On the Measurement of Fragmentation*

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Abstract

In this paper we propose a new party-system fragmentation measure as equivalent number of parties that fully takes into account the pivotal power of parties. The novel approach we adopt in the method of construction for an index of fragmentation allows us to take advantage of the theory of generalized means. We first construct the (class of) quasi-arithmetic mean(s) with Banzhaf power weight function for the party sizes and then, given the average size, we derive an equivalent-number of parties.

Keywords : Fragmentation, Effective number of parties, Voting power, Effective average size, Generalized means

JEL Classification : C65, D71, D72

1 Introduction

In this article, we derive a class of fragmentation measures for party systems as equivalent-number. The method we propose is new in the sense that instead of working directly on the construction of a fragmentation index, we first derive an average size of parties. Moreover, we pay special attention to take into consideration the political power in the *size* used in our measurement of fragmentation. Even if a wealth of literature on party

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systems uses the notion of fragmentation, well contoured definition is still missing. For this reason, we first provide the general feeling drawn from the most influential literature about measuring fragmentation that will serve as foundation for our own measurement approach.

The notion of fragmentation is a fundamental concept to classify party systems with comparative purposes. The notion of fragmentation is often used to study the length of government formation, the stability of government coalitions, the effect of electoral systems on the number of elected parties, etc ... The fragmentation is a key variable in determining the ease with which parties can agree upon a majority governement coalition. Since Laakso and Taagepera [16] devised the notion of *effective number of parties* that became the most widely used notion in comparative studies, the fragmentation of a party system is intimately linked to the effective number of parties. The Laakso-Taagepera effective number of parties is so deeply associated to the measurement of fragmentation that from that time forward both notions have become synonyms.Hence studying the fragmentation of a party system appears to be equivalent to answer the question of how to count the number of political parties in a decision-making assembly ?

It is well known that simply counting the number of parties contesting for office (i.e. in this paper forming a majority government) is often meaningless. Sartori [25] proposes only to count the "relevant" parties *i.e.*, those parties that may form a coalition in order to form a government. This procedure implies that small parties ideologically appealing as coalition partner should be taken into the same account as bigger-sized parties. Despite this principle is pragmatically compelling, it is nevertheless hard to operationalize.

Political alliances and distribution of political power strongly depend on the distribution of relative sizes of parties in a parliament. Laakso and Taagepera [16] address this problem by proposing an *effective* number of parties (abbreviated ENP) that takes into account the relative sizes of the parties, so that smaller parties count less than bigger parties. It is calculated by the following simple formula :

$$ENP = \frac{1}{\sum_{i=1}^{n}(s_i)^2} \tag{1.1}$$

where n is the total number of parties in the parliament under consideration and s_i is the number of seats (in percentage) of party i. An ENP answers the question "how extensive is the degree of fragmentation in a collective decision-making body ?" by "it is rather as if there were x parties of the same size", with x the value taken by the ENP. The interpretation is the following : the ENP is the number of hypothetical equal-sized parties that would have the same effect on fragmentation of the party system as have the actual parties of varying size.

The "sizes" that enter into the calculation of the ENP are the party seat shares. Alas, only looking at seat shares conveys partial information about the decision-making procedure. As Laver and Kato state [14]

> (...) the political implications of an election result must be interpreted in terms of their effect on the post-election government formation process, and that these implications may be much less obvious than they seem at first sight. What is important is not the 'raw' distribution of seats between parties but the decisive structure, the set of winning coalitions generated by this seat distribution.

An important feature of office-seeking models concerns the decisive structure of government formation. A party is said to be pivotal whenever the party can turn a winning coalition into a losing one by leaving or if the party can turn a losing coalition into a winning one by joining. Laver [18] argues that most coalitions tend to exclude nonpivotal parties from government formation. The concern is thus first and foremost put on the decisive structure of a coalition game, that is the set of winning coalitions generated by a given seat distribution and not the seat distribution only. Several different seat distributions can lead to the same decisive structure. As Laver [18] states, "elaborating the decisive structure is by far the most useful way to move from an election result to the strategic complexities of government formation". Recent literature using the notion of fragmentation becomes more and more aware of the importance of the pivotalness power of parties instead of seat distribution (see Dumont and Caulier [8], Grofman [11], Kline [15]). As any fragmentation measure solely defined on the distribution of seats fails to take into consideration the pivotal power of parties, we thus consider as necessary in our work to enrich the domain on which a fragmentation measure is defined to take into consideration power indices in order to properly model the decisive aspect of the structure.

For obvious reasons we now present, our approach is more positive then normative. Drawing inspiration from the measurement of inequality or concentration in industrial organization, two different approaches can be pursued in the measurement of fragmentation. The first one is normative and defines preferences of some benevolent planner over the set of possible seat distributions and derive fragmentation orderings afterwards. The preferences of an independent policy-maker provides an independent normative judgment to assess fragmentation. This approach has a long and venerable tradition in inequality measurement and even in concentration measurement in industries. Nevertheless, despite the attractive features of defining consistent preferences of some independent arbitrator, adopting this approach to the measurement of fragmentation would lead to the somewhat awkward conclusion that the most (or worst) preferred situation is the dictatorship. How can we determine by some independent judgement if less or more fragmentation should be preferred ? If we all agree that dictatorship has to be avoided since it violates the basic foundations of democracy, it has to be considered as the worst situation, but it would then mean that extremely high fragmented situations are the most desirable ! Mainly due to this reason, in this analysis we privilege the second approach, the positive one, that considers measures of fragmentation directly and we attempt to construct it on the basis of desirable properties without any reference to some underlying preferences.

The organization of the paper is the following : in the next section, we define the notion of effective-number or equivalent number and make the link with the notion of effective average size. In the third section, we define more precisely the notion of effective average size and derive a measure of it from the concept of quasideviation. In the section 4, we propose a modeling of the decision-making structure that have to be taken into account when one wants to measure the effective size of a party in a parliament. In section 5, we derive a class of measure of effective number of parties based on properties or axioms that are considered as desirable or unavoidable in this framework. In section 6, we propose to test the new measure on 106 parliaments and compare them with the existing indices. Section 7 concludes.

2 The effective average number of seats

Since the seminal paper of Laakso and Taagepera [16], the notion of fragmentation is better measured as equivalent-number : the effective number of parties (ENP) whose interpretation is the number of *equal-sized* parties in a hypothetical parliament that is considered as equally fragmented to the actual parliament of various size parties. In industrial economics, the notion of equivalent number has gained considerable currency as inverse measure of concentration. The relation between both notions is one-to-one, a decrease of equivalent number indicates an increase of concentration. Adelman [2] goes even further and states that an equivalent-number of equal-sized firms is the most natural and intuitive persuasive unit of concentration. An industry is said to be concentrated whenever a small number of firms own the major part of market shares. The same kind of link is often made with fragmentation of party systems. For example, in 2004 Bogaards [6] proposes to use the ENP to identify the presence of a dominant party in Africa. Caulier and Dumont [7] present and test empirically four measures of fragmentation to identify cases of dominance in various cases of party systems. It is thus self-evident that we will be able to rely upon some properties of concentration measures in industrial economics for the development of some fragmentation index as number equivalent.

The way we develop theoretically our new (class of) equivalent number(s) of parties in this paper is indirect. We will illustrate the method of construction shortly. We first remark that even if we know that in order to properly define a fragmentation measure the domain of definition must not be the distribution of seats, we discard for the moment this argument and provide a very simple example of our method of construction of an equivalent-number of parties based on the distribution of seat shares. Keep thus in mind that we drop this assumption later on for the development of our new class of measures.

Assume a set of elected political parties in a parliament N with cardinality n. The distribution of seats among the n parties is denoted $\mathbf{x} = (x_1, \ldots, x_n)$, with $x_i \in \mathbb{R}$, $i = 1, \ldots, n$. The set of possible seat distributions is thus $\mathbb{R}^n$.

The total number of seats in this assembly is denoted $X \equiv \sum_{i=1}^{n} x_i$. We assume that X is known and fixed for the given assembly.

