would prefer the jointly ex-post efficient contract $(x_1^*(s_1, s_2), u(x_1^*(s_1, s_2)) - u_1)$ given the payoff level u_1 that he is willing to give to the buyer. If he knew that seller 2's signal was s_2', seller 1, with the signal s_1, would prefer the jointly ex-post efficient contract $(x_1^*(s_1, s_2'), u(x_1^*(s_1, s_2')) - u_1)$. Because the seller's preferences over those contracts along the buyer's indifference curve depend on the vector of signals, it may create incentives for a seller to offer a menu of contracts to the buyer in which multiple jointly ex-post efficient contracts are optimal for the buyer, making it possible for the buyer to choose the one most preferred by the seller given the buyer's correct perception on the vector of signals in equilibrium. As will be explained later, with an absence of the buyer's commitment on what to choose, a menu with multiple optimal contracts for the buyer also leaves multiple continuation equilibria.

3 Non-Committed Procurement

In non-committed procurement, the buyer advertises an open invitation for potential sellers because she cannot commit herself to a scoring rule. Sellers then simultaneously submit menus of contracts. After reviewing and evaluating all menus submitted by sellers, the buyer accepts one or none of menus. If she accepts a menu, she then chooses a contract (x, t) from the menu. Those sellers whose menus are not accepted receive zero payoffs. Formally, a menu m_i that seller i submits is a closed subset of $\mathbb{X} \times \mathbb{R}$. Let M_i be the set of all feasible menus available to each seller i.

By reviewing and evaluating menu m_i, the buyer learns the maximum payoff that she can achieve from accepting m_i. For any menu m_i, define

$$D(m_i) \equiv \{(x, t) \in m_i : u(x) - t \geq u(x') - t', \forall (x', t') \in m_i\}.$$

The payoff from choosing any contract in $D(m_i)$ is the maximum payoff that the buyer can achieve once she accepts m_i. Given $m = [m_1, \ldots, m_N]$, let $\sigma_i(m) \in \Delta(m_i)$ be the probability distribution over the contracts in the menu m_i that the buyer choose a contract (x, t) from. The buyer's strategy for contract choices $\sigma = [\sigma_1, \ldots, \sigma_N]$ is a continuation equilibrium if the support of $\sigma_i(m)$ is a subset of $D(m_i)$ for all m and all i.[8] Let $\mathcal{C}$ be the set of all continuation equilibria. Given $m = [m_1, \ldots, m_N]$, $s = [s_1, \ldots, s_N]$, and $\sigma = [\sigma_1, \ldots, \sigma_N] \in \mathcal{C}$, seller i's payoff is

$$v_i(m, s, \sigma_i) = \int_{(x,t) \in D(m_i)} (t - c_i(x, s))\, d\sigma_i(m_i, m_{-i})$$

upon the buyer's acceptance of m_i.

The buyer's menu acceptance behavior in non-committed procurement is as follows. She accepts the menu that gives her the highest payoff among all submitted menus if the highest payoff is non-negative. If there are two or more such

[8] A continuation equilibrium consists of (i) the buyer's contracting decision strategies $\sigma = [\sigma_1, \ldots, \sigma_N]$ and (ii) her belief on sellers' signals contingent on their menu offers, which is formed according to Bayes' rule whenever possible. A continuation equilibrium is referred as only the buyer's contracting decision strategies unless specified because it is straightforward to assign an admissible belief for a truthful monotone equilibrium defined in Definition 1.

menus, she accepts either of them with equal probability. Let $\mathbf{m}_j : [0, 1] \to M_j$ denote seller j's menu strategy. Let $\mathbf{u}_j(s_j)$ denote the maximum payoff that the buyer can achieve by accepting $\mathbf{m}_j(s_j)$. That is, $\mathbf{u}_j(s_j)$ is the payoff that the buyer can achieve by choosing a contracting decision in $D(\mathbf{m}_j(s_j))$. Let u_i be the maximum payoff that the buyer can achieve by accepting seller i's menu m_i. Let $k_i(u_i, \mathbf{u}_{-i}(s_{-i}))$ be the number of sellers such that

$$k_i(u_i, \mathbf{u}_{-i}(s_{-i})) = \#\{j : \mathbf{u}_j(s_j) = u_i = \max_n \mathbf{u}_n(s_n) \geq 0\}.$$

Let $\lambda_i(u_i, \mathbf{u}_{-i}(s_{-i}))$ denote the probability that seller i wins the procurement when he offers a menu m_i that induces a maximum payoff u_i to the buyer, given the other sellers' menu offers $\mathbf{m}_{-i}(s_{-i})$:

$$\lambda_i(u_i, \mathbf{u}_{-i}(s_{-i})) = \begin{cases} 1/k_i(u_i, \mathbf{u}_{-i}(s_{-i})) & \text{if } k_i(u_i, \mathbf{u}_{-i}(s_{-i})) \geq 1, \\ 0 & \text{otherwise.} \end{cases}.$$

When the other sellers employ menu strategies $\mathbf{m}_{-i}$, seller i's interim payoff associated with his signal s_i and his menu offer m_i is

$$V_i(m_i, \mathbf{m}_{-i}|s_i, \sigma_i) = \mathbb{E}[v_i(m_i, \mathbf{m}_{-i}(S_{-i}), S, \sigma_i)\lambda_i(u_i, \mathbf{u}_{-i}(s_{-i}))|s_i].$$

Definition 1 $\{\mathbf{m}, \sigma\}$ *is a truthful monotone (pure-strategy) equilibrium if*

1. *for all $i \in \mathcal{N}$, all $m_i \in M_i$, and a.e. $s_i \in [0, 1]$, $\mathbf{m}_i$ satisfies*

$$V_i(\mathbf{m}_i(s_i), \mathbf{m}_{-i}|s_i, \sigma_i) \geq V_i(m_i, \mathbf{m}_{-i}|s_i, \sigma_i),$$

   given a continuation equilibrium $\sigma \in \mathcal{C}$.

2. *for all $i \in \mathcal{N}$, $\mathbf{m}_i(s_i) \neq \mathbf{m}_i(s'_i)$ if $s_i \neq s'_i$ and*

3. *for all $i \in \mathcal{N}$, $\mathbf{u}_i(s'_i) \geq \mathbf{u}_i(s_i)$ if $s'_i \geq s_i$.*

Definition 1.1 is self-explanatory in that $\mathbf{m}_i(s_i)$ is the best reply for seller i with signal s_i given the other sellers' strategies and the continuation equilibrium that the buyer chooses to follow for her choice of a contract upon accepting a menu. Definition 1.2 implies the truthfulness of each seller's menu strategy in the sense that the buyer can correctly infer each seller's signal from his menu offer. A menu is sufficiently general for a seller to reveal his signal on production costs. For example, even when seller i is willing to give the same level of payoff u_i to the buyer under two different signals, he can construct two different menus m_i and m'_i that can induce the same maximum payoff u_i.[9] Definition 1.3 implies

[9]Although m_i and m'_i induce the same maximum payoff u_i for the buyer, the sets of optimal contracts for the buyer can be different ($D(m_i) \neq D(m'_i)$) so that $m_i \neq m'_i$. Alternatively, m_i and m'_i may have the same set of optimal contracts for the buyer ($D(m_i) = D(m'_i)$) but they can have different contracts that yield payoffs lower than u_i so that $m_i \neq m'_i$.

the monotonicity of each seller's menu strategy in the sense that if his signal is higher, a seller offers a menu that yields a higher payoff to the buyer.

The potential difficulty in analyzing the equilibrium is the multiplicity of continuation equilibrium that will arise when a menu leaves multiple optimal contracts available to the buyer. Not only should a seller consider competing sellers' menu strategies, but he should also form a correct belief over the continuation equilibrium that the buyer would choose to follow from among the multiple continuation equilibria. Given the other sellers' non-decreasing strategies, let $A(u_i)$ be the event that the maximum payoff that the other sellers' menus can give to the buyer is no greater than u_i:

$$A(u_i) = \left\{ s_{-i} \in [0,1]^{N-1} : \max_{j \neq i} \mathbf{u}_j(s_j) \leq u_i \right\}, \tag{1}$$

where $\mathbf{u}_j(s_j)$ is the maximum payoff that the buyer can achieve by accepting the menu $\mathbf{m}_j(s_j)$.

If $\Pr(A(u_i))|s_i) = 0$, then $\mathbb{E}[\,(R(x,S) - u_i)\,\lambda_i(u_i, \mathbf{u}_{-i}(S_{-i}))|A(u_i), s_i] = 0$ because $\lambda_i(u_i, \mathbf{u}_{-i}(s_{-i})) = 0$ for all s_{-i}, given s_i and u_i. When seller i's menu m_i induces a maximum payoff u_i to the buyer and she chooses the contract (x,t) with $t = u(x) - u_i$, upon accepting m_i, seller i's interim payoff is

$$\begin{aligned}
&\mathbb{E}[\,(t - c_i(x,S))\,\lambda_i(u_i, \mathbf{u}_{-i}(S_{-i}))|s_i] = \\
&\qquad \mathbb{E}[\,(u(x) + c_i(x,S) - u_i)\,\lambda_i(u_i, \mathbf{u}_{-i}(S_{-i}))|s_i] = \\
&\qquad\qquad \Pr(A(u_i))|s_i)\mathbb{E}[\,(R(x,S) - u_i)\,\lambda_i(u_i, \mathbf{u}_{-i}(S_{-i}))|A(u_i), s_i].
\end{aligned}$$

Consider the jointly interim efficient and the jointly ex-post efficient characteristics of the good when seller i sells the good to the buyer given the other sellers' non-decreasing strategies. When u_i is the maximum payoff that the buyer can achieve from a menu m_i offered by seller i with signal s_i, and she chooses (x,t) with $t = u(x) - u_i$ from m_i, the joint interim surplus conditional on $(A(u_i), s_i)$ becomes $\mathbb{E}[R_i(x,S)\lambda_i(u_i, \mathbf{u}_{-i}(S_{-i}))|A(u_i), s_i]$. Given $(A(u_i), s_i, \mathbf{u}_{-i})$, let

$$X_i^e(A(u_i), s_i, \mathbf{u}_{-i}) = \arg\max_{x \in \mathbb{X}} \mathbb{E}[R_i(x,S)\lambda_i(u_i, \mathbf{u}_{-i}(S_{-i}))|A(u_i), s_i] \tag{2}$$

be the set of jointly interim efficient characteristics of the good.

Because $R_i(x,s)$ is continuous in x at each s and $\lambda_i(u_i, \mathbf{u}_{-i}(s_{-i}))$ is a real number in $[0,1]$, $R_i(x,s)\lambda_i(u_i, \mathbf{u}_{-i}(s_{-i}))$ is continuous in x at each s. It implies that $\mathbb{E}[R_i(x,S)\lambda_i(u_i, \mathbf{u}_{-i}(S_{-i}))|A(u_i), s_i]$ is continuous in x. Because $\mathbb{X}$ is compact and $\mathbb{E}[R_i(x,S)\lambda_i(u_i, \mathbf{u}_{-i}(S_{-i}))|A(u_i), s_i]$ is continuous in x, $X_i^e(A(u_i), s_i, \mathbf{u}_{-i})$ is non-empty by Weierstrass' Theorem. Let $x_i^e(u_i, s_i, \mathbf{u}_{-i})$ denote a typical element in $X_i^e(A(u_i), s_i, \mathbf{u}_{-i})$. If $\mathbb{E}[(R(x,S) - u_i)\lambda_i(u_i, \mathbf{u}_{-i}(S_{-i}))|A(u_i), s_i] = 0$ for all $x \in \mathbb{X}$, then let $X_i^e(u_i, s_i, \mathbf{u}_{-i}) = \{x_\circ\}$, where $x_\circ \in \mathbb{X}$ denotes the status-quo action such as no trading, that makes $R_i(x_\circ, s) = v_i(x_\circ, s) + u(x_\circ) = 0$ for all $s \in [0, 1]$. Let $x \geq x_\circ$ for all $x \in \mathbb{X}$.

Suppose that the buyer will always choose the same contract from a seller's menu regardless of her perception on all sellers' production cost signals after she evaluates all menus. In equilibrium, the same contract that the buyer chooses must be a jointly interim efficient contract because it is always feasible for a seller to submit a degenerate menu consisting of a single contract. Therefore, the equilibrium is only jointly interim efficient even though sellers' signals are fully revealed to the buyer in equilibrium.

Contrarily, suppose that the buyer will always choose a jointly ex-post efficient contract from a seller's menu when it is available in the menu and optimal to the buyer. Consider the jointly ex-post efficient contract $(x_i^*(s), u(x_i^*(s)) - \mathbf{u}_i(s_i))$ associated with the maximum payoff $\mathbf{u}_i(s_i)$ that seller i is willing to give to the buyer at each s_i , where $s = (s_i, s_{-i})$. Given s_i and $\mathbf{u}_i(s_i)$, seller i may include every jointly ex-post efficient contract $(x_i^*(s), u(x_i^*(s)) - \mathbf{u}_i(s_i))$ for every s_{-i}. Because the buyer is fully aware of sellers' production cost signals after evaluating their menus in a truthful monotone equilibrium, she can choose the jointly ex-post efficient contract $(x_i^*(s), u(x_i^*(s)) - \mathbf{u}_i(s_i))$ from seller i's menu at every s as long as $\mathbf{u}_i(s_i)$ is the maximum payoff that she can achieve from accepting seller i's menu. When all sellers believe that the buyer would always choose such a jointly ex-post efficient contract, they will submit their menus accordingly and the equilibrium is jointly ex-post efficient.[10]

When a menu includes multiple optimal contracts for the buyer, it leads to multiple continuation equilibria. If it was optimal, the buyer could always choose a jointly ex-post efficient contract from a menu. Alternatively, the buyer may choose the same optimal contract regardless of her perception on sellers' signals. In this case, the contract that the buyer chooses must be jointly interim efficient because a seller can always offer a degenerate menu consisting of a single contract. Finally, the buyer may even randomize her choice between a jointly ex-post efficient contract and a jointly interim efficient one in a menu.

4 Truthful Monotone Equilibria

Because the buyer cannot commit herself to a scoring rule, sellers' menu offers depend on their beliefs over how likely the buyer will choose a jointly ex-post efficient contract in a continuation equilibrium when it is one of the optimal contracts for her within a menu. We establish the existence of a truthful monotone

[10] As in first-price auctions with asymmetric bidders, the equilibrium may not be fully ex-post efficient with asymmetric sellers even when the buyer always chooses a jointly ex-post efficient contract from the winning seller's menu. This is because the buyer may not accept a menu offered by a seller with whom she can maximize the joint ex-post surplus.

equilibrium by taking two steps. As the first step, this section considers a modified procurement in which (i) each seller i directly submits the buyer's payoff bid u_i along with the characteristics x_i of the good and (ii) the buyer knows sellers' signals but each seller knows only his own signal. When the buyer accepts contract (u_i, x_i) from seller i in this modified procurement, she chooses x_i and makes the monetary payment $t_i = u(x_i) - u_i$ with probability $1 - \tau_i$ so that $(x_i, u(x_i) - u_i)$ generates her a payoff that exactly matches the payoff bid u_i that seller i submits. With probability τ_i, the buyer chooses the jointly ex-post efficient contract $(x_i^*(s), u(x_i^*(s)) - u_i)$, at each s, that generates the payoff bid u_i submitted by seller i. We complete the first step by showing the existence of a monotone equilibrium in this modified procurement with any arbitrary $\tau = [\tau_1, \ldots, \tau_N] \in [0, 1]^N$. In the second step, we show that for a monotone equilibrium in this modified procurement with any given τ, there exists a payoff-equivalent truthful monotone equilibrium in the non-committed procurement in which the buyer chooses, upon accepting a menu, a jointly ex-post efficient contract with probability τ_i and a jointly interim efficient contract with probability $1 - \tau_i$ from each seller i's equilibrium menu. Because there exists a truthful monotone equilibrium for any given $\tau \in [0, 1]^N$, there exists a continuum of truthful monotone equilibria that spans the entire space of $[0, 1]^N$.

