On the not-preference-based Hoede-Bakker index*

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Abstract. The paper concerns a certain modification of the generalized Hoede-Bakker index - a notion defined for a social network of players. In the original Hoede-Bakker set up, preferences of players are involved. It is assumed that a player has an inclination either to accept or to reject a proposal, but due to the influence of others, his final decision may be different from his original inclination. In this paper, we propose the not-preference-based (NPB) generalized Hoede-Bakker index, where feasible strategies instead of players' inclinations are considered. We show that if all feasible strategy profiles are equally probable, then the NPB generalized Hoede-Bakker index is a 'net' Success, i.e., 'Success - Failure', where Success and Failure of a player is defined as the probability that the player is successful and fails, respectively. Moreover, under the assumption of equal probabilities of all feasible strategy profiles, we show that the probability that a player is lucky (Luck) equals the probability that he fails (Failure). Since Success - Luck = Decisiveness, it follows that, under the same assumption, the NPB generalized Hoede-Bakker index is equal to the probability that a player is decisive.

Keywords: Hoede-Bakker index, feasible strategy, success, failure, decisiveness

JEL Classification: C7, D7

1 Introduction

The concept of the *Hoede-Bakker index* has been introduced in the Journal of Mathematical Sociology in 1982 (Hoede and Bakker 1982). Some properties of the Hoede-Bakker index, including its sensitivity to voting power paradoxes, have been analyzed by Rusinowska and De Swart (2007). In Rusinowska and De Swart (2006), a natural generalization of the Hoede-Bakker index, coinciding with the absolute Banzhaf index (Banzhaf 1965) has been presented. Moreover, the authors proposed several modifications of the Hoede-Bakker index that coincide with the Rae index (Rae 1969), the Coleman indices (Coleman 1971, 1986), and the König-Bräuninger index (König and Bräuninger 1998).

In the Hoede-Bakker set up, a social network is considered in which players have to make a decision about accepting or rejecting a given proposal. It is assumed that each player has an inclination either to accept or to reject the proposal. The players may influence each other, and as a consequence of this influence, the final decision of a player may be different from his original inclination. Following Braham and Holler (2005), where the impossibility of a preference-based power index is discussed, we look 'critically' at the Hoede-Bakker index based on players' inclinations. The aim of this paper is to bring the concept of the (generalized) Hoede-Bakker index closer to the measurement of power defined as the ability of a player to affect an outcome. We introduce a not-preference-based

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(NPB) version of the (generalized) Hoede-Bakker index and investigate what precisely this NPB index measures.

There are basically two approaches to power indices: the axiomatic approach and the probabilistic one. In Straffin (1977, 1978), two closely related probabilistic models which give the Shapley-Shubik index (Shapley and Shubik 1954) and the absolute Banzhaf index (Banzhaf 1965), respectively, are proposed. Straffin (1977, 1978) considers two different assumptions about voting behavior: the *independence assumption*, which says that 'players vote completely independently' (which is equivalent to assuming that each player will vote 'yes' with probability one half on any given proposal), and the *homogeneity assumption*, which means that 'a common set of values tends to influence the choices of all players'. Furthermore, Straffin asks the *question of effect on outcome*, i.e., what is the probability that the vote of a player will make a difference in the outcome. It is proven (Straffin 1977) that under the independence assumption, the answer to a player's question of effect on outcome is given by the absolute Banzhaf index of the player, while under the homogeneity assumption, the answer to this question is given by the Shapley-Shubik index of the player. The model also suggests modifications leading to other measures of voting power. One modification concerns the *partial homogeneity assumption*, according to which there are various blocs of players who are homogeneous among themselves and behave independently of other players. Straffin (1977, 1978) defines a power index which gives the answer to a player's question of effect on outcome under the partial homogeneity assumption. Another modification is to vary the probability distribution, i.e., to consider a probability distribution different from the uniform one, and to impose the *general independence assumption* and the *general homogeneity assumption*. This leads to a number of 'intermediate' indices. Straffin (1978) also asks the *question of group-individual agreement*, i.e., what is the probability that the group decision will agree with a player's decision. This is, of course, the question originally considered by Rae (1969). In Straffin (1978), power indices which give the answer to the question of group-individual agreement under the independence assumption and the homogeneity assumption, respectively, are defined.

In this paper, we follow the probabilistic approach, but apply another probabilistic model to the Hoede-Bakker set up, i.e., the model proposed by Laruelle and Valenciano (2005). The model will be recapitulated later on. We consider the concepts of (ex ante) *Success*, (ex ante) *Luck*, (ex ante) *Decisiveness*, and (ex ante) *Failure* of a player, meaning the probability that the player is successful, lucky, decisive, and fails, respectively. We prove that if all vote configurations are equally probable, then (ex ante) Luck is equal to (ex ante) Failure. Consequently, we give an argument against pessimistic thinking that 'It is more likely that I will fail than be successful', by showing that under the assumption of equal probability of all vote configurations, (ex ante) Success is not smaller than (ex ante) Failure. Subsequently, we apply the probabilistic approach to an analysis of the NPB generalized Hoede-Bakker index. We show that if all strategy profiles are equally probable, then the NPB generalized Hoede-Bakker index is a 'net Success', i.e., (ex ante) Success − (ex ante) Failure. Moreover, the relation between (ex ante) Luck and (ex ante) Failure is applied to prove that the NPB generalized Hoede-Bakker index is also equal to (ex ante) Decisiveness if all strategy profiles are equally probable.

