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Abstract: In the framework of Galichon, Henry-Labordère and Touzi [9], we consider the model-free no-arbitrage bound of variance option given the marginal distributions of the underlying asset. We first make some approximations which restrict the computation on a bounded domain. Then we propose a gradient projection algorithm together with a finite difference scheme to approximate the bound. The general convergence result is obtained. We also provide a numerical example on the *variance swap* option.

Key-words: Variance option, model-free price bound, gradient projection algorithm.

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Un borne de valeur sans arbitrage, indépendante d'un modèle, d'options sur variance

Résumé : Dans le cadre de Galichon, Henry-Labordère et Touzi [9], nous considérons la borne sans arbitrage, indépendante d'un modèle, étant donné la distribution marginale du sous-jacent. Nous restreignons d'abord le calcul à un domaine borné. Puis nous proposons un algorithme de gradient avec projection, combiné à un schéma de différences finies, pour approcher la borne. Nous obtenons un résultat général de convergence, puis traitons un exemple numérique d'option sur swap.

Mots-clés : Option sur variance, borne de prix indépendante d'un modèle, algorithme de gradient avec projection.

1 Introduction

In a recent work of Galichon, Henry-Labordère and Touzi [9], the authors proposed a framework to compute the optimal model-free no-arbitrage price bound of exotic options in a vanilla-liquid market. Let $\Omega^d := C([0,T],\mathbb{R}^d)$ be the canonical space with canonical process X and canonical filtration $\mathbb{F}^d = (\mathcal{F}^d_t)_{0\le t\le T}$, S_0 be a constant. We denote by $\mathcal{P}(\delta_{S_0})$ the collection of all probability measures $\mathbb{P}$ on $(\Omega^d,\mathcal{F}^d_T)$ under which X is a $\mathbb{F}^d-$martingale and $X_0 = S_0$ $\mathbb{P}-$a.s. As indicated in [9], there is a progressively measurable process $\langle X\rangle_t$ which is pathwise defined and coincides with the $\mathbb{P}$-quadratic variation of X, $\mathbb{P}$-a.s. for every $\mathbb{P}\in\mathcal{P}(\delta_{S_0})$.
The process X is a candidate of underlying stock price, we do not impose any dynamic assumptions on X, but only suppose that it is a martingale. Then for an option with payoff $G\in\mathcal{F}^d_T$, the upper bound of model-free no-arbitrage price is given by

$$\sup_{\mathbb{P}\in\mathcal{P}(\delta_{S_0})} \mathbb{E}^{\mathbb{P}}\big[G\big].$$

Suppose in addition that we are in a market where the vanilla options with maturity T are liquid, so that the investor can identify the marginal distribution μ of X_T. In other words, let $\phi\in\mathbb{L}^1(\mathbb{R}^d,\mu)$, the $T-$maturity European option with payoff $\phi(X_T)$ has a unique no-arbitrage price

$$\mu(\phi) \;\;:=\;\; \int_{\mathbb{R}^d}\phi(x)\mu(dx).$$

Let us use the vanilla option portfolio to hedge G. By buying a portfolio $\phi(X_T)$, we spend $\mu(\phi)$ and so the payoff at maturity T becomes $G-\phi(X_T)$. Therefore, we get a new upper bound of model-free price: $\sup_{\mathbb{P}\in\mathcal{P}(\delta_{S_0})}\mathbb{E}^{\mathbb{P}}\big[G-\phi(X_T)\big]+\mu(\phi)$. By minimizing on the vanilla option portfolio ϕ, the optimal upper bound is then given by

$$\inf_{\phi\in\mathbb{L}^1(\mu)}\sup_{\mathbb{P}\in\mathcal{P}(\delta_{S_0})}\big\{\mathbb{E}^{\mathbb{P}}\big[G-\phi(X_T)\big]+\mu(\phi)\big\}. \tag{1.1}$$

As another motivation, we observe that the upper bound (1.1) is formally the conjugate dual formulation of problem

$$\sup_{\mathbb{P}\in\mathcal{P}(\delta_{S_0},\mu)}\mathbb{E}^{\mathbb{P}}\big[G\big] \;=\; \sup_{\mathbb{P}\in\mathcal{P}(\delta_{S_0})}\inf_{\phi\in\mathbb{L}^1(\mu)}\big\{\mathbb{E}^{\mathbb{P}}\big[G-\phi(X_T)\big]+\mu(\phi)\big\}, \tag{1.2}$$

where $\mathcal{P}(\delta_{S_0},\mu)$ denotes the collection of all martingale probability measures $\mathbb{P}\in\mathcal{P}(\delta_{S_0})$ such that $X_T\sim^{\mathbb{P}}\mu$. We remark that the above equality holds since

$$\inf_{\phi\in\mathbb{L}^1(\mu)}\big\{\mathbb{E}^{\mathbb{P}}\big[G-\phi(X_T)\big]+\mu(\phi)\big\} \;=\; \begin{cases}\mathbb{E}^{\mathbb{P}}[G] & \text{if } X_T\sim^{\mathbb{P}}\mu,\\ -\infty & \text{otherwise.}\end{cases}$$

In this paper, we shall consider in particular the no-arbitrage price bound of variance option in a similar framework. Let us restrict to the one-dimensional case $d=1$ and $T_1>T_0\ge 0$ be two constants. We define the corresponding canonical space as $\Omega := C([0,T_1],\mathbb{R})$ and denote still by X the canonical process, $\mathbb{F}=(\mathcal{F}_t)_{0\le t\le T_1}$ the canonical filtration and by $\langle X\rangle$ the progressively measurable process which coincides with the quadratic variation of X under every martingale probability measure $\mathbb{P}$. Suppose that the vanilla options of maturities T_0, T_1 are liquid such that we can identify the marginal distribution μ_0 (resp. μ_1) for X_{T_0} (resp. X_{T_1}). We shall consider the variance option with payoff

$$G \;:=\; g(\langle X\rangle_{T_0,T_1}, X_{T_1}) \text{ at maturity } T_1 \text{ for some appropriate function } g,$$

where $\langle X\rangle_{T_0,T_1} := \langle X\rangle_{T_1} - \langle X\rangle_{T_0}$. Let $\mathcal{P}^2(\mu_0)$ denotes the set of all the probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F}_{T_1})$ such that $X_{T_0} \sim^{\mathbb{P}} \mu_0$ and $\mathbb{E}^{\mathbb{P}}\big[\langle X\rangle_{T_0,T_1}\big|\mathcal{F}_{T_0}\big] < \infty$, $\mathbb{P}-$a.s., we define the no-arbitrage price upper bound of variance option $G = g(\langle X\rangle_{T_0,T_1}, X_{T_1})$ by

$$\inf_{\phi\in\text{Quad}} \sup_{\mathbb{P}\in\mathcal{P}^2(\mu_0)} \left\{\mathbb{E}^{\mathbb{P}}\big[g(\langle X\rangle_{T_0,T_1}, X_{T_1}) - \phi(X_{T_1})\big] + \mu_1(\phi)\right\}, \tag{1.3}$$

where Quad denotes the set of functions satisfying a quadratic growth condition, i.e.

$$\text{Quad} \quad := \quad \Big\{ \phi : \mathbb{R} \to \mathbb{R} \text{ such that } \sup_{x\in\mathbb{R}} \frac{|\phi(x)|}{1+|x|^2} < \infty \Big\}. \tag{1.4}$$

Remark 1.1. *The main reason to choose* Quad *is from the observation of Dupire [7] that variance swap is equivalent to a European option option with payoff* X_T^2*, see also Remark 2.3 and Corollary 3.9.*

By the time-change martingale theorem (see e.g. Theorem 3.4.6 of Karatzas and Shreve [12]), we can establish a correspondence between the set of martingale probability measures on $(\Omega, \mathcal{F}_{T_1})$ and the set of stopping times on a Brownian motion. In fact, a local martingale Y can be represented as a time-changed Brownian motion, i.e. $Y_t = W_{\langle Y\rangle_t}$ with a Brownian motion W. On the other hand, given a stopping time τ on W, the process Y defined by $Y_t := W_{\tau\wedge\frac{t}{T-t}}$ is a local martingale between 0 and T. Therefore, (1.3) can be formulated as

$$\overline{U} \quad := \inf_{\phi\in\text{Quad}} \bar{u}(\phi) \quad \text{with} \quad \bar{u}(\varphi) \;:=\; \sup_{\tau\in\mathcal{T}} \mathbb{E}\big[g(\tau, W_\tau) - \phi(W_\tau)\big] \;+\; \mu_1(\phi), \tag{1.5}$$

where W is a Brownian motion such that $W_0 \sim \mu_0$ and

$$\mathcal{T} \quad := \quad \big\{\tau \text{ stopping times such that } \mathbb{E}\big[\tau\big|W_0\big] < \infty, \text{ a.s.}\big\}. \tag{1.6}$$

We can also derive a dual formulation for (1.5) following the same arguments as for deriving (1.2). Let $\mathcal{T}(\mu_1)$ denote the set of all stopping times $\tau \in \mathcal{T}$ such that $W_\tau \sim \mu_1$, then the dual formulation of (1.5) becomes

$$\sup_{\tau\in\mathcal{T}} \inf_{\phi\in\text{Quad}} \mathbb{E}\Big[g(\tau, W_\tau) - \phi(W_\tau)\Big] + \mu_1(\phi) \quad = \quad \sup_{\tau\in\mathcal{T}(\mu_1)} \mathbb{E}\big[g(\tau, W_\tau)\big]. \tag{1.7}$$

Given a Brownian motion W and a distribution μ_1, the problem of finding stopping time τ such that $W_\tau \sim \mu_1$, i.e. $\tau \in \mathcal{T}(\mu_1)$, is called the Skorokhod Embedding Problem (SEP). Then our formulation (1.5) is consistent with Hobson's [10] observation of the connection between the SEP and the problem of optimal no-arbitrage bounds of exotic options in a vanilla-liquid market.