4.1 Modified Procurement

In the modified procurement, each seller i submits a bid (u_i, x_i) from $U_i \times \mathbb{X}$, where $U_i = [0, \infty) \cup \{u_\circ\}$ with $u_\circ < 0$. Let u_i be a payoff bid and x_i be a bid for the characteristics of the good. When seller i submits $u_\circ$, he must submit $x_\circ$ along with it. Let $(u_\circ, x_\circ)$ be the losing bid regardless of the bids submitted by other sellers. If $(u_i, x_i) \neq (u_\circ, x_\circ)$, it is called a serious bid. Assume that each seller knows only his own signal but the buyer knows every seller's signal. The buyer chooses a seller whose payoff bid is the highest non-negative bid from among all sellers' payoff bids. If there are two or more sellers who submit the highest non-negative payoff bid, the buyer chooses either of them with equal probability. When seller i wins the procurement with $(u_i, x_i) \in U_i \times \mathbb{X}$, the buyer buys the good with characteristics x_i from seller i by paying $t_i = u(x_i) - u_i$ with probability $1 - \tau_i$, but she buys the good with jointly ex-post efficient characteristics $x_i^*(s) \in X_i^*(s)$ by paying $t_i = u(x_i^*(s)) - u_i$ for all $s \in [0, 1]^N$ with probability τ_i. We fix $\tau = [\tau_1, \ldots, \tau_N]$ as part of the procurement rule and it is known to sellers.

We now examine the existence of a monotone equilibrium in the modified procurement with any given $\tau = [\tau_1, \ldots, \tau_N] \in [0, 1]^N$. Seller i's strategy is a pair consisting of a payoff bidding function, $\mathbf{u}_i : [0, 1] \to U_i$, and a bidding function for the characteristics of the good, $\mathbf{x}_i : [0, 1] \to \mathbb{X}$. Suppose that the other sellers' strategies are non-decreasing: i.e., for all $j \neq i$, $\mathbf{u}_j(s_j') \geq \mathbf{u}_j(s_j)$ and

$\mathbf{x}_j(s'_j) \geq \mathbf{x}_j(s_j)$ if $s'_j \geq s_j$. Seller i's interim payoff associated with submitting (u_i, x_i) is

$$\begin{aligned} &V_i(u_i, x_i, \mathbf{u}_{-i}|s_i) = \\ &\quad (1-\tau_i)\Pr(A(u_i)|s_i)\mathbb{E}\left[(R_i(x_i, S) - u_i)\lambda_i(u_i, \mathbf{u}_{-i}(S_{-i}))|A(u_i), s_i\right] + \\ &\qquad \tau_i \Pr(A(u_i)|s_i)\mathbb{E}\left[(R_i(x_i^*(S), S) - u_i)\,\lambda_i(u_i, \mathbf{u}_{-i}(S_{-i}))|A(u_i), s_i\right]. \end{aligned} \quad (3)$$

Alternatively, we can express seller i's interim payoff as follows. First, define the surplus between seller i and the buyer as

$$R_i^\tau(x_i, s) = (1-\tau_i)\left[v_i(x_i, s) + u(x_i)\right] + \tau_i\left[v_i(x_i^*(s), s) + u(x_i^*(s))\right] \quad (4)$$

when seller i wins the procurement with a characteristics bid x_i and an arbitrary payoff bid. Because of Assumption 1.(v)-(vi), $R_i^\tau(x_i, s)$ is supermodular in x_i and $R_i^\tau(x'_i, s) - R_i^\tau(x_i, s)$ is non-decreasing in s whenever $x'_i \geq x_i$. Seller i's interim payoff associated with submitting (u_i, x_i) can be expressed as

$$\begin{aligned} &V_i(u_i, x_i, \mathbf{u}_{-i}|s_i) = \\ &\qquad \Pr(A(u_i)|s_i)\mathbb{E}\left[(R_i^\tau(x_i, S) - u_i)\,\lambda_i(u_i, \mathbf{u}_{-i}(S_{-i}))|A(u_i), s_i\right]. \end{aligned} \quad (5)$$

Theorem 1 below establishes the existence of a monotone equilibrium in the modified procurement with any given $\tau = [\tau_1, \ldots, \tau_N] \in [0,1]^N$.

Theorem 1 *For any given $\tau = [\tau_1, \ldots, \tau_N] \in [0,1]^N$, the modified procurement possesses a monotone equilibrium $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$.*

Theorem 1 is closely related to Reny (2011) because a bid (u_i, x_i) is multidimensional. Theorem 4.1 in Reny (2011) demonstrates that if certain conditions (G.1-G.5 in Reny 2011) on players' actions, payoff functions, and types are satisfied and each player's set of monotone best replies is nonempty and join-closed whenever the others employ monotone pure strategies, then a Bayesian game possesses a monotone equilibrium. In our modified procurement, let $B_i(s_i)$ be the set of best replies for seller i with signal s_i when the other sellers employ monotone strategies so that $B_i(s_i)$ includes every (u_i, x_i) that maximizes seller i's interim payoff given the other sellers' monotone strategies.

$B_i(\cdot)$ is monotone if for any monotone strategies of the other sellers $(\mathbf{u}_{-i}, \mathbf{x}_{-i})$, whenever $(u_i, x_i) \in B_i(s_i)$ and $(u'_i, x'_i) \in B_i(s'_i)$ for $s'_i \geq s_i$, then $(u_i \vee u'_i, x_i \vee x'_i) \in B_i(s'_i)$. This monotonicity is strictly weaker than the increasing property of best replies in the strong order set (Milgrom and Shannon 1994). $B_i(\cdot)$ is join-closed if $(u_i, x_i) \in B_i(s_i)$, and $(u'_i, x'_i) \in B_i(s_i)$ implies that $(u_i \vee u'_i, x_i \vee x'_i) \in B_i(s_i)$.

Reny establishes the existence of a monotone equilibrium ingeniously by utilizing a fixed-point theorem based on contractibility rather than the convexity of best replies (Athey 2001, McAdams 2006). Proposition 4.4 in Reny (2011) provides a convenient sufficient condition for the existence of a monotone equilibrium: If the set of bids for each player is a lattice and each player's interim payoff function is weakly single crossing and weakly quasisupermodular, then each player' set of monotone best replies is non-empty and join-closed. To see this point, fix the other sellers' monotone strategies. Each seller i's interim payoff function is weakly single crossing if, for all pairs of bids $(u_i', x_i') \geq (u_i, x_i)$ and all pairs of signals $s_i' \geq s_i$,

$$V_i(u_i', x_i', \mathbf{u}_{-i}|s_i) \geq V_i(u_i, x_i, \mathbf{u}_{-i}|s_i) \Longrightarrow V_i(u_i', x_i', \mathbf{u}_{-i}|s_i') \geq V_i(u_i, x_i, \mathbf{u}_{-i}|s_i').$$

Each seller i's interim payoff function is weakly quasisupermodular if, for all $(u_i, x_i), (u_i', x_i') \in U_i \times \mathbb{X}$ and all $s_i \in [0, 1]$,

$$V_i(u_i, x_i, \mathbf{u}_{-i}|s_i) \geq V_i(u_i \wedge u_i', x_i \wedge x_i', \mathbf{u}_{-i}|s_i) \Longrightarrow \\ V_i(u_i \vee u_i', x_i \vee x_i', \mathbf{u}_{-i}|s_i) \geq V_i(u_i', x_i', \mathbf{u}_{-i}|s_i).$$

The idea behind these conditions is straightforward. Consider any pair of best replies such that $(u_i, x_i) \in B_i(s_i)$ and $(u_i', x_i') \in B_i(s_i')$ for $s_i' \geq s_i$. Because $(u_i, x_i) \in B_i(s_i)$, we have

$$V_i(u_i, x_i, \mathbf{u}_{-i}|s_i) \geq V_i(u_i \wedge u_i', x_i \wedge x_i', \mathbf{u}_{-i}|s_i).$$

By weakly single crossing, the inequality above implies

$$V_i(u_i, x_i, \mathbf{u}_{-i}|s_i') \geq V_i(u_i \wedge u_i', x_i \wedge x_i', \mathbf{u}_{-i}|s_i').$$

Applying weakly quasisupermodularity, this inequality implies

$$V_i(u_i \vee u_i', x_i \vee x_i', \mathbf{u}_{-i}|s_i') \geq V_i(u_i', x_i', \mathbf{u}_{-i}|s_i').$$

Because $(u_i', x_i') \in B_i(s_i')$, the last inequality implies $(u_i \vee u_i', x_i \vee x_i') \in B_i(s_i')$ so that $B_i(\cdot)$ is monotone. By setting up $s_i = s_i'$, the join-closedness follows as well.

However, a seller's interim payoff function in the modified procurement may fail to be weakly single crossing and/or weakly quasisupermodular at irrational bids or rational bids with a positive probability of ties, as a seller's interim payoff function in first price auctions with single dimensional bids (i.e., real numbers) may fail to satisfy Athey's (2001) single crossing property.[11] Reny and Zamir (2004) avoid this problem and establish the existence of a monotone equilibrium in a first price auction with single dimensional bids by considering limits of ever finer finite bid sets such that no two sellers have a common serious bid and by

[11] See the examples in Reny and Zamir (2004) that show the failure of the single crossing property.

recalling that single crossing is needed only at individually rational bids, i.e., "individually rational tieless single-crossing condition" (IRT-SCC).

For the existence of a monotone equilibrium in the modified procurement, this paper proposes the "tieless supermodular condition" (TLS-SMC) and extends Reny and Zamir's IRT-SCC. TLS-SMC, together with IRT-SCC, ensures both the weakly single crossing condition and the weakly quasisupermodular condition at individually rational tieless bids. We formally define TLS-SMC in Definition 2 below.

Definition 2 *The modified procurement satisfies TLS-SMC if, for each seller i, any (u_i, x_i), (u'_i, x'_i) with $\Pr[u_\circ < \max_{j\neq i} \mathbf{u}_j(s_j) = u_i \text{ or } u'_i] = 0$ given any non-decreasing payoff bidding functions $\mathbf{u}_{-i}$ of the other sellers, the following inequality holds:*

$$V_i(u_i \vee u'_i, x_i \vee x'_i, \mathbf{u}_{-i}|s_i) - V_i(u'_i, x'_i, \mathbf{u}_{-i}|s_i) \geq V_i(u_i, x_i, \mathbf{u}_{-i}|s_i) - V_i(u_i \wedge u'_i, x_i \wedge x'_i, \mathbf{u}_{-i}|s_i). \quad (6)$$

TLS-SMC implies that for any two tieless serious bids (u_i, x_i) and (u'_i, x'_i),

$$V_i(u_i, x_i, \mathbf{u}_{-i}|s_i) \geq V_i(u_i \wedge u'_i, x_i \wedge x'_i, \mathbf{u}_{-i}|s_i) \Longrightarrow V_i(u_i \vee u'_i, x_i \vee x'_i, \mathbf{u}_{-i}|s_i) \geq V_i(u'_i, x'_i, \mathbf{u}_{-i}|s_i) \quad (7)$$

and

$$V_i(u'_i, x'_i, \mathbf{u}_{-i}|s_i) \geq V_i(u_i \vee u'_i, x_i \vee x'_i, \mathbf{u}_{-i}|s_i) \Longrightarrow V_i(u_i \wedge u'_i, x_i \wedge x'_i, \mathbf{u}_{-i}|s_i) \geq V_i(u_i, x_i, \mathbf{u}_{-i}|s_i).$$

The extension of Reny and Zamir's IRT-SCC is given below.

Definition 3 *The modified procurement satisfies IRT-SCC if, for each seller i, all pairs of $(\overline{u}_i, \overline{x}_i)$ and $(\underline{u}_i, \underline{x}_i)$ such that (i) $\overline{u}_i \geq \underline{u}_i$ and $\overline{x}_i \geq \underline{x}_i$ and (ii) $\Pr[u_\circ < \max_{j\neq i} \mathbf{u}_j(s_j) = \underline{u}_i \text{ or } \overline{u}_i] = 0$ given any non-decreasing payoff bidding functions $\mathbf{u}_{-i}$ for the other sellers, the following condition is satisfied: If $V_i(\overline{u}_i, \overline{x}_i, \mathbf{u}_{-i}|s_i) \geq 0$, then*

$$V_i(\overline{u}_i, \overline{x}_i, \mathbf{u}_{-i}|s_i) \geq V_i(\underline{u}_i, \underline{x}_i, \mathbf{u}_{-i}|s_i) \Longrightarrow V_i(\overline{u}_i, \overline{x}_i, \mathbf{u}_{-i}|s'_i) \geq V_i(\underline{u}_i, \underline{x}_i, \mathbf{u}_{-i}|s'_i) \quad (8)$$

for all $s'_i \geq s_i$. If $V_i(\underline{u}_i, \underline{x}_i, \mathbf{u}_{-i}|s'_i) \geq 0$ for any $s'_i \geq s_i$, then

$$V_i(\underline{u}_i, \underline{x}_i, \mathbf{u}_{-i}|s'_i) \geq V_i(\overline{u}_i, \overline{x}_i, \mathbf{u}_{-i}|s'_i) \Longrightarrow V_i(\underline{u}_i, \underline{x}_i, \mathbf{u}_{-i}|s_i) \geq V_i(\overline{u}_i, \overline{x}_i, \mathbf{u}_{-i}|s_i).$$

TLS-SMC requires that bids be serious and tieless but the individual rationality of bids (i.e., $V_i(\overline{u}_i, \overline{x}_i, \mathbf{u}_{-i}|s_i) \geq 0$, $V_i(\underline{u}_i, \underline{x}_i, \mathbf{u}_{-i}|s'_i) \geq 0$) is also required for IRT-SCC. Lemma 1 enables us to apply Theorem 4.1 in Reny (2011) when proving the existence of a monotone equilibrium.

Lemma 1 *The modified procurement satisfies TLS-SMC and IRT-SCC.*

We first consider the finite modified procurement in which (i) the set of feasible bids for each seller i is given by $U_i^n \times \mathbb{X}$, where U_i^n is a finite set including $u_\circ$ and satisfies that for any $u_i \in U_i^n$, $u_i \neq 0$ implies that $u_i \geq 0$, and (ii) U_i^n and U_j^n do not have any common serious payoff bids for any $i \neq j$. Therefore, TLS-SMC and IRT-SCC are satisfied in the finite modified procurement. As in Reny and Zamir (2004) and Athey (2001), each seller is restricted to submit the losing bid $(u_\circ, x_\circ)$ whenever his signal is in $[0, \varepsilon)$, where $\varepsilon = 1/n$ with n being a natural number. In the finite modified procurement, n is fixed so that $[0, \varepsilon)$ has positive measure but the measure of $[0, \varepsilon)$ converges to zero as $n \to \infty$.

Lemma 2 *The finite modified procurement possesses a monotone equilibrium.*

Proof. First, we show that the set of each seller i's best replies is monotone. Consider $B_i(s_i)$ and $B_i(s_i')$ for seller i, one under s_i and the other with s_i' such that $s_i' \geq s_i \geq \varepsilon$ given the other sellers' monotone strategies. For Reny's monotonicity of the set of seller i's best replies, it is sufficient to show that whenever a best reply (u_i, x_i) in $B_i(s_i)$ is not $(u_\circ, x_\circ)$, then $(u_i \vee u_i', x_i \vee x_i') \in B_i(s_i')$ for any $(u_i', x_i') \in B_i(s_i')$.

Suppose that $(u_i', x_i') \neq (u_\circ, x_\circ)$. Because $(u_i, x_i) \neq (u_\circ, x_\circ)$ is in $B_i(s_i)$ and $(u_\circ, x_\circ)$ is always feasible, we have

$$V_i(u_i, x_i, \mathbf{u}_{-i}|s_i) \geq V_i(u_i \wedge u_i', x_i \wedge x_i', \mathbf{u}_{-i}|s_i). \tag{9}$$

Note that both (u_i, x_i) and (u_i', x_i') are not $(u_\circ, x_\circ)$. Because no two payoff bid sets have any serious bid in common, $\Pr[u_\circ < \max_{j \neq i} \mathbf{u}_j(s_j) = u_i \text{ or } u_i'] = 0$. Furthermore, (u_i, x_i) is individually rational because it is in $B_i(s_i)$. Invoking (8) in IRT-SCC, (9) yields

$$V_i(u_i, x_i, \mathbf{u}_{-i}|s_i') \geq V_i(u_i \wedge u_i', x_i \wedge x_i', \mathbf{u}_{-i}|s_i'). \tag{10}$$

Applying (7) from TLS-SMC, (10) implies

$$V_i(u_i \vee u_i', x_i \vee x_i', \mathbf{u}_{-i}|s_i') \geq V_i(u_i', x_i', \mathbf{u}_{-i}|s_i') \tag{11}$$

so that $(u_i \vee u_i', x_i \vee x_i')$ is in $B_i(s_i')$.