The structure of the paper is as follows. In Section 2, we recapitulate the concepts of the Hoede-Bakker index (Hoede and Bakker 1982) and the generalized Hoede-Bakker index (Rusinowska and de Swart 2006). Section 3 concerns a modification of the Hoede-Bakker index, introducing the not-preference-based version of this index. In Section 4, we recapitulate the probabilistic model developed in Laruelle and Valenciano (2005), and we prove the relation between Success and Failure. Using the probabilistic approach, in particular, the relation between Success and Failure proved in Section 4, we show in Section 5 what precisely is measured by the not-preference-based Hoede-Bakker index. Finally, in Section 6, we draw some conclusions. In particular, we discuss the importance of the NPB generalized Hoede-Bakker index and its added value in contrast to the work done by Rae (1969) and Straffin (1978).

2 The Hoede-Bakker index

Let us first recapitulate the definition of the *Hoede-Bakker index* (Hoede and Bakker, 1982). We consider the situation in which $n \geq 1$ players (actors, voters) have to make an acceptance-rejection decision. Let $N = \{1, ..., n\}$ be the set of all players. Each actor is supposed to have an inclination either to say 'yes' (denoted by 1) or 'no' (denoted by 0). Let i be an *inclination vector* (i.e., an n-vector consisting of ones and zeros and indicating the inclinations of the actors), and let I denote the set of all inclination n-vectors. Of course, $|I| = 2^n$. Players may influence each other. Due to influences of the other actors, the final decision of a player may be different from his original inclination. Each inclination vector $i \in I$ is then transformed into a *decision vector* b which is also an n-vector consisting of ones and zeros and indicating the decisions made by all the players. Formally, there is an operator $B: \ I \rightarrow \ B(I)$, that is, $b = Bi$, where $B(I)$ denotes the set of all decision vectors. We also introduce the *group decision* $gd : B(I) \rightarrow \{+1, -1\}$ which is a function defined on the decision vectors b, having the value $+1$ if the group decision is 'yes', and the value -1 if the group decision is 'no'.
Let $i^c = (i_1^c, ..., i_n^c)$ denote the *complement* of $i \in I$, that is, for each $k \in \{1, ..., n\}$

$$i_k^c = \begin{cases} 1 \, if \; i_k = 0 \\ 0 \, if \; i_k = 1 \end{cases}. \tag{1}$$

Moreover, let

$$i \leq i' \iff \{k \in N \mid i_k = 1\} \subseteq \{k \in N \mid i'_k = 1\}. \tag{2}$$

Let $gd(B)$ be the composition of B and gd. Hoede and Bakker (1982) adopted the following two axioms:

$$\forall i \in I \; [gd(Bi^c) = -gd(Bi)], \tag{3}$$

$$\forall i \in I \; \forall i' \in I \; [i \leq i' \;\Rightarrow\; gd(Bi) \leq gd(Bi')]. \tag{4}$$

Axiom (3) means that changing all inclinations leads to the opposite group decision. According to axiom (4), the group decision 'yes' cannot be changed into 'no' if the set of players with inclination 'yes' is enlarged.

Given $gd(B)$, the *Hoede-Bakker index of player* $k \in N$ is defined as

$$HB_k(gd(B)) = \frac{1}{2^{n-1}} \cdot \sum_{\{i:\ i_k=1\}} gd(Bi). \tag{5}$$

In Rusinowska and De Swart (2006), instead of imposing (3), the authors adopted the following two axioms (together with (4)):

$$gd(B(i_1, ..., i_n)) = +1,\ if\ i_k = 1\ for\ each\ k = 1, 2, ..., n \tag{6}$$

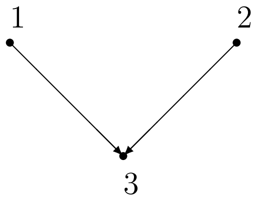
$$gd(B(i_1, ..., i_n)) = -1,\ if\ i_k = 0\ for\ each\ k = 1, 2, ..., n. \tag{7}$$

If all players have the same inclination, there is no reason to assume that the group decision may be different from this common inclination. Moreover, given $gd(B)$, the *generalized Hoede-Bakker index of player* $k \in N$ is defined by

$$GHB_k(gd(B)) = \frac{1}{2^n} \cdot (\sum_{\{i:\ i_k=1\}} gd(Bi) - \sum_{\{i:\ i_k=0\}} gd(Bi)). \tag{8}$$

Example 1 In order to illustrate the notions mentioned above, we consider a three-player social network given in Figure 1.

