Measuring Power and Satisfaction in Societies with Opinion Leaders: Dictator and Opinion Leader Properties

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Abstract. A well known and established model in communication policy in sociology and marketing is that of opinion leadership. Opinion leaders are actors in a society who are able to affect the behavior of other members of the society called followers. Hence, opinion leaders might have a considerable impact on the behavior of markets and other social agglomerations being made up of individual actors choosing among a number of alternatives. For marketing or policy purposes it appears to be interesting to investigate the effect of different opinion leader-follower structures in markets or any other collective decision-making situations in a society.

We study a two-action model in which the members of a society are to choose one action, for instance, *to buy* or *not to buy* a certain joint product, or to vote *yes* or *no* on a specific proposal. Each of the actors has an inclination to choose one of the actions. By definition opinion leaders have some power over their followers, and they exercise this power by influencing the behavior of their followers, i.e. their choice of action. After all actors have chosen their actions, a decision-making mechanism determines the collective choice resulting out of the individual choices. Making use of bipartite digraphs we introduce novel satisfaction and power scores which allow us to analyze the actors' satisfaction and power with respect to the collective choice for societies with different opinion leader-follower structures. Moreover, we study common dictator and opinion leader properties of the above scores and illustrate our findings for a society with five members.

JEL Classification: C7, D7

Keywords: Bipartite digraph, influence, inclination, collective choice, opinion leader, follower, satisfaction, power, dictator properties, opinion leader properties

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1 Introduction

The concept of opinion leadership received considerable attention in sociology and marketing. It rose out of the *two-step flow of communication* theory introduced by the 'Lazarsfeld group' (see e.g. Katz and Lazarsfeld, 1955, and Lazarsfeld et al., 1968, [29, 31]). In its most rudimentary form it claims that 'ideas often flow *from* radio and print *to* the opinion leaders and *from* them to the less active sections of the population' (Lazarsfeld et al., 1968, [31]). For instance, Lazarsfeld et al. (1968, [31]) investigated the influence of mass communication on the 1940 presidential election campaign in the US. They found that the voters' choices were more influenced by actors which they called opinion leaders than by mass communication and concluded that the communication process is not a one- but a two-step process. According to this model information distributed by mass media first reaches the so-called *opinion leaders*. These are actors who are specified as highly self-confident with strong opinions. In Lazarsfeld et al. (1968, [31]) they act as intermediaries between the mass media and the recipients. In general, the latter actors are called *followers*. They feel attracted by the opinion leaders holding them in high esteem and are prepared to accept their opinion for their own behavior. Hence, a major characteristic of opinion leaders is their exercised power over their followers. After critiques of the model by the 'Lazarsfeld group' (see e.g. Bostian, 1970, [6]), Troldahl (1966, [41]) introduced a modified version of their model called the *two-cycle flow of communication* model which corresponded to other results in the field (see e.g. Deutschmann and Danielson, 1960, [13]). Troldahl's model distinguishes between two phases in the communication process. Phase one is a *flow of information* from the mass media to the members of the society which is assumed to be a *one-step process*, i.e. the information goes directly to all members of the society. Phase two is the *flow of influence* on beliefs and behavior which is assumed to be a *two-step process*. In a first step opinion leaders form their own opinion based on additional information provided by experts, such as academics, while in a subsequent second step they try to influence the behavior of their followers. Since Troldahl's contribution the literature on opinion leadership has provided a strong body of knowledge of how and why opinion leaders influence followers choices (see Hoyer and Stockburger-Sauer, 2007, [27]).

Opinion leaders form an attractive group for marketing and policy purposes (see e.g. Hoyer and Stockburger-Sauer, 2007, [27]) as the existence (or non-existence) of opinion leaders in a society and their relations to their followers may have a considerable impact on market behavior (such as consumer or financial markets), and other social agglomerations being made up of individual actors choosing among a number of alternatives (open to them at a given time). Hence, it appears to be interesting to investigate the effect of different *opinion leader-follower structures* in markets or other collective decision-making situations in a society. This includes questions such as whether it would be worthwhile to establish a new opinion leader in a society or whether a change in an existing opinion leader-follower structure can be expected to make a difference to the society. However, to our best knowledge there exists no study which addresses such issues on bare theoretical grounds. In this paper we lay the foundation to fill out this lacunae by introducing novel *power* and *satisfaction* scores for societies with opinion leaders.[1] The former informs us about the power distribution among the members of the society with respect to their ability to affect the state of the society concerning a specific outcome, while the latter tells us to which degree members of the society can be expected to end up with an outcome that they like. Moreover, in this paper we study common dictator and opinion

[1] Note that this research is in some respects also related to work on opinion leaders and the Condorcet Jury Theorem (see, e.g., Estlund, 1994, [15]), threshold models of collective behavior (see, e.g., Granovetter, 1978, and Granovetter and Soong, 1986, [23, 24]) and, in more general terms to the literature on network externalities.

leader properties of the above scores and illustrate our findings for a society with five members.

For our analysis we consider the example of binary choice as it can also be found in Sinha and Raghavendra (2006, [37]) who study the effect of opinion leaders on market outcomes. It is assumed that an actor can choose among two alternatives. For instance, this can be a market in which the actors have to decide whether they should *buy* or *not buy* a joint product, or a voting situation in which the members of the society have to choose to vote either *yes* or *no* on a specific proposal. From now onwards we will refer to a voting situation only. However, all results presented in this paper also apply to markets. We assume that the actors in a society have to decide whether they would like to remain with the status quo or whether a specific *exogenous* proposal leading to a new state of the society should be adopted. We assume that the proposal has been distributed among all actors. Each actor has to form its own opinion on the proposal, i.e. without being influenced by any other actor. We will call this the actor's inclination. The society is partitioned into *opinion leaders*, *followers*, and *independent actors*. Followers are free to choose their own opinion leaders. Then, in line with the inherent idea of opinion leadership we suppose that via informal discussions of the proposal the inclinations of the opinion leaders are becoming public information prior to the real decision. Only after these discussions all actors will (secretly) choose their action which coincide with the actors inclination if it is an opinion leader or independent actor. Concerning the followers we assume that for their choice of action they - independently of their own inclination - adopt the inclination of a certain fraction of its opinion leaders if these have the same inclination. Finally, based on the individual choices of all actors a decision-making mechanism determines the collective choice, i.e. whether the proposal is adopted or not.

In the literature we can find several scores and measures being introduced for analyzing collective decision-making situations with a possible influence between the actors.[2] For instance, some measures for arbitrary digraphs have been studied in van den Brink and Borm (2002, [10]) and van den Brink and Gilles (2000, [11]). Since in the present paper we consider an opinion leader-follower structure which can be represented by a bipartite digraph, our model is related to the studies done in [10, 11]. Coming from a slightly different direction are the works presented in the voting power literature. One of the traditional measures is the Rae index (Rae, 1969, [35]) which measures the success of an actor in a voting situation. An actor is said to be successful if its vote coincides with the voting outcome. Such a successful actor can be additionally powerful. For the calculation of the voting power of an actor a number of measures have been suggested. The most prominent measures are the Banzhaf measure (Banzhaf, 1965, [1], see also Dubey and Shapley, 1979, and Owen, 1975, [14, 34]), and the Shapley-Shubik index (Shapley and Shubik, 1954, [36]). They ascribe power to an actor if its vote coincides with the voting outcome, but this outcome would have been different if the actor changed its vote.[3] As we are also concerned with measuring *satisfaction* and *power* distributions, our research is also related to the work on measurement of voting power. In the present paper we analyze satisfaction in a digraph by the number of times the collective choice is the same

[2] For the distinction between scores, measures, and indices, see Felsenthal and Machover (1998, [17]).

[3] Both measures can also be derived from a probabilistic framework. See, e.g., Straffin (1977, 1978, [38, 39]) and, more recently, Laruelle and Valenciano (2005, [30]).

as the inclination of an actor. Power of an actor in a bipartite digraph is measured by the number of times the actor has a *swing*, where by a swing we mean that an actor by changing its inclination, given the inclinations of the others, enforces a change in the collective choice. We show that the power and satisfaction scores that we introduce satisfy some common dictator and opinion leader properties.

The paper is structured as follows. In Section 2 we describe the model, and in Section 3 we introduce novel satisfaction and power scores for actors in societies with opinion leaders. In Section 4, we suggest some desirable properties of satisfaction and power for such societies and prove that they are satisfied by our scores. In Section 5, we illustrate the introduced scores and their properties for a society represented by five-actors digraphs. In Section 6, we draw some conclusions, discuss some possible extensions of the model and a future research agenda. Proofs are presented in the Appendix.