Suppose that $(u_i', x_i') = (u_\circ, x_\circ)$. The interim payoff for seller i with signal s_i, associated with $(u_i, x_i) \in B_i(s_i)$ such that $(u_i, x_i) \neq (u_\circ, x_\circ)$, is

$$V_i(u_i, x_i, \mathbf{u}_{-i}|s_i) = \Pr(A(u_i)|s_i)\mathbb{E}[R_i^\tau(x_i, S) - u_i|A(u_i), s_i] \geq 0, \tag{12}$$

where the inequality holds because $(u_i, x_i) \in B_i(s_i)$, the losing bid $(u_\circ, x_\circ)$ is always feasible for seller i, and $u_i \neq u_\circ$ does not tie with the other sellers' payoff bids. Because every other seller j submits $(u_\circ, x_\circ)$ when his signal is in $[0, \varepsilon)$ and the joint density of signals is strictly positive on $[0, 1]^N$, $(u_i, x_i) \neq (u_\circ, x_\circ)$ wins the procurement with positive probability, i.e., $\Pr(A(u_i)|s_i) > 0$, for any s_i. It follows that (12) implies

$$\mathbb{E}[R_i^\tau(x_i, S) - u_i|A(u_i), s_i] \geq 0. \tag{13}$$

Because $R_i^\tau(x_i, s) - u_i$ is non-decreasing in s_i, Theorem 5 in Milgrom and Weber leads (13) to

$$\mathbb{E}[R_i^\tau(x_i, S) - u_i|A(u_i), s_i'] \geq 0. \tag{14}$$

From (14), we have

$$V_i(u_i, x_i, \mathbf{u}_{-i}|s_i') = \Pr(A(u_i)|s_i')\mathbb{E}[R_i^\tau(x, S) - u|A(u_i), s_i'] \geq 0. \tag{15}$$

Because $(u_i', x_i') = (u_\circ, x_\circ) \in B_i(s_i')$, (15) implies that

$$V_i(u_i \vee u_i', x_i \vee x_i', \mathbf{u}_{-i}|s_i') = V_i(u_i, x_i, \mathbf{u}_{-i}|s_i') \geq V_i(u_i', x_i', \mathbf{u}_{-i}|s_i') = 0 \tag{16}$$

so that $(u_i \vee u_i', x_i \vee x_i') = (u_i, x_i) \in B_i(s_i')$

(11) and (16) show that whenever a best reply (u_i, x_i) in $B_i(s_i)$ is not $(u_\circ, x_\circ)$, then $(u_i \vee u_i', x_i \vee x_i') \in B_i(s_i')$ for any $(u_i', x_i') \in B_i(s_i')$ and any s_i' and s_i such that $s_i' \geq s_i \geq \varepsilon$. Therefore, Reny's monotonicity of the set of seller i's best replies goes through.

Examine the join-closedness of $B_i(s_i)$. Reny's monotonicity of the set of seller i's best replies states that if $s_i' \geq s_i$, $(u_i, x_i) \in B_i(s_i)$ and $(u_i', x_i') \in B_i(s_i')$ imply $(u_i \vee u_i', x_i \vee x_i') \in B_i(s_i')$. Setting $s_i' = s_i$ yields that $(u_i, x_i) \in B_i(s_i)$ and $(u_i', x_i') \in B_i(s_i)$ imply $(u_i \vee u_i', x_i \vee x_i') \in B_i(s_i)$. Therefore, $B_i(s_i)$ is join-closed.

Examine the non-emptiness of $B_i(s_i)$. A characteristics bid x_i that seller i submits along with a payoff bid u_i does not affect the winning event, and it is chosen with probability τ_i conditional on seller i winning the procurement. Furthermore, the buyer takes the monetary payment $t_i = u(x_i) - u_i$ given, in this case, seller i's bid (u_i, x_i). If seller i with signal s_i submits a payoff bid u_i for any $u_i \in U_i^n$, it is optimal for him to submit the jointly interim efficient characteristics $x_i^e(A(u_i), s_i, \mathbf{u}_{-i})$ in $X_i^e(A(u_i), s_i, \mathbf{u}_{-i})$, as defined in (2). Note that $X_i^e(A(u_i), s_i, \mathbf{u}_{-i})$ is non-empty. Because U_i^n is a finite set and $V_i(u_i, x_i^e(A(u_i), s_i, \mathbf{u}_{-i}), \mathbf{u}_{-i}|s_i)$ is bounded, there exists a payoff bid in U_i^n that maximizes $V_i(u_i, x_i^e(A(u_i), s_i, \mathbf{u}_{-i}), \mathbf{u}_{-i}|s_i)$. Therefore, $B_i(s_i)$ is non-empty.

Finally, U_i^n is a finite set and a lattice. This property and assumption 1 lead $U_i^n \times \mathbb{X}$ to satisfy G.3 and G.4 in Reny (2011). Assumption 3.(i) and 2.(i)-(ii) satisfy G.1, G.2, and G.5; therefore, Reny's conditions (G.1-G.5) on players' actions, payoff functions and types are all satisfied. The existence of a monotone equilibrium is established by Theorem 4.1 in Reny (2011). ■

Let $\{(\mathbf{u}_1^n, \mathbf{x}_1^n), \ldots, (\mathbf{u}_N^n, \mathbf{x}_N^n)\}$ be a monotone equilibrium in the modified procurement game G^n, with an arbitrary $\tau = [\tau_1, \ldots, \tau_N] \in [0,1]^N$, in which seller i's finite set of payoff bids is denoted by U_i^n and, hence, the set of payoff and characteristics bids is $U_i^n \times \mathbb{X}$. We assume that $U_i^{n-1} \subseteq U_i^n$ and that $\bigcup_n U_i^n$ is dense in U_i. The proof of Theorem 1 is completed by showing that the limit $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$ of $\{(\mathbf{u}_1^n, \mathbf{x}_1^n), \ldots, (\mathbf{u}_N^n, \mathbf{x}_N^n)\}$ is a monotone equilibrium in the modified procurement without restrictions on the sets of payoff bids. The proof also shows that the probability that, under the limit of sellers' equilibrium strategies, two or more sellers simultaneously submit the highest payoff bid above $u_\circ$ is zero. In a monotone equilibrium $\{(\mathbf{u}_1^n, \mathbf{x}_1^n), \ldots, (\mathbf{u}_N^n, \mathbf{x}_N^n)\}$ in the finite modified procurement, $\mathbf{x}_i^n(s_i)$ is necessarily jointly interim efficient conditional on $(A(\mathbf{u}_i^n(s_i)), s_i, \mathbf{u}_{-i}^n)$ given no possibility of ties at any payoff bid above $u_\circ$. Because the probability that two or more sellers simultaneously submit the highest payoff bid above $u_\circ$ is zero in the monotone equilibrium $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$, the limit $\hat{\mathbf{x}}_i(s_i)$ of $\mathbf{x}_i^n(s_i)$ is also jointly interim efficient conditional on $(A(\hat{\mathbf{u}}_i(s_i)), s_i, \hat{\mathbf{u}}_{-i})$, given no possibility of ties at the equilibrium payoff bid $\hat{\mathbf{u}}_i(s_i)$ above $u_\circ$.[12]

4.2 Existence of Truthful Monotone Equilibria

Theorem 2 below demonstrates that for any monotone equilibrium in the modified procurement with any given $\tau = [\tau_1, \ldots, \tau_N] \in [0,1]^N$, we can find a payoff-equivalent truthful monotone equilibrium in the non-committed procurement in which each seller i believes that the buyer, upon accepting his equilibrium menu, would optimally choose a jointly ex-post efficient contract with probability τ_i and a jointly interim efficient contract with probability $1 - \tau_i$.

Theorem 2 *For a monotone equilibrium $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$ in the modified procurement with any given $\tau = [\tau_1, \ldots, \tau_N] \in [0,1]^N$, there exists a truthful monotone equilibrium $\{\mathbf{m}_1^\tau, \ldots, \mathbf{m}_N^\tau, \sigma^\tau\}$ in the non-committed procurement such that*

$$V_i(\mathbf{m}_i^\tau(s_i), \mathbf{m}_{-i}^\tau | s_i, \sigma_i^\tau) = V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i} | s_i)$$

for each i and all s_i.

Fix a monotone equilibrium $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$ in the modified procurement with any given $\tau = [\tau_1, \ldots, \tau_N] \in [0,1]^N$. Menus are sufficiently general to make it possible for each seller i to reveal his signal by offering a signal-contingent menu. Let each seller i with signal s_i choose a menu $\mathbf{m}_i^\tau(s_i)$ that satisfies

(i) the maximum payoff that the buyer can achieve by accepting $\mathbf{m}_i^\tau(s_i)$ is the same as $\hat{\mathbf{u}}_i(s_i)$ for all $s_i \in [0,1]$ and,

[12]Each bidder may face at most a finite number of ties with positive probability off the equilibrium path.

(ii) $(x_i^*(s), u(x_i^*(s)) - \hat{\mathbf{u}}_i(s_i)) \in D(\mathbf{m}_i^\tau(s_i))$ for all $s \in [0,1]^N$ and, $(\hat{\mathbf{x}}_i(s_i), u(\hat{\mathbf{x}}_i(s_i)) - \hat{\mathbf{u}}_i(s_i)) \in D(\mathbf{m}_i^\tau(s_i))$ for all $s_i \in [0,1]$ and,

(iii) $\mathbf{m}_i^\tau(s_i) \neq \mathbf{m}_i^\tau(s_i')$ if $s_i \neq s_i'$.

When each seller i offers a menu according to the strategy $\mathbf{m}_i^\tau$, the buyer can correctly infer the seller's true signal, allowing the buyer to optimally choose a jointly ex-post efficient contract $(x_i^*(s), u(x_i^*(s)) - \hat{\mathbf{u}}_i(s_i)) \in D(\mathbf{m}_i^\tau(s_i))$ upon accepting $\mathbf{m}_i^\tau(s_i)$. Note that if a seller submits the losing bid in the modified procurement, it is equivalent to offering menus that are not acceptable to the buyer in the non-committed procurement. Even when it is optimal for a seller not to win the procurement given the other sellers' menu strategies, he can reveal his true signal in the non-committed procurement by making a non-acceptable menu offer contingent on his signal.[13]

The key to Theorem 2 is to assign a continuation equilibrium (i.e., the buyer's optimal contracting decision choice rule upon accepting a menu) in the non-committed procurement. Suppose that each seller i with signal s_i believes that if the buyer accepted his menu $\mathbf{m}_i^\tau(s_i)$, she would optimally choose a jointly ex-post efficient contract in $D(\mathbf{m}_i^\tau(s_i))$ with probability τ_i and a jointly interim efficient contract in $D(\mathbf{m}_i^\tau(s_i))$ with probability $1 - \tau_i$. It is certainly an optimal contracting decision choice rule for the buyer because the contracting decisions are all in $D(\mathbf{m}_i^\tau(s_i))$. Each seller i also believes that if he offered a menu $m_i \neq \mathbf{m}_i^\tau(s_i')$ for all s_i' and the buyer accepted it, she would optimally choose an arbitrary optimal contract in $D(m_i')$ with probability one.[14]

Given this continuation equilibrium, it is straightforward to show that the interim payoff for each seller i with signal s_i associated with offering $\mathbf{m}_i^\tau(s_i)$ in the non-committed procurement is the same as the one associated with submitting $(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i))$ in the modified procurement:

$$V_i(\mathbf{m}_i^\tau(s_i), \mathbf{m}_{-i}^\tau | s_i, \sigma_i^\tau) = V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i} | s_i). \tag{17}$$

In order to show that there is no profitable deviation for a seller in the non-committed procurement, note that two types of deviations are available for each seller i.

[13] One can consider the cases where a seller simply does not offer a menu when it is optimal for him not to win the auction. If this happens for a positive measure of his signal, there is a strictly positive probability that the buyer does not fully know all sellers' signals. In this case, equilibrium will be only partially revealing. We believe that it is still possible to establish the existence of partially revealing equilibria in which the buyer chooses a jointly ex-post efficient contract with probability τ_i from seller i's menu only if all sellers submit menus. The modified procurement rule should be also properly modified so that the buyer chooses a jointly ex-post efficient contract with probability τ_i for seller i only if all sellers submit serious bids.

[14] In general, many continuation equilibria that the buyer chooses to follow off the equilibrium path can prevent sellers from deviating. This is one of them.

First of all, seller i with signal s_i can deviate to offer the menu $\mathbf{m}_i^\tau(s_i')$ that he would offer if he had a different signal, say s_i'. If the buyer accepted $\mathbf{m}_i^\tau(s_i')$, she would choose $(x_i^*(s_i', s_{-i}), u(x_i^*(s_i', s_{-i}) - \hat{\mathbf{u}}_i(s_i'))$ at each s_{-i} with probability τ_i and $(\hat{\mathbf{x}}_i(s_i'), u(\hat{\mathbf{x}}_i(s_i')) - \hat{\mathbf{u}}_i(s_i'))$ with probability $1-\tau_i$ as if seller i's signal were s_i'.

In the modified procurement, seller i can deviate to submit the correct jointly interim efficient characteristics $x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i})$ along with the maximum payoff bid $\hat{\mathbf{u}}_i(s_i')$ that the menu $\mathbf{m}_i^\tau(s_i')$ would induce for the buyer. Note that the winning event $A(\hat{\mathbf{u}}_i(s_i'))$ is the same whether seller i with signal s_i deviates to $\mathbf{m}_i^\tau(s_i')$ in the non-committed procurement or to $(\hat{\mathbf{u}}_i(s_i'), x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i}))$ in the modified procurement. Furthermore, in the modified procurement, the buyer takes the correct jointly interim efficient contract

$$(x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i}), u(x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i})) - \hat{\mathbf{u}}_i(s_i'))$$

with probability $1-\tau_i$ and a correct jointly ex-post efficient contract

$$(x_i^*(s_i, s_{-i}), u(x_i^*(s_i, s_{-i}) - \hat{\mathbf{u}}_i(s_i'))$$

at each s_{-i} with probability τ_i, knowing that seller i's true signal is s_i. Therefore, seller i's interim payoff upon this deviation to $(\hat{\mathbf{u}}_i(s_i'), x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i}))$ in the modified procurement is no less than his interim payoff upon deviation to $\mathbf{m}_i^\tau(s_i')$ in the non-committed procurement:

$$V_i(\mathbf{m}_i^\tau(s_i'), \mathbf{m}_{-i}^\tau | s_i, \sigma_i^\tau) \leq V_i(\hat{\mathbf{u}}_i(s_i'), x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i}), \hat{\mathbf{u}}_{-i} | s_i). \tag{18}$$

Because $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$ is an equilibrium in the modified procurement, the following inequality relation holds

$$V_i(\hat{\mathbf{u}}_i(s_i'), x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i}), \hat{\mathbf{u}}_{-i} | s_i) \leq V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i} | s_i) \tag{19}$$

for a.e. s_i. From (17) - (19), we have

$$V_i(\mathbf{m}_i^\tau(s_i'), \mathbf{m}_{-i}^\tau | s_i, \sigma_i^\tau) \leq V_i(\mathbf{m}_i^\tau(s_i), \mathbf{m}_{-i}^\tau | s_i, \sigma_i^\tau)$$

for a.e. s_i so that it is not profitable for seller i with a.e. s_i to deviate to any menu that he would offer if he had a different signal.