Figure 1 A three-player social network



Players 1 and 2 are not influenced, i.e., they always decide according to their own inclinations, but they do influence player 3. Player 3 is assumed to follow players 1 and 2 only if the influencing players have the same inclinations. Otherwise, player 3 just follows his own inclination. Concerning the group decision, in order to stress that the (generalized) Hoede-Bakker index depends on $gd(B)$, we consider two cases. First, we consider a group decision which is equal to the decision of a majority of players (the group decision denoted by gd). This means, of course, that the group decision is 'yes' if and only if at least two players decide for 'yes'. Next, we consider unanimity, i.e., we assume that a group decision is 'yes' if and only if all three players say 'yes' (the group decision denoted by gd'). Table 1 presents the group decision in both cases.

Table 1 Group decision for Figure 1

i	Bi	$gd(Bi)$	$gd'(Bi)$	i	Bi	$gd(Bi) = gd'(Bi)$
(1,1,1)	(1,1,1)	$+1$	$+1$	(0,0,0)	(0,0,0)	-1
(1,1,0)	(1,1,1)	$+1$	$+1$	(0,0,1)	(0,0,0)	-1
(1,0,1)	(1,0,1)	$+1$	-1	(0,1,0)	(0,1,0)	-1
(0,1,1)	(0,1,1)	$+1$	-1	(1,0,0)	(1,0,0)	-1

Note that axioms (4), (6) and (7) are satisfied in both cases. This means that we can calculate the generalized Hoede-Bakker index (see definition (8)) for both cases. Axiom (3) holds if we consider the group decision gd, but it is violated in the case of gd'. Hence, we are not able to determine the (original) Hoede-Bakker index if we consider the group decision gd'. This shows the restrictiveness of the (original) Hoede-Bakker index as already noticed in Rusinowska and de Swart (2006). By virtue of formulas (5) and (8), we get the following results:

$$GHB_k(gd(B)) = \frac{3-1+3-1}{8} = \frac{1}{2} = \frac{3-1}{4} = HB_k(gd(B)) \quad for \quad k = 1,2,3$$

$$GHB_1(gd'(B)) = GHB_2(gd'(B)) = \frac{2-2+4-0}{8} = \frac{1}{2}$$

$$GHB_3(gd'(B)) = \frac{1-3+3-1}{8} = 0.$$

As one can see, switching from gd to gd' did not change the generalized Hoede-Bakker index of either player 1 or player 2, but it did decrease the index of player 3.

3 The not preference-based decisional power index

In Braham and Holler (2005), the authors discuss the impossibility of a preference-based power index. According to the 'core theorem of the measurement of power' (Braham and Holler 2005), 'if power is the ability of k to affect an outcome, then a measure of k's power must exclude any reference to k's preference with respect to affecting that outcome'. Hence, the (generalized) Hoede-Bakker index as based on voters' inclinations cannot be treated as a measure of power (defined as the ability of a player to affect an outcome) if we treat player k's *inclination* to say 'yes' or 'no' as player k's *preference* to say 'yes' or 'no'.

In this section, we propose the not-preference-based version of the Hoede-Bakker index. Let $N = \{1, ..., n\}$ denote, as before, the set of all players (actors). Instead of assuming that player $k \in N$ has 'inclination' i_k either to say 'yes' ($i_k = 1$) or to say 'no' ($i_k = 0$), we assume that player k chooses a *feasible strategy* (which we will call just a *strategy*) f_k from among two possibilities: *I will say 'yes'* (i.e., $f_k = 1$), or *I will say 'no'* ($f_k = 0$). Choosing strategy f_k by player k is equivalent to player k's preliminary decision (i.e., his decision before any influence of others) either to say 'yes' or to say 'no'. As soon as player k has made a preliminary decision about which strategy (s)he chooses, a confrontation with real life takes place, which, as everybody can experience, may bring a lot of influences of the others. Due to influences of other players, a final decision of player k may be different from his feasible strategy (meant as his preliminary decision). Let $f = (f_1, ..., f_n)$ denote a *strategy profile*. It is an n-vector consisting of ones and zeros, and indicating strategies (preliminary decisions) of the players. Let F denote the set of all possible strategy profiles. Of course, $|F| = 2^n$. Similarly as in Hoede and Bakker (1982), we define an operator $B : F \to B(F)$, assigning to each strategy profile $f \in \{1, 0\}^n$, a (final) decision vector $B(f) \in \{1, 0\}^n$. This covers situations where a player, due to the influence of others, makes a final decision which is different from his preliminary decision (strategy).

In order to see that the preferences (inclinations) of a player do not have to coincide with his strategies, let us consider the following example. A husband has to make a decision about buying an expensive and beautiful house. He would love to buy it, his inclination for buying is 'yes', but he decides not to do it. His strategy, meant as a preliminary decision, is 'no', since such a purchase would bring serious financial problems. He already tries to enjoy the thought how many 'substitutes' he can buy for the money 'saved'. Unfortunately, or fortunately, his wife is quite determined and she uses a typical blackmail. She just says 'either this house or I disappear'. Although the man's preliminary decision was 'no', under the strong influence of his wife, and evaluating all 'for and against', his final decision is 'yes'.