2 The model

We consider the following model. There is a specific exogenous proposal on which n actors have to decide upon either by choosing the *yes*- or *no*-action. Let 1 and 0 stand for the choice of the *yes* and *no*-action, respectively, and let $N = \{1, ..., n\}$ denote the set of all actors forming the society which is partitioned into opinion leaders, followers, and independent actors. We assume that the opinion leader-follower relations are given, i.e. it is given which actors might influence the choice of action of certain other actors by exercising some power over them.[4] Furthermore, we suppose that each actor has already formed an *inclination* to choose either the *yes*- or *no*-action. These are given by an inclination vector $I = (I_1, ..., I_n) \in \{1, 0\}^n$. This is a vector which k^{th} component, I_k, is 1 if actor k has the inclination to choose the *yes*-action, and 0 if it is inclined to choose the *no*-action. It is assumed that via informal discussions of the proposal the inclinations of the opinion leaders have already become public information, i.e. each follower is aware of the inclination of its opinion leader(s). While we allow that these discussions change the inclinations of some actors, we suppose that when choosing their action and after that inclinations do not alter. Moreover, we suppose that the choice of action is done secretly which implies that we consider a simultaneous decision-making situation.

The structure of such 'opinion leader-follower' relations is represented by a *bipartite directed graph* (or *bipartite digraph*) (N, D) with set of nodes N representing the actors and $D \subset N \times N$ a binary relation on N. Since we take the set of actors N fixed, we represent a digraph (N, D) just by its binary relation D. We denote the collection of all bipartite digraphs on N, represented by their binary relation, by $\mathcal{D}^N$.

Let $S_D(k)$ and $P_D(k)$ denote the set of successors and predecessors of actor k in digraph D, respectively, i.e., for each $k \in N$,

$$S_D(k) = \{j \in N : (k, j) \in D\}$$

$$P_D(k) = \{j \in N : (j, k) \in D\}.$$

[4] As a result of this *influence*, the ability of the followers to determine the outcome of the collective choice, i.e. their power to do something (with respect to the outcome of the collective choice) might be affected.

As we assume that each actor is either an opinion leader, follower or independent actor, we consider digraphs D such that

$$|S_D(k)| \cdot |P_D(k)| = 0 \text{ for each } k \in N \tag{1}$$

where $|X|$ denotes the cardinality of set X. Let $OL(D)$, $FOL(D)$, and $IND(D)$ denote the sets of all opinion leaders, followers, and independent actors in digraph D, respectively, i.e.

$$OL(D) = \{k \in N : S_D(k) \neq \emptyset\}$$

$$FOL(D) = \{k \in N : P_D(k) \neq \emptyset\}$$

$$IND(D) = N \setminus (OL(D) \cup FOL(D)).$$

Therefore, by assumption (1) we have that

$$OL(D) \cap FOL(D) = \emptyset,$$

and thus the sets $OL(D)$, $FOL(D)$ and $IND(D)$ form a partition of the set N.

We refer to a pair (I, D) with $I \in \{0,1\}^n$ and a bipartite digraph $D \in \mathcal{D}^N$ as described above as an *opinion leader-follower collective choice situation*. We assume that the actors in $OL(D) \cup IND(D)$ make their simultaneous choice of action according to their own inclinations. If a follower has just one opinion leader, it will choose according to the inclination of its opinion leader. If a follower has more than one opinion leader, then it will not follow its own inclination if and only if a certain fraction of its opinion leaders have the same inclination and this differs from its own inclination.

Let $V = V(I, D) \in \{0,1\}^n$ denote the choice vector, that is, a vector which k^{th} component, V_k, is 1 if actor k has chosen the *yes*-action, and 0 if k has chosen the *no*-action. We assume that for every follower it is specified what fraction (more than one half) of its predecessors it will follow. Assuming this fraction to be uniform over the actors, there exists $q \in [\frac{1}{2}, 1)$ such that the choice vector $V = V(I, D) \in \{0,1\}^n$ is recursively given by:

$$V_k = I_k \; if \; k \in OL(D) \cup IND(D),$$

and for $k \in FOL(D)$:

$$V_k = \begin{cases} x & \text{if } |\{j \in P_D(k) : I_k = x\}| > [q \cdot |P_D(k)|] \\ I_k & \text{otherwise,} \end{cases} \tag{2}$$

where $x \in \{0,1\}$ and $[Z]$ denotes the largest integer not greater than Z. According to (2), given (I, D), if more than the fraction q of the opinion leaders of follower k has the same inclination, then k will follow these opinion leaders, otherwise k will decide according to its own inclination. In particular, if $q = \frac{1}{2}$, and half of the opinion leaders of k are inclined to choose the *yes*-action and half to choose the *no*-action, the follower k will follow its own inclination. For sufficiently large q, actor k follows its opinion leaders independently of its own inclination only if they are all unanimous in their inclinations.

After all actors have chosen their actions, a collective choice is resulting according to the decision-making mechanism in use. The decision-making mechanism is given by the *collective decision function* $C\colon \{0,1\}^n \times \mathcal{D}^N \to \{0,1\}$ which assigns an outcome to every pair $(I, D) \in \{0,1\}^n \times \mathcal{D}^N$, that is, the value 0 if the collective decision is *no*, and the value 1 if the collective decision is *yes*. Usually one only considers collective decision functions C that are neutral[5] and anonymous[6]. In this paper, we consider the collective decision function by simple majority voting. Let for an action $x \in \{0,1\}$ and $V = V(I, D) \in \{0,1\}^n$

$$n_x(V(I,D)) = |\{k \in N : V_k = x\}|$$

be the number of actors choosing the action x. We restrict our analysis to situations in which the number of actors is odd. The collective decision function is defined, for each (I, D), as follows:

$$C(I,D) = \begin{cases} 1 & \text{if } n_1(V(I,D)) > n_0(V(I,D)) \\ 0 & \text{if } n_0(V(I,D)) > n_1(V(I,D)). \end{cases} \tag{3}$$

3 Measuring satisfaction and power

In this section we define satisfaction and power scores for bipartite digraphs which represent a collective decision-making situation as described above. In general, a score or measure for a bipartite digraph is a function $f : \mathcal{D}^N \to \mathbb{R}^n$ which assigns an n-dimensional real vector to every bipartite digraph on N.

Success of an actor in a voting game means that the actor's vote coincides with the voting outcome. Since in our model actors have their *inclinations* to choose either the *yes*- or *no*-action before they actually choose, we propose to measure the *satisfaction* of an actor which is related to how often an actor's *inclination* prior to its actual choice coincides with the collective choice.[7] First, we define a score of satisfaction of an actor under the given inclination vector, i.e., for each $(I, D) \in \{0,1\}^n \times \mathcal{D}^N$ and $k \in N$

$$\overline{SAT}_k(I,D) = \begin{cases} 1 & \text{if } C(I,D) = I_k \\ 0 & \text{otherwise.} \end{cases}$$

Next, based on satisfaction of an actor under each inclination vector, we define a *score of satisfaction* of actor k in a bipartite digraph, $SAT : \mathcal{D}^N \to \mathbb{R}^n$, by

$$SAT_k(D) = \sum_{I \in \{0,1\}^n} \overline{SAT}_k(I,D) \quad \text{for each } k \in N. \tag{4}$$

[5] A collective decision function C is *neutral* if $[C(I,D) = 1$ if and only if $C(I^c, D) = 0]$, where $I^c_k = 1$ if and only if $I_k = 0$.

[6] A collective decision functions C is *anonymous* if for every permutation $\pi : N \to N$, $C(I,D) = C(\pi(I), \pi(D))$ with $\pi(I)_k = I_{\pi(k)}$ and $(\pi(k), \pi(j)) \in \pi(D)$ if and only if $(k, j) \in D$.

[7] Note, that this implies the general belief that (at least from the followers' perspective) the existence of opinion leaders is beneficial for a society. As already said above followers are prepared to accept the views of their chosen opinion leaders even if this implies to choose an action against their own inclination. This is comparable with the behavior of voters in democracy. They usually also do not question the existence of democracy if the outcome of the collective choice is not in line with their view. Another example are fashion goods. Here followers usually choose opinion leaders from which they assume that they correctly predict what will be popular in the future. Followers often adopt their view even if they do not like the style they declare to be trendy in the future. Hence, we can distinguish between two types of inclinations. The above type of inclinations given by I being related to the outcome of the collective choice, and inclinations related to the procedure, i.e. that under certain conditions followers have the inclination to align their individual choice with the view of their chosen opinion leaders even if this is against their own inclinations. This also implies the existence of two notions of satisfaction: one with respect to the outcome and one with respect to the procedure. In this paper we suggest a score for the former one.