Secondly, seller i can deviate to a menu m_i that he would not offer under any possible signal. According to the continuation equilibrium, the buyer would choose an arbitrary optimal contract, say $(x_i, t_i) \in D(m_i)$, with probability one when she accepts the menu. Let u_i be the payoff that (x_i, t_i) induces for the buyer. In the modified procurement, seller i can deviate to submit the bid (u_i, x_i). Seller i's interim payoff upon deviating to m_i in the non-committed procurement is no higher than his interim payoff upon deviating to (u_i, x_i) because the buyer chooses a jointly ex-post efficient contract with probability τ_i in the modified procurement but she never chooses a jointly ex-post efficient contract in the non-committed procurement:

$$V_i(m_i, \mathbf{m}^{\tau}_{-i}|s_i, \sigma^{\tau}_i) \leq V_i(u_i, x_i, \hat{\mathbf{u}}_{-i}|s_i). \tag{20}$$

Seller i's interim payoff upon deviating to (u_i, x_i) in the modified procurement is no less than his equilibrium interim payoff

$$V_i(u_i, x_i, \hat{\mathbf{u}}_{-i}|s_i) \leq V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i) \tag{21}$$

for a.e. s_i. (17), (20), and (21) yield

$$V_i(m_i, \mathbf{m}^{\tau}_{-i}|s_i, \sigma^{\tau}_i) \leq V_i(\mathbf{m}^{\tau}_i(s_i), \mathbf{m}^{\tau}_{-i}|s_i, \sigma^{\tau}_i)$$

for a.e. s_i so that it is not profitable for seller i with a.e. s_i to deviate to a menu that he would not offer under any possible signal.

The existence of the payoff-equivalent truthful monotone equilibrium in the non-committed procurement follows the existence of the corresponding monotone equilibrium in the modified procurement. Because there exists a truthful monotone equilibrium for any given $\tau \in [0,1]^N$, there exists the continuum of truthful monotone equilibria that spans $[0,1]^N$, the entire space of sellers' beliefs on how likely the buyer would choose either a jointly ex-post efficient contract or a jointly interim efficient contract.

5 Menus and Interdependent Values

Under private values, no seller has an incentive to offer a menu of contracts because a jointly ex-post efficient contract between a seller and the buyer depends only on the seller's own signal. Given a seller's signal on his production costs and the maximum payoff that he is willing to yield to the buyer, it is (weakly) dominant for him to offer a single jointly ex-post efficient contract even when he can offer a menu. When sellers' signals have only private values, there is no additional equilibrium allocation in the non-committed procurement where sellers are allowed to offer menus and joint ex-post efficiency is always ensured in the non-committed procurement equilibrium where sellers offer single contracts. The non-committed procurement where sellers offer single contracts is essentially a first scoring auction in which each seller submits a single contract, the seller calculates scores for submitted contracts based on her payoff function, and then rewards procurement to the seller with the highest score.[15]

When sellers' signals on production costs have interdependent values, the first scoring auction in which each seller is allowed to submit only a single contract

[15]Che (1993) pointed out that the first scoring auction can be implemented even when the buyer has no commitment power.

cannot reach the jointly ex-post efficient equilibrium because jointly ex-post efficient characteristics of the good depend on all sellers' signals on production costs. Given the maximum payoff that a seller is willing to give to the buyer, the best single contract that he can submit in the first scoring auction is only jointly interim efficient under interdependent values. Therefore, equilibrium in the first scoring auction is always only jointly interim efficient under interdependent values. This is why sellers should be able to offer menus of contracts to achieve joint ex-post efficiency. By reviewing and evaluating menus submitted by sellers, the buyer would develop a solid idea about sellers' signals in a truthful monotone equilibrium. Given the seller's signal and the maximum payoff that he is willing to give to the buyer, he can include all possible jointly ex-post efficient contracts as the buyer's optimal contracts in his menu. In this way, when the buyer accepts the seller's menu, she can choose the jointly ex-post efficient contract as her optimal contract from the menu given her belief on all sellers signals. However, the lack of the buyer's commitment results in multiple continuation equilibria, leading to a continuum of truthful monotone equilibria in the non-committed procurement in which sellers offer menus under interdependent values.

Given the continuum of truthful monotone equilibria, equilibrium allocation is bounded below by jointly interim efficiency and above by joint ex-post efficiency. In following subsections, we study the stability of the equilibria in two fronts. First of all, we study which equilibrium gives no incentives for *jointly ex-post renegotiation* to the buyer and the winning seller. Secondly, even before the buyer chooses a winning seller, a seller may consider deviating from his menu (i.e., change his menu). Prior to the buyer's choice of a winning seller, sellers' incentives to deviate depend on the continuation equilibrium that they believe the buyer would follow following their deviations. Some equilibrium may be supported only through a particular continuation equilibrium because a seller may want to deviate under any other continuation equilibrium. In this sense, we study how sensitive an equilibrium is to sellers' beliefs on the continuation equilibrium or how *ex-ante robust* it is to sellers' beliefs on the continuation equilibrium.

5.1 Ex-post Renegotiation

We have demonstrated that the degree of efficiency in the non-committed procurement with menus is dependent on sellers' beliefs on how the buyer will use her information on production costs when choosing a contract from the menu. Because the buyer cannot commit herself to scoring rules, there exists a continuum of truthful monotone equilibria. It is important to find out whether there is an equilibrium that is more stable than others.

Fix a truthful monotone equilibrium $\{\mathbf{m}_1^\tau, \dots, \mathbf{m}_N^\tau, \sigma^\tau\}$ given a vector $\tau = [\tau_1, \dots, \tau_N]$. Let $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \dots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$ be its corresponding payoff-equivalent monotone equilibrium in the modified procurement. Let seller i be the winning seller when the signal vector is $s = [s_1, \dots, s_N]$. Seller i's expected ex-post payoff is then

$$R_i^\tau(\hat{\mathbf{x}}_i(s_i), s) - \hat{\mathbf{u}}_i(s_i) = (1-\tau_i)R_i(\hat{\mathbf{x}}_i(s_i), s) + \tau_i R_i(x_i^*(s), s) - \mathbf{u}_i(s_i),$$

where $\hat{\mathbf{x}}_i(s_i) = x_i^e(A(\hat{\mathbf{u}}_i(s_i)), s_i, \hat{\mathbf{u}}_{-i})$ so that $R_i(\hat{\mathbf{x}}_i(s_i), s)$ is the ex-post joint surplus associated with the jointly interim efficient characteristics of the good given the signal vector $s = [s_i, s_{-i}]$ and $R_i(x_i^*(s), s) = v_i(x_i^*(s), s) + u(x_i^*(s))$ is the jointly ex-post efficient surplus.

Suppose that the buyer chooses seller i as the winning seller given the signal vector $s = [s_i, s_{-i}]$ in a truthful monotone equilibrium $\{\mathbf{m}_1^\tau, \dots, \mathbf{m}_N^\tau, \sigma^\tau\}$. The buyer receives the ex-post payoff of $\hat{\mathbf{u}}_i(s_i)$ for certain. However, the winning seller's ex-post payoff is $R_i(\hat{\mathbf{x}}_i(s_i), s) - \hat{\mathbf{u}}_i(s_i)$ with probability $1-\tau_i$ and $R_i(x_i^*(s), s) - \hat{\mathbf{u}}_i(s_i)$ with probability τ_i, so that his expected ex-post payoff is $R_i^\tau(\hat{\mathbf{x}}_i(s_i), s) - \mathbf{u}_i(s_i)$. After the buyer selects the winning seller, the buyer and the winning seller may agree to renegotiate the contract if it is mutually beneficial given a signal vector s. If there is no contract that is mutually beneficial to the buyer and the winning seller given the signal vector, then the equilibrium is said to be *jointly ex-post renegotiation-proof.*

It is clear that $R_i(x_i^*(s), s) \geq R_i(\hat{\mathbf{x}}_i(s_i), s)$ for all $s = [s_i, s_{-i}]$ because $x_i^*(s)$ is jointly ex-post efficient and $\hat{\mathbf{x}}_i(s_i)$ is only jointly interim efficient. We assume that there exists $Z_i \subseteq [0,1]$ and $Z_{-i} \subseteq [0,1]^{N-1}$, each with positive measure such that (i) for all $s_i \in Z_i$ and all $s_{-i} \in Z_{-i}$,

$$R_i(x_i^*(s_i, s_{-i}), s_i, s_{-i}) > R_i(\hat{\mathbf{x}}_i(s_i), s_i, s_{-i}). \tag{22}$$

and (ii) for all $s_i \in Z_i$,

$$\Pr(A(\hat{\mathbf{u}}_i(s_i))|s_i) > 0 \tag{23}$$

If this assumption is not satisfied, then there is no sensible distinction between interdependent values and private values in equilibrium. If the buyer chooses a jointly interim efficient contract with positive probability $1-\tau_i$ from the winning seller i, $\tau_i < 1$, then the ex-post surplus $R_i(\hat{\mathbf{x}}_i(s_i), s_i, s_{-i})$ between the winning seller i and the buyer is strictly less than the jointly ex-post efficient surplus for all $s_i \in Z_i$ and all $s_{-i} \in Z_{-i}$. It implies that with positive probability, the winning seller i and the buyer can renegotiate in such a way that both of them are strictly better off by agreeing on a jointly ex-post efficient contract. When a truthful monotone equilibrium $\{\mathbf{m}_1^\tau, \dots, \mathbf{m}_N^\tau, \sigma^\tau\}$ is jointly ex-post efficient (i.e., $\tau_i = 1$ for all i), one cannot find an alternative contract on which the buyer and the winning seller can mutually agree at any realized signal vector given the payoffs that they would receive from the buyer's original choice of a contract from the menu. It implies that only the jointly ex-post efficient equilibrium, i.e., $\{\mathbf{m}_1^\tau, \dots, \mathbf{m}_N^\tau, \sigma^\tau\}$ with $\tau_i = 1$ for all i, is jointly ex-post renegotiation-proof.

5.2 Ex-Ante Robustness

Joint ex-post renegotiation is a notion for examining whether the buyer and the winning seller can improve upon their renegotiation after the winning seller is determined. A seller may consider deviation from his menu even before the buyer chooses the winning seller. Given the multiplicity of continuation equilibrium, suppose that a truthful equilibrium is based on a continuation equilibrium in which the buyer always chooses a jointly interim efficient contract from seller 1's menu upon accepting it but always chooses a jointly ex-post efficient contract from other sellers' menus upon accepting one of them. Given this particular continuation equilibrium, no sellers have incentives to deviate from their menus. However, if seller 1 believed, for example, that the buyer would in fact always choose a jointly ex-post efficient contract from his menu upon his deviation, he would deviate to submit a menu more aggressively in the sense that his new menu offers a higher payoff to the buyer than his original menu does. Not only does such a deviation show that equilibrium menus (and subsequently equilibrium contracts) in some equilibria are not robust to the possibility that the buyer would choose an alternative continuation equilibrium for her contract choice, but it also implies that the equilibrium prediction on how likely each seller would win is not ex-ante robust because more aggressive menu offers from seller 1 make it more likely for him to win and less likely for other sellers to win. In this sense, it is important to examine how ex-ante robust an equilibrium is to (sellers' beliefs on) the continuation equilibrium.

A truthful monotone equilibrium $\{\mathbf{m}_1^\tau, \ldots, \mathbf{m}_N^\tau, \sigma^\tau\}$ is *ex-ante robust*[16] to some alternative continuation equilibria if there exists a non-empty set of alternative continuation equilibria $C \subseteq \mathcal{C}$ with $\sigma^\tau = [\sigma_1^\tau, \ldots, \sigma_N^\tau] \notin C$ such that, for all i, a.e. s_i, all m_i, all $\sigma' = [\sigma_1', \ldots, \sigma_N'] \in C$

$$V_i(\mathbf{m}_i^\tau(s_i), \mathbf{m}_{-i}^\tau | s_i, \sigma_i^\tau) \geq V_i(m_i, \mathbf{m}_{-i}^\tau | s_i, \sigma_i'). \tag{24}$$

Fix a truthful monotone equilibrium $\{\mathbf{m}_1^\tau, \ldots, \mathbf{m}_N^\tau, \sigma^\tau\}$. Suppose that seller i with s_i considers a deviation. He does not have incentives to deviate when he continues to hold the equilibrium belief on the continuation equilibrium, σ^τ. Because there are multiple continuation equilibria, a seller's incentives for deviation differ across his beliefs on the continuation equilibrium. Even when seller i with signal s_i believes that the buyer might not follow the continuation equilibrium σ^τ, he does not have an incentive to deviate as long as he believes the buyer would follow an alternative continuation equilibrium σ' in C that satisfies (24).

Theorem 3 *Any truthful monotone equilibrium* $\{\mathbf{m}_1^\tau, \ldots, \mathbf{m}_N^\tau, \sigma^\tau\}$ *with* $\tau_i > 0$ *for all* i *is ex-ante robust to some alternative continuation equilibria.*

[16]The notion of robustness follows the strong robustness adopted for competing mechanism games (Han 2007)

Proof. Fix an arbitrary truthful monotone equilibrium $\{\mathbf{m}_1^\tau, \ldots, \mathbf{m}_N^\tau, \sigma^\tau\}$ with $\tau_i > 0$ for all i. Let $\{\mathbf{m}_1^\tau, \ldots, \mathbf{m}_N^\tau, \sigma^\tau\}$ be the payoff-equivalent equilibrium for a monotone equilibrium $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$ in the modified procurement.

Consider the situation in which seller i with signal s_i is contemplating a deviation to a menu m_i that can induce the maximum payoff u_i for the buyer in the modified procurement. Suppose that the buyer follows an alternative continuation equilibrium σ'. When seller i with signal s_i deviates to a menu m_i and the array of the other sellers' menus are $\mathbf{m}_{-i}^\tau(s_{-i})$ given s_{-i}, the buyer chooses a contract upon accepting m_i, following $\sigma_i'(m_i, \mathbf{m}_{-i}^\tau(s_{-i}))$. Let $\sigma_i'(m_i, \mathbf{m}_{-i}^\tau(s_{-i}))$ satisfy the following conditions for each s_{-i}: $\sigma_i'(m_i, \mathbf{m}_{-i}^\tau(s_{-i}))$ puts probability $1 - \tau_i'$ to an arbitrary contract $(x_i, u(x_i) - u_i)$ in $D(m_i)$ and probability τ_i' to $(\tilde{x}_i(s_{-i}), u(\tilde{x}_i(s_{-i})) - u_i) \in D(m_i)$, where $\tau_i' \in [0, \tau_i)$ and $\tilde{x}_i : [0,1]^N \to \mathbb{X}$ is an arbitrary mapping satisfying $(\tilde{x}_i(s_{-i}), u(\tilde{x}_i(s_{-i})) - u_i) \in D(m_i)$ for each s_{-i}.