The SEP and the optimality property of its solutions as well as their applications in finance are studied in several papers recently, we refer to Obłój [15] and Hobson [11] for a survey. In particular, for the optimization problem (1.7), if $g(x,t) = f(t)$ for some function f defined on $\mathbb{R}^+$, it is proved that the maximum is achieved by Root's embedding when f is concave and by Röst's embedding when f is convex (see Root [16] and Rost [17]). However, for general payoff function g, there is no systematic method to find the optimal value of such problems. That is also our main motivation to develop a numerical method to solve these problems.

Our main contribution is then to provide a numerical scheme to approximate the bounds for general variance options.

The rest of the paper is organized as follows: In Section 2, we give an equivalent formulation for the bound $\overline{U}$ in (1.5). Then in Section 3 we provide an asymptotic analysis of our approximation, which restrict the calculation of $\overline{U}$ to a bounded domain. In Section 4, we propose a numerical scheme which combines the gradient projection algorithm and the finite difference method, and we give a general convergence result. Finally, Section 5 provides a numerical example on *variance swap*.

Notations: Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we define

$$\mu(\phi) \ := \ \int_{\mathbb{R}} \phi(x)\mu(dx), \quad \text{for every } \phi \in \mathbb{L}^1(\mu).$$

2 An equivalent formulation of the bound

We will fix the payoff function $g : (t, x) \in \mathbb{R}^+ \times \mathbb{R} \mapsto g(t, x) \in \mathbb{R}$ of the variance option as well as the marginal distributions μ_0, μ_1, and then reformulate the price bound problem (1.5). To make the problem be well posed, let us first make some assumptions on the marginal distributions μ_0, μ_1 and the payoff function g.

Assumption 1. *The probability measures μ_0, μ_1 on $\mathbb{R}$ have finite second moment, i.e.*

$$\mu_0(\phi_0) \ + \ \mu_1(\phi_0) \ < \ \infty, \quad \textit{with} \quad \phi_0(x) \ := \ x^2.$$

Moreover, $\mu_0 \le \mu_1$ in the convex order, i.e.

$$\mu_0(\phi) \ \le \mu_1(\phi), \quad \textit{for every convex function } \phi \textit{ defined on } \mathbb{R}. \tag{2.1}$$

Remark 2.1. *It is shown in Strassen [18] that the convex order inequality* (2.1) *is a sufficient and necessary condition for the existence of a martingale with marginal distributions μ_0 and μ_1 at time T_0 and T_1 such that $T_0 < T_1$.*
In particular, since the identity function I (where $I(x) := x$) and its opposite $-I$ are both convex, it follows immediately from (2.1) *that μ_0 and μ_1 have the same first moment, i.e. $\mu_0(I) = \mu_1(I)$.*

Assumption 2. *The payoff function $g(t, x)$ is L_0-Lipschitz in (t, x) with constant $L_0 \in \mathbb{R}^+$.*

Example 2.2. *The most popular variance option is the "variance swap", whose payoff function is $g(t, x) = t$. There exist also "volatility swap" with payoff $g(t, x) = \sqrt{t}$, and calls (puts) on variance, or volatility, where the payoff function are $(t - K)^+$ $((K - t)^+)$, or $(\sqrt{t} - K)^+$ $((K - \sqrt{t})^+)$.*

In addition to Assumption 2, we give another assumption on the payoff function g.

Assumption 3. *The function $g(t, x)$ increases in t, and convex in x for every fixed $t \in \mathbb{R}^+$. Moreover, for every fixed $t \in \mathbb{R}^+$, $g(t, 0) = \min_{x\in\mathbb{R}} g(t, x)$ and $g(t, x)$ is affine in x on $[M_0, \infty)$ and $(-\infty, -M_0]$ with constant $M_0 \in \mathbb{R}^+$.*

Remark 2.3. *Assumption 3 may not be crucial given Assumptions 1 and 2. As we shall see later in Corollary 3.9, let $K \in \mathbb{R}$ and ψ be defined on $\mathbb{R}$, denote $g_{K,\psi}(t, x) := g(t, x)+Kt+\psi(x)$, we then have*

$$\overline{U}(g_{K,\psi}) \quad = \quad \overline{U}(g) + KC_0 + \mu_1(\psi),$$

where $\overline{U}(g)$ (resp. $\overline{U}(g_{K,\psi})$) denotes the upper bound of (1.5) *associated with the payoff function g (resp. $g_{K,\psi}$), and*

$$C_0 \ := \ \mu_1(\phi_0) - \mu_0(\phi_0), \quad \textit{with} \quad \phi_0(x) \ := \ x^2. \tag{2.2}$$

Therefore, for an arbitrary payoff function g, we can consider the payoff function $g(t, x) + Kt$ which is increasing in t. And this does not change the nature of the upper bound problem (1.5).

Now we shall give an equivalent formulation of the problem (1.5). Let $B = (B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion such that $B_0 = 0$, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be its natural filtration and $\mathcal{T}^\infty$ be a set of $\mathbb{F}-$stopping times defined by

$$\mathcal{T}^\infty \quad := \quad \big\{ \mathbb{F}-\text{stopping time } \tau \text{ such that } \mathbb{E}(\tau) < \infty \big\}. \tag{2.3}$$

Given a strategy function $\phi \in \text{Quad}$ which is given by (1.4), we denote

$$g^\phi(t,x) \quad := \quad g(t,x) \; - \; \phi(x), \tag{2.4}$$

and define functions $\lambda^\phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $\lambda_0^\phi : \mathbb{R} \to \mathbb{R}$ by

$$\lambda^\phi(t,x) \quad := \quad \sup_{\tau\in\mathcal{T}^\infty} \; \mathbb{E}\; \big[\; g^\phi(t+\tau, x+B_\tau) \; \big], \qquad \text{and} \;\; \lambda_0^\phi(\cdot) \; := \; \lambda^\phi(0,\cdot). \tag{2.5}$$

Then the new formulation of the model-free no-arbitrage price upper bound is given by

$$U \quad := \quad \inf_{\phi\in\text{Quad}} u(\phi), \qquad \text{with} \qquad u(\phi) \quad := \quad \mu_0(\lambda_0^\phi) \; + \; \mu_1(\phi). \tag{2.6}$$

We notice that $\mu_0(\lambda_0^\phi)$ is well defined under Assumptions 1 and 2, by the fact that $\lambda_0^\phi(x) \geq g^\phi(0,x) = g(0,x) - \phi(x) \geq -C(1+x^2)$ for some positive constant C and that $\lambda^\phi(t,x)$ is measurable from the following Lemma.

Lemma 2.4. *Let Assumptions 1 and 2 hold, then for every $\phi \in \text{Quad}$, the function $\lambda^\phi(t,x)$ is lower-semicontinuous and hence measurable.*

Proof. By Assumption 2, for a fixed $\phi \in \text{Quad}$, there is a constant $C \in \mathbb{R}^+$ such that

$$\big| \; g^\phi(t+\tau, x+B_\tau) \; \big| \quad \leq \quad C\big(\; 1+t+\tau+x^2+B_\tau^2 \; \big).$$

Thus for a fixed $\tau \in \mathcal{T}^\infty$, $(t,x) \mapsto \mathbb{E}\big[g^\phi(t+\tau, x+B_\tau)\big]$ is continuous by the dominated convergence theorem together with (3.14) proved below. It follows immediately by its definition in (2.5) that λ^ϕ is lower-semicontinuous since it is represented as the supremum of a family of continuous function. □

Theorem 2.5. *Let Assumptions 1, 2 and 3 hold. Then the problem* (1.5) *and* (2.6) *are equivalent, i.e.* $\overline{U} = U$.

The proof is a simple consequence of the dynamic programming, we shall report it in Appendix.