Even more frequently than measuring success, the voting literature studies scores and measures of *power* of an actor. Roughly speaking, we ascribe power to a successful actor in a voting game if given the votes of the others, by changing its vote the actor changes the voting outcome, i.e. the actor has a swing. Since we consider situations where the actors have their inclinations before they choose their action, and the power of an actor is related to its ability to alter the collective choice, we can relate the power of an actor in this model to its inclination. Consequently, we ascribe power to an actor if the actor by changing its inclination alters the collective choice.

Actor $k \in N$ has a *swing* in (I, D) according to collective decision function C if $C(I, D) \neq C(I', D)$ with $I'_k \neq I_k$ and $I'_j = I_j$ for all $j \in N \setminus \{k\}$. We measure power under the given inclination vector by the number of swings:

$$\overline{POW}_k(I, D) = \begin{cases} 1 & \text{if } k \text{ has a swing in } (I, D) \\ 0 & \text{otherwise.} \end{cases}$$

Then, the *power score* $POW \colon \mathcal{D}^N \to \mathbb{R}^n$ is given by

$$POW_k(D) = \sum_{I \in \{0,1\}^n} \overline{POW}_k(I, D) \quad \text{for each } k \in N. \tag{5}$$

Note that both the satisfaction and power scores depend on the quota q, and hence they should be formally denoted by $SAT^q_k(D)$ and $POW^q_k(D)$, respectively. Nevertheless, for notational convenience we will always write $SAT_k(D)$ and $POW_k(D)$ as denoted in (4) and (5). In the next section we will study common dictator and opinion leader properties of the satisfaction and power scores given by (4) and (5).

4 Properties for satisfaction and power scores based on following a qualified majority of opinion leaders

Let us consider the scores SAT and POW defined in (4) and (5), respectively. Each follower is assumed to follow a qualified majority of its opinion leaders, i.e. the choice vector V is as defined in (2). We consider the neutral and anonymous collective decision function C as defined in (3).

First, we state some properties for a score (or measure) $f \colon \mathcal{D}^N \to \mathbb{R}^n$. We start with the property which says that the score for actors with a symmetric position in the bipartite digraph is the same.

Property 1 (Symmetry) *If $S_D(k) = S_D(j)$ and $P_D(k) = P_D(j)$ then $f_k(D) = f_j(D)$.*

Next, we consider two dictator properties. Clearly, a dictator, i.e. a unique opinion leader who is followed by all other actors, has the power to change the outcome for any voting profile by changing its own vote and, if the dictator votes according to its inclination then the outcome will be the inclination of the dictator. Therefore the *dictator property* states that, if there is a dictator, then the score of the dictator is equal to the total number of possible inclination vectors. Note, that since we assume that no actor can be at the same time a follower and an opinion leader, the dictator as defined above cannot be a follower.

Property 2 (Dictator property) *If* $D \in \mathcal{D}^N$ *and* $h \in N$ *is such that* $S_D(h) = N\backslash\{h\}$, *then* $f_h(D) = 2^n$.

Secondly, since a follower who has only one opinion leader has always to follow this opinion leader, *dictated independence* states that the score of a follower with one opinion leader does not change as long as this follower is dictated by a sole opinion leader.

Property 3 (Dictated independence) *If* $D, D' \in \mathcal{D}^N$ *and* $k \in N$ *are such that* $|P_D(k)| = |P_{D'}(k)| = 1$, *then* $f_k(D) = f_k(D')$.

Next we present the three opinion leader properties saying something about the changes in the score for different actors when the opinion leader-follower structure changes, in particular when an actor gets a new opinion leader. The properties that we consider are inspired by similar properties for solutions in cooperative game theory, where studying these kinds of properties has a longer history. The most widely studied and applied property in this field is *fairness* introduced by Myerson (1977, [33]) in the context of cooperative games in which the players belong to some binary communication structure, stating that deleting a communication link between two players changes their individual payoffs by the same amount. In van den Brink (1997, [8]) this type of equal gain/loss property is stated in terms of games in which the players belong to some hierarchical structure, the so-called games with a permission structure, where it is stated that deleting a domination arc between a successor and a predecessor changes the payoffs of these two players by the same amount. In this paper we consider such kind of property for the opinion leader-follower collective choice situation. Suppose that a follower gets one more opinion leader. The *equal absolute change property* states that the changes in scores of this follower and of its new opinion leader are either the same or are opposite, but the same in absolute values.

Property 4 (Equal absolute change property) *Let* $D, D' \in \mathcal{D}^N$, $j \in FOL(D)$, $h \in OL(D) \cup IND(D)$, *and* $D' = D \cup \{(h, j)\}$. *Then either* $f_h(D') - f_h(D) = f_j(D') - f_j(D)$ *or* $f_h(D') - f_h(D) = f_j(D) - f_j(D')$.

Although it is not obvious whether the scores of a follower and new opinion leader should change in the same or in opposite direction, it seems reasonable that these changes are in opposite direction in case the follower was an independent actor before the change. Taking this into account, *the opposite gain property* states that, if an actor becomes a sole opinion leader of another actor who was previously independent, then the sum of the scores of these two actors does not change. This implies that in case the opinion leader gains then this goes fully at the cost of the follower.

Property 5 (Opposite gain property) *Let* $D, D' \in \mathcal{D}^N$, $j \in IND(D)$, $h \in OL(D) \cup IND(D)$, *and* $D' = D \cup \{(h, j)\}$. *Then* $f_h(D') - f_h(D) = f_j(D) - f_j(D')$.

Finally, in the context of cooperative TU-games, Lehrer (1988, [32]) and Haller (1994, [25]) introduced properties that consider collusion of players. In particular, Haller (1994, [25]) considers different types of collusion neutrality properties requiring that the sum of the payoffs of two colluding players does not change. In van den Brink (2010, [9]), these properties are applied to the above mentioned games with a hierarchical permission structure. In that model, deleting a domination link between a successor and a predecessor is interpreted as a collusion between this predecessor and another predecessor with respect to the influence over the successor, leading to a power neutrality property stating that the sum of payoffs of the two colluding predecessors should not change. A similar reasoning in our underlying opinion leader-follower collective choice situation is reflected in *power neutrality for two opinion leaders* which states that, if a follower with one opinion leader gets a second opinion leader, then the sum of scores of the 'old' and 'new' opinion leaders does not change. In other words, the change for the new (second) opinion leader is opposite but in absolute value equal to the change for the old (first) opinion leader.

Property 6 (Power neutrality for two opinion leaders) *Let $D, D' \in \mathcal{D}^N$, $j \in FOL(D)$ and $g \in OL(D)$ such that $P_D(j) = \{g\}$, $h \in OL(D) \cup IND(D)$, and $D' = D \cup \{(h, j)\}$. Then $f_h(D') + f_g(D') = f_h(D) + f_g(D)$.*

Next, we state the first main result which shows that the satisfaction score satisfies all the six properties mentioned above, if the choice vector is defined by (2).

Theorem 1 *Let the choice vector V be defined by (2). The satisfaction score $SAT : \mathcal{D}^N \to \mathbb{R}^n$ defined in (4) satisfies symmetry, the dictator property, dictated independence, the equal absolute change property, the opposite gain property, and power neutrality for two opinion leaders.*

The proof of this theorem can be found in the appendix. We note that from the proof that SAT satisfies the equal absolute change property we obtain as a corollary that whenever the satisfaction of two actors changes in the same direction after one becomes opinion leader of the other, then in the new situation the follower chooses its action according to its own inclination.

Corollary 1 *For every $D, D' \in \mathcal{D}^N$ and $h, j \in N$ such that $P_D(j) \neq \emptyset$, $h \in (OL(D) \cup IND(D)) \setminus P_D(j)$ and $D' = D \cup \{(h, j)\}$, whenever $SAT_h(I, D') - SAT_h(I, D) = SAT_j(I, D') - SAT_j(I, D) = 1$, we have $V_j(I, D') = I_j$.*

It turns out that the power score POW also satisfies all six axioms, if the choice vector is defined by (2).

Theorem 2 *Let the choice vector V be defined by (2). The power score $POW : \mathcal{D}^N \to \mathbb{R}^n$ defined in (5) satisfies symmetry, the dictator property, dictated independence, the equal absolute change property, the opposite gain property, and power neutrality for two opinion leaders.*

The proof of this theorem also can be found in the appendix.