Suppose that seller i with signal s_i deviates to a menu m_i. If the buyer follows the continuation equilibrium σ', seller i's interim payoff upon deviating to m_i is

$$\begin{aligned} &V_i(m_i, \mathbf{m}_{-i}^\tau | s_i, \sigma_i') = \\ &\quad (1 - \tau_i') \Pr(A(u_i)|s_i) \mathbb{E}\left[(R_i(x_i, S) - u_i)\lambda_i(u_i, \hat{\mathbf{u}}_{-i}(S_{-i}))|A(u_i), s_i\right] + \\ &\quad\quad \tau_i' \Pr(A(u_i)|s_i) \mathbb{E}\left[\left(R_i(\tilde{x}_i(S_{-i}), S) - u_i\right) \lambda_i(u_i, \hat{\mathbf{u}}_{-i}(S_{-i}))|A(u_i), s_i\right]. \end{aligned} \tag{25}$$

$V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i)$ is the equilibrium interim payoff in the modified procurement and hence we have, for a.e. s_i,

$$\begin{aligned} &V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i) \geq V_i(u_i, x_i, \hat{\mathbf{u}}_{-i}|s_i) \\ &\quad = (1 - \tau_i) \Pr(A(u_i)|s_i) \mathbb{E}\left[(R_i(x_i, S) - u_i)\lambda_i(u_i, \hat{\mathbf{u}}_{-i}(S_{-i}))|A(u_i), s_i\right] + \\ &\quad\quad \tau_i \Pr(A(u_i)|s_i) \mathbb{E}\left[\left(R_i(x_i^*(S), S) - u_i\right) \lambda_i(u_i, \hat{\mathbf{u}}_{-i}(S_{-i}))|A(u_i), s_i\right]. \end{aligned} \tag{26}$$

Because $\tau_i' < \tau_i$ and $x_i^*(s)$ is a BEE action, we have

$$\begin{aligned} &V_i(u_i, x_i, \hat{\mathbf{u}}_{-i}|s_i) \\ &\quad \geq (1 - \tau_i') \Pr(A(u_i)|s_i) \mathbb{E}\left[(R_i(x_i, S) - u_i)\lambda_i(u_i, \hat{\mathbf{u}}_{-i}(S_{-i}))|A(u_i), s_i\right] + \\ &\quad\quad \tau_i' \Pr(A(u_i)|s_i) \mathbb{E}\left[\left(R_i(x_i^*(S), S) - u_i\right) \lambda_i(u_i, \hat{\mathbf{u}}_{-i}(S_{-i}))|A(u_i), s_i\right]. \end{aligned} \tag{27}$$

Because $x_i^*(s)$ is a BEE action, (25) leads to

$$\begin{aligned} &(1 - \tau_i') \Pr(A(u_i)|s_i) \mathbb{E}\left[(R_i(x_i, S) - u_i)\lambda_i(u_i, \hat{\mathbf{u}}_{-i}(S_{-i}))|A(u_i), s_i\right] + \\ &\quad \tau_i' \Pr(A(u_i)|s_i) \mathbb{E}\left[\left(R_i(x_i^*(S), S) - u_i\right) \lambda_i(u_i, \hat{\mathbf{u}}_{-i}(S_{-i}))|A(u_i), s_i\right] \\ &\quad\quad \geq V_i(m_i, \mathbf{m}_{-i}^\tau | s_i, \sigma_i'). \end{aligned} \tag{28}$$

(26), (27) and (28) imply that, for a.e. s_i,

$$V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i) \geq V_i(m_i, \mathbf{m}_{-i}^\tau | s_i, \sigma_i'). \tag{29}$$

Because $V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i) = V_i(\mathbf{m}_i^\tau(s_i), \mathbf{m}_{-i}^\tau|s_i, \sigma_i^\tau)$, (29) implies that, for a.e. s_i,

$$V_i(\mathbf{m}_i^\tau(s_i), \mathbf{m}_{-i}^\tau|s_i, \sigma_i^\tau) \geq V_i(m_i, \mathbf{m}_{-i}^\tau|s_i, \sigma_i').$$

■

Theorem 3 shows that a truthful monotone equilibrium is ex-ante robust to a set of alternative continuation equilibria as long as the buyer chooses a jointly ex-post efficient contract from the winning seller's menu with positive probability.[17] However, not every truthful monotone equilibrium is ex-ante robust to all continuation equilibria. To see this point, consider a truthful monotone equilibrium $\{\mathbf{m}_1^\tau, \ldots, \mathbf{m}_N^\tau, \sigma^\tau\}$ that does not induce joint ex-post efficiency, i.e., $\tau_i < 1$ for some i. Let $\{\mathbf{m}_1^\tau, \ldots, \mathbf{m}_N^\tau, \sigma^\tau\}$ be the payoff-equivalent equilibrium for a monotone equilibrium $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$ in the modified procurement. Given the assumption for (22) and (23), it is clear to see that there exists $Z_i \subseteq [0,1]$ with positive measure that satisfies, for all $s_i \in Z_i$,

$$\begin{aligned} &\mathbb{E}\left[(R_i(\hat{\mathbf{x}}_i(S_i), S) - \hat{\mathbf{u}}_i(S_i))\lambda_i(\hat{\mathbf{u}}(S))|A(u_i), s_i\right] < \\ &\qquad\qquad \mathbb{E}\left[(R_i(x_i^*(S), S) - \hat{\mathbf{u}}_i(S_i))\,\lambda_i(\hat{\mathbf{u}}(S))|A(u_i), s_i\right] \end{aligned} \tag{30}$$

and $\Pr(A(\hat{\mathbf{u}}_i(s_i))|s_i) > 0$. The right-hand side of (30) is seller i's interim payoff when the buyer chooses a jointly ex-post efficient contract upon accepting seller i's menu and the left-hand side is seller i's interim payoff when the buyer chooses a jointly interim efficient contract. Theorem 3 implies that even when a truthful monotone equilibrium fails to induce joint ex-post efficiency, it is ex-ante robust to a set of alternative continuation equilibria if $\tau_i > 0$ for all i. However, if a truthful monotone equilibrium does not induce joint ex-post efficiency, we can identify alternative continuation equilibria in which seller i can gain upon deviation. For seller i, whose signal s_i is in Z_i, consider the following deviation to $m_i \neq \mathbf{m}_i^\tau(s_i)$ such that

(a) the maximum payoff that the buyer can achieve by accepting m_i is the same as the one $\hat{\mathbf{u}}_i(s_i)$ that she would have achieved by accepting $\mathbf{m}_i^\tau(s_i)$ and

(b) $(\hat{\mathbf{x}}_i(s_i), u(\hat{\mathbf{x}}_i(s_i)) - \hat{\mathbf{u}}_i(s_i)) \in D(m_i)$ and $(x_i^*(s_i, s_{-i}), u(x_i^*(s_i, s_{-i})) - \hat{\mathbf{u}}_i(s_i)) \in D(m_i)$ for all $s_{-i} \in [0,1]^{N-1}$.

Suppose that the buyer plays an alternative continuation equilibrium σ'. Upon seller i's deviation to $m_i \neq \mathbf{m}_i^\tau(s_i)$ satisfying (a) and (b), $\sigma_i'(m_i, \mathbf{m}_{-i}^\tau(s_{-i}))$ puts probability $1-\tau_i'$ on $(\hat{\mathbf{x}}_i(s_i), u(\hat{\mathbf{x}}_i(s_i)) - \hat{\mathbf{u}}_i(s_i))$ and probability τ_i' on $(x_i^*(s), u(x_i^*(s)) -$

[17] Note that a truthful monotone equilibrium is based on a uniform-tie breaking rule. We can show that it is always robust to alternative tie-breaking rules in which the buyer does not choose a deviating bidder's menu in the case of ties.

$\hat{\mathbf{u}}_i(s_i))$ with $\tau_i' > \tau_i$. In this case, the interim payoff $V_i(m_i, \mathbf{m}^\tau_{-i}|s_i, \sigma_i')$ for seller i with $s_i \in Z_i$ associated with m_i under σ_i' is

$$\begin{aligned} &V_i(m_i, \mathbf{m}^\tau_{-i}|s_i, \sigma_i') = \\ &\quad (1-\tau_i') \Pr(A(u_i)|s_i) \mathbb{E}\left[(R_i(\hat{\mathbf{x}}_i(S_i), S) - \hat{\mathbf{u}}_i(S_i))\lambda_i(\hat{\mathbf{u}}(S))|A(u_i), s_i\right] + \\ &\quad\quad \tau_i' \Pr(A(u_i)|s_i) \mathbb{E}\left[(R_i(x_i^*(S), S) - \hat{\mathbf{u}}_i(S_i))\, \lambda_i(\hat{\mathbf{u}}(S))|A(u_i), s_i\right]. \end{aligned} \tag{31}$$

Note that the equilibrium interim payoff for seller i is

$$\begin{aligned} &V_i(\mathbf{m}^\tau_i(s_i), \mathbf{m}^\tau_{-i}|s_i, \sigma^\tau_i) = V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i) = \\ &\quad (1-\tau_i) \Pr(A(u_i)|s_i) \mathbb{E}\left[(R_i(\hat{\mathbf{x}}_i(S_i), S) - \hat{\mathbf{u}}_i(S_i))\lambda_i(\hat{\mathbf{u}}(S))|A(u_i), s_i\right] + \\ &\quad\quad \tau_i \Pr(A(u_i)|s_i) \mathbb{E}\left[(R_i(x_i^*(S), S) - \hat{\mathbf{u}}_i(S_i))\, \lambda_i(\hat{\mathbf{u}}(S))|A(u_i), s_i\right]. \end{aligned} \tag{32}$$

Applying $\tau_i' > \tau_i$ and (30) yields, for all $s_i \in Z_i$,

$$V_i(m_i, \mathbf{m}^\tau_{-i}|s_i, \sigma_i') > V_i(\mathbf{m}^\tau_i(s_i), \mathbf{m}^\tau_{-i}|s_i, \sigma^\tau_i).$$

Therefore, when the buyer chooses a jointly ex-post efficient contract from the winning seller's menu with a probability less than one in a continuation equilibrium, one can always find an alternative continuation equilibrium in which the buyer chooses a jointly ex-post efficient contract with higher probability and it provides the seller with incentives to deviate.

Theorem 4 shows that if a truthful monotone equilibrium is based on the continuation equilibrium in which the buyer always chooses a jointly ex-post efficient contract from the winning seller's menu, it is ex-ante robust to every continuation equilibrium that the buyer chooses to follow for her contract choice.

Theorem 4 *A truthful monotone equilibrium* $\{\mathbf{m}^\tau_1, \ldots, \mathbf{m}^\tau_N, \sigma^\tau\}$ *with* $\tau_i = 1$ *for all* i *is ex-ante robust to every continuation equilibrium.*

Proof. Consider the truthful monotone equilibrium $\{\mathbf{m}^{\mathbf{1}}_1, \ldots, \mathbf{m}^{\mathbf{1}}_N, \sigma^{\mathbf{1}}\}$ with $\mathbf{1} = [1, 1, \ldots, 1]$. Note that it corresponds to a monotone equilibrium $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$ in the modified procurement with $\tau_i = 1$ for all i. Suppose that seller i with signal s_i deviates to an arbitrary menu m_i that can induce the maximum payoff u_i to the buyer in the non-committed procurement. When seller i deviates to the payoff bid u_i along with an action bid x_i in the modified procurement, his interim payoff upon such a deviation satisfies

$$V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i) \geq V_i(u_i, x_i, \hat{\mathbf{u}}_{-i}|s_i) \tag{33}$$

for a.e. s_i because $(\hat{\mathbf{u}}_i(\cdot), \hat{\mathbf{x}}_i(\cdot))$ is bidder i's equilibrium strategy. Note that the buyer always chooses a jointly ex-post efficient contract in the modified procurement and that the winning event for seller i is the same whether he deviates to m_i

in the non-committed procurement or to the corresponding payoff bid u_i along with an action bid x_i in the modified procurement. Therefore, we have that, for any continuation equilibrium $\sigma' = [\sigma'_1, \dots, \sigma'_N]$ and all m_i,

$$V_i(u_i, x_i, \hat{\mathbf{u}}_{-i}|s_i) \geq V_i(m_i, \mathbf{m}^{\mathbf{0}}_{-i}|s_i, \sigma'_i). \tag{34}$$

Because

$$V_i(\mathbf{m}^{\mathbf{1}}_i(s_i), \mathbf{m}^{\mathbf{1}}_{-i}|s_i, \sigma^{\mathbf{1}}_i) = V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i),$$

(33) and (34) imply that, for a.e. s_i,

$$V_i(\mathbf{m}^{\mathbf{1}}_i(s_i), \mathbf{m}^{\mathbf{1}}_{-i}|s_i, \sigma^{\mathbf{1}}_i) \geq V_i(m_i, \mathbf{m}^{\mathbf{0}}_{-i}|s_i, \sigma'_i)$$

given any continuation equilibrium $\sigma' = [\sigma'_1, \dots, \sigma'_N]$ and all m_i. Therefore, the truthful monotone equilibrium $\{\mathbf{m}^\tau_1, \dots, \mathbf{m}^\tau_N, \sigma^\tau\}$ is ex-ante robust to every continuation equilibrium if $\tau = [1, \dots, 1]$. ■

Not only is the jointly ex-post efficient equilibrium jointly ex-post renegotiation-proof, it provides no incentives for sellers to deviate regardless of the continuation equilibrium that they believe the buyer would choose. Hence the equilibrium contract that the buyer chooses in the jointly ex-post efficient equilibrium is both ex-ante and ex-post stable and subsequently the jointly ex-post efficient equilibrium provides a stable prediction on how likely each seller would be to win procurement.

The jointly ex-post efficient truthful monotone equilibrium is jointly ex-post renegotiation-proof regardless of the tie-breaking rule because it considers whether there exists a mutually beneficial alternative contract for the winning seller and the buyer after the winning seller is determined. While it, like any other truthful monotone equilibria, is ex-ante robust to some alternative tie-breaking rules, it may not be ex-ante robust to every alternative tie-breaking rule under interdependent values. The tie-breaking rule does not have any bite if a deviating seller's menu has no possibility of ties with the other bidders' menus in terms of the buyer's maximum payoff that the menus can induce. However, it can affect a seller's incentive to deviate to a menu that could tie with the other sellers' menus with positive probability. Suppose that the the buyer follows an alternative tie-breaking rule in which she chooses the deviating seller's menu for sure in "good news" events in which a lot of the other sellers' menus may tie with the deviating bidder's menu with positive probability. If the payoff upon always winning a tie conditional on "good news" events is sufficiently high and a seller believes that the buyer would follow such an alternative continuation equilibrium, he may have incentives to deviate to a menu with the possibility of ties.[18]

[18]The existence of a monotone equilibrium under an alternative tie-breaking rule is yet to be established. The jointly ex-post efficient truthful monotone equilibrium is robust to both all

6 Discussion

Equilibrium analysis of non-committed procurement with asymmetric sellers, interdependent values, and affiliated signals is technically very challenging because it is not easy to establish the existence of an (monotone) equilibrium given the complexity of each seller's strategy space (i.e., the set of all possible menus of contracts). By extending Reny and Zamir's IRT-SCC and introducing TLS-SMC, this paper demonstrates that both the weakly single crossing condition and the weakly quasisupermodular condition in Reny (2011) are ensured at individually rational tieless bids. This leads us not only to establish the existence of truthful monotone equilibria but also enables us to study the nature of equilibrium allocations in non-committed procurement under interdependent values in very general environments.

For the jointly ex-post efficient equilibrium under interdependent values, it is essential for potential sellers to be able to submit a menu of contracts because the equilibrium is only jointly interim efficient when a seller is restricted to submit a single contract. However, by allowing sellers to submit menus of contracts, the buyer induces a continuum of truthful monotone equilibria in such a way that each truthful monotone equilibrium depends on the continuation equilibrium that sellers believe the buyer would follow for her choice of a contract from the winning seller's menu. We show that the truthful monotone equilibrium is bounded above by the joint ex-post efficiency and below by the joint interim efficiency.

In the jointly ex-post efficient equilibrium, sellers believe that the buyer will always choose a jointly ex-post efficient contract if it is available as an optimal contract in the menu, and they submit menus accordingly. Because the jointly ex-post efficient equilibrium is both jointly ex-post renegotiation-proof and ex-ante robust to any continuation equilibrium, it is very appealing and we may expect it to be the most plausible equilibrium. With a lack of the buyer's commitment, it should be important in practice for the buyer to establish her *reputation* in a way that leads sellers to believe she would choose a jointly ex-post efficient contract when negotiating with the winning seller.

Abstracting from reality, our paper provides a positive theory of non-committed procurement in which each seller submits a menu of contracts. In practice, in non-committed procurement, each seller may submit a lengthy proposal that includes many different aspects of procurement and many possible potential contracts Furthermore it may be a time-consuming process for the buyer (e.g., government) to review and evaluate sellers' proposals. By reviewing and evaluating sellers' proposals, the buyer develops an accurate idea about their signals on production costs in equilibrium. Therefore, the buyer can act as an information collector

continuation equilibria and all tie-breaking rules only if every bidder's feasible menus have no possibility of ties with the other bidders' equilibrium menus. However, it is not known when this condition is satisfied.

in the non-committed procurement. Even without commitment on the buyer's side, sellers' proposals can lead to the jointly ex-post efficient equilibrium under interdependent values if they believe that the buyer will exploit her information during her negotiation with the winning seller. As an information collector, it is practically important for the buyer to establish a reputation that the acquired information on all seller's production costs would be used when negotiating a contract with the winning seller. It suggests sensitive roles for *regulations* (e.g., Federal Acquisition Regulation in the U.S.) or *oversight agencies* (e.g., Office of the Procurement Ombudsman in Canada). Even when the government may not commit to any mechanisms or scoring rules, it can establish regulations or oversight agencies for well-defined procedures of non-committed procurement. They are important not only in preventing favoritism or corruption, and ensuring competitive bidding, but also in providing and maintaining the government's reputation that its additional information on production costs would be used when negotiating a contract with the winning seller. This makes potential sellers submit their proposals, expecting the jointly ex-post efficient equilibrium.