An equivalent-number of parties can be defined as a function $Q^n : \mathbb{R}^n \to \mathbb{R}_+$. Note that an equivalent-number is a real-valued function and is thus an abstract concept. It may not exist in the real life, just as it is the case for the concept of average number of children. For a distribution $\mathbf{x} \in \mathbb{R}^n$, the equivalent number of parties $Q^n(\mathbf{x})$ tells us that the actual situation is equivalent to a fictious situation **where all the $Q^n(\mathbf{x})$ parties have the same**

number of seats ! Now, corresponding to the equivalent number of parties $Q^n(\mathbf{x})$, we can construct $A^n(\mathbf{x}) \equiv X/Q^n(\mathbf{x})$, the number of seats each of the $Q^n(\mathbf{x})$ parties would have. In Hannah and Kay [12], they regard $A^n(\mathbf{x})$ as the *effective average size* of firms. It is defined as

$$A^n(\mathbf{x}) \equiv X/Q^n(\mathbf{x}) \tag{2.1}$$

where x is a vector of market shares in $\mathbb{R}^n_+$, $X = \sum_{i=1}^n x_i$ and $Q^n(\mathbf{x})$ is the equivalent number of firms. They call it *effective* in order to distinguish $A^n(\mathbf{x})$ from the classical average, that is the arithmetic mean. In the computation of an effective average size, not all parties have the same weight in the formula. Following the well-established literature in mathematics, such an "effective average" is called a "mean" or a "generalized mean".

Thanks to formula (2.1), we see that any property possessed by a number-equivalent index will be transmitted through this simple relationship to the effective average size. In this paper we adopt the detour of determining a set of desirable properties for an effective average size that then will be transmitted through the linkage (2.1) to an equivalent number of parties. Since X is fixed for a given assembly, the relation will be straightforward. In the last section, we present all the properties seemed as desirable for an equivalent number of parties to display and show that they all are deduced from the properties displayed by the effective average size. Following the literature on the subject (see *e.g*. Hardy *et al*. [13]), as stated above, an "effective average" is called a "generalized mean". In our task to clarify the notion of fragmentation or equivalent number of parties through the notion of generalized mean, we introduce first some aggregation principles that lead naturally to a broad class of means. On this class, we impose certain properties and narrow down the set of allowable means considerably. We next show that the yardstick measure of *typical size* we obtain gives rise to a measure of equivalent number of parties that performs well in the context of fragmentation measurement.

3 Quasideviation means

In this section, we recall some useful definitions about means that will be used to build an effective average size.

The empirical object under study is a finite set N of cardinality n representing a decision-making body such as a parliament or any assembly making collective decisions. The set N is composed of elements to which are attached some measures. Even if we work on a greater generality so that our measurement of fragmentation can be used to any decision-making body, we assume that N is a parliament whose elements are political parties, and to each party is associated a mathematical entity : the *size* of the party.

The distributions of *sizes* under study are of the following form : let $I \subset \mathbb{R}_+$ be an arbitrary interval in the nonnegative part of the real line. Let $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ denote a distribution of sizes, with x_i is the *size* of party i.[1] The set of possible distributions is $\mathcal{D}(I) = \bigcup_{n=1}^{\infty} I^n$, *i.e.* each party possesses a nonnegative size. The reason why we talk about sizes instead of seats (or seat shares), is because we have to build up our fragmentation index on a richer domain than seat shares distributions that can only take rational values when the number of parties is small. As we will see in subsequent sections, seat shares only are not enough to evaluate the fragmentation. Hence, the *sizes* of *parties* we want to aggregate in the computation of fragmentation may assume any (nonnegative) real value inside an interval and the *sizes* we will aggregate **are not** seat shares.

An effective average, or mean, is a function $M : \mathcal{D}(I) \to I$ that associates to each distribution a value on the interval I. The simplest mean one can think about is the *arithmetic mean* $\mathcal{U}(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$ with $\mathbf{x} \in \mathbb{R}_+^n$.[2] In order to specify the notion of effective average size in the particular context of collective decision-making, we need to depart from the most basic notion of mean. This will lead us to the notion of quasideviation. We first give a definition for a mean and give an interesting result exemplifying how we can restrict the class of acceptable means by imposing some conditions.

[1] Interchangeably we could also call the elements whose sizes are under consideration shareholder i, agent i, coalition i, ... Because we are mainly interested in the fragmentation of political party systems, we stick without loss of generality to the term *party*.

[2] Actually the arithmetic mean is also defined for any distribution of real numbers, not only for distribution of sizes that only achieve nonnegative real values.

3.1 Discrete symmetric means

For any $\mathbf{x} \in I^n$, let

$$\langle \mathbf{x} \rangle = \langle x_1, \ldots, x_n \rangle = \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid \lambda_i > 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

$\langle \mathbf{x} \rangle$ is the interval between $\min_i x_i$ and $\max_i x_i$.

We see that

$$\langle \mathbf{x} \rangle = \{x_1\}$$

if $x_1 = \cdots = x_n$ and

$$\langle \mathbf{x} \rangle =]\min_{1 \le i \le n} x_i, \max_{1 \le i \le n} x_i[$$

otherwise.

Definition 3.1. *A function $M : \mathcal{D}(I) \to I$ is an averaging aggregation if it is monotone non-decreasing in each variable and satisfies*

$$M(\mathbf{x}) \in \langle \mathbf{x} \rangle$$

for any $\mathbf{x} \in \mathcal{D}(I)$.

Definition 3.2. *A function $M : \mathcal{D}(I) \to I$ is symmetric if*

$$M(\mathbf{x}) = M(\pi \mathbf{x})$$

for any $\mathbf{x} \in \mathcal{D}(I)$ and any permutation matrix[3] π of conformable size.

Definition 3.3. *A function $M : \mathcal{D}(I) \to I$ is a discrete symmetric mean if and only if $M(.)$ is an averaging aggregator and symmetric.*

Classical examples of discrete symmetric means are the arithmetic mean $\mathcal{U}(\mathbf{x})$, the geometric mean $O(\mathbf{x}) = (x_1 x_2 \ldots x_n)^{1/n}$ and the quasi-arithmetic mean :

$$K(\mathbf{x}) = f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f(x_i) \right)$$

[3] A permutation matrix has only one coefficient in each row equal to 1 and only one coefficient in each column equal to 1.

with f a strictly monotonic continuous real-valued function on I.

The definition (3.3) gives us the basic requirements for a function to be considered as a mean.

Considering definition (3.3) as a stepstone, we now define a very broad class of generalized means that will be used to define an effective average size : the quasideviation means.

3.2 Quasideviations and quasideviation means

The following generalization of the concept of mean was first given by Páles [23] and uses the concept of quasideviation. This generalization encompasses as special cases several classes of means and turns out to be very useful in their characterization.

Definition 3.4. *Let $I \subseteq \mathbb{R}_+$ be an arbitrary interval. The function $E : I^2 \to \mathbb{R}_+$ is a quasideviation on I if :*

(Q1) for all $x, t \in I$,

$$\operatorname{sgn} E(x,t) = \operatorname{sgn}(x-t)$$

where sgn *represents the sign function.*

(Q2) the function $t \to E(x,t)$, $t \in I$ is continuous on I for each fixed $x \in I$,

(Q3) the function

$$t \to \frac{E(y,t)}{E(x,t)}, \qquad t \in]x,y[$$

is strictly monotone increasing on $]x,y[$ for any fixed $x, y \in I$ with $x < y$.

The class of quasideviations on I is denoted by $\mathcal{E}(I)$. A quasideviation is not a mean since it does not satisfy definition (3.3). For example, for any $x \in I$, any quasideviation $E \in \mathcal{E}(I)$ gives $\operatorname{sgn} E(x,x) = 0$ by condition $Q1$ and thus violates definition 3.1. Nevertheless, as we will see, to each quasideviation is uniquely associated a class of means : the quasideviation means.

Definition 3.5. *Let $E \in \mathcal{E}(I)$, $\mathbf{x} \in I^n$. The unique solution $t \in I$ of the equation*

$$\sum_{i=1}^{n} E(x_i, t) = 0 \tag{3.1}$$

is called the quasideviation mean of $\mathbf{x}$ *generated by* E *and is denoted* $M_E(\mathbf{x})$*.*

The most important examples for quasideviations are $E_1(x,y) = x - y$ and $E_2(x,y) = f(x) - f(y)$, where $f : I \to \mathbb{R}$ is a strictly increasing and continuous function. Then the E_1-quasideviation mean is the arithmetic mean $\mathcal{U}(\mathbf{x})$ and the E_2-quasideviation mean generates a quasi-arithmetic mean $K(\mathbf{x})$.