Remark 2.6. *Here we only give the upper bound formulation. By the symmetry of the set* Quad *defined in* (1.4)*, if we reverse the payoff function to $-g(t,x)$, then with the upper bound $U(-g)$ associated to payoff $-g$, the value $-U(-g)$ is the lower bound for the payoff g.*

When $g(t,x) = (t-K)^+$, i.e. the option is the variance call, Dupire [7], Carr and Lee [6] proposed a systematic scheme to find a non-optimal bound as well as the associated strategy ϕ in a similar context. In their implemented examples, they showed that their bounds are quite close to the optimal bounds from Root's embedding solution.
For general payoff functions $g(t,x)$, when there is no systematic method to solve the problem (2.6), we shall propose a numerical scheme to approximate the optimal ϕ as well as the optimal upper bound U. In fact, we can easily observe that $\phi \mapsto \lambda^\phi$ is convex since it is represented as the supremum of a family of linear mapping in (2.5). Thus $\phi \mapsto u(\phi)$ is a convex function and the problem of U in (2.6) turns out to be a minimization problem of a convex function, as expected for a dual formulation of (1.7). We propose to use the finite difference scheme to solve $u(\phi)$ with every given ϕ, and then approximate the minimization problem on ϕ by an iterative algorithm.

3 Analytic approximation

In order to make the numerical resolution of U in (2.6) possible, we shall first restrict the calculations to a bounded domain by some analytic approximations.

3.1 The analytic approximation in four steps

Let us present the analytic approximation in four steps. The first step is to introduce a subset of Quad defined by

$$\mathrm{Quad}_0 \quad := \quad \big\{ \ \phi \in \mathrm{Quad} \text{ non negative, convex, such that } \phi(0) = 0 \ \big\},$$

and then to prove that it is equivalent to optimize on Quad_0 for problem (2.6).

Proposition 3.1. *Let Assumptions 1, 2 and 3 hold true, then $|U| < \infty$, and*

$$U \quad = \quad \inf_{\phi \in \mathrm{Quad}_0} \ u(\phi). \tag{3.1}$$

Our second approximation is on the growth coefficient of ϕ in Quad_0. Let K be a positive constant, we denote

$$U^K \ := \inf_{\phi \in \mathrm{Quad}_0^K} u(\phi) \ \text{ with } \ \mathrm{Quad}_0^K \ := \ \big\{ \phi \in \mathrm{Quad}_0 \ : \ \phi(x) \le K(|x| \vee x^2) \ \big\}. \tag{3.2}$$

By the convexity of functions in Quad_0, we see that every $\phi \in \mathrm{Quad}_0$ is in fact locally Lipschitz continuous, and hence $\mathrm{Quad}_0 = \cup_{K>0} \mathrm{Quad}_0^K$. Then it follows immediately that

$$U^K \ \searrow \ U \quad \text{as} \quad K \ \longrightarrow \ \infty. \tag{3.3}$$

The third approximation is on the tail of functions in Quad_0^K. Given a constant $M \ge M_0$, where M_0 is given in Assumption 1, we denote

$$\mathrm{Quad}_0^{K,M} \quad := \quad \big\{ \ \phi \in \mathrm{Quad}_0^K \text{ such that } \phi(x) = Kx^2 \text{ for } |x| \ge 2M \ \big\}, \tag{3.4}$$

and

$$U^{K,M} \quad := \quad \inf_{\phi \in \mathrm{Quad}_0^{K,M}} \ u(\phi). \tag{3.5}$$

Proposition 3.2. *Let Assumptions 1, 2 and 3 hold, then*

$$0 \quad \le \quad U^{K,M} - U^K \ \le \ \mu_1(\phi_{K,M}), \tag{3.6}$$

where

$$\phi_{K,M}(x) \quad := \quad 4KM(|x| - M)\mathbf{1}_{M \le |x| \le 2M} \ + \ Kx^2 \mathbf{1}_{|x| > 2M}. \tag{3.7}$$

Clearly, $\phi_{K,M} \in \mathrm{Quad}_0^{K,M}$ and for every fixed $K > 0$, $\mu_1(\phi_{K,M}) \to 0$ as $M \to \infty$ when μ_1 satisfies Assumption 1.

For the fourth step of the analytic approximation, we first introduce

$$\lambda^{\phi,T}(t,x) \quad := \quad \sup_{\tau \in \mathcal{T}^\infty, \ \tau \le T-t} \ \mathbb{E}[g^\phi(t+\tau, x + B_\tau)], \qquad \lambda_0^{\phi,T}(\cdot) \ := \ \lambda^{\phi,T}(0,\cdot),$$

$$\lambda^{\phi,\tau_R}(t,x) \quad := \quad \sup_{\tau\in\mathcal{T}^\infty,\ \tau\le\tau^R_x} \mathbb{E}[g^\phi(t+\tau,x+B_\tau)], \tag{3.8}$$

and

$$\lambda^{\phi,T,R}(t,x) \ := \ \sup_{\tau\in\mathcal{T}^\infty,\ \tau\le\tau^R_x\wedge(T-t)} \mathbb{E}[g^\phi(t+\tau,x+B_\tau)], \tag{3.9}$$

where

$$\tau^R_x \quad := \quad \inf\{s \ : \ x+B_s \notin (-R,R)\}.$$

Lemma 3.3. *Let Assumptions 2 and 3 holds true, and L_0, M_0 are given in the Assumptions. Suppose that $K > L_0$, $M \ge M_0$ and $R \ge \big(1+\sqrt{\frac{K}{K-L_0}}\big)M$. Then for every $\phi \in \mathrm{Quad}^{K,M}_0$,*

$$\lambda^\phi(t,x) = \lambda^{\phi,\tau_R}(t,x), \quad \textit{and} \quad \lambda^{\phi,T}(t,x) = \lambda^{\phi,T,R}(t,x), \quad \forall (t,x)\in[0,T]\times\mathbb{R}.$$

Given $\phi \in \mathrm{Quad}^{K,M}_0$, we define

$$U^{K,M,T} \quad := \quad \inf_{\phi\in\mathrm{Quad}^{K,M}_0} u^T(\phi), \quad \text{with} \ \ u^T(\phi) \ := \ \mu_0(\lambda^{\phi,T}_0) \ + \ \mu_1(\phi). \tag{3.10}$$

Proposition 3.4. *Let Assumptions 1, 2 and 3 hold, M_0 and L_0 be constants given in Assumption 2, $K > L_0$, $M \ge M_0$, $R = \big(1+\sqrt{\frac{K}{K-L_0}}\big)M$ and $L = 2(K+2L_0)(R^2\vee 1)$, we denote*

$$\delta \ := \ -\log(q(R)) > 0, \quad \textit{where} \ \ q(R) \ := \ \frac{1}{\sqrt{2\pi}}\int_{-2R}^{2R} e^{-x^2/2}\,dx.$$

Then

$$0 \quad \le \quad U^{K,M} - \ U^{K,M,T} \quad \le \quad Le^{-\delta(T-1)}. \tag{3.11}$$

Finally, we just remark that $U^{K,M,T}$ in (3.10) is defined via $\lambda^{\phi,T}$ which is equivalent to $\lambda^{\phi,T,R}$ from Lemma 3.3. Then by Theorem 6.7 of Touzi [19], we can characterized $\lambda^{\phi,T,R}$ as the viscosity solution of a variational inequality.

Proposition 3.5. *The function $\lambda^{\phi,T,R}$ defined in* (3.9) *is the unique viscosity solution of variational inequality*

$$\min\Big(\ \lambda - g^\phi, \ \ -\frac{1}{2}\frac{\partial^2\lambda}{\partial x^2} - \frac{\partial\lambda}{\partial t}\ \Big)(t,x) \ = \ 0, \qquad \textit{on} \ \ [0,T)\times(-R,R), \tag{3.12}$$

with boundary condition

$$\lambda(t,x) \ = \ g^\phi(t,x), \qquad \textit{on} \ \ \big([0,T]\times\{\pm R\}\big) \ \cup \ \big(\{T\}\times[-R,R]\big).$$

3.2 A first analysis

Before proving the convergence results given in Propositions 3.1, 3.2 and 3.4, we first give two well-known properties of the stopping times on a Brownian motion and report their proofs for completeness. We then provide also a first analysis on $u(\phi)$ and U in (2.6).