The result also gives us new insight into why multiple open bidding is important in practice especially under interdependent values. Recently, Canada announced a $9 billion plan to purchase sixty five F-35 fighter jets from Lockheed Martin. It was heavily criticized because the federal government chose F-35 fighter jets through exclusive bargaining with Lockheed Martin without competitive bidding from other potential sellers. The potential cost of such an exclusive bargaining goes much deeper. The announced plan includes not only the simple purchase of the fighter jets but also the modification of the jets, long-term maintenance, and training that are tailored specific to the needs of Canadian Air Force. The costs of modification, long-term maintenance, and training may not be fully known to the buyer or a single seller. As the buyer invites proposals from many potential sellers and evaluates the proposals, she learns about costs that are not necessarily known to the winning seller. Subsequently, the buyer can negotiate with the winning seller with the knowledge acquired from other sellers' proposals. The federal government of Canada missed such a valuable opportunity by exclusively bargaining with Lockheed Martin.[19] In this light, this paper can also be viewed as offering a new aspect of competitive bidding under interdependent values, in the sense that as the buyer reviews and evaluates sellers' proposals, she learns more about production costs and can use this knowledge

[19]The minister of defense defended the exclusive bargaining with Lockheed Martin by pointing out that they knew that the F-35 fighter jet is the best fighter jet on the market so that it was not necessary to consider any other fighter jets. Not only does such a remark reflect the lack of understanding of procurement under interdependent values, but it also does not help the government to establish a reputation for gathering and learning additional information from many sellers' proposals and using that information during the negotiation with the winning seller.

during negotiation with the winning seller.

APPENDIX

Proof of Lemma 1. Following Reny and Zamir (2004), call a product of N real intervals in $\mathbb{R}^k$ with $k \geq 1$ - each of which can be closed, open or half-open - a cell. For any cells A and A' in $\mathbb{R}^k$, $A' \geq A$ if the lower (upper) endpoint of each interval in the product defining A is no greater than the lower (upper) endpoint of the corresponding interval in the product defining A'.

TLS-SMC: Consider any (u_i, x_i) and (u'_i, x'_i) such that $\Pr[u_\circ < \max_{j\neq i} \mathbf{u}_j(s_j) = u_i \text{ or } u'_i] = 0$ given any non-decreasing payoff bidding functions $\mathbf{u}_{-i}$. In this way, we can ensure that if seller i's payoff bid is u_i or u'_i, then the probability that seller i's payoff bid is the highest non-negative payoff bid is equal to the probability that seller i's payoff bid is the unique highest non-negative payoff bid. Because $f > 0$ and the event that seller i's payoff bid is uniquely the highest depends only on the other sellers' signals, this event has positive probability if and only if it has positive probability on every $s_i \in [0,1]$. Hence we can respectively define the event that u'_i is a winning payoff bid and the event that u_i is a winning payoff bid as $A' = \left\{s_{-i} \in [0,1]^N : \max_{j\neq i} \mathbf{u}_j(s_j) < u'_i\right\}$ and $A = \{s_{-i} \in [0,1]^N : \max_{j\neq i} \mathbf{u}_j(s_j) < u_i\}$. Without loss of generality, let $u'_i \geq u_i$ and hence $u_i \vee u'_i = u'_i$ and $u_i \wedge u'_i = u_i$. Because u_i or u'_i is the unique payoff bids, the differences in seller i's interim payoffs are

$$\begin{aligned} V_i(u_i \vee u'_i, x_i \vee x'_i, \mathbf{u}_{-i}|s_i) - V_i(u'_i, x'_i, \mathbf{u}_{-i}|s_i) &= \Pr(A'|s_i)\mathbb{E}\left[R_i^\tau(x_i \vee x'_i, S) - R_i^\tau(x'_i, S)|A', s_i\right], \\ V_i(u_i, x_i, \mathbf{u}_{-i}|s_i) - V_i(u_i \wedge u'_i, x_i \wedge x'_i, \mathbf{u}_{-i}|s_i) &= \Pr(A|s_i)\mathbb{E}\left[R_i^\tau(x_i, S) - R_i^\tau(x_i \wedge x'_i, S)|A, s_i\right]. \end{aligned}$$

Therefore, $V_i(u_i \vee u'_i, x_i \vee x'_i, \mathbf{u}_{-i}|s_i) - V_i(u'_i, x'_i, \mathbf{u}_{-i}|s_i)$ is the difference in seller i's interim payoffs associated with two action bids $x_i \vee x'_i$ and x'_i given his payoff bid u'_i. $V_i(u_i, x_i, \mathbf{u}_{-i}|s_i) - V_i(u_i \wedge u'_i, x_i \wedge x'_i, \mathbf{u}_{-i}|s_i)$ is the difference in seller i's interim payoffs associated with two action bids x_i and $x_i \wedge x'_i$ given his payoff bid u_i. Because $\Pr(A'|s_i) \geq \Pr(A|s_i)$, TLS-SMC holds if

$$\mathbb{E}\left[R_i^\tau(x_i \vee x'_i, S) - R_i^\tau(x'_i, S)|A', s_i\right] \geq \mathbb{E}\left[R_i^\tau(x_i, S) - R_i^\tau(x_i \wedge x'_i, S)|A, s_i\right]. \tag{35}$$

We first compare $\mathbb{E}\left[R_i^\tau(x_i \vee x'_i, S) - R_i^\tau(x'_i, S)|A', s_i\right]$ and $\mathbb{E}[R_i^\tau(x_i \vee x'_i, S) - R_i^\tau(x'_i, S)|A, s_i]$. Because $u'_i \geq u_i$ and the other sellers employ non-decreasing strategies, both A' and A are products of cells with zero lower endpoints, but the upper endpoint of the cell for each seller j's signal in A' is no less than the upper endpoint of the corresponding cell in A. Furthermore, $R_i^\tau(x_i \vee x'_i, s) - R_i^\tau(x'_i, s)$ is non-decreasing in s_{-i}. Therefore, we can directly invoke Theorem 5 in Milgrom and Weber (1982) so that

$$\mathbb{E}\left[R_i^\tau(x_i \vee x'_i, S) - R_i^\tau(x'_i, S)|A', s_i\right] \geq \mathbb{E}\left[R_i^\tau(x_i \vee x'_i, S) - R_i^\tau(x'_i, S)|A, s_i\right]. \tag{36}$$

Consider $\mathbb{E}\left[R_i^\tau(x_i \vee x'_i, S) - R_i^\tau(x'_i, S)|A, s_i\right]$ and $\mathbb{E}[R_i^\tau(x_i, S) - R_i^\tau(x_i \wedge x'_i, S)|A,\ s_i]$. Because R_i^τ is supermodular at each s, it is clear that

$$\mathbb{E}\left[R_i^\tau(x_i \vee x'_i, S) - R_i^\tau(x'_i, S)|A, s_i\right] \geq \mathbb{E}\left[R_i^\tau(x_i, S) - R_i^\tau(x_i \wedge x'_i, S)|A, s_i\right]. \tag{37}$$

Combining (36) and (37) yields (35) and hence TLS-SMC holds.

IRT-SCC: This proof closely follows the proof of IRT-SCC in Reny and Zamir (2004). To show IRT-SCC, fix $(\overline{u}_i, \overline{x}_i)$ and $(\underline{u}_i, \underline{x}_i)$ with $\overline{u}_i \geq \underline{u}_i$ and $\overline{x}_i \geq \underline{x}_i$ and, for all $j \neq i$, fix non-decreasing payoff bidding functions so that $\Pr[u_\circ < \max_{j\neq i} \mathbf{u}_j(S_j) = \overline{u}_i \text{ or } \underline{u}_i] = 0$. As in proof of TLS-SMC, this makes the event that $\overline{u}_i$ is a winning payoff bid as

$$\overline{A} = \left\{ s_{-i} \in [0,1]^N : \max_{j\neq i} \mathbf{u}_j(s_j) < \overline{u}_i \right\}.$$

Suppose that $(\underline{u}_i, \underline{x}_i)$ wins with probability zero: $V_i(\underline{u}_i, \underline{x}_i, \mathbf{u}_{-i}|s_i) = 0$ for every s_i. IRT-SCC holds because $\mathbb{E}\left[R_i^\tau(\underline{x}_i, S) - \underline{u}_i|\overline{A}, s_i\right]$ is non-decreasing in s_i whenever $\overline{A}$ has positive probability.

Suppose that $(\underline{u}_i, \underline{x}_i)$ wins with positive probability. This means that $(\overline{u}_i, \overline{x}_i)$ also wins with positive probability because $\overline{u}_i \geq \underline{u}_i$; hence $\overline{A}$ has positive probability. As in Reny and Zamir (2004), partition $\overline{A}$ as follows. For every subset J of $\{1, \ldots, N\}\backslash\{i\}$, define

$$A(J) = \overline{A} \cap \{s_{-i} \in [0,1]^{N-1} : \forall j \neq i, \mathbf{u}_j(s_j) \geq \underline{u}_i \text{ iff } j \in J\}.$$

Ignoring ties, $A(J)$ is the event that $(\underline{u}_i, \underline{x}_i)$ loses against precisely those sellers in J. Because $A(J)$ is contained in $\overline{A}$, $(\overline{u}_i, \overline{x}_i)$ wins against every $j \neq i$ in each event $A(J)$. Also, $A(\emptyset)$, being the event that $(\underline{u}_i, \underline{x}_i)$ loses against no one, is the event that $(\underline{u}_i, \underline{x}_i)$ wins the procurement and so has positive probability. IRT-SCC holds if the following statement holds for all pairs of $s_i' \geq s_i$: When $\mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i|\overline{A}, s_i\right] \geq 0$,

$$\mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i - (R_i^\tau(\underline{x}_i, S) - \underline{u}_i)I_{A(\emptyset)}|\overline{A}, s_i\right] \geq 0 \Longrightarrow \\ \mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i - (R_i^\tau(\underline{x}_i, S) - \underline{u}_i)I_{A(\emptyset)}|\overline{A}, s_i'\right] \geq 0, \quad (38)$$

and, when $\mathbb{E}\left[R_i^\tau(\underline{x}_i, S) - \underline{u}_i|A(\emptyset), s_i'\right] \geq 0$,

$$\mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i - (R_i^\tau(\underline{x}_i, S) - \underline{u}_i)I_{A(\emptyset)}|\overline{A}, s_i'\right] \leq 0 \Longrightarrow \\ \mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i - (R_i^\tau(\underline{x}_i, S) - \underline{u}_i)I_{A(\emptyset)}|\overline{A}, s_i\right] \leq 0. \quad (39)$$

By Theorem 5 in Milgrom and Weber (1982), $\mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i|\overline{A}, s_i\right] \geq 0$ implies that $\mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i|\overline{A}, s_i'\right] \geq 0$. Subsequently, if $\mathbb{E}[(R_i^\tau(\underline{x}_i, S) - \underline{u}_i)I_{A(\emptyset)}|\overline{A}, s_i'] < 0$, equivalently, $\mathbb{E}\left[R_i^\tau(\underline{x}_i, S) - \underline{u}_i|A(\emptyset), s_i'\right] < 0$, then (38) trivially holds because the second difference is non-negative. Therefore, it is sufficient to establish (38) and (39) when $\mathbb{E}\left[(R_i^\tau(\underline{x}_i, S) - \underline{u}_i)I_{A(\emptyset)}|\overline{A}, s_i'\right] \geq 0$ equivalently, $\mathbb{E}\left[R_i^\tau(\underline{x}_i, S) - \underline{u}_i|A(\emptyset), s_i'\right] \geq 0$. This can be done by showing that when $\mathbb{E}[R_i^\tau(\underline{x}_i, S) - \underline{u}_i|A(\emptyset), s_i'] \geq 0$,

$$\mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i - (R_i^\tau(\underline{x}_i, S) - \underline{u}_i)I_{A(\emptyset)}|\overline{A}, s_i'\right] \geq \\ \mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i - (R_i^\tau(\underline{x}_i, S) - \underline{u}_i)I_{A(\emptyset)}|\overline{A}, s_i\right]. \quad (40)$$

Let $\Delta_i(s) = R_i^\tau(\overline{x}_i, s) - \overline{u}_i - (R_i^\tau(\underline{x}_i, s) - \underline{u}_i)\, I_{A(\emptyset)}$. Note that $\Delta_i(s)$ is non-decreasing in s_i. According to Lemma A.1 in Reny and Zamir (2004), it is therefore enough to show that $\Delta_i(s_i', \cdot)$ is cell-wise non-decreasing with respect to $f(s_{-i}|\overline{A}, s_i')$, where $f(s_{-i}|\overline{A}, s_i')$ is the density function for s_{-i} conditional on $(\overline{A}, s_i')$. By considering the above finite partition, $\{A(J)\}$, of A into cells, we can restrict attention to those subsets of J such that $A(J)$ is non-empty. For any non-empty $A(J)$, $(\underline{u}_i, \underline{x}_i)$ loses against precisely those sellers in J so that

$$\mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i - (R_i^\tau(\underline{x}_i, S) - \underline{u}_i)I_{A(\emptyset)}|A(J), s_i'\right] = \mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i|A(J), s_i'\right]. \quad (41)$$

If $A(J') \geq A(J)$, the inequality relation

$$\mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i | A(J'), s_i'\right] \geq \mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i | A(J), s_i'\right] \tag{42}$$

follows from Theorem 5 in Milgrom and Weber (1982). (41) and (42) imply that for any pair of non-empty $A(J') \geq A(J)$,

$$\mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i - (R_i^\tau(\underline{x}_i, S) - \underline{u}_i) I_{A(\emptyset)} | A(J'), s_i'\right] \geq \\ \mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i - (R_i^\tau(\underline{x}_i, S) - \underline{u}_i) I_{A(\emptyset)} | A(J), s_i'\right]. \tag{43}$$

Furthermore, for every $A(J) \geq A(\emptyset)$, we have

$$\mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i | A(J), s_i'\right] \geq \mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i | A(\emptyset), s_i'\right] \\ \geq \mathbb{E}[R_i^\tau(\overline{x}_i, S) - \overline{u}_i - (R_i^\tau(\underline{x}_i, S) - \underline{u}_i) \,| A(\emptyset), s_i'], \tag{44}$$

where the first inequality follows from Theorem 5 in Milgrom and Weber (1982) and the second follows because $\mathbb{E}[R_i^\tau(\underline{x}_i, S) - \underline{u}_i | A(\emptyset), s_i'] \geq 0$. (41) and (44) imply that, for any non-empty $A(J)$,

$$\mathbb{E}\left[R_i^\tau(\overline{x}_i, S) - \overline{u}_i - (R_i^\tau(\underline{x}_i, S) - \underline{u}_i) I_{A(\emptyset)} | A(J), s_i'\right] \geq \\ \mathbb{E}[R_i^\tau(\overline{x}_i, S) - \overline{u}_i - (R_i^\tau(\underline{x}_i, S) - \underline{u}_i) \,| A(\emptyset), s_i']. \tag{45}$$

Finally, (43) and (45) show that $\Delta_i(\overline{s}_i, \cdot)$ is cell-wise non-decreasing with respect to $f(s_{-i}|A, s_i')$. ∎

Proof of Theorem 1 The existence of a monotone equilibrium in the modified procurement with finite sets of payoff bids is established in Lemma 2. The proof of Theorem 1 extends part 2 of the proof of Theorem 2.1 in Reny and Zamir (2004) to show that the limit of the modified procurements with finite sets of payoff bids has a monotone equilibrium without restrictions on the sets of payoff bids. For $n = 1, 2, \ldots$, let G^n denote the modified procurement in which seller i's finite set of payoff bids is denoted by U_i^n and hence the set of payoff bids and actions is $U_i^n \times \mathbb{X}$. We assume that $U_i^{n-1} \subseteq U_i^n$ and that $\bigcup_n U_i^n$ is dense in U_i. Let $(\mathbf{u}_i^n, \mathbf{x}_i^n)$ be seller i's equilibrium strategy in the modified procurement G^n.

Consider the limit strategies. Because $R^\tau(\cdot, s)$ is bounded above at each $s \in [0,1]^N$ with Assumption 2.(iii), there exists $\tilde{u} > 0$ such that $R^\tau(x, s) - \tilde{u} < 0$ for all $(x, s) \in \mathbb{X} \times [0,1]^N$. Therefore, $\mathbf{u}_i^n(\cdot)$ is bounded above by $\tilde{u}$ and below by $u_\circ$, and it is also non-decreasing in s_i. By Helley's Selection Theorem, we then have that $\mathbf{u}_i^n(s_i) \to \hat{\mathbf{u}}_i(s_i)$ for a.e. $s_i \in [0,1]$, where $\hat{\mathbf{u}}_i(\cdot)$ is a non-decreasing function.