The *quasideviation mean* is the unique solution to the equation (3.1) as the following theorem shows :

Theorem 3.1. *Let* $E \in \mathcal{E}(I)$*,* $n \in \mathbb{N}$*,* $x_1, \ldots, x_n \in I$*, then there exists a value* $t_0 \in I$ *for each* $t \in I$ *such that*

$$\operatorname{sgn}\left(\sum_{i=1}^{n} E(x_i, t)\right) = \operatorname{sgn}(t_0 - t) \tag{3.2}$$

for $t \in I$ *and*

$$t_0 \in \langle x_1, \ldots, x_n \rangle. \tag{3.3}$$

The proof of this theorem can be found in Páles [23], theorem 2.1, p.246. For the sake of completeness, we provide this proof in the appendix.

Thanks to theorem 3.1, we know that $M_E(\mathbf{x})$ exists and is unique. Moreover, it can be checked that $M_E(\mathbf{x})$ fulfills all the requirements to be a discrete symmetric mean on I and that a particular form of E will generate a specific M_E.

A quasideviation is not only a convenient technical tool, it also has a direct interpretation : it tells us exactly how should be compared a given value drawn from a distribution $\mathbf{x}$ from an interval I with all other values of the distribution. A clue to convince you of that fact is to remark that a quasideviation fulfills all properties to be a distance with the exception of positive definitiveness.

Before proceeding to the main aim of this paper, we need to describe some particular aspects of the framework ensuing from the interaction between decision-makers.

4 Collective decision-making procedures

4.1 Simple games, voting rules and voting behavior

In order to model the decisive structure of a collective body, we need to take into account two basic informations of specification : the voting rule and the voting behavior. These natural requirements guarantee some consistency of the procedure and can easily be formalized by the class of simple supperadditive games. In order to model a collective decision-making, we provide the following definitions :

A *simple game* is a pair (N, v), where N is the set of parties represented in the parliament (of cardinality n) and $v : 2^N \rightarrow \{0, 1\}$ is a function whose domain is the power set of N. We assume $v(\emptyset) = 0$. Any element of the power set is called a *coalition*. The number of players of a coalition S is s. We say that v is monotonic when $v(T) \geq v(S)$ if $T \supset S$. The set of winning coalitions is denoted $W(v)$ and is defined by $W(v) = \{S \in 2^N \mid v(S) = 1\}$. Accordingly, the set of losing coalitions is $N \setminus W(v)$ and is defined by $\{S \in 2^N \mid v(S) = 0\}$. We always assume that $N \in W(v)$ and $\emptyset \notin W(v)$ for any v.

A winning coalition S is *minimal* if it does not contain any subset S' such that $v(S') = 1$. A player i is said to be *pivotal* for S such that $i \in S$ if $v(S) = 1$ and $v(S \setminus \{i\}) = 0$. A player $i \in N$ is said to be a *null player* if $v(S) = v(S \setminus \{i\})$ for all S such that $i \in S$. The set of minimal winning coalitions is denoted $W^*(S)$. A simple game (N, v) is *superadditive* if for any $S \subset N : v(S) + v(N \setminus S) \leq 1$. We denote by SG_n the set of all possible simple superadditive n-person games. Remark that any of the sets $W(v)$ or $W^*(v)$ fully characterizes the voting rule provided v is monotonic. Indeed, $W = v^{-1}(\{1\}) \equiv \{S : v(S) = 1\}$ and $W^* \equiv \{S \mid v(S) = 1$ and there is no $S' \subset S$ s.t. $v(S') = 1\}$.

In the particular case under interest in this paper, each elected party $i \in N$ is endowed with some seat (shares) $x_i \in I$. We will thus focus our attention to particular cases of simple games : the weighted voting games. We define a *quota* as a number q such that $q \in \left[\frac{1}{2}, 1\right[$. A simple game (N, v_w) with $v_w : 2^N \rightarrow \{0, 1\}$ such that $W(v_w) \equiv \{S \subseteq N \mid \sum_{i \in S} x_i > q \sum_i x_i\}$ is a *weighted voting game*. For example, the simple game $(N, v_M) \in SG^N$ such that $q = \frac{1}{2}$ is the simple weighted voting majority game.[4] We call the *voting rule* the function v_w in a weighted voting game that determines the quota and henceforth

[4]Note that the way we define a quota does not include the unanimity game.

which coalitions are winning or not. To simplify notation and otherwise stated, we simply denote by v a weighted voting game.

In order to measure the influence of a party i in the collective decision-making procedure, we define a *power index* to be a function $\phi : SG_n \rightarrow \mathbb{R}^n$. For each game $v \in SG_n$, the power of party i is $\phi_i(v)$. The set of all possible power indices (for N given) is denoted $\Phi^n(v)$, for any $v \in SG_n$ and $\Phi(v) = \bigcup_{n=1}^{\infty} \Phi^n(v)$.

In our framework, we assume the following voting behavior : each party has an equal probability to vote yes or no for each proposal. Indeed, in the current state of affair in a parliament, each party may make a law proposal to be voted upon. If we assume that proportional representation ensures heterogeneity of representation, parties are scattered randomly on some ideological space. Without any a priori knowledge about their actual positions, we may suppose that the set of proposals to be voted upon during the legislature is uniformly distributed on the ideological space. If parties are driven by the same rule of accepting or rejecting a proposal by some given distance from their ideal point on the ideological line, then we may suppose that for each proposal, the probability for a given party to vote yes is 1/2.[5]

4.2 The Banzhaf Index of power

Under these assumptions, the most relevant power index $\phi \in \Phi$ we chose is the one named after Banzhaf, whose following characterization is due to Laruelle and Valenciano [17] :

Theorem 4.1. *Let $\phi : SG_n \rightarrow \mathbb{R}^n$, then ϕ satisfies*

(i) Anonymity : for all $v \in SG_n$, any permutation π of N and any $i \in N$,

$$\phi_i(\pi v) = \phi_{\pi(i)}(v)$$

with $(\pi v)(S) \equiv v(\pi(S))$.

(ii) Null Player : for all $v \in SG_n$ and any $i \in N$, if i is a null player for v then

$$\phi_i(v) = 0$$

[5] see Felsenthal and Machover [10] about the principle of insufficient reason and Valenciano and Laruelle [27]: "any information beyond the rule itself should be ignored".

(iii) *Symmetric Gain-Loss : for all $v \in SG_n$ and all $S \in W^*(v), (S \neq N)$, and all $i, j \in S$ (and resp. for all $i, j \in N \setminus S$),*

$$\phi_i(v) - \phi_i(v_S^*) = \phi_j(v) - \phi_j(v_S^*).$$

with v_S^* be the game obtained from $v \in SG_n$ such that $W(v_S^*) = W(v) \setminus S$, $S \in W^*(v)$, $S \neq N$.

(iv) *Average Gain-Loss Balance : for all $v \in SG_n$ and all $S \in W^*(v)(S \neq N)$,*

$$\frac{1}{s}\sum_{i\in S}(\phi_i(v) - \phi_i(v_S^*)) = \frac{1}{n-s}\sum_{j\in N\setminus S}\left(\phi_j(v_S^*) - \phi_j(v)\right)$$

with v_S^* as defined above.

(v) *Unit of Power : for any $i \in N$,*

$$\max_{v\in SG_n}\phi_i(v) - \min_{v\in SG_n}\phi_i(v) = 1.$$

if and only if

$$\phi_i(v) = \frac{1}{2^{n-1}}\sum_{S\subset N, i\in S}(v(S) - v(S\setminus\{i\})) \equiv Bz_i^*(v) \tag{4.1}$$

for all $i \in N$ and $v \in SG_n$.

If we are interested in the assessment of how effective a party is in turning a decision into its favor, we advocate for the use of the Banzhaf power index

$$\mathbf{Bz}^*(v) = (Bz_1^*(v), \ldots, Bz_n^*(v))$$

(definition 4.1), characterized by the set of properties above-mentionned as it embodies how the influence of a given party is affected by some well-defined change in the voting rule v and how should be compared the influence of two given parties.