5 Examples

In order to illustrate the developed scores and obtained results we analyze a society consisting of five actors, i.e. $N = \{1, 2, 3, 4, 5\}$. We assume eight potential opinion leader-follower structures for this society, each represented by a digraph. For the followers we assume three different levels of the qualified majorities of the opinion leaders required to adapt their own behavior,

$$q = \frac{1}{2}, \quad q' = \frac{2}{3}, \quad q'' = \frac{3}{4}.$$

We analyze the following digraphs (see Figures 1a - 1h):

$D_0 = \emptyset$ (Figure 1a, no opinion leaders)

$D_1 = \{(2,1)\}$ (Figure 1b, actor 2 is the sole opinion leader of actor 1)

$D_2 = \{(2,1),(3,1)\}$ (Figure 1c, actors 2 and 3 are the opinion leaders of actor 1)

$D_3 = \{(2,1),(3,1),(4,1)\}$ (Figure 1d, actors 2, 3 and 4 are the opinion leaders of 1)

$D_4 = \{(2,1),(3,1),(4,1),(5,1)\}$ (Figure 1e, actors in $N \setminus \{1\}$ are the opinion leaders of 1)

$D_5 = \{(1,2),(1,3),(1,4),(1,5)\}$ (Figure 1f, actor 1 is a dictator)

$D_6 = \{(1,2),(3,4)\}$ (Figure 1g, actor 2 follows 1, actor 4 follows 3)

$D_7 = \{(1,2),(3,4),(5,4)\}$ (Figure 1h, actor 2 follows 1, actors 3 and 5 are the opinion leaders of 4).

For this setup we show how the number of opinion leaders affects the relations between actors' satisfaction and power scores.

FIGURE 1 ABOUT HERE

Note that in digraph D_3, in which actor 1 has three opinion leaders, when following the qualified majority of its opinion leaders, under $q = \frac{1}{2}$, actor 1 will always decide according to the inclination of at least two of its opinion leaders with the same inclination. Under $q' = \frac{2}{3}$ and $q'' = \frac{3}{4}$, it will follow its opinion leaders independently of its own inclination only when they are unanimous. In digraph D_4, in which actor 1 has four opinion leaders, under q and q' actor 1 will follow at least three unanimous opinion leaders, but it will decide according to its own inclination if two of its opinion leaders have the positive inclination, and two have the negative inclination. Under q'' agent 1 follows its opinion leaders 2, 3, 4, 5, independently of its own inclination only if they are unanimous in their inclinations. When an actor has two opinion leaders, like in D_2 and D_7, following the qualified majority of the opinion leaders means, of course, following the opinion leaders independently of their own inclination only when they are unanimous.

Tables 1 and 2 present the chosen individual actions and the collective choices based on the collective decision function as given in (3) for digraphs D_0 to D_3, and D_4 to D_7, respectively. The sign "-" means that for a given I, the result (either the chosen action or the collective choice) is the same as in D_0.

TABLE 1 ABOUT HERE

TABLE 2 ABOUT HERE

Table 3 shows the satisfaction and power scores for the actors in the situations represented by digraphs D_0 to D_7.

TABLE 3 ABOUT HERE

Note that for digraph D_3 with actor 1 as follower and actors 2, 3, and 4 as its opinion leaders, the satisfaction and power score of the follower (the opinion leaders, respectively) increases (decreases, respectively) when the quota increases from q to q', and each of these two scores remains the same when the quota increases from q' to q''. On the other hand, the scores of each actor in digraph D_4 are the same for q, q', and q''.

Using the scores calculated in Table 3, we can illustrate the properties presented in Section 4. Symmetry is obviously satisfied by both scores in all digraphs. The scores of satisfaction and power of actors in digraph D_5 with actor 1 as a dictator, satisfy the dictator property. Dictated independence can be illustrated with the scores in question of actor 1 in D_1, actors 2 and 4 in D_6, and actor 2 in D_7, with satisfaction always equal to 16, and power always equal to 0.

In order to illustrate the equal absolute change property, let us first consider digraphs D_2 and D_3 with $q = \frac{1}{2}$, where the follower 1, having already two opinion leaders in D_2 (actors 2 and 3), gets his new opinion leader in D_3, actor 4. The following holds:

$$SAT_4(D_3) - SAT_4(D_2) = 24 - 20 = 20 - 16 = SAT_1(D_2) - SAT_1(D_3)$$

and

$$POW_4(D_3) - POW_4(D_2) = 16 - 8 = 8 - 0 = POW_1(D_2) - POW_1(D_3).$$

On the other hand, when considering D_3 and D_4 with $q = \frac{1}{2}$, where actor 5 becomes a new opinion leader of actor 1, we have:

$$SAT_5(D_4) - SAT_5(D_3) = 22 - 16 = SAT_1(D_4) - SAT_1(D_3)$$

and

$$POW_5(D_4) - POW_5(D_3) = 12 - 0 = POW_1(D_4) - POW_1(D_3).$$

The opposite gain property can be seen when comparing D_0 with D_1, and D_1 with D_6. Indeed, for the latter case, we have:

$$SAT_4(D_1) + SAT_3(D_1) = 20 + 20 = 16 + 24 = SAT_4(D_6) + SAT_3(D_6)$$

and

$$POW_4(D_1) + POW_3(D_1) = 8 + 8 = 0 + 16 = POW_4(D_6) + POW_3(D_6).$$

Power neutrality for two opinion leaders can be shown when comparing D_1 with D_2, and D_6 with D_7. Indeed, for the follower 1 in D_1 and D_2, and actors 2 and 3 as actor 1's opinion leaders, we get as follows:

$$SAT_2(D_1) + SAT_3(D_1) = 28 + 20 = 24 + 24 = SAT_2(D_2) + SAT_3(D_2)$$

and

$$POW_2(D_1) + POW_3(D_1) = 24 + 8 = 16 + 16 = POW_2(D_2) + POW_3(D_2).$$

We want to wind up this section with some observations concerning satisfaction, power and resulting outcome effects of different opinion leader-follower structures. Comparing D_0 with D_3 for q' and q'' and D_4 for q, q' and q'' from Tables 1 and 2, we can find that for all actors' satisfaction and power is always identical, i.e. it has no effect whether an actor is an opinion leader, follower, or independent actor or whether the society is structured in one way or the other. Moreover, if actors become opinion leaders or followers their effect on the outcome does not change. Hence, for these structures it does not matter whether there exists no or three or four opinion leaders if there is only up to one follower. However, to establish an opinion leader in a society without any opinion leader (moving from D_0 to D_1) has some effect on the outcome of the collective choice: it changes in 18.75 percent of the cases, but adding another opinion leader to an existing opinion leader (moving from D_1 to D_2) decreases the number of cases where the outcome changes. In contrast to the case without an opinion leader, the outcome alters only in 6.25 percent of the cases. Hence, the existence of only one opinion leader has a stronger effect to society as having a second one if there exists only one follower.

6 Future research

The existence of opinion leaders and their influence over other actors can be seen in every day life situations: in small as well as in large societies be it in politics or business. Both satisfaction and power are the very natural measures of actors' *strength* or *status* in such situations. Although, as mentioned in the introduction, there exist several related theoretical studies in the literature on voting models and on networks, the approach which we use in the paper, i.e. the analysis of *opinion leader-follower* structures and the properties of the scores in question has brought up several innovative elements and can also be regarded to contribute to knowledge, in particular, in marketing.

However, there are several improvements one could bring to this framework in future research. First of all, after stating some initial properties of satisfaction and power scores for opinion leader-follower structures, a next step is to deliver full axiomatic characterizations of the satisfaction and power scores, which would show a difference between the scores from the axiomatic point of view. In van den Brink, Rusinowska and Steffen (2010, [12]) properties presented in this paper are used to axiomatize these scores in the special case of unanimity, i.e. independently from its own inclination a follower will choose the action which corresponds to the inclination of its opinion leader(s), if all of them have the same inclination, but will choose an action which corresponds to its own inclination, if the inclinations of its opinion leader(s) are not all the same. In this axiomatic system the satisfaction and power scores differ only with respect to some normalization axiom.

In this paper we assumed that an actor cannot be at the same time an opinion leader for some actor(s) and a follower of some other actor(s). In a future research on this topic, one could try to relax this assumption and to consider a more general digraph, allowing the sets of predecessors and successors of a given actor to be both nonempty. In terms of Troldahl's (1966, [41]) *two-cycle flow of communication* model this would allow us to include the experts as an additional group of actors into our analysis acting as opinion leaders of the opinion leaders.

Furthermore, we could apply the same approach to some related models on organizational hierarchies based on subordinates and their superiors (see, e.g., Hammond and Thomas, 1990, [26]), where an organizational choice is to be made. Although such topics are naturally related to our present work, we expect that results on the properties of the scores in question in the *superior-subordinate* structures will be quite different from the ones obtained in the present model.