Another example we like to mention concerns a birthday cake. Suppose I am invited for a birthday party of one of my friends. Of course, a very tempting birthday cake will be served during the party. I will have to say either 'yes' (I will try it) or 'no' (I will not). My inclination is positive: I would love to try the birthday cake since the cake has been bought in the best bakery in town. Nevertheless, since I want to realize my New Year decision 'No cakes this year', I choose the strategy 'no'. This is a preliminary decision, made before facing the influence of other players, and with such a strategy I enter my friend's apartment. Nevertheless, since I seem to upset seriously my friend by my strategy, and being influenced additionally by other participants of the party, my final decision is 'yes'.

Similarly as in Hoede and Bakker (1982), we also define a group decision, assigning to each decision vector $B(f)$, a group decision $gdB(f) \in \{+1, -1\}$, where $gdB(f) = +1$ if the group decision is 'yes', and $gdB(f) = -1$ if the group decision is 'no'. Adopting axioms (4), (6), (7) with inclination vector $i = (i_1, ..., i_n) \in I$ replaced by strategy profile $f = (f_1, ..., f_n) \in F$, we propose the following definition:

Definition 1 *Given the composition $gd(B)$, the not-preference-based (NPB) generalized Hoede-Bakker index of player $k \in N$ is given by*

$$\widehat{GHB}_k(gd(B)) = \frac{1}{2^n} \cdot (\sum_{\{f:\ f_k=1\}} gd(Bf) - \sum_{\{f:\ f_k=0\}} gd(Bf)). \tag{9}$$

Let us introduce the following notations. Let for $k \in N$

$$F_k^{1+} := \{f \in F \mid f_k = 1 \ \wedge \ gd(Bf) = +1\} \tag{10}$$

$$F_k^{1-} := \{f \in F \mid f_k = 1 \ \wedge \ gd(Bf) = -1\} \tag{11}$$

$$F_k^{0+} := \{f \in F \mid f_k = 0 \ \wedge \ gd(Bf) = +1\} \tag{12}$$

$$F_k^{0-} := \{f \in F \mid f_k = 0 \ \wedge \ gd(Bf) = -1\}. \tag{13}$$

Using these notations, we can write for $k \in N$

$$\widehat{GHB}_k(gd(B)) = \frac{|F_k^{1+}| - |F_k^{1-}| + |F_k^{0-}| - |F_k^{0+}|}{2^n}. \tag{14}$$

In an analogous way, given the composition $gd(B)$, we can also define the *not preference-based Hoede-Bakker index of player $k \in N$* as

$$\widehat{HB}(k) = \frac{1}{2^{n-1}} \cdot \sum_{\{f:\ f_k=1\}} gd(Bf), \tag{15}$$

and using the notations above, we have for $k \in N$

$$\widehat{HB}(k) = \frac{|F_k^{1+}| - |F_k^{1-}|}{2^{n-1}}. \tag{16}$$

Nevertheless, since there seems to be a serious drawback of the (original) Hoede-Bakker index, we will focus on the NPB generalized Hoede-Bakker index.

In the remainder of this paper, we will need more notations, based on the ones introduced above. Let for $f = (f_1, ..., f_n) \in F$ and $k \in N$, $f^k = (f_1^k, ..., f_n^k)$ be defined as follows:

$$f_j^k = \begin{cases} f_j & if\ j \neq k \\ 1 - f_j & if\ j = k \end{cases} \quad for \quad j = 1, ..., n. \tag{17}$$

Moreover, let for $k \in N$

$$F_k^{1++} := \{f \in F_k^{1+} \mid gd(Bf^k) = +1\} \tag{18}$$

$$F_k^{1+-} := \{f \in F_k^{1+} \mid gd(Bf^k) = -1\} \tag{19}$$

$$F_k^{0--} := \{f \in F_k^{0-} \mid gd(Bf^k) = -1\} \tag{20}$$

$$F_k^{0-+} := \{f \in F_k^{0-} \mid gd(Bf^k) = +1\}. \tag{21}$$

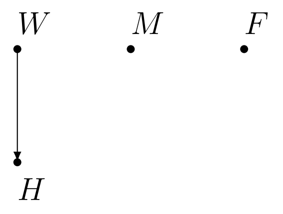
We have, of course, for $k \in N$

$$|F_k^{1+}| = |F_k^{1++}| + |F_k^{1+-}| \tag{22}$$

$$|F_k^{0-}| = |F_k^{0--}| + |F_k^{0-+}|. \tag{23}$$

Example 2 Let us consider the example just mentioned above, concerning a decision about buying an expensive and beautiful house. The situation is modeled as a four-player social network given in Figure 2. There are four players in this game: W - wife (player 1), H - husband (player 2), M - mother of W (player 3), and F - father of W (player 4). All players but the husband are not influenced by anyone and they decide according to their strategies. The husband is fully influenced by his wife, and consequently, he always follows her strategy. The parents of the wife do not influence anybody: they have their own strategies which they will follow, but they do not want to influence their children. The group decision is defined by the majority of the players: they will decide to buy the house if and only if at least three players will say ‘yes’.

Figure 2 A four-player social network



Let $f = (f_W, f_H, f_M, f_F)$ denote the strategy profile. Table 2 presents the group decision for our example.