The condition of *Anonymity* simply states that the index i attached to a given party has no meaning, in weighted voting games it means that what really counts in determining the pivotal power of a given party is its seat shares and the voting rule.

The formulation of the *Null Player* property says that whenever a party is never in position to affect the outcome of a decision, its power should be zero. Laruelle and valenciano [17] precise that this axiom, as stated here and despite its plausibility, only conveys its full meaning together with the other axioms. In their paper, in order to avoid any trivial flat measure of power, they propose to modify the property by stating that being a null player in any game leads to the minimal power measure of power in this game.

(ii) Null Player* : for all $v \in SG_n$, and all $i \in N$,*

$$i \text{ is a null player in } v \Leftrightarrow \text{for all } w \in SG_n,\ \phi_i(v) \leq \phi_i(w).$$

If we add the condition of Anonymity and some Normalization principle, their restatement of the null player property is the same as the one stated here. Without normalizing, null player property as minimal power in a game together with anonymity, imply that two null players in a game have the same power.

The third property is the *Symmetric Gain-Loss*. This property is really important to understand the role of the decision rule. As we say above, a monotonic simple game v is entirely characterized by its set $W(v)$ of winning coalitions. If one minimal coalition is removed from $W(v)$, what would be the effect on the power of players ? The answer is given by the Symmetric Gain-Loss property : any two player belonging to the removed winning coalition from the set of winning coalition suffers equally, and any two players not belonging the removed coalition equally gain from the removal. To keep track of the influence of a party, we list the minimal winning coalitions to which it belongs to. If the player quits a minimal winning coalition, the coalition is losing. Suppose that we remove this minimal winning coalition from the list of winning coalition, then any two players belonging to this coalition have logically to be affected in the same fashion. A similar line reasoning explains why any two players outside a removed coalition have also to be affected equally.

The *Average Gain-loss Balance* gives some insights about the transfer of power among players consecutively to the deletion of a minimal winning coalition. It states that the average loss of the players in the removed minimal winning coalition equals the average gain of the players outside it.

The last property *Unit of Power* is a normalization principles that fixes to unity the range of values taken by the power index.

If we drop this last normalization principle and replace the Null Player property by the Null Player* one, Laruelle and Valenciano [17] (Theorem 2, p.96) obtain the following characterization of a class of power indices :

Theorem 4.2. *A power index* $\mathbf{Bz}^n : SG_n \to \mathbb{R}^n$ *satisfies Anonymity, Null Player*, Symmetric Gain-Loss and Average Gain-Loss Balance if and only if* $\mathbf{Bz}^n = \alpha \mathbf{Bz}^* + \kappa \mathbf{1}$ *for some* $\alpha > 0$, $\kappa \in \mathbb{R}$, $\mathbf{1} \equiv (1, \ldots, 1) \in \mathbb{R}^n$ *and* $\mathbf{Bz}^* \in \mathbb{R}^n$ *the vector of Banzhaf power indices defined by equation (4.1).*

Any two power indices belonging to this class rank the power of any two parties in any two games identically. Laruelle and Valenciano express some indices that have been proposed in the literature belonging to this class of indices : The Banzhaf index is obtained by putting $\alpha = 1$ and $\kappa = 0$, the "raw" Banzhaf with $\alpha = 2^{n-1}$ and $\kappa = 0$ while the Rae index is derived by putting $\alpha = 2^{n-1} = \kappa$ while the normalized Banzhaf index does not belong to this class of indices. In the sequel, we call this class the *class of Banzhaf power* and denote it

Definition 4.1. *In a given parliament N, the class of Banzhaf power* $\mathbf{Bz}^n \in \Phi^n$ *is defined by*

$$\mathbf{Bz}^n(v) \equiv \alpha \mathbf{Bz}^*(v) + \kappa \mathbf{1}$$

for any $v \in SG_n$ *and the same notation as in Theorem 4.2.*

By consequence, we see that to determine the effective average size of parties in some decision-making assembly, it is necessary to combine the seat shares distribution with the voting rule and voting behavior. This is the work done by a power index defined on simple games modeling the decision procedure. A class of power indices at hand, we are now ready to state the main result of our paper, proposing a class of fragmentation measures as equivalent numbers.

5 Characterization of a class of Effective Number of Parties indices

5.1 Effective number of parties as quasideviation means

The discussion in the previous section has proved that more information than seat distribution was needed in order to model collective decision-making : voting rule plays a key role in determining the influence of parties. Properly defining an effective average size of parties, and hence, an effective number of parties, needs to extent its domain of definition in order to take into account the parties power indices in simple weighted voting games. As we have seen, simple weighted games are sufficient to represent voting rules and necessary to measure influence. An effective average size is thus a function

$$A^n : I^n \times \Phi^n(v) \to I$$

for any $n \in \mathbb{N}$ and $v \in SG_n$ is a weighted voting game, with $I \subset \mathbb{R}_+$ an arbitrary interval, and $\Phi^n : SG_n \to \mathbb{R}^n$ the set of possible power indices on v. An effective average size A^n attaches thus a positive real number to each vector of the following form : $(x_1, \ldots, x_n, \phi_1(v), \ldots, \phi_n(v))$ where $x_i \in I$ and $\phi_i(v) \in \mathbb{R}$, $i = 1, \ldots, n$, and accordingly, by equation (2.1), any effective number of parties will be a function $Q^n(\mathbf{x}, \Phi^n)$.

Our objective in this section is to use the notion of quasideviation mean to build up a relevant effective average size which takes into account the size of the parties as well as their associated power indices.

On this purpose, we first provide the following definition and lemmas :

Definition 5.1. *Let $I \subset \mathbb{R}_+$ be an arbitrary interval and $\mathbf{x} \in I^n$ a distribution of sizes among n players. Let a simple weighted voting game v and $\mathbf{Bz}^n(v)$ the corresponding distribution of Banzhaf powers (definition 4.1). We define $\tilde{\mathbf{x}}$ a permutation of $\mathbf{x}$ such that $\tilde{x}_1 \leq \tilde{x}_2 \leq \cdots \leq \tilde{x}_n$ and $\tilde{\mathbf{Bz}}^n(v)$ the corresponding permutation of the vector of Banzhaf powers.*

Then we define the function $h : I \times SG_n \to \mathbf{Bz}^n(v)$ by

$$\begin{aligned} h(x, v) &= \tilde{B}z_1^n(v) \quad \text{if } x \in [0, \tilde{x}_1], \\ h(x, v) &= \tilde{B}z_2^n(v) \quad \text{if } x \in]\tilde{x}_1, \tilde{x}_2], \\ &\cdots \\ h(x, v) &= \tilde{B}z_n^n(v) \quad \text{if } x \in]\tilde{x}_{n-1}, \tilde{x}_n]. \end{aligned}$$

Lemma 5.1. *Let $I \subset \mathbb{R}_+$ be an arbitrary interval, $x, y \in I$, $\mathbf{Bz}^n(v) : SG_n \to \mathbb{R}$ the class of Banzhaf power (definition (4.1)) and $f : I \to \mathbb{R}_+$ is a strictly increasing continuous* *function. Then the function*

$$E_{Bz}(x, y) = h(x, v)\,(f(x) - f(y)) \tag{5.1}$$

is a quasideviation for any $v \in SG_n$ *if* $\kappa \geq 0$ *and* $h(.,.)$ *the function defined in definition 5.1.*

Proof. The function (5.1) is a quasideviation if it satisfies (Q1), (Q2) and (Q3). By the monotonicity of v, $Bz_i^n(v)$ is increasing in x for all $i \in N$. By Anonymity and Null Player* of $\mathbf{Bz}^n(v)$, its lowest value is κ. By assumption sgn(0) $\in \{+, -\}$, then equation (5.1) obvisouly satisfies (Q1). (Q2) follows directly from the continuity of $f(.)$ and condition (Q3) follows from

$$\partial\left(\frac{E_{Bz}(y, t)}{E_{Bz}(x, t)}\right)/\partial t \geqslant 0$$

for any $x, y \in I$ with $x < y$ and $t \in \,]x, y[$. □

The lemma (5.1) gives us the way how should be compared a given $x \in I$ drawn from a distribution in I^n with any other $y \in I$ in the distribution. Whenever $x = y$, the quasideviation between them is zero. If x appears to be a null-player's size in the game, it should count for nothing and can then be disregarded, its quasideviation measure being flat. The triangular inequality is also satisfied by property $Q3$. If two parties belong to the same set of winning coalitions, then only their difference in term of size matters. Equation (5.1) is thus a way to compare two different party size in a given distribution taking into account their voting behavior and the voting rule. We now show that this quasideviation, fulfilling all properties defined in Theorem (4.1) leads to a unique class of generalized means.