Since we consider the two-action model, a natural and useful generalization of the framework will be to enlarge the set of possible actions, i.e., to follow some works on abstention (see, e.g. Braham and Steffen, 2002, Felsenthal and Machover, 1997, 1998, 2001, Tchantcho et al., 2008, [7, 16–18, 40]), and on multi-choice games (see e.g. Grabisch and Rusinowska, 2010, and Hsiao and Raghavan, 1993, [22, 28]). Related models are also games with r alternatives, where the alternatives are not ordered; see Bolger (1986, 1993, 2000, 2002, [2–5]). Also in Freixas (2005a, 2005b, [19, 20]) and Freixas and Zwicker (2003, [21]), the authors consider decision-making situations, i.e. voting systems, with several levels of approval in the input and output, where those levels are qualitatively ordered. They introduce (j, k) simple games, in which each actor expresses one of j possible levels of input support, and the output consists of one of k possible levels of collective support.

Finally, another issue of future research agenda would be to apply the opinion leader-follower collective choice situation to concrete situations of real life.

References

1. J. Banzhaf. Weighted voting doesn't work: a mathematical analysis. *Rutgers Law Review*, 19:317–343, 1965.
2. E. M. Bolger. Power indices for multicandidate voting games. *International Journal of Game Theory*, 14:175–186, 1986.
3. E. M. Bolger. A value for games with n players and r alternatives. *International Journal of Game Theory*, 22:319–334, 1993.
4. E. M. Bolger. A consistent value for games with n players and r alternatives. *International Journal of Game Theory*, 29:93–99, 2000.
5. E. M. Bolger. Characterizations of two power indices for voting games with r alternatives. *Social Choice and Welfare*, 19:709–721, 2002.
6. L. R. Bostian. The two-step flow theory: Cross-cultural implications. *Journalism Quarterly*, 47:109–117, 1970.
7. M. Braham and F. Steffen. Voting power in games with abstentions. In M.J. Holler, H. Kliemt, D. Schmidtchen, and M.E. Streit, editors, *Power and Fairness*, pages 333–348. Mohr-Siebeck, 2002.
8. R. Brink van den. An axiomatization of the disjunctive permission value for games with a permission structure. *International Journal of Game Theory*, 26:27–43, 1997.
9. R. Brink van den. Axiomatizations of Banzhaf permission values for games with a permission structure. *International Journal of Game Theory*, 39:445–466, 2010.
10. R. Brink van den and P. Borm. Digraph competitions and cooperative games. *Theory and Decision*, 53:327–342, 2002.
11. R. Brink van den and R. P. Gilles. Measuring domination in directed networks. *Social Networks*, 22:141–157, 2000.
12. R. Brink van den, A. Rusinowska, and F. Steffen. Measuring power and satisfaction in societies with opinion leaders: An axiomatization. Mimeo, 2010.
13. P. J. Deutschmann and W. Danielson. Diffusion of knowledge of the major new story. *Journalism Quarterly*, 37:345–355, 1960.
14. P. Dubey and L. S. Shapley. Mathematical properties of the Banzhaf power index. *Mathematics of Operations Research*, 4:99–131, 1979.
15. D. M. Estlund. Opinion leaders, independence, and Condorcet's jury theorem. *Theory and Decision*, 36:131–162, 1994.

16. D. Felsenthal and M. Machover. Ternary voting games. *International Journal of Game Theory*, 26:335–351, 1997.
17. D. Felsenthal and M. Machover. *The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes.* London: Edward Elgar Publishers, 1998.
18. D. Felsenthal and M. Machover. Models and reality: the curious case of the absent abstention. In M. J. Holler and G. Owen, editors, *Power Indices and Coalition Formation*, pages 87–103. Kluwer, Dordrecht, 2001.
19. J. Freixas. Banzhaf measures for games with several levels of approval in the input and output. *Annals of Operations Research*, 137:45–66, 2005.
20. J. Freixas. The Shapley-Shubik power index for games with several levels of approval in the input and output. *Decision Support Systems*, 39:185–195, 2005.
21. J. Freixas and W. S. Zwicker. Weighted voting, abstention, and multiple levels of approval. *Social Choice and Welfare*, 21:399–431, 2003.
22. M. Grabisch and A. Rusinowska. A model of influence with an ordered set of possible actions. *Theory and Decision*, 69(4):635–656, 2010.
23. M. Granovetter. Threshold models of collective behavior. *American Journal of Sociology*, 83:1420–1443, 1978.
24. M. Granovetter and R. Soong. Threshold models of interpersonal effects in consumer demand. *Journal of Economic Behavior and Organization*, 7:83–99, 1986.
25. H. Haller. Collusion properties of values. *International Journal of Game Theory*, 23:261–281, 1994.
26. T. H. Hammond and P. Thomas. Invisible decisive coalitions in large hierarchies. *Public Choice*, 66:101–116, 1990.
27. W. D. Hoyer and N. E. Stockburger-Sauer. A comparison of antecedents and consequences of market mavens and opinion leaders. In T. Bayón, A. Herrmann, F. Huber, M. Hammerschmidt, and N. Stockburger-Sauer, editors, *Vielfalt und Einheit in der Marketingwissenschaft - Ein Spannungsverhältnis*, pages 215–236. Gabler, 2007.
28. C. R. Hsiao and T. E. S. Raghavan. Shapley value for multichoice cooperative games. *Games and Economic Behavior*, 5:240–256, 1993.
29. E. Katz and P. F. Lazarsfeld. *Personal Influence: The Part Played by People in the Flow of Mass Communication.* Free Press, 1955.
30. A. Laruelle and F. Valenciano. Assessing success and decisiveness in voting situations. *Social Choice and Welfare*, 24:171–197, 2005.
31. P. F. Lazarsfeld, B. Berelson, and H. Gaudet. *The People's Choice - How the Voter Makes up his Mind in a Presidential Campaign.* Columbia University Press, 1968. 3rd ed., the first edition of this book was published in 1944.
32. E. Lehrer. An axiomatization of the Banzhaf value. *International Journal of Game Theory*, 17:89–99, 1988.
33. R. B. Myerson. Graphs and cooperation in games. *Mathematics of Operations Research*, 2:225–229, 1977.
34. G. Owen. Multilinear extensions and the Banzhaf value. *Naval Research Logistic Quarterly*, 22:741–750, 1975.
35. D. Rae. Decision-rules and individual values in constitutional choice. *American Political Science Review*, 63:40–56, 1969.
36. L. S. Shapley and M. Shubik. A method for evaluating the distribution of power in a committee system. *American Political Science Review*, 48:787–792, 1954.
37. S. Sinha and S. Raghavendra. Emergence of two-phase behavior in markets through interaction and learning in agents with bounded rationality. In *Practical Fruits of Econophysics: Proceedings of the 3rd Nikkei Econophysics Symposium*, pages 200–204. Springer, 2006.
38. P. D. Straffin. Homogeneity, independence, and power indices. *Public Choice*, 30:107–118, 1977.
39. P. D. Straffin. Probability models for power indices. In P. C. Ordeshook, editor, *Game Theory and Political Science*, pages 477–510. New York University Press, 1978.
40. B. Tchantcho, L. Diffo Lambo, R. Pongou, and B. Mbama Engoulou. Voters' power in voting games with abstention: Influence relation and ordinal equivalence of power theories. *Games and Economic Behavior*, 64:335–350, 2008.
41. V. C. Troldahl. A field test of a modified 'two-step flow of communication' model. *Public Opinion Quarterly*, 30:609–623, 1966.

Appendix

In this appendix we provide all proofs of the results in this paper.

Proof of Theorem 1
It is straightforward that SAT satisfies symmetry.

The dictator property follows straightforward since a dictator is followed in all 2^n inclination vectors in $\{0,1\}^n$, i.e., if $S_D(h) = N \backslash \{h\}$, then $C(I,D) = I_h$ for all $I \in \{0,1\}^n$.

To show dictated independence, note that actor k always chooses an action according to j's inclination if $P_D(k) = \{j\}$. That means that the collective choice is independent of actor k's inclination, i.e. $C(I,D) = C(I',D)$ if $I_h = I'_h$ for all $h \in N \setminus \{k\}$. Hence, in half of the inclination vectors $C(I,D) = I_k$ and in the other half $C(I,D) \neq I_k$. So, SAT satisfies dictated independence.