Table 2 Group decision for Figure 2

f	Bf	$gd(Bf)$	f	Bf	$gd(Bf)$
$(1,1,1,1)$	$(1,1,1,1)$	$+1$	$(0,0,0,0)$	$(0,0,0,0)$	-1
$(1,1,1,0)$	$(1,1,1,0)$	$+1$	$(0,0,0,1)$	$(0,0,0,1)$	-1
$(1,1,0,1)$	$(1,1,0,1)$	$+1$	$(0,0,1,0)$	$(0,0,1,0)$	-1
$(1,0,1,1)$	$(1,1,1,1)$	$+1$	$(0,1,0,0)$	$(0,0,0,0)$	-1
$(0,1,1,1)$	$(0,0,1,1)$	-1	$(1,0,0,0)$	$(1,1,0,0)$	-1
$(1,1,0,0)$	$(1,1,0,0)$	-1	$(0,0,1,1)$	$(0,0,1,1)$	-1
$(1,0,1,0)$	$(1,1,1,0)$	$+1$	$(0,1,0,1)$	$(0,0,0,1)$	-1
$(1,0,0,1)$	$(1,1,0,1)$	$+1$	$(0,1,1,0)$	$(0,0,1,0)$	-1

Note that all axioms (4), (6), and (7) (with inclination vector $i = (i_1, ..., i_n) \in I$ replaced by strategy profile $f = (f_1, ..., f_n) \in F$) are satisfied. By virtue of definition (9), we get

$$\widehat{GHB}_W(gd(B)) = \frac{6-2+8-0}{16} = \frac{3}{4}$$

$$\widehat{GHB}_H(gd(B)) = \frac{3-5+5-3}{16} = 0$$

$$\widehat{GHB}_M(gd(B)) = \widehat{GHB}_F(gd(B)) = \frac{4-4+6-2}{16} = \frac{1}{4}.$$

The NPB generalized Hoede-Bakker index, defined in a social network with possible influences between players, seems to be an interesting notion. The question is what precisely this index measures. In order to answer this question, we apply a probabilistic approach. In Section 4, the probabilistic model, describing the notions of Success, Decisiveness, Failure and Luck will be recapitulated. In particular, in Section 4, we will show the relation between Failure and Luck. Next, this relation will be used in Section 5 to prove what precisely the NPB generalized Hoede-Bakker index measures.

4 Is Failure more likely than Success?

Sometimes in common parlance one misunderstands the difference or relation between the concepts of success, luck, and decisiveness. According to Barry (1980), a very simple relation between these notions holds:

$$\text{'Success'} = \text{'Decisiveness'} + \text{'Luck'}.$$

A simple and elegant probabilistic model for measuring 'success' and 'decisiveness' in voting situations has been presented in Laruelle and Valenciano (2005). In this paper, we recapitulate some notions of this model, stressing the concept of 'failure'.

Once a proposal is submitted, voters cast votes, voting either 'yes' (abstention included) or 'no'. The vote configuration S refers to the result of voting where all voters in S vote 'yes', and all voters in $N \setminus S$ vote 'no', with $N = \{1, 2, ..., n\}$. Hence, $k \in S$ means that

k's vote was 'yes'. For n voters, there are, of course, 2^n possible vote configurations. The vote configurations leading to the passage of a proposal are called winning configurations. Let W be the set of winning configurations, representing an N-voting rule. A voting rule is assumed to satisfy the following conditions:

(i) $N \in W$
(ii) $\emptyset \notin W$
(iii) If $S \in W$, then $T \in W$ for any T containing S
(iv) If $S \in W$, then $N \setminus S \notin W$.

Condition (i) says that the unanimous 'yes' vote configuration leads to the passage of the proposal. Condition (ii) means that the unanimous 'no' vote configuration leads to the rejection of the proposal. According to condition (iii), if a vote configuration is winning, then any other vote configuration containing it is also winning. Condition (iv) means that if a vote configuration S leads to the passage of the proposal, then $N \setminus S$ leads to its rejection, which implies, of course, that the game is proper. As mentioned in Laruelle and Valenciano (2005), in certain cases this condition is not necessary. In particular, we do not need condition (iv) in our paper, since all results shown in this paper hold without condition (iv).

We have some 'ex post' concepts that depend on the voting rule used and the resulting vote configurations after a vote is cast (Laruelle and Valenciano 2005). After a decision is made according to an N-voting rule W, if S is the resulting vote configuration and $k \in N$,

– Voter k is said to have been *successful* in (W, S) if the decision coincides with his vote, that is, iff
$$(k \in S \in W) \; or \; (k \notin S \notin W) \tag{24}$$

– Voter k is said to have been *decisive* in (W, S) if he was successful and his vote was critical for it, that is, iff
$$(k \in S \in W \; and \; S \setminus \{k\} \notin W) \; or \; (k \notin S \notin W \; and \; S \cup \{k\} \in W) \tag{25}$$

– Voter k is said to have been *lucky* in (W, S) if he was successful but irrelevant (i.e., his vote was not critical for it), that is, iff
$$(k \in S \in W \; and \; S \setminus \{k\} \in W) \; or \; (k \notin S \notin W \; and \; S \cup \{k\} \notin W) \tag{26}$$