Lemma 5.2. *Let* E_{Bz} *as defined in equation (5.1) and* $\mathbf{x} \in I^n$*, then the unique solution to equation (3.1) is given by*

$$M_{E_{Bz}}(\mathbf{x}, v) = f^{-1}\left(\frac{\sum_{i=1}^{n} Bz_i^n(v) f(x_i)}{\sum_{i=1}^{n} Bz_i^n(v)}\right) \tag{5.2}$$

Proof. Let $E_{Bz}(x_i, y) = Bz_i^n(v)\,(f(x_i) - f(y))$ for all $i \in N$, $x_i, y \in I$. By definition 3.5 and theorem 3.1, there exists a unique solution y^* :

$$\begin{aligned}\sum_{i=1}^{n} E_{Bz}(x_i, y^*) &= \sum_{i=1}^{n} Bz_i^n(v)(f(x_i) - f(y^*)) = 0\\ &\Leftrightarrow \sum_{i=1}^{n} Bz_i^n(v) f(x_i) = \sum_{i=1}^{n} Bz_i^n(v) f(y^*)\\ &\Leftrightarrow \frac{\sum_{i=1}^{n} Bz_i^n(v) f(x_i)}{\sum_{i=1}^{n} Bz_i^n(v)} = f(y^*)\\ &\Leftrightarrow f^{-1}\left(\frac{\sum_{i=1}^{n} Bz_i^n(v) f(x_i)}{\sum_{i=1}^{n} Bz_i^n(v)}\right) = y^*.\end{aligned}$$

By definition 3.5, $y^* \equiv M_{E_{Bz}}(\mathbf{x}, v)$ is the unique quasideviation mean generated by E.

□

An effective average size $A^n(\mathbf{x}, \mathbf{Bz}^n)$ is thus a quasideviation mean with weight function $\mathbf{Bz}^n$ satisfying all the properties described in section 4.2 if and only if it has the form (5.2).

In order to qualify a reasonable measure of effective number of parties, there are still a number of desirable properties that the family

$$Q^n(\mathbf{x}, \mathbf{Bz}^n) = \frac{\sum_{i=1}^{n} x_i}{A^n(\mathbf{x}, \mathbf{Bz}^n)}$$

should satisfy and that will narrow the admissible forms of f.

We now turn to a presentation of these properties considered as unavoidable that will characterize the final form prescribed for $Q^n(\mathbf{x}, \mathbf{Bz}^n)$.

5.2 Null-player independence

For all $\mathbf{x} \in I^n$, $v \in SG_n$ and all $n \in \mathbb{N}$, if there exists a party $i \in N$ such that i is a null-player (see definition in section 4.2) in v, then

$$Q^n(\mathbf{x}, \mathbf{Bz}^n) = Q^n\left(x_1, \ldots, x_i, \ldots, x_n, Bz_1^n(v), \ldots, Bz_{i-1}^n(v), 0, Bz_{i+1}^n(v), \ldots, Bz_n^n(v)\right).$$

The presence of a null-player, a party that is never in position to affect the decision, should not affect the effective number of parties.

Proposition 5.1. *For all* $\mathbf{x} \in I^n$*,* $v \in SG_n$ *and all* $n \in \mathbb{N}$*,* $Q^n(\mathbf{x}, \mathbf{Bz}^n)$ *satisfies the null player independance property with* $A^n(\mathbf{x}, \mathbf{Bz}^n)$ *a quasideviation mean with weight* $\mathbf{Bz}^n$ *if and only if* $\mathbf{Bz}^n$ *satisfies property (ii) in Theorem 4.1, that is if* $\kappa = 0$ *in definition 4.1.*

Proof. Evident. □

The null player property independance is the key property that makes a concentration index different from an inequality index : zero-size parties on the distribution have no impact on the value taken by the concentration index whereas their presence would strongly aggravates inequality. For this reason, it has been argued that a concentration index is a function of both the inequality in the distribution and the number of elements in the distribution. It has also a strong implication : the knowledge of Q^n will be enough to recover the knowledge of Q^{n-1}, as it is sufficient to add a zero-sized party in Q^n to have a measure in Q^{n-1}.

5.3 Homogeneity of degree zero

For all $\mathbf{x} \in I^n$, $v \in SG_n$ and all $n \in \mathbb{N}$, $Q^n(\mathbf{x}, \mathbf{Bz}^n)$ is homogeneous of degree zero (or nullhomogeneous) in $\mathbf{x}$ if

$$Q^n(t.\mathbf{x}, \mathbf{Bz}^n) = Q^n(\mathbf{x}, \mathbf{Bz}^n)$$

for any $t > 0$.

Since an effective number is the inverse of a measure of concentration and that measures of concentration are relative, that is they only depend on shares, an effective number of parties should not be affected by a global rescaling of the seat distribution.

Now, we see directly that for all $\mathbf{x} \in I^n$, $v \in SG_n$ and all $n \in \mathbb{N}$, $Q^n(\mathbf{x}, \mathbf{Bz}^n)$ is nullhomogeneous in $\mathbf{x}$ if and only if $A^n(\mathbf{x}, \mathbf{Bz}^n)$ is homogeneous (of degree 1) in x.

Definition 5.2. *For all* $\mathbf{x} \in I^n$*,* $v \in SG^n$*,* $\mathbf{Bz}^n \in \Phi^n(v)$ *and all* $n \in \mathbb{N}$*,* $A^n : I^n \times \Phi^n(v) \to I$ *is homogeneous of degree 1 in* $\mathbf{x}$ *if and only if*

$$A^n(t.\mathbf{x}, \mathbf{Bz}^n) = t.A^n(\mathbf{x}, \mathbf{Bz}^n) \tag{5.3}$$

for any scalar $t > 0$*.*

In order to characterize the form of $A^n(.,.)$ that is homogeneous among the class of possible $A^n(.,.)$, we need to explore first some properties of the class of Banzhaf power $\mathbf{Bz}^n$.

Definition 5.3 (Isomorphic simple games)**.** *Let v and v' be simple games in SG^n with respective parliaments N and N'. An isomorphism from v to v' is a bijection σ from N to N' such that for any $S \subseteq N$,*

$$S \in W(v) \Leftrightarrow \sigma(S) \in W(v')$$

with $\sigma(S) = \{\sigma(i) : i \in S \subseteq N\}$.

If such a σ exists, v and v' are isomorphic.

An isomoprhism from v to v is an automorphism.

Since an isomorphism does not alter the set of (minimal) winning coalitions, any power index must be invariant under isomorphism. The most evident (and well-accepted) example is the invariance under parties label permutation, known as the anonymity principle. This procedure is obviously an automorphism. If we consider a weighed voting game (v, N), what would be the effect on the distribution of power according to any index belonging to the class $\mathbf{Bz}^n(v)$ of rescaling by a common factor t all the sizes x_i, $i = 1, \ldots, n$? To restate the question, do we expect a change on power distribution if we express the party sizes in shares instead of seats ? Obviously no. We don't feel warmer if the temperature is expressed in Fahrenheit instead of Celsius. The power index has to be insensitive to any rescaling of the sizes of the parties. What really matters is the relative magnitude between parties, not the absolute ones. The power index must be homogeneous of degree zero in the $\mathbf{x}$'s. Any index in $\mathbf{Bz}^n(v)$ is homogeneous of degree zero.

Proposition 5.2. *For all $\mathbf{x} \in I^n$, $v \in SG^n$ any weighted voting game and all $n \in \mathbb{N}$, any $\varphi \in \mathbf{Bz}^n(v)$ is homogeneous of degree zero.*

Proof. Take any $\mathbf{x} \in I^n$, any weighted voting game $v \in SG^n$, fix $n \in \mathbb{N}$ and take any $\varphi \in \mathbf{Bz}^n(v)$. Fix $i \in N$.