To show the equal absolute change property, let $D, D' \in \mathcal{D}^N$, $j \in FOL(D)$, $h \in OL(D) \cup IND(D)$ and $D' = D \cup \{(h,j)\}$. For all inclination vectors I such that $C(I,D) = C(I,D')$, obviously it holds $\overline{SAT}_j(I,D') - \overline{SAT}_j(I,D) = 0 = \overline{SAT}_h(I,D') - \overline{SAT}_h(I,D)$. Let $I \in \{0,1\}^n$ be such that $C(I,D) \neq C(I,D')$. Then, $C(I,D) \neq I_h$ and $C(I,D') = I_h$, and therefore $\overline{SAT}_h(I,D') - \overline{SAT}_h(I,D) = 1$.
Case I: Suppose that $[q \cdot |P_D(j)|] = q \cdot |P_D(j)|$ and an I exists so that $C(I,D) \neq C(I,D')$. In this case $q \cdot |P_D(j)|$ opinion leaders of j have inclination $x = I_h \neq I_j$. Then $V_j(I,D) = I_j$ and $C(I,D) = I_j$. But $V_j(I,D') = I_h$ and $C(I,D') = I_h \neq I_j$. Then, $\overline{SAT}_j(I,D) - \overline{SAT}_j(I,D') = 1 = \overline{SAT}_h(I,D') - \overline{SAT}_h(I,D)$, and with (4) we get $SAT_h(D') - SAT_h(D) = SAT_j(D) - SAT_j(D')$.
Case II: Suppose that $[q \cdot |P_D(j)|] < q \cdot |P_D(j)|$ and an I exists so that $C(I,D) \neq C(I,D')$. In this case $[q \cdot |P_D(j)|] + 1$ opinion leaders of j have inclination $x \neq I_h = I_j$, $[q \cdot |P_D(j)|] + 1 = [q \cdot (|P_D(j)| + 1)]$, $V_j(I,D) \neq I_j$, $C(I,D) \neq I_j$. But $V_j(I,D') = I_j = I_h$ and $C(I,D') = I_j$. Then, $\overline{SAT}_j(I,D') - \overline{SAT}_j(I,D) = 1 = \overline{SAT}_h(I,D') - \overline{SAT}_h(I,D)$, and with (4) we have $SAT_h(D') - SAT_h(D) = SAT_j(D') - SAT_j(D)$.
Hence, SAT satisfies the equal absolute change property.

To show the opposite gain property, let $D, D' \in \mathcal{D}^N$, $j \in IND(D)$, $h \in OL(D) \cup IND(D)$ and $D' = D \cup \{(h,j)\}$. Suppose that $C(I,D) \neq C(I,D')$. Then it must hold that actor j had to deviate from its inclination and follow h, and this must result in a change of collective choice from I_j to I_h, with $I_j \neq I_h$. So, $C(I,D) = I_j \neq I_h$ and $C(I,D') = I_h \neq I_j$. Then, $\overline{SAT}_j(I,D') = \overline{SAT}_j(I,D) - 1$ and $\overline{SAT}_h(I,D') = \overline{SAT}_h(I,D) + 1$. So, $\overline{SAT}_h(I,D') - \overline{SAT}_h(I,D) = -(\overline{SAT}_j(I,D') - \overline{SAT}_j(I,D)) \geq 0$. Obviously, this last equality also holds if $C(I,D) = C(I,D')$. Thus, with (4) we have $SAT_h(D') - SAT_h(D) = -(SAT_j(D') - SAT_j(D)) \geq 0$, showing that SAT satisfies the opposite gain property.

To show power neutrality for two opinion leaders, let $D, D' \in \mathcal{D}^N$, $j \in FOL(D)$ and $g \in OL(D)$ such that $P_D(j) = \{g\}$, $h \in OL(D) \cup IND(D)$ and $D' = D \cup \{(h,j)\}$. Obviously, if $C(I,D) = C(I,D')$, then $\overline{SAT}_h(I,D') - \overline{SAT}_h(I,D) = \overline{SAT}_g(I,D) - \overline{SAT}_g(I,D') = 0$. Suppose now that $C(I,D) \neq C(I,D')$. Then it must hold that actor j initially had to choose an action against its inclination and now can choose an action according to its inclination because its new opinion leader h has the same inclination. So, for $g \in P_D(j)$ we have $C(I,D) = I_g \neq I_j = I_h$ and $C(I,D') = I_j = I_h \neq I_g$.

Then, $\overline{SAT}_h(I,D') - \overline{SAT}_h(I,D) = \overline{SAT}_g(I,D) - \overline{SAT}_g(I,D') = 1$. Thus, with (4) we have $SAT_h(D') - SAT_h(D) = SAT_g(D) - SAT_g(D')$, showing that SAT satisfies power neutrality for two opinion leaders.

■

Proof of Theorem 2

It is straightforward that POW satisfies symmetry.

Since a dictator has a swing in every inclination vector, POW satisfies the dictator property. Since an actor with a unique opinion leader never has a swing, POW satisfies dictated independence.

To show the equal absolute change property, let $D, D' \in \mathcal{D}^N$, $j \in FOL(D)$, $h \in OL(D) \cup IND(D)$ and $D' = D \cup \{(h,j)\}$. Note that $POW_h(I,D) = 1$ implies that $POW_h(I,D') = 1$.

Case I: Suppose that $[q \cdot |P_D(j)|] = q \cdot |P_D(j)|$. We distinguish the following subcases:

(i) If $I \in \{0,1\}^n$ is such that $[q \cdot |P_D(j)|] + 1$ opinion leaders of j have the same inclination, then $POW_j(I,D) = POW_j(I,D') = 0$ and $POW_h(I,D) = POW_h(I,D')$.

(ii) If I is such that less than $[q \cdot |P_D(j)|]$ opinion leaders of j have the inclination $x \in \{0,1\}$ and less than $[q \cdot |P_D(j)|]$ opinion leaders have the inclination $y \in \{0,1\} \setminus \{x\}$, then $POW_j(I,D) = POW_j(I,D')$ and $POW_h(I,D) = POW_h(I,D')$.

(iii) Suppose that $I \in \{0,1\}^n$ is such that $[q \cdot |P_D(j)|]$ opinion leaders have the same inclination, say $x \in \{0,1\}$, and less than $[q \cdot |P_D(j)|]$ have the different inclination $y \in \{0,1\} \setminus \{x\}$.

(a) If $I_j = x \neq y = I_h$, then $POW_j(I,D) = POW_j(I,D')$ and $POW_h(I,D) = POW_h(I,D')$.

(b) Let $x = I_h \neq y = I_j$. If $POW_j(I,D) = 1$, then $POW_j(I,D) - POW_j(I,D') = 1$ and $POW_h(I,D') - POW_h(I,D) = 1$. Moreover, for $I' \in \{0,1\}^n$ such that $x = I'_j = I'_h \neq y$, $I'_k = I_k$ for $k \notin \{j,h\}$, $POW_j(I',D) - POW_j(I',D') = 1$ and $POW_h(I',D') - POW_h(I',D) = 0$, but for $I'' \in \{0,1\}^n$ such that $x \neq I''_j = I''_h = y$, $I''_k = I_k$ for $k \notin \{j,h\}$, $POW_j(I'',D) - POW_j(I'',D') = 0$ and $POW_h(I'',D') - POW_h(I'',D) = 1$.

In a similar way we can consider the case when $[q \cdot |P_D(j)|] + 1$ opinion leaders of j have one inclination, and $[q \cdot |P_D(j)|] + 1$ opinion leaders have the other inclination.

Thus, with (5), we get $POW_h(D') - POW_h(D) = POW_j(D) - POW_j(D')$.

Case II: Suppose that $[q \cdot |P_D(j)|] < q \cdot |P_D(j)|$, and $[q \cdot |P_D(j)|] + 1 = [q \cdot (|P_D(j)| + 1)]$. We distinguish the following cases.

(i) If $I \in \{0,1\}^n$ is such that $[q \cdot |P_D(j)|] + 2$ opinion leaders of j have the same inclination, then $POW_j(I,D) = POW_j(I,D') = 0$ and $POW_h(I,D) = POW_h(I,D')$.

(ii) If $I \in \{0,1\}^n$ is such that $[q \cdot |P_D(j)|]$ (or less) opinion leaders of j have one inclination and less than $[q \cdot |P_D(j)|]$ opinion leaders have the other inclination, then $POW_j(I,D) = POW_j(I,D')$ and $POW_h(I,D) = POW_h(I,D')$.

(iii) Suppose I is such that $[q \cdot |P_D(j)|] + 1$ opinion leaders have the same inclination, say $x \in \{0,1\}$. Then $POW_j(I,D) = 0$. There are several possibilities.

(a) If $I_j = x = I_h \neq y$, then $POW_j(I,D') = 0$ and $POW_h(I,D) = POW_h(I,D')$.