Because $\mathbf{x}_i^n(\cdot)$ is non-decreasing in s_i, and the conditions G.1-G.3 in Reny (2011), a generalized Helley's selection Theorem (Lemma A.10 in Reny 2011) implies that $\mathbf{x}_i^n(s_i) \to \hat{\mathbf{x}}_i(s_i)$ for a.e. $s_i \in [0,1]$, where $\hat{\mathbf{x}}_i(\cdot)$ is a non-decreasing function. We shall prove that $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$ is a monotone equilibrium in the modified procurement. Ties should be carefully handled in two fronts.

In point 1, we show that, given the limit bidding functions $\hat{\mathbf{u}}_{-i}$ of the other sellers, seller i's interim payoff associated with any bid (u_i, x_i) can be approximated arbitrarily well or he can improve upon his payoff by slightly increasing his payoff bid, given the same action bid x_i, that does not tie the other sellers' payoff bids with probability one. This is illustrated in (46).

In point 2, we are concerned that the possibility of ties may lead the limiting payoffs to differ from the payoffs at the limit strategies. It is shown that the probability that, under $\hat{\mathbf{u}}$, two or more sellers simultaneously submit the highest payoff bid above $u_\circ$ is zero so that the limiting payoff for seller i with s_i is always obtained by employing the limit bids $(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i))$ given the other sellers' limit bidding functions, $\hat{\mathbf{u}}_{-i}$.

Point 1: Given the other sellers' non-decreasing payoff bidding functions $\hat{\mathbf{u}}_{-i}$, let $A(u_i) = \{s_{-i} \in [0,1]^{N-1} : \max_{j \neq i} \hat{\mathbf{u}}_j(s_j) \leq u_i\}$. Suppose that seller i with s_i submits a bid (u_i, x_i) such that $V_i(u_i, x_i, \hat{\mathbf{u}}_{-i}|s_i) \geq 0$. Let $\mathbb{E}[\cdot|s_i, H_i] = 0$ if $\Pr(H_i|s_i) = 0$. Then, the following relations hold:

$$
\begin{aligned}
0 &\leq V_i(u_i, x_i, \hat{\mathbf{u}}_{-i}|s_i) \\
&= \Pr(A(u_i)|s_i)\mathbb{E}[(R_i^\tau(x_i, S) - u_i)\, \lambda_i(u_i, \hat{\mathbf{u}}_{-i}(S_{-i}))|A(u_i), s_i] \\
&\leq \Pr(A(u_i)|s_i)\mathbb{E}\left[R_i^\tau(x_i, S) - u_i|A(u_i), s_i\right] \mathbb{E}\left[\lambda_i(u_i, \hat{\mathbf{u}}_{-i}(S_{-i}))|A(u_i), s_i\right] \\
&\leq \Pr(A(u_i)|s_i)\mathbb{E}\left[R_i^\tau(x_i, S) - u_i|A(u_i), s_i\right] \\
&= \lim_{u_i' \downarrow u_i} V_i(u_i', x_i, \hat{\mathbf{u}}_{-i}|s_i),
\end{aligned}
\tag{46}
$$

where the inequality on the third line follows from Theorem 23 in Milgrom and Weber (1982) because both $R_i^\tau(x_i, s) - u_i$ and $1 - \lambda_i(u_i, \hat{\mathbf{u}}_{-i}(s_{-i}))$ are non-decreasing in s_{-i} given the uniform tie-breaking rule. The inequality on the fourth line holds because

$$
\Pr(A(u_i)|s_i) > 0 \Longrightarrow 0 < \mathbb{E}\left[\lambda_i(u_i, \hat{\mathbf{u}}_{-i}(S_{-i}))|A(u_i), s_i\right] \leq 1
$$

given the uniform tie-breaking rule. The equality on the fifth line holds because seller i can approximate his payoff arbitrarily well by submitting a slightly higher payoff bid u_i', with the same characteristics bid x_i, that is never one of the at most countably many mass points of $\max_{j \neq i} \hat{\mathbf{u}}_j(S_j)$.

Point 2: Recall that $R^\tau(x_\circ, s) - u_\circ = 0$ for all i and all s. When $u_i = u_\circ$, define $u_i' \downarrow u_i$ to mean $u_i' = u_\circ$. Note that $u_\circ \in U_i^n$. Therefore, for every i and a.e. s_i such that $\hat{\mathbf{u}}_i(s_i) > u_\circ$ and $\Pr[\max_{j \neq i} \hat{\mathbf{u}}_j(S_j) \leq \hat{\mathbf{u}}_i(S_i)|s_i] > 0$, the following holds when n is large enough:

$$
\begin{aligned}
0 &\leq \mathbb{E}[R_i^\tau(\mathbf{x}_i^n(s_i), S) - \mathbf{u}_i^n(s_i)|s_i, \max_{j \neq i} \mathbf{u}_j^n(S_j) \leq \mathbf{u}_i^n(s_i)] \\
&\leq \mathbb{E}[R_i^\tau(\mathbf{x}_i^n(s_i), S) - \mathbf{u}_i^n(s_i)|s_i, \max_{j \neq i} \mathbf{u}_j^n(S_j) \leq \hat{\mathbf{u}}_i(s_i) + \delta] \\
&\to \mathbb{E}[R_i^\tau(\hat{\mathbf{x}}_i(s_i), S) - \hat{\mathbf{u}}_i(s_i)|s_i, \max_{j \neq i} \hat{\mathbf{u}}_j(S_j) \leq \hat{\mathbf{u}}_i(s_i)].
\end{aligned}
\tag{47}
$$

$\hat{\mathbf{u}}_i(s_i) > u_\circ$ implies that $\mathbf{u}_i^n(s_i)$ is a serious payoff bid when n is large enough. This implies that (i) it wins with positive probability given the strategy restriction and (ii) the right hand side in the first line is seller i's payoff because ties in G^n cannot occur at serious payoff bids. Therefore, the first line of (47) holds. The second line follows for any $\delta > 0$ by Theorem 5 in Milgrom and Weber (1982). For the third line, note that $R_i^\tau(\cdot, s)$ is continuous by Assumption 1.(i)-(ii). The third line follow by taking the limit first as $n \to \infty$ and then as $\delta \downarrow 0$ along a sequence such that $\hat{\mathbf{u}}_i(s_i) + \delta$ is never one of the at most countably many mass points of $\max_{j \neq i} \hat{\mathbf{u}}_j(S_j)$. This ensures that the first limit in n exists for each such δ.

Now we consider the limit payoffs. Because $\lambda_i(u_i, u_{-i})$ is non-decreasing in u_i and non-increasing in u_{-i}, $\lambda_i(\mathbf{u}^n(s))$ is a sequence of functions each of which is monotone in each of its arguments, $s_1, \ldots, s_N$, and is non-decreasing in s_i and non-increasing in s_{-i}. Hence, by Helley's Theorem, there exists $\alpha_i : [0,1]^N \to [0,1]$ that is non-decreasing in s_i and non-increasing in s_{-i} such that $\lambda_i(\mathbf{u}^n(s)) \to \alpha_i(s)$ for a.e. $s \in [0,1]^N$. Consequently, we have $\sum_i \alpha_i(s) = 1$ for a.e. $s \in [0,1]^N$. One can think of $\alpha_i(s)$ as a surrogate tie-breaking rule that is a function of the vector of signals alone. Then, the equilibrium interim payoff for seller i in G^n, $V_i(\mathbf{u}_i^n(s_i), \mathbf{x}_i^n(s_i), \mathbf{u}_{-i}^n|s_i) = \mathbb{E}[(R_i^\tau(\mathbf{x}_i^n(s_i), S) - \mathbf{u}_i^n(s_i))\lambda_i(\mathbf{u}^n(S))|s_i]$ converges to $\mathbb{E}[(R_i^\tau(\hat{\mathbf{x}}_i(s_i), S) - \hat{\mathbf{u}}_i(s_i))\, \alpha_i(S)|s_i]$ by the dominated convergence theorem.

Because each $\hat{\mathbf{u}}_j(S_j)$ has at most countably many mass points and U_i^n becomes dense in U_i, for every $u_i \in U_i$, every $\varepsilon > 0$, and a.e. s_i, there exist $\bar{n} \geq 1$ and $\bar{u} \in U_i^{\bar{n}}$ such that

$$\begin{aligned} \lim_{u_i' \downarrow u_i} V_i(u_i', x_i, \hat{\mathbf{u}}_{-i}|s_i) &\leq V_i(\bar{u}_i, x_i, \hat{\mathbf{u}}_{-i}|s_i) + \varepsilon \\ &\leq V_i(\bar{u}_i, x_i, \mathbf{u}_{-i}^n|s_i) + 2\varepsilon \\ &\leq V_i(\mathbf{u}_i^n(s_i), \mathbf{x}_i^n(s_i), \mathbf{u}_{-i}^n|s_i) + 2\varepsilon \end{aligned} \tag{48}$$

for $n \geq \bar{n}$. The first and second lines in (48) hold because we can choose $\bar{u}_i$ such that the probability that any $\hat{\mathbf{u}}_j(S_j)$ is equal to $\bar{u}_i$ is arbitrarily small. The third line holds because $\bar{u}_i$ is feasible in U_i^n for $n \geq \bar{n}$ and $\mathbf{u}_i^n$ is the equilibrium bidding function for seller i. Because ε is arbitrarily small and $(u_\circ, x_\circ)$ yields a payoff of zero, we obtain the following relation for all i and a.e. s_i:

$$\begin{aligned} \sup_{u_i, x_i} V_i(u_i, x_i, \hat{\mathbf{u}}_{-i}|s_i) \leq \lim_{n \uparrow \infty} V_i(\mathbf{u}_i^n(s_i), \mathbf{x}_i^n(s_i), \mathbf{u}_{-i}^n|s_i) = \\ \mathbb{E}[(R_i^\tau(\hat{\mathbf{x}}_i(s_i), S) - \hat{\mathbf{u}}_i(s_i))\, \alpha_i(S)|s_i]. \end{aligned} \tag{49}$$

Letting $A(\hat{\mathbf{u}}_i(s_i)) = \{s_{-i} \in [0,1]^{N-1} : \max_{j \neq i} \hat{\mathbf{u}}_j(s_j) \leq \hat{\mathbf{u}}_i(s_i)\}$, that for a.e. s_i such that $\hat{\mathbf{u}}_i(s_i) > u_\circ$,

$$\begin{aligned} 0 &\leq \mathbb{E}[(R_i^\tau(\hat{\mathbf{x}}(s_i), S) - \hat{\mathbf{u}}_i(s_i))\, \alpha_i(S)|s_i] \\ &= \Pr(A(\hat{\mathbf{u}}_i(s_i))|s_i)\mathbb{E}[(R_i^\tau(\hat{\mathbf{x}}_i(s_i), S) - \hat{\mathbf{u}}_i(s_i))\, \alpha_i(S)|A(\hat{\mathbf{u}}_i(s_i)), s_i] \\ &\leq \Pr(A(\hat{\mathbf{u}}_i(s_i))|s_i)\mathbb{E}[R_i^\tau(\hat{\mathbf{x}}(s_i), S) - \hat{\mathbf{u}}_i(s_i)|A(\hat{\mathbf{u}}_i(s_i)), s_i]\mathbb{E}[\alpha_i(S)|A(\hat{\mathbf{u}}_i(s_i)), s_i] \\ &\leq \Pr(A(\hat{\mathbf{u}}_i(s_i))|s_i)\mathbb{E}[R_i^\tau(\hat{\mathbf{x}}(s_i), S) - \hat{\mathbf{u}}_i(s_i)|A(\hat{\mathbf{u}}_i(s_i)), s_i] \\ &= \lim_{\varepsilon \downarrow 0} V_i(\hat{\mathbf{u}}_i(s_i) + \varepsilon, \hat{\mathbf{x}}(s_i), \hat{\mathbf{u}}_{-i}|s_i) \\ &\leq \sup_{u_i, x_i} V_i(u_i, x_i, \hat{\mathbf{u}}_{-i}|s_i). \end{aligned} \tag{50}$$

The inequality on the first line in (50) follows from (49). The equality on the second line is immediately apparent because $\alpha_i(s) = 0$ for a.e. $s_{-i} \notin A(\hat{\mathbf{u}}_i(s_i))$. The inequality on the third line follows from Theorem 23 in Milgrom and Weber (1982). The inequality on the fourth line holds because $\alpha_i(s) \in [0,1]$, and by (47), $\Pr(A(\hat{\mathbf{u}}_i(s_i))|s_i) > 0$ implies that $\mathbb{E}[R_i^\tau(\hat{\mathbf{x}}(s_i), S) - \hat{\mathbf{u}}_i(s_i)|A(\hat{\mathbf{u}}_i(s_i)), s_i] \geq 0$. The equality on the fifth line holds, as on the fifth line in (46), because seller i can approximate his payoff arbitrarily well by submitting a slightly higher payoff bid $\hat{\mathbf{u}}_i(s_i)$ that, with probability one, does not tie the others' payoff bids, along with $\hat{\mathbf{x}}(s_i)$. By the last inequality in (50) and the inequality in (49), the second, third, and fourth inequalities in (50) must be equalities.

Now we prove that $\mathbb{E}[\alpha_i(S)|A(\hat{\mathbf{u}}_i(s_i)), s_i] = 1$ for a.e. s_i that satisfies $\Pr(A(\hat{\mathbf{u}}_i(s_i))|s_i) > 0$. Because the third inequality in (50) holds with equality, we have, for a.e. s_i with $\Pr(A(\hat{\mathbf{u}}_i(s_i))|s_i) > 0$, we have

$$\begin{aligned} 0 \leq \mathbb{E}[R_i^\tau(\hat{\mathbf{x}}_i(s_i), S) - \hat{\mathbf{u}}_i(s_i)|A(\hat{\mathbf{u}}_i(s_i)), s_i]\mathbb{E}[\alpha_i(S)|A(\hat{\mathbf{u}}_i(s_i)), s_i] \\ = \mathbb{E}[R_i^\tau(\hat{\mathbf{x}}_i(s_i), S) - \hat{\mathbf{u}}_i(s_i)|A(\hat{\mathbf{u}}_i(s_i)), s_i]. \end{aligned} \tag{51}$$

For a.e. s_i and s_i' such that $s_i > s_i'$, $\Pr(A(\hat{\mathbf{u}}_i(s_i))|s_i) > 0$ and $\Pr(A(\hat{\mathbf{u}}_i(s_i'))|s_i') > 0$, the following relation holds:

$$0 \leq \mathbb{E}[R_i^\tau(\hat{\mathbf{x}}_i(s_i'), S) - \hat{\mathbf{u}}_i(s_i')|A(\hat{\mathbf{u}}_i(s_i')), s_i'] < \mathbb{E}[R_i^\tau(\hat{\mathbf{x}}(s_i'), S) - \hat{\mathbf{u}}_i(s_i')|A(\hat{\mathbf{u}}_i(s_i')), s_i]. \tag{52}$$

The first inequality in (52) holds by (51). Because $R_i^\tau(x_i, s) - u_i$ is strictly increasing in s_i, Theorem 5 in Milgrom and Weber (1982) implies the second inequality in (52).