– Voter k is said to have *failed* in (W, S) if he was not successful in (W, S) (i.e., the decision was different from his vote), that is, iff
$$(k \notin S \in W) \; or \; (k \in S \notin W). \tag{27}$$

A probability distribution over all possible vote configurations is incorporated into the model. A probability distribution may be represented by a map $p : 2^N \to [0, 1]$, associating with each vote configuration $S \subseteq N$ its probability $p(S)$ of occurring, where $\sum_{S \subseteq N} p(S) = 1$. That is, $p(S)$ is the probability that all voters in S vote 'yes', and all voters in $N \setminus S$ vote 'no'. Laruelle and Valenciano (2005) consider also the following 'ex ante' concepts, where by 'ex ante' they mean: before the voters cast their votes.
Let (W, p) be an N-voting situation, where W is the voting rule to be used and p is the probability distribution over vote configurations, and let $k \in N$. Then:

– Voter k's *(ex ante) Success* is the probability that k is successful:

$$SUC_k(W,p) := Prob(k\ is\ successful) = \sum_{S:k\in S\in W} p(S) + \sum_{S:k\notin S\notin W} p(S) \tag{28}$$

– Voter k's *(ex ante) Decisiveness* is the probability that k is decisive:

$$D_k(W,p) := Prob(k\ is\ decisive) = \sum_{\substack{S\,:\,k\in S\in W \\ S\setminus\{k\}\notin W}} p(S) + \sum_{\substack{S\,:\,k\notin S\notin W \\ S\cup\{k\}\in W}} p(S) \tag{29}$$

– Voter k's *(ex ante) Luck* is the probability that k is lucky:

$$L_k(W,p) := Prob(k\ is\ lucky) = \sum_{\substack{S\,:\,k\in S \\ S\setminus\{k\}\in W}} p(S) + \sum_{\substack{S\,:\,k\notin S \\ S\cup\{k\}\notin W}} p(S) \tag{30}$$

– Voter k's *(ex ante) Failure* is the probability that k fails:

$$FAIL_k(W,p) := Prob(k\ fails) = \sum_{S:k\notin S\in W} p(S) + \sum_{S:k\in S\notin W} p(S). \tag{31}$$

For any voting rule W, probability distribution p and voter k, we have

$$SUC_k(W,p) = D_k(W,p) + L_k(W,p) \tag{32}$$

$$FAIL_k(W,p) + SUC_k(W,p) = \sum_{S\subseteq N} p(S) = 1. \tag{33}$$

From the last two equations, one may immediately derive, for an arbitrary probability distribution p, that $FAIL_k(W,p) \leq 1 - D_k(W,p)$ and $FAIL_k(W,p) \leq 1 - L_k(W,p)$. This means, of course, that not being decisive is at least as likely as failing, and not being lucky is at least as likely as failing.

Let us assume that all vote configurations are equally probable, that is,

$$p^*(S) := \frac{1}{2^n}\ for\ each\ S \subseteq N. \tag{34}$$

The following theorem holds.

Theorem 1 *If all vote configurations are equally probable, then (ex ante) Luck of a voter is equal to (ex ante) Failure of this voter, and it is not greater than $\frac{1}{2}$, that is, for any voting rule W and any voter $k \in N$*

$$L_k(W,p^*) = FAIL_k(W,p^*) \leq \frac{1}{2}. \tag{35}$$

Proof. Note that, for each $k \in N$

$$|\{S:\ k\in S\ \wedge\ S\setminus\{k\}\in W\}| = |\{S:\ k\notin S\ \wedge\ S\in W\}| \tag{36}$$

$$|\{S:\ k\notin S\ \wedge\ S\cup\{k\}\notin W\}| = |\{S:\ k\in S\ \wedge\ S\notin W\}|. \tag{37}$$

Hence, from equations (34), (36) and (37), we get for any W and k,

$$\sum_{\substack{S\,:\,k\in S\\ S\setminus\{k\}\in W}} p^*(S) = \sum_{S:k\notin S\in W} p^*(S), \tag{38}$$

$$\sum_{\substack{S\,:\,k\notin S\\ S\cup\{k\}\notin W}} p^*(S) = \sum_{S:k\in S\notin W} p^*(S), \tag{39}$$

and therefore, by virtue of (30) and (31), $L_k(W,p^*) = FAIL_k(W,p^*)$. Moreover, by virtue of (32) and (33), $2FAIL_k(W,p^*) + D_k(W,p^*) = 1$, and since $D_k(W,p^*) \geq 0$, we get $FAIL_k(W,p^*) \leq \frac{1}{2}$. □

Since $FAIL_k(W,p^*) \leq \frac{1}{2}$ for any W and k, then using (33), we immediately conclude that $SUC_k(W,p^*) \geq \frac{1}{2}$ which means that

Conclusion 1 *If all vote configurations are equally probable, then (ex ante) Failure is not greater than (ex ante) Success, that is, for any voting rule W and any voter $k \in N$*

$$FAIL_k(W,p^*) \leq SUC_k(W,p^*). \tag{40}$$

This conclusion is an argument against all pessimists who say that it is more likely to fail than to be successful. The situation is, in fact, the opposite if all vote configurations are equally probable. In Section 5, we will apply the probabilistic model recapitulated in this section, in particular, Theorem 1, to the analysis of the NPB generalized Hoede-Bakker index.