Lemma 3.6. *Let $\psi : (t,x) \in \mathbb{R}^+ \times \mathbb{R} \mapsto \psi(t,x) \in \mathbb{R}$ be a function Lipschitz in t and satisfying $\sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}} \frac{|\psi(t,x)|}{1+x^2} < \infty$. Then for every $\tau \in \mathcal{T}^\infty$,*

$$\mathbb{E}\left[\ \psi(\tau, B_\tau)\ \right] \quad = \quad \lim_{t\to\infty} \mathbb{E}\left[\ \psi(\tau\wedge t, B_{\tau\wedge t})\ \right]. \tag{3.13}$$

In particular,

$$\mathbb{E}[B_\tau^2] \ = \ \lim_{t\to\infty} \mathbb{E}[B_{\tau\wedge t}^2] \ = \ \lim_{t\to\infty} \mathbb{E}[\tau\wedge t] \ = \ \mathbb{E}[\tau] \quad \text{and} \quad \mathbb{E}[B_\tau] \ = \ 0. \tag{3.14}$$

Proof. Given a stopping time $\tau \in \mathcal{T}^\infty$, let $Y_t := B_{\tau\wedge t}$. Then by assumptions on ψ, there is a constant $C > 0$ such that

$$\psi(B_{\tau\wedge t}, \tau\wedge t) \ \le \ C\big(1 + Y_t^2 + \tau\big) \ \le \ C\Big(1 + \sup_{s\ge 0} Y_s^2 + \tau\Big), \quad \forall t \ge 0.$$

We notice that $(Y_t)_{t\ge 0}$ is a continuous uniformly integrable martingale by its definition, and $\mathbb{E}\big[\sup_{s\ge 0} Y_s^2\big] \ \le \ 4\mathbb{E}[\tau] \ < \ \infty$ by Doob's inequality. And hence it follows by the dominated convergence theorem that (3.13) holds true. □

Given $T > 0$, we denote by $\mathcal{T}^T$ the collection of all $\mathbb{F}-$stopping times taking value in $[0,T]$, i.e.

$$\mathcal{T}^T \quad := \quad \big\{\tau\wedge T : \tau \in \mathcal{T}^\infty\big\}. \tag{3.15}$$

Lemma 3.7. *Let $\psi \in \mathrm{Quad}$ and denote by ψ^{conv} its convex envelope, then*

$$\inf_{\tau\in\mathcal{T}^T} \mathbb{E}\ \psi(B_\tau) \ \to \ \inf_{\tau\in\mathcal{T}^\infty} \mathbb{E}\ \psi(B_\tau) \ = \ \psi^{conv}(0), \quad \text{as} \quad T \to \infty.$$

Proof. Let $a \le 0 \le b$ be two constants and $\tau_{a,b} := \inf\big\{t : B_t \notin (a,b)\big\}$. We first notice that $\tau_{a,b} \in \mathcal{T}^\infty$ since $\mathbb{E}[\tau_{a,b}] = \lim_{t\to\infty} \mathbb{E}[\tau_{a,b}\wedge t] = \lim_{t\to\infty} \mathbb{E}[B^2_{\tau_{a,b}\wedge t}] \le (a^2 + b^2) < \infty$. Hence by (3.14), $\mathbb{E}[B_{\tau_{a,b}}] = 0$, which implies that $\mathbb{P}(B_{\tau_{a,b}} = a) = \frac{b}{b-a}$ and $\mathbb{P}(B_{\tau_{a,b}} = b) = \frac{-a}{b-a}$. Therefore,

$$\inf_{\tau\in\mathcal{T}^\infty} \mathbb{E}\psi(B_\tau) \ \le \ \inf_{a<0<b} \mathbb{E}\psi(B_{\tau_{a,b}}) \ = \ \inf_{a<0<b} \Big(\frac{b}{b-a}\psi(a) + \frac{-a}{b-a}\psi(b)\Big) \ = \ \psi^{conv}(0).$$

On the other side, for every $\tau \in \mathcal{T}^\infty$, by Jensen's inequality together with the fact that $\mathbb{E}[B_\tau] = 0$ from (3.14), it follows that $\psi^{conv}(x) \le \mathbb{E}[\psi^{conv}(x + B_\tau)] \le \mathbb{E}[\psi(x + B_\tau)]$, and therefore,

$$\inf_{\tau\in\mathcal{T}^\infty} \ \mathbb{E}\psi(B_\tau) \quad = \quad \psi^{conv}(0).$$

Finally, the convergence of $\inf_{\tau\in\mathcal{T}^T} \mathbb{E}\psi(B_\tau)$ to $\inf_{\tau\in\mathcal{T}^\infty} \mathbb{E}\psi(B_\tau)$ as $T \to \infty$ is a direct consequence of (3.13) in Lemma 3.6. □

With the above two lemmas, we can now give a first analysis on $u(\phi)$ as well as U defined in (2.6).

Corollary 3.8. *Let $\phi \in \mathrm{Quad}$ and $(a,b) \in \mathbb{R}^2$, then $u(\phi) = u(\phi_{a,b})$, where $\phi_{a,b}$ is given by $\phi_{a,b}(x) := \phi(x) + ax + b$.*

Proof. By the definition of λ_0^ϕ in (2.5) together with Lemma 3.6, it follows that $\lambda_0^{\phi_{a,b}}(x) = \lambda_0^\phi(x) + ax + b$. Moreover, as discussed in Remark 2.1, $\mu_0(I) = \mu_1(I)$ for the identity function I. Then we get $u(\phi) = u(\phi_{a,b})$ by their definitions in (2.6). □

The next result can be viewed as a consequence of Dupire's [7] observation that *variance swap* is equivalent to a European option with payoff function $g(x) = x^2$. We give it in our context.

Corollary 3.9. *Let Assumptions 1, 2 hold true, $\psi \in$ Quad, $K \in \mathbb{R}$ and $g(t,x)$ be the payoff function, we define another payoff function $g_{K,\psi}$ by $g_{K,\psi}(t,x) := g(t,x) + Kt + \psi(x)$. Denote by $U(g)$ (resp. $U(g_{K,\psi})$) the no-arbitrage price upper bound defined in* (2.6) *associated with the payoff function g (resp. $g_{K,\psi}$). Then*

$$U(g_{K,\psi}) \;\; = \;\; U(g) \; + \; KC_0 \; + \; \mu_1(\psi), \tag{3.16}$$

where C_0 is given by (2.2). *In particular, the upper bound of "variance swap" option is C_0, and the bound of a European option with payoff function $\psi(x)$ is given by $\mu_1(\psi)$.*

Proof. Given $\phi \in$ Quad, we denote $\phi_{K,\psi}(x) := \phi(x) + \psi(x) + Kx^2$ which also belongs to Quad, then by (3.14)

$$\mathbb{E}\big[g_{K,\psi}(t+\tau, x+B_\tau) - \phi_{K,\psi}(x+B_\tau)\big] \;\; = \;\; \mathbb{E}\big[g^\phi(t+\tau, x+B_\tau)\big] - Kx^2, \;\; \forall \tau \in \mathcal{T}^\infty.$$

It follows by the definition of U in (2.6) that $U(g_{K,\psi}) \geq U(g) + KC_0 + \mu_1(\psi)$. And moreover, by the arbitrariness of $K \in \mathbb{R}$, $\psi \in$ Quad and symmetric relationship between g and $g_{K,\psi}$, we proved (3.16).

For the last statement, it follows by (3.16) that we only need to prove that $U(g^0) = 0$ with $g^0 \equiv 0$. Indeed, with the payoff function $g^0 \equiv 0$, we get immediately from (2.5) and (2.6) as well as Lemma 3.7 that

$$u(\phi) \;\; = \;\; - \; \mu_0(\phi^{conv}) \; + \; \mu_1(\phi) \;\; \geq \;\; \mu_1(\phi^{conv}) - \mu_0(\phi^{conv}) \;\; \geq \;\; 0,$$

where the last inequality comes from Assumption 1. Finally, we conclude with $U(g^0) = 0$ by the fact that $u(g^0) = 0$. □

Remark 3.10. *Let us consider the formulation of $\overline{U}$ in* (1.5). *From the definition of $\mathcal{T}$ in* (1.6), *we see that every stopping time $\tau \in \mathcal{T}$ conditioned on W_0 belongs to $\mathcal{T}^\infty$ defined in* (2.3). *Then by the same arguments, we have under the same conditions as in Corollary 3.9 that*

$$\overline{U}(g_{K,\psi}) \;\; = \;\; \overline{U}(g) \; + \; KC_0 \; + \; \mu_1(\psi),$$

where $\overline{U}(g)$ (resp. $\overline{U}(g_{K,\psi})$) denotes the price bound associated with payoff function g (resp. $g_{K,\psi}$) given in (1.5).

3.3 Proofs of the convergence

Now we are ready to give the proof of the convergence results in Propositions 3.1, 3.2 and 3.4.

Proof of Proposition 3.1. First, with the positive constant L_0 given in Assumption 1, we have

$$g(0,x) \;\; \leq \;\; g(t,x) \;\; \leq \;\; g(0,x) \; + \; L_0 t.$$

Moreover, it is clear that U is monotone w.r.t. the payoff function g by its definition in (2.6). Then it follows by Corollary 3.9 that

$$\mu_1(g(0,\cdot)) \; \leq \; U \; \leq \; \mu_1(g(0,\cdot)) \; + \; L_0 C_0, \;\; \text{with} \;\; C_0 \;\; \text{defined in} \;\;\; (2.2).$$

Next, let us prove the equality (3.1) for U. Let $T \in \mathbb{R}^+$, $\tau_0 \in \mathcal{T}^T$ and $\phi \in$ Quad. By the dominated convergence theorem, it is easy to see that $x \mapsto \inf_{\tau \in \mathcal{T}^T} \mathbb{E}\phi(x + B_\tau)$ is continuous.