By definition (4.2) and equation (4.1) in Theorem (4.1),

$$\varphi_i(v) = \frac{\alpha}{2^{n-1}} \sum_{\substack{S \subseteq N \\ S \ni i}} [v(S) - v(S \setminus \{i\})] + \kappa \mathbf{1}$$

for some $\alpha > 0$, $\kappa \in \mathbb{R}$, $\mathbf{1} \equiv \underbrace{(1, ..., 1)}_{n}$, with

$$v(S) = \begin{cases} 1 & \text{if } \sum_{j \in S} x_j \geq q. \sum_{i \in N'} x_i \\ 0 & \text{otherwise} \end{cases}$$

for any $S \subseteq N$ and $0 < q \in \left[\frac{1}{2}, 1\right[$.

Define $\sigma : N \to N'$ such that $tx_i = x_{\sigma i}$, $t > 0$, $i \in N$, $\sigma i \in N'$.

Let $v'_w \in SG^n$ be such that

$$v'_w(S) = \begin{cases} 1 & \text{if } \sum_{j \in S} x_j \geq q. \sum_{i \in N'} x_i \\ 0 & \text{otherwise} \end{cases}$$

for any $S \subseteq N'$. By construction, v and v'_w are isomorphic.

Then

$$\begin{aligned} \varphi_{\sigma i}(v'_w) &= \frac{\alpha}{2^{n-1}} \sum_{\substack{S \subseteq N' \\ S \ni \sigma i}} \left[v'_w(S) - v'_w(S \setminus \{\sigma i\})\right] + \kappa \mathbf{1} \\ &= \frac{\alpha}{2^{n-1}} \sum_{\substack{\sigma^{-1} S \subseteq N \\ \sigma^{-1} S \ni i}} \left[v(\sigma^{-1} S) - v(\sigma^{-1} S \setminus \{i\})\right] + \kappa \mathbf{1} \\ &= \varphi_i(v) \end{aligned}$$

□

To prove that $A^n(\mathbf{x}, \mathbf{Bz}^n)$ is homogeneous, by the Theorem 2 of Pàles [24] (p.137), we only have to show that $E_{Bz}(tx_i, ty)/E_{Bz}(x_i, y)$ is constant on $\{x_i \in I \mid tx_i \in I, x_i \neq y\}$, for any fixed $y \in I$, $ty \in I$, $t > 0$ and $A^n(\mathbf{x}, \mathbf{Bz}^n) = M_{E_{Bz}}$.

Proposition 5.3. *Let $I \subset \mathbb{R}_+$ be an arbitrary interval, $x_i, y \in I$, $\mathbf{Bz}^n(v) : SG_n \to \mathbb{R}$, $f : I \to \mathbb{R}_+$ is a strictly increasing continuous function and v a simple weighted voting game. Define the following quasideviation :*

$$E_{Bz}(x_i, y) = Bz_i^n(v)\,(f(x_i) - f(y))$$

and let $A^n(\mathbf{x}, \mathbf{Bz}^n) = M_{E_{Bz}}$ be the quasideviation mean generated by E_{Bz}. Then, $M_{E_{Bz}}$ is homogeneous of degree 1 if and only if the function f in E_{Bz} has one of the following forms :

1. $f(x_i) = ax_i^c + b$
with $a \neq 0$*,* $c \neq 0$ *and b constants, or*

2. $f(x_i) = a \log x_i + b$
with $a \neq 0$ *and b constants*

for any $x_i \in I$*.*

Proof. Denote the value of $E_{Bz}(tx_i, ty)/E_{Bz}(x_i, y)$ by $\alpha(y)$ for any $y \in I$, $ty \in I$. We also assume that $t \in I$ and $1 \in I$.

Then

$$\frac{Bz_i^n(tx_i, v)\,(f(tx_i) - f(ty))}{Bz_i^n(x_i, v')\,(f(x_i) - f(y))} = \frac{f(tx_i) - f(ty)}{f(x_i) - f(y)} = \alpha(y)$$

by proposition 5.2.

$$\Leftrightarrow f(tx_i) - f(ty) = \alpha(y)\,(f(x_i) - f(y)) \tag{5.4}$$

Let $\theta(t) = f(ty) - \alpha(t)f(y)$, then (5.4) becomes

$$f(tx_i) = \alpha(t)f(x_i) + \theta(t) \tag{5.5}$$

We pose $x_i = 1$ in (5.5) to get

$$f(t) = \alpha(t)f(1) + \theta(t) \tag{5.6}$$

By substracting (5.6) from (5.5), and with

$$g(x_i) = f(x_i) - f(1) \tag{5.7}$$

we get

$$g(tx_i) = \alpha(t)g(x_i) + g(t) \tag{5.8}$$

We distinguish two cases :

First, if $\alpha(t) = 1$ for all $t > 0$,

$$g(tx_i) = g(x_i) + g(t) \tag{5.9}$$

The general solution of the functional equation (5.9) is given by

$$g(x_i) = a \log x_i + b$$

with a and b arbitrary constants, $a \neq 0$.

From (5.7) we get

$$f(x_i) = a \log x_i + b$$

The remaining case is when $\alpha(t) \not\equiv 1$ for all $t > 0$. Then the general non-constant and continuous solution of the functional equation (5.8), by (5.7) is given by

$$f(x_i) = ax_i^c + b$$

with $a \neq 0$, $c \neq 0$ and b constants (see for example Aczél and Daróczy [1], p.25).

□

The properties we have so far proposed appear to be enough to characterize a satisfying class of measure of fragmentation, that we present in the next result.

Theorem 5.1. *For all* $\mathbf{x} \in I^n$, $v \in SG_n$ *and all* $n \in \mathbb{N}$, $Q^n(\mathbf{x}, \mathbf{Bz}^n)$ *is a nullhomogeneous effective number of parties satisfying independence to null players with* $A^n(\mathbf{x}, \mathbf{Bz}^n)$ *an effective average size with weights in the class of Banzhaf power if and only if*

$$A^n(\mathbf{x}, \mathbf{Bz}^n) = \left(\frac{\sum_{i=1}^n \phi_i(v)(x_i)^p}{\sum_{i=1}^n \phi_i(v)} \right)^{\frac{1}{p}} \equiv A_H^n \tag{5.10}$$

$p \neq 0$ *and* $\phi_i(v)$ *a power index belonging to the class of Banzhaf power with* $\kappa = 0$*, that is* $\phi_i(v)$ *is proportional to the non normalized Banzhaf index* $Bz_i(v)$ *as defined in equation (4.1).*

The proof of the theorem follows logically from the above propositions. We thus see that A_H^n is the only family of mean measures homogeneous and independent from the presence of null-player parties. Moreover, it lets some choice to the researcher with the parameter p, the higher the latter, the less small parties are taken into consideration in the computation.

To fully validate $Q^n \equiv X/A_H^n$ among all possible measures, we now show that Q_H^n may also satisfy other basic requirements for a measure of effective number of political parties. The next property will restrain our choice of the parameter p in A_H^n.

5.4 The transfer principle

In the context of (income) inequality measurement, the *transfer principle* or *regressive transfer* requires inequality not to fall when a transfer of income is performed from a poor person to a richer one, without altering their ranking and affecting the other persons. The principle is also known as the *Pigou-Dalton* transfer principle by reference to the first two authors who proposed such concept. If the sole concern of inequality measurement is the distribution of income, the idea that inequality should be aggravated by a regressive transfer is very well founded ; but the transposition of this principle in the context of fragmentation measurement is not without difficulty. Two main reasons can be put forward for explaining this difficulty. The first difficulty is to interpret *à la lettre* the principle. A distribution of seats is usually the result of elections and one party is thus not legally allowed to secede some of its seats to another party. Of course this difficulty is not hard to surmount as in fact what the principle tells us is how to compare two distributions that differ only on two entries ; it does not tell that one distribution must be deduced from the other one from an actual transfer. Both distributions can be the result of elections and still be compared according to the principle. A second difficulty, the most important one according to us, refers to the interdependence between the seat distribution and power distribution. Requiring fragmentation no to fall whenever a small party transfers some seats to a bigger party implies that seat and power distribution are linearly related to each other. If this is the case, what would be the reason of existence of power indices ? Fortunately, as the following simple example shows, power indices remain an invaluable tool that cannot (always) be inferred from the seat distribution.