(b) Suppose $x = I_h \neq y = I_j$. Then, $POW_j(I,D') = 0$ and if, moreover, $POW_h(I,D) = 1$, we have $POW_h(I,D') = 1$. Further, for $I' \in \{0,1\}^n$ such that $x \neq I'_j = I'_h = y$, $I'_k = I_k$ for $k \notin \{j,h\}$, $POW_h(I',D) = POW_h(I',D') = 1$ and $POW_j(I',D') =$

$POW_j(I',D)=0$, and for $I'' \in \{0,1\}^n$ such that $x = I''_j \neq I''_h = y$, $I''_k = I_k$ for $k \notin \{j,h\}$, $POW_j(I'',D) = POW_j(I'',D') = 0$ and $POW_h(I'',D') = POW_h(I'',D)$. Suppose now that $POW_h(I,D)=0$. If $POW_h(I,D')=0$ as well, then $POW_h(I,D') - POW_h(I,D) = POW_j(I,D') - POW_j(I,D)$, and the same holds for I' and I''. If $POW_h(I,D')=1$, then $POW_h(I,D') - POW_h(I,D) = 1$ and $POW_j(I,D') - POW_j(I,D) = 0$, $POW_h(I',D') - POW_h(I',D) = 1$ and $POW_j(I',D') - POW_j(I',D) = 1$, but
$POW_h(I'',D') - POW_h(I'',D) = 0$ and $POW_j(I'',D') - POW_j(I'',D) = 1$.
Thus, with (5), $POW_h(D') - POW_h(D) = POW_j(D') - POW_j(D)$.
Hence, POW satisfies the equal absolute change property.

To show the opposite gain property, let $D, D' \in \mathcal{D}^N$, $j \in \mathit{IND}(D)$, $h \in \mathit{OL}(D) \cup \mathit{IND}(D)$ and $D' = D \cup \{(h,j)\}$. Since in D' actor j has to choose an action according to its unique opinion leader h, j has never a swing in D', i.e. $POW_j(D') = 0$. So, we have to show that $POW_h(D') - POW_h(D) = POW_j(D) \geq 0$. We distinguish the following three cases.

(i) If h does not have a swing in (I,D) but j has a swing in (I,D), then h has a swing in (I,D'), i.e. if $POW_h(I,D)=0$ and $POW_j(I,D)=1$ then $POW_h(I,D')=1$.

(ii) If h has a swing in (I,D), then h has a swing in (I,D'), i.e. if $POW_h(I,D)=1$ then $POW_h(I,D')=1$. If, moreover, also j has a swing in (I,D) then h has also a swing in (I',D') with $I'_j = I'_h \neq I_h = I_j$, i.e. if $POW_h(I,D)=1$ and $POW_j(I,D)=1$ then $POW_h(I',D')=1$.

(iii) Finally, if h does not have a swing in (I,D) and j does not have a swing in (I,D), then the only possibility for h to have a swing in (I,D') is as described in the last case before. So, $POW_h(D') = \sum_{I \in \{0,1\}^n} \overline{POW}_h(I,D') = \sum_{I \in \{0,1\}^n} (\overline{POW}_h(I,D) + \overline{POW}_j(I,D)) \geq 0$, showing that POW satisfies the opposite gain property.

To show power neutrality for two opinion leaders, let $D, D' \in \mathcal{D}^N$, $j \in \mathit{FOL}(D)$ and $g \in \mathit{OL}(D)$ such that $P_D(j) = \{g\}$, $h \in \mathit{OL}(D) \cup \mathit{IND}(D)$ and $D' = D \cup \{(h,j)\}$. Note that $POW_g(I,D)=0$ implies that $POW_g(I,D')=0$. Moreover, $POW_h(I,D)=1$ implies $POW_h(I,D')=1$.
Note that it is impossible that at the same time $POW_h(I,D') + POW_g(I,D') \geq 1$ and $POW_h(I,D) + POW_g(I,D) = 0$. Moreover, it is impossible that at the same time $POW_h(I,D') + POW_g(I,D') \leq 1$ and $POW_h(I,D) + POW_g(I,D) = 2$.
Furthermore, note that for $I \in \{0,1\}^n$
$[POW_h(I,D') + POW_g(I,D') = 2$ and $POW_h(I,D) + POW_g(I,D) = 1]$ if and only if $[I_h = I_g \neq I_j$ and for $I' \in \{0,1\}^n$ given by $I'_h = I'_g \neq I_g (= I_h)$, and $I'_k = I_k$ for all $k \in N \setminus \{g,h\}$ (and thus $I'_j = I'_h = I'_g$), satisfies $POW_h(I',D') + POW_g(I',D') = 0$ and $POW_h(I',D) + POW_g(I',D) = 1]$.
Hence, POW satisfies power neutrality for two opinion leaders. ■