This, together with the weak dynamic programming in Theorem 4.1 of Bouchard and Touzi [4], implies the dynamic programming principle:

$$\inf_{\tau_0\le\tau\le T} \mathbb{E}\phi(x+B_\tau) \;=\; \mathbb{E}\Big[\inf_{\tau_0\le\tau\le T}\mathbb{E}\big[\phi(x+B_\tau)\big|\mathcal{F}_{\tau_0}\big]\Big].$$

Then for constants $\hat{T}>T$,

$$\lambda_0^\phi(x) \;=\; \sup_{\tau\in\mathcal{T}^\infty}\mathbb{E}\big[\, g^\phi(\tau,x+B_\tau)\,\big] \;\ge\; \sup_{\tau_0\le\tau\le\hat{T}}\mathbb{E}\big[\, g(\tau,x+B_\tau)-\phi(x+B_\tau)\,\big].$$

By the increase of g in t and its convexity in x from Assumption 3, we have

$$\mathbb{E}\big[g(\tau,x+B_\tau)\big|\mathcal{F}_{\tau_0}\big] \;\ge\; \mathbb{E}\big[g(\tau_0,x+B_\tau)\big|\mathcal{F}_{\tau_0}\big] \;\ge\; g(\tau_0,x+B_{\tau_0}),$$

and hence

$$\lambda_0^\phi(x) \;\ge\; \mathbb{E}\big[g(\tau_0,x+B_{\tau_0})\big] \;-\; \mathbb{E}\Big[\inf_{\tau_0\le\tau\le\hat{T}}\mathbb{E}\big[\phi(x+B_\tau)\big|\mathcal{F}_{\tau_0}\big]\Big].$$

Sending $\hat{T}$ to $+\infty$, by Lemma 3.7, it follows that

$$\lambda_0^\phi(x) \;\ge\; \mathbb{E}\,\big[\, g(\tau_0,x+B_{\tau_0}) \;-\; \phi^{conv}(x+B_{\tau_0})\,\big].$$

Thus, by arbitrariness of τ_0 in $\mathcal{T}^T$ as well as that of $T\in\mathbb{R}^+$, we get

$$\begin{aligned}\lambda_0^\phi(x) \;&\ge\; \lim_{T\to\infty}\sup_{\tau_0\in\mathcal{T}^T}\mathbb{E}\,\big[\, g(\tau_0,x+B_{\tau_0}) \;-\; \phi^{conv}(x+B_{\tau_0})\,\big],\\ &=\; \sup_{\tau_0\in\mathcal{T}^\infty}\mathbb{E}\,\big[\, g(\tau_0,x+B_{\tau_0}) \;-\; \phi^{conv}(x+B_{\tau_0})\,\big],\end{aligned}$$

where the last equality is a direct consequence of Lemma 3.6 since ϕ^{conv} is either of quadratic growth or equals to $-\infty$.
Finally, since $\phi\ge\phi^{conv}$, by the definition of u and U in (2.6), it is clear that the infimum in (2.6) can be taken on the collection of all convex functions in Quad. Moreover, by the property of $u(\phi)$ in Corollary 3.8, the infimum can be then taken on the collection of all positive convex functions ϕ in Quad such that $\phi(0)=0$, i.e. $U=\inf_{\phi\in\mathrm{Quad}_0}u(\phi)$. We then proved (3.1). □

Proof of Proposition 3.2. Let us first recall that every function $\phi\in\mathrm{Quad}_0^K$ is nonnegative, convex such that $\phi(0)=0$ and $\phi(x)\le K(|x|\vee x^2)$. Given $\phi\in\mathrm{Quad}_0^K$, we denote $\phi_M:=\phi\vee\phi_{K,M}$. Clearly, ϕ_M lies in $\mathrm{Quad}_0^{K,M}$ and $\lambda^{\phi_M}\le\lambda^\phi$ since $\phi_M\ge\phi$. It follows from the definition of $u(\phi)$ in (2.6) and positivity of ϕ that

$$u(\phi_M) \;-\; u(\phi) \;\le\; \mu_1(\phi_M) \;-\; \mu_1(\phi) \;\le\; \mu_1(\phi_{K,M}).$$

This, together with the arbitrariness of $\phi\in\mathrm{Quad}_0^K$ and the fact that $\phi_M\in\mathrm{Quad}_0^{K,M}$, concludes the proof for (3.6). □

In preparation of the proof for Lemma 3.3 and Proposition 3.4, we first give a property for functions in $\mathrm{Quad}_0^{K,M}$.

Lemma 3.11. *Let Assumptions 2 and 3 hold true, L_0, M_0 be the constants given in Assumption 2, $K>L_0$, $M\ge M_0$ and $R=\big(1+\sqrt{\frac{K}{K-L_0}}\,\big)M$. Given fixed $t\in\mathbb{R}^+$ and $\phi\in\mathrm{Quad}_0^{K,M}$, we denote*

$$\psi(x) \;:=\; -\,g^\phi(t,x) \;-\; L_0x^2 \;=\; \phi(x) \;-\; g(t,x) \;-\; L_0x^2.$$

Then $\psi^{conv}(x)=\psi(x)$ when $x\notin[-R,R]$.

Proof. By Assumption 2, we know that there are constants C_1, C_2 such that $x \mapsto g(t,x)$ is affine with derivative C_1 when $x \geq M$, and affine with derivative C_2 when $x \leq -M$. For fixed $t \in \mathbb{R}^+$, let χ be a continuous function defined on $\mathbb{R}$ by the following: χ is affine on intervals $[-2M,-M]$, $[-M,0]$, $[0,M]$, $[M,2M]$ and

$$\begin{cases} \chi(0) & := & -\,g(t,0), \\ \chi(\pm M) & := & -\,L_0M^2 \;-\; g(t,\pm M), \\ \chi(\pm 2M) & := & 4(K-L_0)M^2 \;-\; g(t,\pm 2M), \\ \chi(x) & := & (K-L_0)x^2 \;-\; g(t,2M) \;-\; C_1(x-2M), & x \geq 2M, \\ \chi(x) & := & (K-L_0)x^2 \;-\; g(t,-2M) - C_2(x+2M), & x \leq -2M. \end{cases}$$

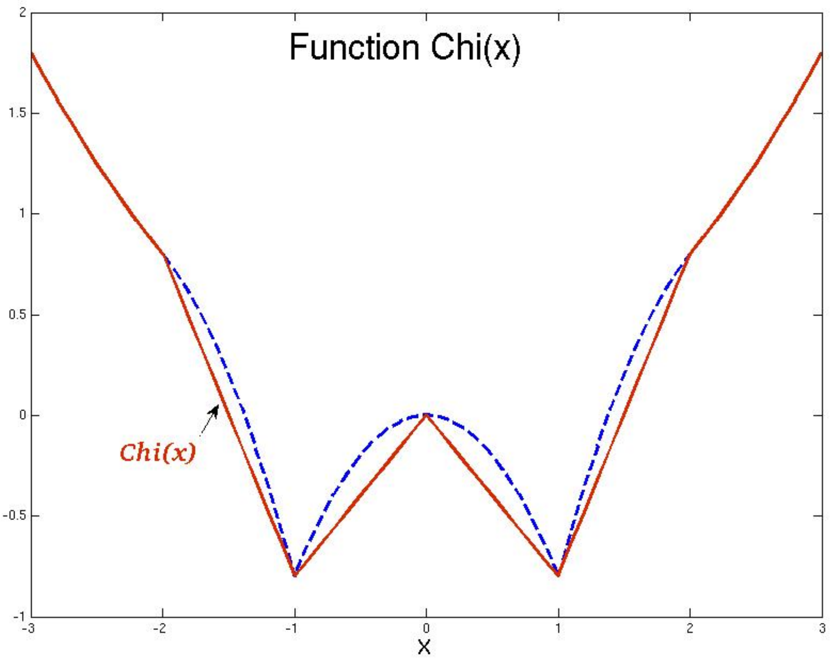

Figure 1: An example of function χ when $M = 1$.

By Assumptions 2 and 3, we can verify that for every $\phi \in \mathrm{Quad}_0^{K,M}$ and the corresponding ψ defined in the statement of the lemma,

$$\psi(x) \begin{cases} \geq \ \chi(x), & \text{when } x \in [-2M,2M], \\ = \ \chi(x), & \text{when } x \notin [-2M,2M]. \end{cases}$$

Then given $x \notin [-R,R]$, it follows by a simple calculation that $\chi(y) \geq \chi(x) + \chi'(x)(y-x)$ for every $y \in \mathbb{R}$, which implies that $\chi^{conv}(x) = \chi(x)$. And hence $\psi(x) \geq \psi^{conv}(x) \geq \chi^{conv}(x) = \chi(x) = \psi(x)$ for $x \notin [-R,R]$. □

Proof of Lemma 3.3. We shall just show that $\lambda^{\phi} = \lambda^{\phi,\tau_R}$, since $\lambda^{\phi,T} = \lambda^{\phi,T,R}$ holds with the same arguments. Moreover, to prove $\lambda^{\phi} = \lambda^{\phi,\tau_R}$, it is enough to show that $\lambda^{\phi} \leq \lambda^{\phi,\tau_R}$ since its inverse inequality is obvious from the definition of λ^{ϕ,τ_R} in (3.8).