Example 5.1. *let N be an eight-party assembly with the following seat shares distribution :* $(45, 30, 20, 1, 1, 1, 1, 1)$*. If we set the quota at* 51 *we observe a total of* 196 *swing positions out of which the first party has* 65 *(hence* $33,16\%$ *of the total) and last five parties have only* 1 *swing position (*$0,51\%$*). Suppose now that the first party gives away one percentage of its seats to the third one, a "progressive transfer", such that inequality should not be aggravated. The new distribution is now :* $(44, 30, 21, 1, 1, 1, 1, 1)$*. With the same quota* (51) *we now observe a total of* 194 *swing positions out of which* 64 *is for the first party (hence* $33,33\%$*) and last five parties don't hold any more a swing position.*

The striking aspect of this example is that by giving away some seats to a smaller party, the biggest party diminishes the total number of swing possibilities and hence, even if the first party looses a swing opportunity, having 64 instead of 65, it can nevertheless improve its relative position by going from $33,16\%$ to $33,33\%$ of the total swing positions after the transfer. This situation is explained by the fact that due to the transfer, the five small parties become dummies. In conclusion we see that seat and power don't always behave in the same way !

Do we thus have to discard any transfer principle for a fragmentation measure ? Of course not. An alternative way to state the transfer principle can be found in the literature under the name of *majorization* (see e.g. Marshall and Olkin [20]).

Definition 5.4 (Majorization)**.** *A vector* $\mathbf{x}$ *is majorized by a vector* $\mathbf{y}$ *if and only if there exists a bistochastic matrix*[6] P *such that* $\mathbf{x} = \mathbf{y}P$*.*

Equivalently, a vector $\mathbf{x}$ *is majorized by a vector* $\mathbf{y}$ *if and only if* $\mathbf{y}$ *is obtained from* $\mathbf{x}$ *by a finite number of regressive transfers (or* $\mathbf{x}$ *is obtained by* $\mathbf{y}$ *by a finite number of progressive transfers).*

If two vectors can be compared through majorization then undoubtedly one vector is less unequally distributed than the other one. If $\mathbf{x}$ is majorized by $\mathbf{y}$, then $x_1, \dots, x_n$ is less "spread out" than $y_1, \dots, y_n$. Post-multiplication of a vector by a bistochastic matrix is an averaging operator i.e. has the effect to move some vector entries closer to the arithmetic mean of all entries. In the context of fragmentation measurement, if a given vector $\mathbf{x}$ in the domain of definition of a fragmentation measure, is majorized by a vector $\mathbf{y}$ and f is a fragmentation measure, then we want $f(\mathbf{x} = \mathbf{y}P) \geq f(\mathbf{y})$. Such function is said to be S-concave. Formally, a function $f : I^n \to I$ is S-concave if and only if for any bistochastic matrix P and $\mathbf{x} \in I^n$, we have $f(\mathbf{x}P) \geq f(\mathbf{x})$.

We are now able to formally state the transfer principle in the context of fragmentation measurement :

Definition 5.5. *For all* $\mathbf{x} \in I^n$*,* $v \in SG_n$ *and all* $n \in \mathbb{N}$*,* $Q^n(\mathbf{x}, \mathbf{Bz}^n)$ *satisfies the Pigou-Dalton transfer principle if and only if*

[6] or *doubly stochastic* matrix. A bistochastic matrix P has only nonnegative entries and each row and each column sums to 1.

$$Q^n\left((\mathbf{x},\mathbf{Bz}^n)P\right) \geq Q^n\left(\mathbf{x},\mathbf{Bz}^n\right)$$

for any $2n \times 2n$ bistochastic matrix P,

that is, if and only if $Q^n(\mathbf{x},\mathbf{Bz}^n)$ is S-concave.

The explanation of this definition is simple : a fragmentation measure depends on a $2n$-long vector consisting of a distribution of seats and the corresponding distribution of power indices. When comparing two different $2n$-long vectors, if one can be deduced from the other one by post-multiplication of a bistochastic matrix, then the former vector is obviously less "spread-out" than the latter one and has to be considered as more fragmented by any measure of fragmentation consistent with the transfer principle. To guarantee that $Q^n \equiv X/A^n_H$ with A^n_H defined by equation (5.10) is S-concave we need to impose some restrictions on the parameter p, as shown in the next theorem.

Theorem 5.2. *For all $\mathbf{x} \in I^n$, $v \in SG_n$ and all $n \in \mathbb{N}$,*

$$Q^n(\mathbf{x},\mathbf{Bz}^n) = \frac{X}{A^n_H(\mathbf{x},\mathbf{Bz}^n)}$$

with

$$A^n(\mathbf{x},\mathbf{Bz}^n) = \left(\frac{\sum_{i=1}^n \phi_i(v)(x_i)^p}{\sum_{i=1}^n \phi_i(v)}\right)^{\frac{1}{p}}$$

and $\phi_i(v)$ proportional to the non normalized Banzhaf index $Bz_i(v)$ (equation (4.1))

is S-concave if and only if $p \geq 1$.

Proof. To have that $Q^n((\mathbf{x},\mathbf{Bz}^n)P) \geq Q^n(\mathbf{x},\mathbf{Bz}^n)$ for any bistochastic matrix P, since X is fixed, we must have $A^n_H((\mathbf{x},\mathbf{Bz}^n)P) \leq A^n_H(\mathbf{x},\mathbf{Bz}^n)$, that is we need to have A^n_H S-convex.

By theorem 4 in Berge [5], p.220, to have A^n_H S-convex it is sufficient to have A^n_H symmetric and convex. That A^n_H is symmetric is evident. Finally, to have

$$A^n_H(\mathbf{x},\mathbf{Bz}^n) = \left(\frac{\sum_{i=1}^n \phi_i(v)(x_i)^p}{\sum_{i=1}^n \phi_i(v)}\right)^{\frac{1}{p}}$$

convex, denote $\sum_{i=1}^n \phi_i(v)$ by Φ and $\phi_i(v)(x_i)$ by y_i. Then $A^n_H(\mathbf{x},\mathbf{Bz}^n) = \frac{1}{\Phi}\left(\sum_{i=1}^n y_i^p\right)^{\frac{1}{p}}$, which is convex if and only if $p \geq 1$ (see Berge [5] p.214).

□

5.5 Normalization

The normalization principle just imposes some conveniant value that a fragmentation measure should take under particular circumstances : when all the political parties have the same size, an effective number should be equal to the actual number of parties.

Proposition 5.4. *For all* $\mathbf{x} \in I^n$*,* $v \in SG_n$ *and all* $n, k \in \mathbb{N}$*,* $Q^n(\mathbf{x}, \mathbf{Bz}^n)$ *satisfies normalization.*

Proof. We have $Q^n(\mathbf{x}, \mathbf{Bz}^n) = \frac{X}{A_H^n(\mathbf{x}, \mathbf{Bz}^n)}$ for any $n \in \mathbb{N}$.

Since $A_H^n(.,.)$ is a mean by definition (3.3), if $\mathbf{x} \in I^n$ is composed of n identical entries, that is if $\mathbf{x} = (\underbrace{x, \ldots, x}_{n})$, then $A_H^n(\mathbf{x}, \mathbf{Bz}^n) = x$ and thus

$$Q_H^n(\mathbf{x}, \mathbf{Bz}^n) = \frac{X}{x} = \frac{n.x}{x} = n$$

□

We then conclude by remarking that since the form we propose for $Q_H^n(\mathbf{x}, \mathbf{Bz}^n)$ satisfies normalization, the transfer principle, is homogenous and symmetric, then it is bounded where the lower and upper bounds are given by 1 and n. Any value less than n indicates a unequal distribution of seats.

6 Empirical comparison of the main effective number of parties indices

As an illustration of the computation and working of the newly introduced class of effective number of parties Q_H, we calculate Q_H for two different values of the parameter p on 106 post World War II parliaments in 5 selected countries.[7] We also calculate the *ENP* (see equation (1.1)) and the *Effective Number of Relevant Parties* (ENRP) as proposed by Dumont and Caulier [8], which is simply the ENP using the relative Banzhaf power (that sum to one) of parties instead of their seat shares. We summarize the formula we use in the following table :

[7] Data's for Japan are taken from Mackie and Rose [19] and for the others countries from Müller and Strøm [22].