Now we prove the following relation:

$$\begin{aligned} 0 < \lim_{\varepsilon\downarrow 0} V_i(\hat{\mathbf{u}}_i(s_i') + \varepsilon, \hat{\mathbf{x}}_i(s_i'), \hat{\mathbf{u}}_{-i}|s_i) &\leq \sup_{u_i, x_i} V_i(u_i, x_i, \hat{\mathbf{u}}_{-i}|s_i) \\ &= \lim_{\varepsilon\downarrow 0} V_i(\hat{\mathbf{u}}_i(s_i) + \varepsilon, \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i). \end{aligned} \tag{53}$$

Note that $\Pr(A(\hat{\mathbf{u}}_i(s_i'))|s_i') > 0$ implies $\Pr(A(\hat{\mathbf{u}}_i(s_i')|s_i) > 0$ by Assumption 3.(i). Therefore, (52) leads to

$$0 < \Pr(A(\hat{\mathbf{u}}_i(s_i')|s_i)\mathbb{E}[R_i^\tau(\hat{\mathbf{x}}_i(s_i'), S) - \hat{\mathbf{u}}_i(s_i')|A(\hat{\mathbf{u}}_i(s_i')), s_i]. \tag{54}$$

Because seller i can approximate his conditional interim payoff arbitrarily well to the right hand side of (54) by submitting a slightly higher payoff bid that, with probability one, does not tie the others' payoff bids, along with submitting $\hat{\mathbf{x}}_i(s_i')$, (54) implies that the inequality on the first line of (53) holds. The inequality on the second line follows the definition of $\sup_{u_i,x_i} V_i(u_i, x_i, \hat{\mathbf{u}}_{-i}|s_i)$ and the equality on the third line holds because the fourth inequality in (50) is in fact an equality. (53) shows that $0 < \lim_{\varepsilon\downarrow 0} V_i(\hat{\mathbf{u}}_i(s_i) + \varepsilon, \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i)$. Because $\Pr(A(\hat{\mathbf{u}}_i(s_i)|s_i) > 0$, it implies that

$$\mathbb{E}[R_i^\tau(\hat{\mathbf{x}}_i(s_i), S) - \hat{\mathbf{u}}_i(s_i)|A(\hat{\mathbf{u}}_i(s_i)), s_i] > 0.$$

Therefore, the inequality in (51) must be strict for a.e. s_i such that $\Pr(A(\hat{\mathbf{u}}_i(s_i))\ |s_i) > 0$ so that $\mathbb{E}[\alpha_i(S)|A(\hat{\mathbf{u}}_i(s_i)), s_i] = 1$ for a.e. s_i such that $\Pr(A(\hat{\mathbf{u}}_i(s_i))|s_i) > 0$.

Let $T_I = \{s : \hat{\mathbf{u}}_i(s_i) = \max_j \hat{\mathbf{u}}_j(s_j) > u_\circ, \forall i \in I\}$ for any non-empty subset $I \subseteq \{1, \ldots, N\}$. Consequently, if $\Pr(T_I) > 0$, then every $i \in I$, $\alpha_i(s) = 1$ for a.e. $s \in T_I$. However, $\sum_{i\in I} \alpha_i(s) \leq 1$ for a.e. $s \in [0,1]^N$. This implies that $\#(I) = 1$. Therefore, the probability that under payoff bidding functions $\hat{\mathbf{u}}$, two or more sellers simultaneously submit the highest bid above $u_\circ$ is zero. Then, for every i and a.e. s_i, $V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i)$ is continuous at $(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}(s_i), \hat{\mathbf{u}}_{-i})$, being continuous whenever $\hat{\mathbf{u}}_i(s_i) = u_\circ$ because $u_\circ$ is isolated. Therefore, we have $\lim_{n\uparrow\infty} V_i(\mathbf{u}_i^n(s_i), \mathbf{x}_i^n(s_i), \mathbf{u}_{-i}^n|s_i) = V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), , \hat{\mathbf{u}}_{-i}|s_i)$ for a.e. s_i and (49) implies that $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$ is a monotone equilibrium of the modified procurement.■

Proof of Theorem 2. Fix a monotone equilibrium $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$ in the modified procurement for any given $\tau = [\tau_1, \ldots, \tau_N] \in [0,1]^N$. For any menu m_i in the non-committed procurement, let $\tilde{u}_i(m_i)$ be the maximum payoff level that the buyer can achieve by choosing m_i. Let each seller i choose a menu $\mathbf{m}_i^\tau(s_i)$ that satisfies (i) $\tilde{u}_i(\mathbf{m}_i^\tau(s_i)) = \hat{\mathbf{u}}_i(s_i)$ for all $s_i \in [0,1]$, (ii) $(x_i^*(s), u(x_i^*(s)) - \hat{\mathbf{u}}_i(s_i)) \in D(m_i)$ for all $s \in [0,1]^N$, (iii) $(\hat{\mathbf{x}}_i(s_i), u(\hat{\mathbf{x}}_i(s_i)) - \hat{\mathbf{u}}_i(s_i)) \in D(m_i)$ for all $s_i \in [0,1]$, and (iv) $\mathbf{m}_i^\tau(s_i) \neq \mathbf{m}_i^\tau(s_i')$ if $s_i \neq s_i'$. Once the buyer accepts seller i's menu m_i, given the other sellers' menu strategies $\mathbf{m}_{-i}^\tau$, she chooses a contract from m_i in the following manner:

1. If $m_i = \mathbf{m}_i^\tau(s_i)$ for some s_i

$$\sigma_i^\tau(m_i, \mathbf{m}_{-i}^\tau(s_{-i})) = \begin{cases} (\hat{\mathbf{x}}_i(s_i), u(\hat{\mathbf{x}}_i(s_i)) - \hat{\mathbf{u}}_i(s_i)) \in D(m_i) & \text{with prob } 1 - \tau_i \\ (x_i^*(s), u(x_i^*(s)) - \hat{\mathbf{u}}_i(s_i)) \in D(m_i) & \text{with prob. } \tau_i \end{cases},$$

2. otherwise, $\sigma_i^\tau(m_i, \mathbf{m}_{-i}^\tau(s_{-i})) = (x_i, u(x_i) - \tilde{u}_i(m_i))$ with prob. one, where x is some arbitrary characteristics such that $(x_i, u(x_i) - \tilde{u}_i(m_i)) \in D(m_i)$.

Because $D(m_i)$ is the set of optimal contracts for the buyer once she accepts m_i, $\sigma^\tau = [\sigma_1^\tau, \ldots, \sigma_N^\tau]$ characterized by 1 and 2 is a continuation equilibrium. Suppose that seller i chooses a strategy $\mathbf{m}_i^\tau$. When his signal is s_i and he offers the menu $\mathbf{m}_i^\tau(s_i)$, his interim payoff is

$$
\begin{aligned}
&V_i(\mathbf{m}_i^\tau(s_i), \mathbf{m}_{-i}^\tau | s_i, \sigma_i^\tau) = \\
&\quad (1-\tau_i)\Pr(A(\hat{\mathbf{u}}_i(s_i))|s_i)\mathbb{E}\left[(R_i(\hat{\mathbf{x}}_i(S_i), S) - \hat{\mathbf{u}}_i(S_i))\lambda_i(\hat{\mathbf{u}}(S))|A(\hat{\mathbf{u}}_i(s_i)), s_i\right] + \\
&\quad \tau_i \Pr(A(\hat{\mathbf{u}}_i(s_i))|s_i)\mathbb{E}\left[(R_i(x_i^*(S), S) - \hat{\mathbf{u}}_i(S_i))\lambda_i(\hat{\mathbf{u}}(S))|A(\hat{\mathbf{u}}_i(s_i)), s_i\right] = \\
&\quad V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i).
\end{aligned} \tag{55}
$$

There are two types of deviations in the non-committed procurement. First of all, consider that seller i deviates to a menu $\mathbf{m}_i^\tau(s_i')$ for some s_i' ($s_i' \neq s_i$). When seller i with s_i deviates to the menu $\mathbf{m}_i^\tau(s_i')$, the buyer believes that his signal is s_i'. Once she accepts the menu $\mathbf{m}_i^\tau(s_i')$, she will take the characteristics $\hat{\mathbf{x}}_i(s_i')$ with probability $1-\tau_i$ and the characteristics $x_i^*(s_i', s_{-i})$ with probability τ_i when the other sellers' menus are $\mathbf{m}_{-i}^\tau(s_{-i})$. Therefore, seller i's interim payoff $V_i(\mathbf{m}_i^\tau(s_i'), \mathbf{m}_{-i}^\tau|s_i, \sigma_i^\tau)$ becomes

$$
\begin{aligned}
&V_i(\mathbf{m}_i^\tau(s_i'), \mathbf{m}_{-i}^\tau|s_i, \sigma_i^\tau) = (1-\tau_i)\Pr(A(\hat{\mathbf{u}}_i(s_i'))|s_i)\times \\
&\quad \mathbb{E}\left[(R_i(\hat{\mathbf{x}}_i(s_i'), S) - \hat{\mathbf{u}}_i(s_i'))\lambda_i(\hat{\mathbf{u}}_i(s_i'), \hat{\mathbf{u}}_{-i}(S_{-i}))|A(\hat{\mathbf{u}}_i(s_i')), s_i\right] + \\
&\quad \tau_i \Pr(A(\hat{\mathbf{u}}_i(s_i')|s_i)\times \\
&\quad \mathbb{E}\left[(R_i(x_i^*(s_i', S_{-i}), S) - \hat{\mathbf{u}}_i(s_i'))\,\lambda_i(\hat{\mathbf{u}}_i(s_i'), \hat{\mathbf{u}}_{-i}(S_{-i}))|A(\hat{\mathbf{u}}_i(s_i')), s_i\right].
\end{aligned} \tag{56}
$$

This interim payoff $V_i(\mathbf{m}_i^\tau(s_i'), \mathbf{m}_{-i}^\tau|s_i, \sigma_i^\tau)$ satisfies

$$
\begin{aligned}
&V_i(\mathbf{m}_i^\tau(s_i'), \mathbf{m}_{-i}^\tau|s_i, \sigma_i^\tau) \le (1-\tau_i)\Pr(A(\hat{\mathbf{u}}_i(s_i'))|s_i)\times \\
&\quad \mathbb{E}\left[(R_i(x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i}), S) - \hat{\mathbf{u}}_i(s_i'))\lambda_i(\hat{\mathbf{u}}_i(s_i'), \hat{\mathbf{u}}_{-i}(S_{-i}))|A(\hat{\mathbf{u}}_i(s_i')), s_i\right] + \\
&\quad \tau_i \Pr(A(\hat{\mathbf{u}}_i(s_i')|s_i)\times \\
&\quad \mathbb{E}\left[(R_i(x_i^*(S), S) - \hat{\mathbf{u}}_i(s_i'))\,\lambda_i(\hat{\mathbf{u}}_i(s_i'), \hat{\mathbf{u}}_{-i}(S_{-i}))|A(\hat{\mathbf{u}}_i(s_i')), s_i\right] = \\
&\quad V_i(\hat{\mathbf{u}}_i(s_i'), x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i}), \hat{\mathbf{u}}_{-i}|s_i).
\end{aligned} \tag{57}
$$

We show why (57) holds. Note that the equality in (57) follows the definition of the interim payoff for seller i with s_i, i.e., $V_i(\hat{\mathbf{u}}_i(s_i'), x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i}), \hat{\mathbf{u}}_{-i}|s_i)$, in the modified procurement when he deviates to submit the bid $(\hat{\mathbf{u}}_i(s_i'), x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i}))$. Consider the inequality in (57). The winning event associated with the bid $(\hat{\mathbf{u}}_i(s_i'), x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i}))$ in the modified procurement and the winning event associated with $\mathbf{m}_i^\tau(s_i')$ in the non-committed procurement are both $A(\hat{\mathbf{u}}_i(s_i'))$. When seller i of signal s_i deviates to $\mathbf{m}_i^\tau(s_i')$ in the non-committed procurement, the buyer chooses $(\hat{\mathbf{x}}_i(s_i'), u(\hat{\mathbf{x}}_i(s_i')) - \hat{\mathbf{u}}_i(s_i'))$ with probability $1-\tau_i$ but $(x_i^*(s_i', s_{-i}), u(x_i^*(s_i', s_{-i})) - \hat{\mathbf{u}}_i(s_i'))$ at each s_{-i} with probability τ_i as if seller i's signal was s_i'. However, when seller i of signal s_i deviates to the bid $(\hat{\mathbf{u}}_i(s_i'), x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i}))$ in the modified procurement, the buyer takes the correct jointly interim efficient contract

$$(x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i}), u(x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i})) - \hat{\mathbf{u}}_i(s_i'))$$

with probability $1-\tau_i$ and a correct jointly ex-post efficient contract

$$(x_i^*(s_i, s_{-i}), u(x_i^*(s_i, s_{-i}) - \hat{\mathbf{u}}_i(s_i'))$$

at each s_{-i} with probability τ_i, knowing that seller i's true signal is s_i. Therefore, the inequality in (57) holds.

Because $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$ is a monotone equilibrium in the modified procurement, we have

$$V_i(\hat{\mathbf{u}}_i(s_i'), x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i}), \hat{\mathbf{u}}_{-i}|s_i) \leq V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i) \quad (58)$$

for a.e. s_i. Because $V_i(\mathbf{m}_i^\tau(s_i'), \mathbf{m}_{-i}^\tau|s_i, \sigma_i^\tau) \leq V_i(\hat{\mathbf{u}}_i(s_i'), x_i^e(\hat{\mathbf{u}}_i(s_i'), s_i, \hat{\mathbf{u}}_{-i}), \hat{\mathbf{u}}_{-i}|s_i)$ by (57) and $V_i(\mathbf{m}_i^\tau(s_i), \mathbf{m}_{-i}^\tau|s_i, \sigma_i^\tau) = V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i)$ by (55), (58) implies, for a.e. s_i,

$$V_i(\mathbf{m}_i^\tau(s_i'), \mathbf{m}_{-i}^\tau|s_i, \sigma_i^\tau) \leq V_i(\mathbf{m}_i^\tau(s_i), \mathbf{m}_{-i}^\tau|s_i, \sigma_i^\tau).$$

Therefore, it is not profitable for seller i with a.e. s_i to deviate to $\mathbf{m}_i^\tau(s_i')$ with $s_i' \neq s_i$ in the non-committed procurement.

Secondly, suppose that seller i with an arbitrary signal s_i deviates to a menu $m_i \neq \mathbf{m}_i^\tau(s_i')$ for all $s_i' \in [0,1]$. In continuation equilibrium σ^τ, the buyer takes characteristics x_i along with the monetary payment $u(x_i) - \tilde{u}_i(m_i)$ with probability one upon accepting m_i. Suppose that seller i deviates to submit $(\tilde{u}_i(m_i), x_i)$ in the modified procurement. The winning event for seller i in the modified procurement is the same as the one in the non-committed procurement. While the buyer always chooses $(x_i, u(x_i) - \tilde{u}_i(m_i))$ in the non-committed procurement upon accepting m_i, she chooses $(x_i^*(s), u(x_i^*(s)) - \tilde{u}_i(m_i))$ with probability τ_i and $(x_i, u(x_i) - \tilde{u}_i(m_i))$ with probability $1 - \tau_i$ in the modified auction. Because $x_i^*(s)$ is jointly ex-post efficient, seller i's interim payoff upon deviation to a menu m_i in the non-committed procurement is no greater than the one associated with the deviation to $(\tilde{u}_i(m_i), x_i)$ in the modified procurement. Therefore, we have

$$V_i(m_i, \mathbf{m}_{-i}^\tau|s_i, \sigma_i^\tau) \leq V_i(\tilde{u}_i(m_i), x_i, \hat{\mathbf{u}}_{-i}|s_i). \quad (59)$$

Because $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$ is a monotone equilibrium in the modified procurement, we have

$$V_i(\tilde{u}_i(m_i), x_i, \hat{\mathbf{u}}_{-i}|s_i) \leq V_i(\hat{\mathbf{u}}_i(s_i), \hat{\mathbf{x}}_i(s_i), \hat{\mathbf{u}}_{-i}|s_i). \quad (60)$$

for a.e. s_i. From (55), (59) and (60), we can conclude, for a.e., s_i,

$$V_i(m_i, \mathbf{m}_{-i}^\tau|s_i, \sigma_i^\tau) \leq V_i(\mathbf{m}_i^\tau(s_i), \mathbf{m}_{-i}^\tau|s_i, \sigma_i^\tau).$$

Therefore, it is also not profitable for seller i with a.e. s_i deviates to a menu $m_i \neq \mathbf{m}_i^\tau(s_i')$ for all $s_i' \in [0,1]$.

Because there is no profitable deviation for each seller i in the non-committed procurement, the existence of the truthful monotone equilibrium $\{\mathbf{m}_1^\tau, \ldots, \mathbf{m}_N^\tau, \sigma^\tau\}$ follows immediately from the existence of the corresponding monotone equilibrium $\{(\hat{\mathbf{u}}_1, \hat{\mathbf{x}}_1), \ldots, (\hat{\mathbf{u}}_N, \hat{\mathbf{x}}_N)\}$ in the modified procurement. ■

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