First, let us fix $t \in \mathbb{R}^+$ and $x \notin (-R, R)$, we denote $\psi_x(y) := -g^\phi(t,y) - L_0 y^2 + L_0 x^2$. Then by Lemma 3.11, we have $\psi_x^{conv}(x) = \psi_x(x) = -g^\phi(t,x)$. And it follows that for every $\tau \in \mathcal{T}^\infty$,

$$\begin{aligned} \mathbb{E}\left[\ g^\phi(t+\tau, x+B_\tau)\ \right] &\leq \mathbb{E}\left[\ g^\phi(t, x+B_\tau)\ +\ L_0\tau\ \right] \\ &= \mathbb{E}\left[\ g^\phi(t, x+B_\tau)\ +\ L_0(x+B_\tau)^2\ -\ L_0 x^2\ \right] \\ &= -\,\mathbb{E}\ \psi_x(x+B_\tau)\ \leq\ -\psi_x^{conv}(x)\ =\ g^\phi(t,x), \end{aligned} \tag{3.17}$$

which implies that $\lambda^\phi(t,x) \leq \lambda^{\phi,\tau_R}(t,x)$ for every $x \notin (-R,R)$ since in this case $\tau_x^R = 0$. Next, for every $\tau \in \mathcal{T}^\infty$ and $x \in [-R,R]$, we have according to (3.17) that

$$\begin{aligned} &\mathbb{E}\left[\ g^\phi(t+\tau, x+B_\tau)\ \right] \\ =\ &\mathbb{E}\left[g^\phi(t+\tau, x+B_\tau)\mathbf{1}_{\tau \leq \tau_x^R} \right]\ +\ \mathbb{E}\left[\ \mathbb{E}\left[g^\phi(t+\tau, x+B_\tau)\mathbf{1}_{\tau > \tau_x^R}\ \middle|\ \mathcal{F}_{\tau\wedge\tau_x^R}\ \right] \right] \\ \leq\ &\mathbb{E}\left[\ g^\phi(t+\tau\wedge\tau_x^R, x+B_{\tau\wedge\tau_x^R})\ \right], \end{aligned}$$

which implies that $\lambda^\phi(t,x) \leq \lambda^{\phi,\tau_R}(t,x)$ for all $x \in [-R,R]$. □

Proof of Proposition 3.4. We first derive an estimate on stopping times inferior to τ_x^R, borrowed from Carlier and Galichon's [5] Lemma 5.2. Let $x \in [-R,R]$, then for every stopping time $\tau \leq \tau_x^R$, we have

$$\mathbb{P}(\tau \geq T)\ \leq\ \mathbb{P}\big(\tau_x^R \geq T\big)\ \leq\ \Pi_{1\leq n\leq T}\mathbb{P}\big(|B_n - B_{n-1}| \leq 2R\big)\ \leq\ e^{-\delta(T-1)}. \tag{3.18}$$

Recall that $\mathbb{E}\big[(x+B_\tau)^2\big] = x^2 + \mathbb{E}[\tau]$, $\forall \tau \leq \tau_x^R$ from (3.14). Then by the definitions of λ^{ϕ,τ_R} and $\lambda^{\phi,T,R}$ in (3.9), for every $\phi \in \mathrm{Quad}_0^{K,M}$,

$$\begin{aligned} \lambda^{\phi,\tau_R}(0,x) - \lambda^{\phi,T,R}(0,x) &\leq \sup_{\tau\leq\tau_x^R} \mathbb{E}\left[\ g^\phi(\tau, x+B_\tau)\ -\ g^\phi(\tau\wedge T, x+B_{\tau\wedge T})\ \right] \\ &= \sup_{\tau\leq\tau_x^R} \mathbb{E}\left[\ \psi(\tau\wedge T, x+B_{\tau\wedge T}) - \psi(x+B_\tau, \tau)\ \right], \end{aligned}$$

where $\psi(t,x) := -g^\phi(t,x) - L_0x^2 + L_0 t$. Clearly, ψ increases in t and $|\psi(t,x_1) - \psi(t,x_2)| \leq 2(K+2L_0)(R^2 \vee 1)$, $\forall x_1,\ x_2 \in [-R,R]$ by Assumptions 2 and 3, therefore,

$$\begin{aligned} \lambda^{\phi,\tau_R}(0,x) - \lambda^{\phi,T,R}(0,x) &\leq \sup_{\tau\leq\tau_x^R} \mathbb{E}\left[\ \big|\psi(\tau\wedge T, x+B_{\tau\wedge T})\ -\ \psi(\tau\wedge T, x+B_\tau)\big|\ \right] \\ &= \sup_{\tau\leq\tau_x^R} \mathbb{E}\left[\ \big|\psi(T, x+B_T)\ -\ \psi(T, x+B_\tau)\big|\mathbf{1}_{\tau\geq T}\ \right] \\ &\leq \sup_{\tau\leq\tau_x^R}\ 2\ (K+2L_0)\ (R^2\vee 1)\ \mathbb{P}(\tau \geq T) \\ &\leq L e^{-\delta(T-1)}, \end{aligned}$$

where the last inequality is from (3.18). Finally, by arbitrariness of $\phi \in \mathrm{Quad}_0^{K,M}$ together with Lemma 3.3, we prove (3.11). □

4 The numerical approximation

We shall propose a numerical method to approximate $U^{K,M,T}$. The idea is to compute $\lambda^{\phi,T,R}$ with a finite differences numerical scheme, and then solve the minimization problem (3.10) with an iterative algorithm. Concretely, we shall first propose a discrete system characterized by $h = (\Delta t, \Delta x)$, on which there is a discrete optimization problem with value $U_h^{K,M,T}$ close to $U^{K,M,T}$. Then we use the gradient projection algorithm to solve the discrete optimization problem of $U_h^{K,M,T}$.

4.1 A finite difference approximation

Let T, $R > 2M$ be constants in $\mathbb{R}^+$ and $(l, r, m) \in \mathbb{N}^3$, $h = (\Delta x, \Delta t) \in (\mathbb{R}^+)^2$ such that $l\Delta t = T$, $r\Delta x = R$ and $m\Delta x = M$. Denote $x_i = i\Delta x$ and $t_k = k\Delta t$ and define the discrete grid:

$$\mathcal{N} \;:=\; \{x_i \;:\; i \in \mathbb{Z}\}, \quad \mathcal{N}_R \;:=\; \mathcal{N} \cap [-R, R],$$

$$\mathcal{M}_{T,R} \;:=\; \big\{(t_k, x_i) \;:\; (k,i) \in \mathbb{Z}^+ \times \mathbb{Z}\big\} \cap \big([0,T] \times [-R,R]\big),$$

The terminal set, boundary set as well as interior set of $\mathcal{M}_{T,R}$ are denoted by

$$\partial_T \mathcal{M}_{T,R} \;:=\; \big\{ \;(T, x_i) \;: -r \le i \le r \;\big\}, \quad \partial_R \mathcal{M}_{T,R} \;:=\; \big\{ \;(t_k, \pm R) \;:\; 0 \le k \le l \;\big\},$$

$$\mathring{\mathcal{M}}_{T,R} \;:=\; \mathcal{M}_{T,R} \setminus \big(\partial_R \mathcal{M}_{T,R} \cup \partial_T \mathcal{M}_{T,R}\big).$$

Given a function $w(t,x)$ defined on $\mathcal{M}_{T,R}$, we introduce the discrete derivative of w:

$$D^2 w(t_k, x_i) \;:=\; \frac{w(t_k, x_{i+1}) - 2w(t_k, x_i) + w(t_k, x_{i-1})}{\Delta x^2}.$$

Then with function φ defined on $\mathcal{N}_R$ and the notation

$$g^{\varphi}(t_k, x_i) \;:=\; g(t_k, x_i) \;-\; \varphi(x_i) \tag{4.1}$$

as well as $\theta \in [0,1]$, we define $\lambda_h^{\varphi,T,R}$ as the solution of the finite difference scheme of variational inequality (3.12) on $\mathcal{M}_{T,R}$:

$$\begin{cases} \lambda_h^{T,R}(t_{k+1}, x_i) \;-\; \tilde{\lambda}_h^{T,R}(t_k, x_i) \\ \qquad +\; \frac{1}{2}\Delta t \Big(\theta\; D^2 \tilde{\lambda}_h^{T,R}(t_k, x_i) \;+\; (1-\theta)\; D^2 \lambda_h^{T,R}(t_{k+1}, x_i)\Big) \;=\; 0, \\ \lambda_h^{T,R}(t_k, x_i) \;=\; \max\Big(\; g^{\varphi}(t_k, x_i)\;,\; \tilde{\lambda}_h^{T,R}(t_k, x_i)\;\Big), \qquad (t_k, x_i) \in \mathring{\mathcal{M}}_{T,R}, \\ \lambda_h^{T,R}(t_k, x_i) \;=\; g^{\varphi}(t_k, x_i), \qquad (t_k, x_i) \in \partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}. \end{cases} \tag{4.2}$$