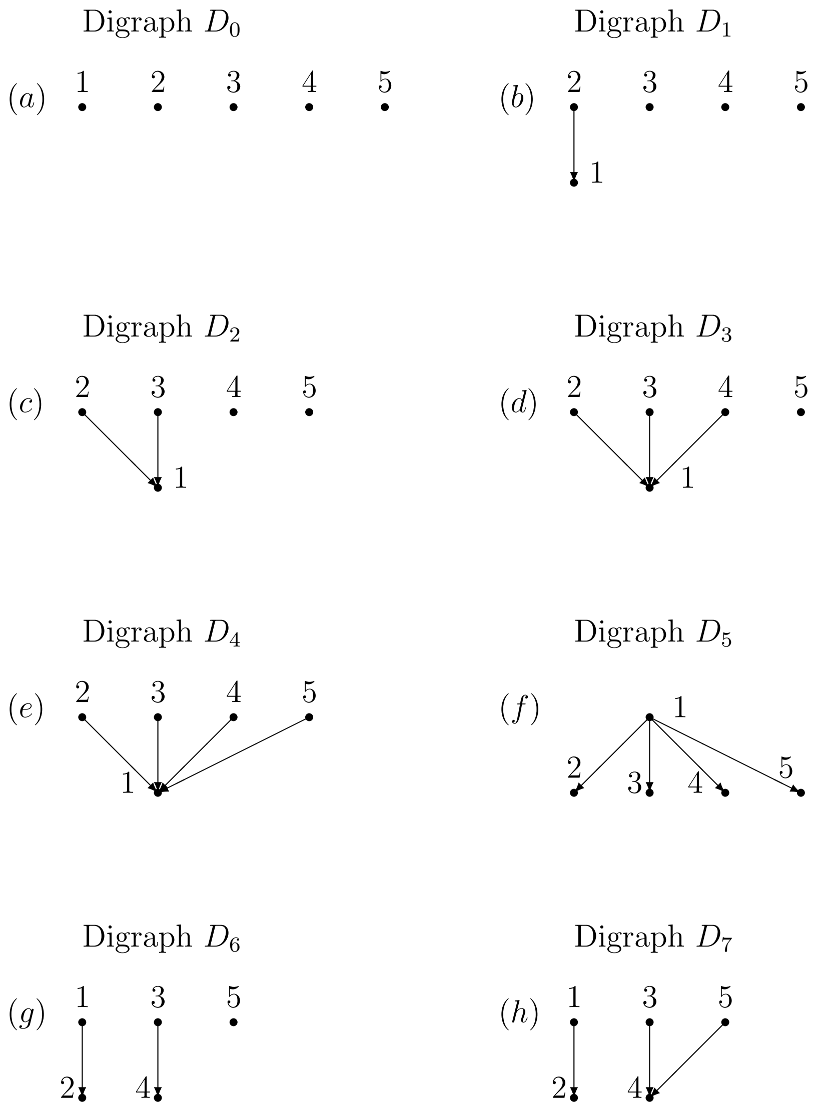

Figure 1: Digraphs D_0 - D_7

$I=V(I,D_0)$	$C(I,V)$	$V(I,D_1)$	$C(I,V)$	$V(I,D_2)$	$C(I,V)$	$V(I,D_3)$ with q	$C(I,V)$ with q	$V(I,D_3)$ with q',q''	$C(I,V)$ with q',q''
(0,0,0,0,0)	0	-	-	-	-	-	-	-	-
(1,0,0,0,0)	0	(0,0,0,0,0)	-	(0,0,0,0,0)	-	(0,0,0,0,0)	-	(0,0,0,0,0)	-
(0,1,0,0,0)	0	(1,1,0,0,0)	-	-	-	-	-	-	-
(0,0,1,0,0)	0	-	-	-	-	-	-	-	-
(0,0,0,1,0)	0	-	-	-	-	-	-	-	-
(0,0,0,0,1)	0	-	-	-	-	-	-	-	-
(1,1,0,0,0)	0	-	-	-	-	(0,1,0,0,0)	-	-	-
(1,0,1,0,0)	0	(0,0,1,0,0)	-	-	-	(0,0,1,0,0)	-	-	-
(1,0,0,1,0)	0	(0,0,0,1,0)	-	(0,0,0,1,0)	-	(0,0,0,1,0)	-	-	-
(1,0,0,0,1)	0	(0,0,0,0,1)	-	(0,0,0,0,1)	-	(0,0,0,0,1)	-	(0,0,0,0,1)	-
(0,1,1,0,0)	0	(1,1,1,0,0)	1	(1,1,0,0,0)	1	(1,1,1,0,0)	1	-	-
(0,1,0,1,0)	0	(1,1,0,1,0)	1	-	-	(1,1,0,1,0)	1	-	-
(0,1,0,0,1)	0	(1,1,0,0,1)	1	-	-	-	-	-	-
(0,0,1,1,0)	0	-	-	-	-	(1,0,1,1,0)	1	-	-
(0,0,1,0,1)	0	-	-	-	-	-	-	-	-
(0,0,0,1,1)	0	-	-	-	-	-	-	-	-
(1,1,1,0,0)	1	-	-	-	-	-	-	-	-
(1,1,0,1,0)	1	-	-	-	-	-	-	-	-
(1,1,0,0,1)	1	-	-	-	-	(0,1,0,0,1)	0	-	-
(1,0,1,1,0)	1	(0,0,1,1,0)	0	-	-	-	-	-	-
(1,0,1,0,1)	1	(0,0,1,0,1)	0	-	-	(0,0,1,0,1)	0	-	-
(1,0,0,1,1)	1	(0,0,0,1,1)	0	(0,0,0,1,1)	0	(0,0,0,1,1)	0	-	-
(0,1,1,1,0)	1	(1,1,1,1,0)	-	(1,1,1,1,0)	-	(1,1,1,1,0)	-	(1,1,1,1,0)	-
(0,1,1,0,1)	1	(1,1,1,0,1)	-	(1,1,1,0,1)	-	(1,1,1,0,1)	-	-	-
(0,1,0,1,1)	1	(1,1,0,1,1)	-	-	-	(1,1,0,1,1)	-	-	-
(0,0,1,1,1)	1	-	-	-	-	(1,0,1,1,1)	-	-	-
(1,1,1,1,0)	1	-	-	-	-	-	-	-	-
(1,1,1,0,1)	1	-	-	-	-	-	-	-	-
(1,1,0,1,1)	1	-	-	-	-	-	-	-	-
(1,0,1,1,1)	1	(0,0,1,1,1)	-	-	-	-	-	-	-
(0,1,1,1,1)	1	(1,1,1,1,1)	-	(1,1,1,1,1)	-	(1,1,1,1,1)	-	(1,1,1,1,1)	-
(1,1,1,1,1)	1	-	-	-	-	-	-	-	-

Table 1. Individual and collective choices for digraphs D_0 - D_3

I	$V(I,D_4)$ with q,q'	$C(I,V)$ with q,q'	$V(I,D_4)$ with q''	$C(I,V)$ with q''	$V(I,D_5)$	$C(I,V)$	$V(I,D_6)$	$C(I,V)$	$V(I,D_7)$	$C(I,V)$
(0,0,0,0,0)	-	-	-	-	(0,0,0,0,0)	-	-	-	-	-
(1,0,0,0,0)	(0,0,0,0,0)	-	(0,0,0,0,0)	-	(1,1,1,1,1)	1	(1,1,0,0,0)	-	(1,1,0,0,0)	-
(0,1,0,0,0)	-	-	-	-	(0,0,0,0,0)	-	(0,0,0,0,0)	-	(0,0,0,0,0)	-
(0,0,1,0,0)	-	-	-	-	(0,0,0,0,0)	-	(0,0,1,1,0)	-	(0,0,1,0,0)	-
(0,0,0,1,0)	-	-	-	-	(0,0,0,0,0)	-	(0,0,0,0,0)	-	(0,0,0,0,0)	-
(0,0,0,0,1)	-	-	-	-	(0,0,0,0,0)	-	-	-	-	-
(1,1,0,0,0)	(0,1,0,0,0)	-	-	-	(1,1,1,1,1)	1	-	-	-	-
(1,0,1,0,0)	(0,0,1,0,0)	-	-	-	(1,1,1,1,1)	1	(1,1,1,1,0)	1	(1,1,1,0,0)	1
(1,0,0,1,0)	(0,0,0,1,0)	-	-	-	(1,1,1,1,1)	1	(1,1,0,0,0)	-	(1,1,0,0,0)	-
(1,0,0,0,1)	(0,0,0,0,1)	-	-	-	(1,1,1,1,1)	1	(1,1,0,0,1)	1	(1,1,0,0,1)	1
(0,1,1,0,0)	-	-	-	-	(0,0,0,0,0)	-	(0,0,1,1,0)	-	(0,0,1,0,0)	-
(0,1,0,1,0)	-	-	-	-	(0,0,0,0,0)	-	(0,0,0,0,0)	-	(0,0,0,0,0)	-
(0,1,0,0,1)	-	-	-	-	(0,0,0,0,0)	-	(0,0,0,0,1)	-	(0,0,0,0,1)	-
(0,0,1,1,0)	-	-	-	-	(0,0,0,0,0)	-	-	-	-	-
(0,0,1,0,1)	-	-	-	-	(0,0,0,0,0)	-	(0,0,1,1,1)	1	(0,0,1,1,1)	1
(0,0,0,1,1)	-	-	-	-	(0,0,0,0,0)	-	(0,0,0,0,1)	-	-	-
(1,1,1,0,0)	-	-	-	-	(1,1,1,1,1)	-	(1,1,1,1,0)	-	-	-
(1,1,0,1,0)	-	-	-	-	(1,1,1,1,1)	-	(1,1,0,0,0)	0	(1,1,0,0,0)	0
(1,1,0,0,1)	-	-	-	-	(1,1,1,1,1)	-	-	-	-	-
(1,0,1,1,0)	-	-	-	-	(1,1,1,1,1)	-	(1,1,1,1,0)	-	(1,1,1,1,0)	-
(1,0,1,0,1)	-	-	-	-	(1,1,1,1,1)	-	(1,1,1,1,1)	-	(1,1,1,1,1)	-
(1,0,0,1,1)	-	-	-	-	(1,1,1,1,1)	-	(1,1,0,0,1)	-	(1,1,0,1,1)	-
(0,1,1,1,0)	(1,1,1,1,0)	-	-	-	(0,0,0,0,0)	0	(0,0,1,1,0)	0	(0,0,1,1,0)	0
(0,1,1,0,1)	(1,1,1,0,1)	-	-	-	(0,0,0,0,0)	0	(0,0,1,1,1)	-	(0,0,1,1,1)	-
(0,1,0,1,1)	(1,1,0,1,1)	-	-	-	(0,0,0,0,0)	0	(0,0,0,0,1)	0	(0,0,0,1,1)	0
(0,0,1,1,1)	(1,0,1,1,1)	-	-	-	(0,0,0,0,0)	0	-	-	-	-
(1,1,1,1,0)	-	-	-	-	(1,1,1,1,1)	-	-	-	-	-
(1,1,1,0,1)	-	-	-	-	(1,1,1,1,1)	-	(1,1,1,1,1)	-	(1,1,1,1,1)	-
(1,1,0,1,1)	-	-	-	-	(1,1,1,1,1)	-	(1,1,0,0,1)	-	-	-
(1,0,1,1,1)	-	-	-	-	(1,1,1,1,1)	-	(1,1,1,1,1)	-	(1,1,1,1,1)	-
(0,1,1,1,1)	(1,1,1,1,1)	-	(1,1,1,1,1)	-	(0,0,0,0,0)	0	(0,0,1,1,1)	-	(0,0,1,1,1)	-
(1,1,1,1,1)	-	-	-	-	(1,1,1,1,1)	-	-	-	-	-

Table 2. Individual and collective choices for digraphs D_4 - D_7

$f(D) =$ $D =$	$SAT(D)$	$POW(D)$
$D_\emptyset$	(22,22,22,22,22)	(12,12,12,12,12)
D_1	(16,28,20,20,20)	(0,24,8,8,8)
D_2	(20,24,24,20,20)	(8,16,16,8,8)
D_3 with q	(16,24,24,24,16)	(0,16,16,16,0)
D_3 with q',q''	(22,22,22,22,22)	(12,12,12,12,12)
D_4 with q,q'	(22,22,22,22,22)	(12,12,12,12,12)
D_4 with q''	(22,22,22,22,22)	(12,12,12,12,12)
D_5	(32,16,16,16,16)	(32,0,0,0,0)
D_6	(24,16,24,16,24)	(16,0,16,0,16)
D_7	(24,16,24,16,24)	(16,0,16,0,16)

Table 3. Satisfaction and power in digraphs D_0 - D_7