ENP	$1/\sum s_i^2$
ENRP	$1/\sum Bz_i^{*2}$
Q_H^1	$\sum x_i / \frac{\sum \phi_i x_i}{\sum \phi_i}$
Q_H^2	$\sqrt{\sum x_i / \frac{\sum \phi_i x_i^2}{\sum \phi_i}}$

Table 1: Formula's used : ENP = the Laakso and Taagepera Effective Number of parties, ENRP : the Effective Number of relevant parties and Q_H^p the newly introduced index for parameter $p = 1$ and $p = 2$.

with x_i the number of seats of party i, s_i the seat shares of party i, Bz_i^* the normalized Banzhaf power of party i and ϕ_i the power of party i measured by any index in the class of Banzhaf power such that $\kappa = 0$.

All the four indices share the common characteristic to be equal to n whenever all parties have the same number of seats. The ENRP takes the special value of 1 whenever a party owns more than 51% of seat shares. In the table 2 we display summary statistics of our calculations.

	Mean	St Deviation	MIN	MAX
ENP	2,95	0,66	1,98	4,41
ENRP	2,03	0,98	1	4,19
Q_H^1	2,6	0,75	1,58	4,04
Q_H^2	2,41	0,6	1,58	3,85

Table 2: Summary statistics

In the table 3, we display the arithmetic mean values for each index by country.

	ENP	ENRP	Q_H^1	Q_H^2
UK	2,11	1,34	2	1,92
Ireland	2,83	2,22	2,69	2,45
Sweden	3,32	2,33	2,9	2,64
France	3,46	2,55	3,06	2,81
Japan	2,8	1,33	2,08	2,01

Table 3: Mean values by country

The ENRP is systematically the measure that has to the lower mean value over all observations (by country or for a given country over time) since it reaches the value of 1 when there is a majority party. Since we have especially selected countries for having dominant parties during the time-span chosen, this result is not surprising. The ENP generally leads the larger values and confirms Dunleavy and Boucek [9] or Molinar [21] that the ENP overestimates the fragmentation especially in presence of a dominant party. The case of Japan is illustrative. In the 15 seat distributions examined, we observe 10 cases in which a party owns more than the majority of the seat shares. Nevertheless, the average value of ENP is 2,8 (it even takes a value of more than 3 under a given majority party configuration). If the ENRP is an adequate tool to identify the majority cases, it nevertheless does not provide the intensity of dominance of the majority party : if a party has 51% or 99% of the shares leads to the same value of ENRP. In that case, Taagepera [26] suggested to supplement the ENP by the seat shares of the largest party (or its inverse). This index has also the advantage not to overestimate fragmentation in such cases. But this is precisely the value given by any member of the family Q_H^p for any p when a party owns a majority of seats ! It is thus more convenient to use directly Q_H^p as we don't have to look at two different indices and moreover in term of interpretation, Q_H^p gives us an idea of the degree of dominance of the majority party : the lower the value, the less dominant the party. It is well accepted that a party with bare majorities (slightly higher than the majority) are in weaker dominant position than one with a supermajority : if the party losses some seats on the next election, the larger its majority, the more chance it has to remain dominant.

The difference between Q_H^1 and Q_H^2 is observed in the presence of small parties : Q_H^1 accords more importance to them and Q_H^2 disregards them more. This is why Q_H^2 leads generally to smaller values, since it gives less weight to small parties.

Finally, any Q_H^p is a continuous value, responding to any change in the distribution of seats. It is thus a more precise (in a sense) measure than the ENRP that remains stuck at some values for some changes in the configurations.

To summarize, both Q_H^1 and Q_H^2 are continuous values that don't overestimate the fragmentation where the ENP does. Moreover, they give us directly the intensity of dominance of a majority party without the need of another measure, or remaining fixed at some value as the ENRP does. By the way they have been constructed, they also take into consideration the coalitional potential of the parties in the parliament, they thus embody more information, as the voting behavior and voting rule, than any other measure whose domain of definition is solely the seat shares distribution.

Given these optimistic preliminary results, we advocate for more tests on the adequacy of Q_H^p to depict party systems.

7 Conclusion

In this paper, we derive an effective number of parties from an average effective size of parties in a parliament. The effective average size is built as a quasideviation mean with weight function taking into account the decisive structure of the assembly. Imposing properties that we see as relevant in this framework, we find a class of measures that resemble to the general class of inequality measures of Atkinson [3] but weighted by the Banzhaf index of power of the parties (see [4] or [17]). By the way it has been constructed, whenever all parties have the same number of seats, the effective number of parties corresponds to the actual number of parties. Some preliminary tests on actual data's we have conducted show that even in its simplest form (for $f(x) = x$), it seems that our new class of indices outperforms the existing measures in the sense that all criticisms raised so far look like to be answered.

8 Appendix

Proof of Theorem (3.1). Without loss of generality, we may assume $x_1 \leq \cdots \leq x_n$. If $x_1 = x_n$, then $t_0 = x_1 = x_n$ and (3.2) follows from $(Q1)$. Thus, we may assume that $x_1 < x_n$.

Using $(Q1)$, we have

$$\sum_{i=1}^{n} E(x_i, x_1) = \sum_{i=1}^{n} E(x_i, \min_{1\leq i\leq n} x_i) > 0, \sum_{i=1}^{n} E(x_i, x_n) = \sum_{i=1}^{n} E(x_i, \max_{1\leq i\leq n} x_i) < 0. \tag{8.1}$$

By $(Q2)$, $\sum_{i=1}^{n} E(x_i, .)$ is a continuous function, hence there exists a value $t_0 \in]x_1, x_n[= \langle x_1, \ldots, x_n \rangle$ such that $\sum_{i=1}^{n} E(x_i, t_0) = 0$. Thus (3.3) is satisfied. To prove (3.2) let $t < t_0$

be an arbitrary element of I. If $t \leq x_1$ then the inequality $\sum_{i=1}^{n} E(x_i, t) > 0$ (equivalent to (3.2)) follows from $(Q1)$. Otherwise we may assume that

$$x_1 \leq \cdots \leq x_k < t \leq x_{k+1} \leq \cdots \leq x_l \leq t_0 < x_{l+1} \leq \cdots \leq x_n. \tag{8.2}$$

Let $1 \leq i \leq k$ and $l+1 \leq j \leq n$. By $(Q3)$ the function

$$t \to \frac{E(x_j, t)}{E(x_i, t)}, t \in]x_i, x_j[$$

is strictly monotone increasing, hence,

$$\frac{E(x_j, t)}{E(x_i, t)} < \frac{E(x_j, t_0)}{E(x_i, t_0)}$$

because $x_i < t < t_0 < x_j$. Rearranging the last inequality, we have

$$E(x_i, t_0)E(x_j, t) < E(x_j, t_0)E(x_i, t). \tag{8.3}$$

Using $(Q1)$ it can be verified that (8.3) is valid if $k+1 \leq i \leq l$ and $l+1 \leq j \leq n$. Adding the inequalities obtained and applying the equation $\sum_{i=1}^{n} E(x_i, t_0) = 0$ we have

$$\sum_{i=1}^{l} E(x_i, t_0) \sum_{j=l+1}^{n} E(x_j, t) < \sum_{j=l+1}^{n} E(x_j, t_0) \sum_{i=l}^{l} E(x_i, t) \quad =$$
$$\sum_{i=l+1}^{n} E(x_i, t_0) \sum_{j=1}^{l} E(x_j, t) = \left(- \sum_{i=1}^{l} E(x_i, t_0) \right) \sum_{j=1}^{l} E(x_j, t).$$

Thus

$$\sum_{i=1}^{l} E(x_i, t_0) \sum_{j=1}^{n} E(x_j, t) < 0 \tag{8.4}$$

By $(Q1)$ and (8.2), $\sum_{i=1}^{l} E(x_i, t_0) < 0$ thus by (8.4) we get $\sum_{i=1}^{n} E(x_i, t) > 0$.

When $t_0 < t$ it can be analogously be seen that $\sum_{i=1}^{n} E(x_i, t) < 0$. This completes the proof.

□

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