We notice that the above $\theta-$scheme has clearly a unique solution. And it is a consistant scheme for (3.12) in sense of Barles and Souganidis [2]. To see this, it is enough to rewrite the second equation of (4.2) as

$$\min\Big(\; \lambda_h^{T,R} - g^{\varphi}, \;\; \frac{\lambda_h^{T,R} - \tilde{\lambda}_h^{T,R}}{\Delta t} \;\Big)(t_k, x_i) \;=\; 0$$

We shall assume in addition that the discretization parameters $h = (\Delta t, \Delta x)$ satisfy the CFL condition

$$(1-\theta)\,\frac{\Delta t}{\Delta x^2} \;\le\; 1. \tag{4.3}$$

Then the finite difference scheme (4.2) is monotone in sense of [2], and the numerical solution $\lambda_h^{\varphi,T,R}$ converges to $\lambda^{\phi,T,R}$ given $\varphi := \phi|_{\mathcal{N}}$ by the results of [2].

Remark 4.1. *The discrete system* (4.2) *is the θ-scheme for variational inequality* (3.12) *with Dirichlet boundary condition $g(x,t) - \varphi(x)$ on $\partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}$. It is well-known that when the finite difference scheme is explicit (i.e. $\theta = 0$) and the CFL condition $\frac{\Delta t}{\Delta x^2} \le 1$ holds, it can be interpreted as the dynamic programming principle for a system on a Markov chain Λ (see e.g. Kushner [14]). This interpretation holds also true for general θ-scheme, as we shall see later in the proof of Proposition 4.4.*

We next introduce a natural approximation of $u_T(\phi)$ in (3.10):

$$u_{h,T}(\varphi) \;\; := \;\; \mu_0\big(\mathrm{lin}^R[\lambda_{h,0}^{\varphi,T,R}]\big) \; + \; \mu_1\big(\mathrm{lin}^R[\varphi]\big), \tag{4.4}$$

where $\lambda_{h,0}^{\varphi,T,R}(\cdot) := \lambda_h^{\varphi,T,R}(0,\cdot)$, and for every function φ defined on $\mathcal{N}_R$, we denote by $\mathrm{lin}^R[\varphi]$ its linear interpolation extended by zero outside $[-R,R]$.

Assumption 4. *There are constants $(\rho_1,\rho_2,L_{K,M,T}) \in (\mathbb{R}^+)^3$ which are independent of $h = (\Delta t,\Delta x)$ such that*

$$\mu_0\Big(\Big|\lambda_0^{\phi,T,R}\mathbf{1}_{[-R,R]} - \mathrm{lin}^R[\lambda_{h,0}^{\varphi,T,R}]\Big| \Big) \;\; \le \;\; L_{K,M,T}\,\big(\Delta x^{\rho_1} + \Delta t^{\rho_2}\big), \tag{4.5}$$
$$\text{for every } \phi \in \mathrm{Quad}_0^{K,M} \text{ and } \varphi = \phi|_{\mathcal{N}_R}.$$

Remark 4.2. *When $\theta = 1$, (4.2) is the implicit scheme for (3.12), then Assumption 4 holds true with $\rho_1 = \frac{1}{2}$ and $\rho_2 = \frac{1}{4}$ in sprirt of the analysis of Krylov [13].*
When $\theta = 0$ and the CFL condition (4.3) is true, (4.2) is a monotone explicit scheme, then in spirit of Barles and Jakobsen [1], Assumption 4 holds with $\rho_1 = \frac{1}{10}$ and $\rho_2 = \frac{1}{5}$.

Let $\mathrm{Quad}_{0,h}^{K,M}$ be the collection of all functions on the grid $\mathcal{N}_R$ defined as restrictions of functions in $\mathrm{Quad}_0^{K,M}$:

$$\mathrm{Quad}_{0,h}^{K,M} \;\; := \;\; \big\{ \; \varphi := \phi|_{\mathcal{N}_R} \text{ for some } \phi \in \mathrm{Quad}_0^{K,M} \; \big\}. \tag{4.6}$$

We can then provide a discrete approximation for $U^{K,M,T}$ in (3.10):

$$U_h^{K,M,T} \;\; := \;\; \inf_{\varphi \in \mathrm{Quad}_{0,h}^{K,M}} \; u_{h,T}(\varphi). \tag{4.7}$$

Let $B(\mathcal{N}_R)$ be the set of all bounded functions defined on the grid $\mathcal{N}_R$, then clearly

$$\mathrm{Quad}_{0,h}^{K,M} = \Big\{\varphi \in B(\mathcal{N}_R) \text{ nonnegative, convex satisfying } \varphi(0) = 0,\; \varphi(x_i) = Kx_i^2,$$
$$\text{for all } 2m \le |i| \le r, \text{ and } |\varphi(x_{i+1}) - \varphi(x_i)| \le 4KM\Delta x,\; \forall -2m < i \le 2m\Big\}. \tag{4.8}$$

Proposition 4.3. *Let Assumptions 2, 4 hold, then with the same constants $L_{K,M,T}$, ρ_1, ρ_2 introduced in Assumption 4,*

$$\Big|U^{K,M,T} - U_h^{K,M,T}\Big| \;\; \le \;\; L_{K,M,T}\big(\Delta x^{\rho_1} + \Delta t^{\rho_2}\big) + 4KR\Delta x + \; (\mu_0 + \mu_1)(\phi_K^R), \tag{4.9}$$

where $\phi_K^R(x) := Kx^2\mathbf{1}_{|x|>R}$.

Proof. First, given $\phi \in \mathrm{Quad}_0^{K,M}$ which is $4KR-$Lipschitz, we introduce $\varphi := \phi|_{\mathcal{N}_R} \in \mathrm{Quad}_{0,h}^{K,M}$ so that $\big|\mathrm{lin}^R[\varphi] - \phi\big|_{L^\infty([-R,R])} \le 4KR\Delta x$. Then it follows by Assumption 4 that $|u_T(\phi) - u_{h,T}(\varphi)| \; \le L_{K,M,T}(\Delta x^{\rho_1} + \Delta t^{\rho_2}) + 4KR\Delta x + \; (\mu_0 + \mu_1)(\phi_K^R)$, and hence

$$U^{K,M,T} - U_h^{K,M,T} \;\; \le \;\; L_{K,M,T}\,\big(\Delta x^{\rho_1} + \Delta t^{\rho_2}\big) \; + \; 4KR\Delta x + \; (\mu_0 + \mu_1)(\phi_K^R).$$

Next, given $\varphi \in \mathrm{Quad}_{0,h}^{K,M}$, we take $\phi \; := \; \mathrm{lin}^R[\varphi] \; + \; \phi_K^R \; \in \mathrm{Quad}_0^{K,M}$. It follows by Assumption 4 that $|u_T(\phi) - u_{h,T}(\varphi)| \; \le L_{K,M,T}(\Delta x^{\rho_1} + \Delta t^{\rho_2}) + \; (\mu_0 + \mu_1)(\phi_K^R)$, and therefore,

$$U_h^{K,M,T} - U^{K,M,T} \;\; \le \;\; L_{K,M,T}\,\big(\Delta x^{\rho_1} + \Delta t^{\rho_2}\big) + \; (\mu_0 + \mu_1)(\phi_K^R).$$

□

4.2 Gradient projection algorithm

As we can easily observe from its definition in (2.6) that $\phi \mapsto u(\phi)$ is convex since it is represented as the supremum of a family of linear map, we shall show that $\varphi \mapsto u_{h,T}(\varphi)$ is also convex, then a natural candidate for the resolution of $U_h^{K,M,T} = \inf_{\varphi \in \mathrm{Quad}_{0,h}^{K,M}} u_{h,T}(\varphi)$ in (4.7) is the gradient projection algorithm. Recall that $B(\mathcal{N}_R)$ denotes the collection of all bounded function on $\mathcal{N}_R$.