5 What do we measure?

The question arises what precisely is measured by the NPB generalized Hoede-Bakker index. Is it Decisiveness, Success, or maybe something else? In order to answer this question, we apply the probabilistic approach (Laruelle and Valenciano 2005) to the notions introduced in this paper. As recapitulated in the previous section, we assume vote configurations S (possible results of voting) and the set W of winning configurations, representing an N-voting rule. In our analysis, we do not consider $B(f)$ (decision vector) and gd (group decision) separately, but we consider strategy profiles f and the composition $gd(B)$. While the 'ex post' concepts considered in Laruelle and Valenciano (2005) depend on the voting rule and the resulting vote configuration after a vote is cast, our 'ext post' concepts depend on the composition $gd(B)$ and the strategy profile f. Similarly, while the 'ex ante' concepts in Laruelle and Valenciano (2005) concern the situation before the voters cast their votes, our 'ex ante' notions concern the situation before the voters choose their feasible strategies. We propose the following equivalent concepts:

– k's vote is 'yes' iff k's strategy is 'yes' / $k \in S$ iff $f_k = 1$.
– W is equivalent to F^+ representing $gd(B)$, where

$$F^+ = \{f \in F \mid gd(Bf) = +1\}. \tag{41}$$

- We impose a condition equivalent to condition (34):

$$p^*(f) := \frac{1}{2^n} \ for\ each\ f \in F. \tag{42}$$

- Conditions (i), (ii), (iii), and (iv) in Section 4 are by definition equivalent to axioms (6), (7), (4), and (3), but with $f \in F$ instead of $i \in I$, respectively.
- Voter k is said to have been *successful* under $(gd(B), f)$ iff

$$f \in F_k^{1+} \quad or \quad f \in F_k^{0-}. \tag{43}$$

- Voter k is said to have *failed* under $(gd(B), f)$ iff

$$f \in F_k^{0+} \quad or \quad f \in F_k^{1-}. \tag{44}$$

- Voter k is said to have been *decisive* under $(gd(B), f)$ iff

$$f \in F_k^{1+-} \quad or \quad f \in F_k^{0-+}. \tag{45}$$

- Voter k is said to have been *lucky* under $(gd(B), f)$ iff

$$f \in F_k^{1++} \quad or \quad f \in F_k^{0--}. \tag{46}$$

- Let us assume that all strategy profiles are equally probable. Then:
  Voter k's *(ex ante) Success* is equal to:

$$SUC_k(gd(B), p^*) = \frac{|F_k^{1+}| + |F_k^{0-}|}{2^n}, \tag{47}$$

  Voter k's *(ex ante) Failure* is equal to:

$$FAIL_k(gd(B), p^*) = \frac{|F_k^{0+}| + |F_k^{1-}|}{2^n}, \tag{48}$$

  Voter k's *(ex ante) Decisiveness* is equal to:

$$D_k(gd(B), p^*) = \frac{|F_k^{1+-}| + |F_k^{0-+}|}{2^n}, \tag{49}$$

  Voter k's *(ex ante) Luck* is equal to:

$$L_k(gd(B), p^*) = \frac{|F_k^{1++}| + |F_k^{0--}|}{2^n}. \tag{50}$$

  For any $gd(B)$ and $k \in N$, we have

$$SUC_k(gd(B), p^*) = D_k(gd(B), p^*) + L_k(gd(B), p^*), \tag{51}$$

$$FAIL_k(gd(B), p^*) + SUC_k(gd(B), p^*) = 1. \tag{52}$$

- It is also true (see Theorem 1) that for any $gd(B)$ and $k \in N$:

$$L_k(gd(B), p^*) = FAIL_k(gd(B), p^*). \tag{53}$$

By virtue of (14), (47), and (48), we get that if all strategy profiles are equally probable, then the NPB generalized Hoede-Bakker index is equal to

$$(ex\ ante)\ Success - (ex\ ante)\ Failure, \tag{54}$$

that is,

Corollary 1 *Given $gd(B)$, for each $k \in N$*

$$\widehat{GHB}_k(gd(B)) = SUC_k(gd(B), p^*) - FAIL_k(gd(B), p^*). \tag{55}$$

Moreover, we have

Corollary 2 *Given $gd(B)$, for each $k \in N$*

$$\widehat{GHB}_k(gd(B)) = D_k(gd(B), p^*). \tag{56}$$

Proof. By virtue of (51), (53), and (55), we have for each $k \in N$

$$D_k(gd(B), p^*) = SUC_k(gd(B), p^*) - L_k(gd(B), p^*) =$$

$$SUC_k(gd(B), p^*) - FAIL_k(gd(B), p^*) = \widehat{GHB}_k(gd(B)).$$

□

This result is in the line with the result obtained in Rusinowska and De Swart (2006), where using a different method it was shown that the generalized Hoede-Bakker index of a player is equal to the probability that the player is decisive, and consequently, that the generalized Hoede-Bakker index coincides with the absolute Banzhaf index. This result was shown under assumptions that all vote configurations (coalitions) as well as all inclination vectors are equally probable.

6 Conclusions

The original Hoede-Bakker index is based on players' inclinations which may be understood as players' preferences. In the NPB (generalized) Hoede-Bakker index we do not refer to players' inclinations, but to players' strategies treated as preliminary decisions. For many reasons, my real preferences do not have to coincide with my plans (strategies). My final decision may be also different from my strategy if someone is able to influence my preliminary decision.

What does our not-preference-based index measure? In order to answer this question we consider the concepts of (ex ante) Success, (ex ante) Decisiveness, (ex ante) Luck, and (ex ante) Failure of a player, meaning the probability that the player is successful, decisive, lucky, and fails, respectively. We show that if all strategy profiles are equally probable, then the NPB Hoede-Bakker index is equal to '(ex ante) Success − (ex ante) Failure' which we can call the 'net Success'. On the other hand, if all strategy profiles are equally probable, '(ex ante) Luck' is equal to '(ex ante) Failure', and consequently, '(ex ante) Success − (ex ante) Failure' is equal to '(ex ante) Decisiveness'. Hence, the NPB (generalized) Hoede-Bakker index of a player is equal to the probability that the player is decisive if all strategy profiles are assumed to be equally probable. Of course, strategy profiles which count for (ex ante) Luck of a player are different from the strategy profiles which count for (ex ante) Failure of the player, since being lucky means that the player is successful but irrelevant, while failing means that the player is not successful. But what we have proved is that the probability that a player is lucky is equal to the probability that the player fails if all strategy profiles are equally probable.

What is the added value of the NPB generalized Hoede-Bakker index in contrast to other established measures? First of all, our NPB Hoede-Bakker index is defined for a social network, where players may influence each other when making decisions. The fact that the NPB Hoede-Bakker index takes into account the distinction between the strategy of a player (his preliminary decision) and his final decision, contributes to the originality of the NPB Hoede-Bakker index. From this point of view, it seems to be interesting and innovative to deliver a deeper analysis of the measure defined in a set up different from the one in which most measures have been defined. Since many real situations may be represented by a social network with influences between actors, the analysis of different measures in a social network is of importance.

Apart from the evident relation to the model introduced by Laruelle and Valenciano (2005), the paper is, of course, also related to the works done by Rae (1969) and Straffin (1978). In Rae (1969) and Straffin (1978), indices which give the answer to the question of group-individual agreement, or differently speaking, to the question of Success of a player, are introduced and analyzed. Straffin (1977, 1978) also introduces indices which give the answer to the question of effect on the outcome, or differently speaking, to the question of Decisiveness of a player. Although related to the notion of Success, our NPB generalized Hoede-Bakker index happens to be 'net Success = Success − Failure', if all strategy profiles are equally probable. Hence, the NPB generalized Hoede-Bakker index is a measure still different from the notion of Success. The question appears, why one should be interested in measuring 'net Success'. First of all, we agree that 'the question of effect on the outcome may not be the best one for political theorists and legislators to ask, and they may be more interested in the question of group-individual agreement' (Straffin 1978). This would support the motivation for the interest not only in Decisiveness, but also in Success. Nevertheless, we find looking only at Success somewhat 'too optimistic'. If a player only focuses on the measure which tells him whether the group decision coincides with his strategy, he does not have to see explicitly when the group decision was different from his strategy. From this point of view, the NPB generalized Hoede-Bakker index seems to show reality better than Success. We may have reasons to be happy, because of our Success, but we also must see our Failure. Another interesting aspect of the NPB generalized Hoede-Bakker index is its relation with Decisiveness. We have proved that if all strategy profiles are equally probable, then the NPB generalized Hoede-Bakker index of a player is equal to the Decisiveness, defined as the probability that a player is decisive. First of all, this gives us technical support when calculating different measures in a social network. As one may easily realize, calculating Success is technically easier than calculating Decisiveness. When we determine Success of a player, for each strategy profile, we must only check whether the group decision resulting from this profile coincides with the strategy of the given player or not. We do not have to worry about whether this would also be the case if the player changed his strategy on his own. With our result that the NPB generalized Hoede-Bakker index is equal to Decisiveness (under the assumption of equal probability of all strategy profiles), the only thing which we have to count is Success. When we have calculated Success, we get immediately Failure, since Success and Failure together cover all possibilities, and next, thanks to our result on the NPB generalized Hoede-Bakker index, we get immediately Decisiveness.

Summarizing, there are at least three components which contribute to the importance of the NPB generalized Hoede-Bakker index. First of all, this not-preference-based and refined index is defined for a social network, and hence, may be useful for an analysis of many real life situations. Next, as a 'net Success' it seems to be more realistic than Success, since it takes into account both Success and Failure. Finally, it simplifies calculations when determining different measures in a social network, giving, in particular, an immediate answer to the question of Decisiveness.

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