Proposition 4.4. *Under the CFL condition* (4.3)*, the function $\varphi \mapsto u_{h,T}(\varphi)$ is convex.*

Proof. Let us first rewrite the finite differences scheme (4.2) into a vector system. Denote $\alpha := \frac{\Delta t}{2\Delta x^2}$, $\lambda_k := \big(\lambda_h^{\varphi,T,R}(t_k,x_i)\big)_{-r\le i\le r}$, $\tilde{\lambda}_k := \big(\tilde{\lambda}_h^{\varphi,T,R}(t_k,x_i)\big)_{-r\le i\le r}$ and $q_k := \big(g^\varphi(t_k,x_i)\big)_{-r\le i\le r} \in \mathbb{R}^{2r+1}$. Let I_{2r+1} denote the $(2r+1)\times(2r+1)$ identity matrix, Π and $b_k \in \mathbb{R}^{2r+1}$ be defined by

$$\Pi := \begin{pmatrix} 0 & 0 & 0 & 0 & & & 0 \\ 1 & -2 & 1 & 0 & & & \\ 0 & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & 0 \\ & & & 0 & 1 & -2 & 1 \\ 0 & & & 0 & 0 & 0 & 0 \end{pmatrix}, \quad b_k := \begin{pmatrix} q_k(-r) - \lambda_{k+1}(-r) \\ 0 \\ \vdots \\ 0 \\ q_k(r) - \lambda_{k+1}(r) \end{pmatrix},$$

and $\Theta := \big[I_{2r+1} - \theta\alpha\Pi\big]^{-1}\big[I_{2r+1} + (1-\theta)\alpha\Pi\big]$, then scheme (4.2) can be rewritten as

$$\tilde{\lambda}_k \;=\; \Theta\lambda_{k+1} \;+\; b_k, \quad \text{and} \quad \lambda_k \;=\; \tilde{\lambda}_k \vee q_k. \tag{4.10}$$

Under CFL condition (4.3), we can verify that the above scheme is monotone, i.e. every element of Θ is positive, and moreover, $\Theta\mathbf{1} = \mathbf{1}$, where $\mathbf{1} := (1,\cdots,1)^T \in \mathbb{R}^{2r+1}$. It follows that Θ can be the probability transition matrix of some Markov chain Λ, whose state space is the grid $\mathcal{N}_R$ with absorbing boundary. Let $\mathcal{T}_h^R$ denote the collection of all stopping times τ on Λ such that $\Lambda_t \in \mathcal{N}_R$ for $t \le \tau$, then $\lambda_h^{\varphi,T,R}$ can be represented as solutions of an optimal stopping problem on Λ:

$$\lambda_h^{\varphi,T,R}(t_k,x_i) \quad = \quad \sup_{\tau\in\mathcal{T}_h^R,\ \tau\ge t_k} \mathbb{E}\,\Big[\, g^\varphi(\Lambda_\tau,\tau)\;\Big|\;\Lambda_{t_k} = x_i\,\Big].$$

Now given a family of stopping times $\tau_h = (\tau_h^i)_{-r\le i\le r}$ in $\mathcal{T}_h^R$, we introduce the function $\lambda_{h,0}^{\varphi,T,R,\tau_h}$ defined on $\mathcal{N}_R$:

$$\lambda_{h,0}^{\varphi,T,R,\tau_h}(x_i) \quad := \quad \mathbb{E}\,\Big[\, g^\varphi(\Lambda_\tau,\tau)\;\Big|\;\Lambda_0 = x_i\,\Big].$$

Then $u_{h,T}$ has an equivalent representation:

$$u_{h,T}(\varphi) = \sup_{\tau_h\in(\mathcal{T}_h^R)^{2r+1}} \bar{u}_{h,T}^{\tau_h}(\varphi) \;:=\; \sup_{\tau_h\in(\mathcal{T}_h^R)^{2r+1}} \mu_0\big(\mathrm{lin}^R[\lambda_{h,0}^{\varphi,T,R,\tau_h}]\big) \;+\; \mu_1\big(\mathrm{lin}^R[\varphi]\big). \tag{4.11}$$

Clearly, for every τ_h, $\varphi \mapsto \bar{u}_{h,T}^{\tau_h}(\varphi)$ is linear, and finally it follows by (4.11) that $\varphi \mapsto u_{h,T}(\varphi)$ is convex. □

Remark 4.5. *In the above Markov chain system* (4.11)*, given $\varphi \in B(\mathcal{N}_R)$, let us define an optimal stopping time $\tau_h(\varphi)$ by*

$$\tau_h(\varphi) \quad := \quad \inf\Big\{\; t_k \;:\; \lambda_h^{\varphi,T,R,\tau_h}(t_k,\Lambda_{t_k}) \;=\; g^\varphi(t_k,\Lambda_{t_k})\;\Big\}, \tag{4.12}$$

then clearly,

$$u_{h,T}(\varphi) \;=\; \sup_{\tau_h \in (\mathcal{T}_R^h)^{2r+1}} \bar{u}_{h,T}^{\tau_h}(\varphi) \;=\; \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi). \tag{4.13}$$

Now we are ready to give the gradient projection algorithm for $U_h^{K,M,T}$ in (4.7). Given $\varphi \in B(\mathcal{N}_R)$, we denote by $P_{\mathrm{Quad}_{0,h}^{K,M}}[\varphi]$ its projection on $\mathrm{Quad}_{0,h}^{K,M}$. Of course, such a projection depends on the norm equipped on $B(\mathcal{N}_R)$, which is an important issue to be discussed later. Let $\gamma = (\gamma_n)_{n\geq 0}$ be a sequence of positive real numbers, we propose the following algorithm:

Algorithm 1. *For optimization problem* (4.7)*:*

- *1, Let* $\varphi_0 := \phi_{K,M}|_{\mathcal{N}_R}$*, where* $\phi_{K,M}$ *is defined in* (3.7)*.*
- *2, Given* φ_n*, compute* $u_{h,T}(\varphi_n)$ *and a sub-gradient* $\nabla u_{h,T}(\varphi_n)$*.*
- *3, Let* $\varphi_{n+1} := P_{\mathrm{Quad}_{0,h}^{K,M}}\big[\varphi_n - \gamma_n \nabla u_{h,T}(\varphi_n)\big]$*.*
- *4, Go back to step 2.*

In the following, we shall discuss essentially three issues: the computation of sub-gradient $\nabla u_{h,T}(\varphi)$, the projection from $B(\mathcal{N}_R)$ to $\mathrm{Quad}_{0,h}^{K,M}$ and the convergence of the above gradient projection algorithm.

4.2.1 Computation of sub-gradient

Let us fix $\varphi \in B(\mathcal{N}_R)$, we then denote by $(p^j, \tilde{p}^j)$ the unique solution of the following linear system on $\mathcal{M}_{T,R}$:

$$\begin{cases} p^j(t_k,x_i) \;=\; -\,\delta_{i,j}, & (t_k,x_i) \in \partial_T\mathcal{M}_{T,R} \cup \partial_R\mathcal{M}_{T,R}, \\ p^j(t_{k+1},x_i) - \tilde{p}^j(t_k,x_i) + \frac{1}{2}\Delta t\,\big(\theta D^2\tilde{p}^j(t_k,x_i) + (1-\theta)D^2 p^j(t_{k+1},x_i)\big) = 0, & \\ p^j(t_k,x_i) = \begin{cases} \tilde{p}^j(t_k,x_i), & \text{if } \lambda_h^{\varphi,T,R}(t_k,x_i) > g^{\varphi}(t_k,x_i), \\ -\,e_j(x_i), & \text{otherwise.} \end{cases} & (t_k,x_i) \in \mathring{\mathcal{M}}_{T,R}. \end{cases} \tag{4.14}$$

where $e_j \in B(\mathcal{N}_R)$ is defined by $e_j(x_i) := \delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$ Let $p_0^j := p^j(0,\cdot)$.

Proposition 4.6. *Let CFL condition* (4.3) *hold true, then the vector*

$$\nabla u_{h,T}(\varphi) \;:=\; \Big(\, \mu_0(\mathrm{lin}^R[p_0^j]) \;+\; \mu_1(\mathrm{lin}^R[e_j])\,\Big)_{-2m\leq j\leq 2m} \tag{4.15}$$

is a sub-gradient of map $\varphi \mapsto u_{h,T}(\varphi)$*.*

Proof. Let us first consider the Markov chain Λ introduced in the proof of Proposition 4.4. By (4.13), we have for every perturbation $\Delta\varphi \in B(\mathcal{N}_R)$,

$$u_{h,T}(\varphi + \Delta\varphi) \;=\; \bar{u}_{h,T}^{\tau_h(\varphi+\Delta\varphi)}(\varphi+\Delta\varphi) \;\geq\; \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi+\Delta\varphi).$$

It follows still by (4.13) that

$$u_{h,T}(\varphi+\Delta\varphi) \;-\; u_{h,T}(\varphi) \;\geq\; \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi+\Delta\varphi) \;-\; \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi),$$

which implies that

$$\Big(\ \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi + e_j)\ -\ \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi)\ \Big)_{-r\le j\le r} \tag{4.16}$$

is a sub-gradient of $u_{h,T}$ at φ since $\psi \mapsto \bar{u}_{h,T}^{\tau(\varphi)}(\psi)$ is linear by its definition in (4.11).
Finally, by the definition of $\tau_h(\varphi)$ in (4.12) as well as (4.2) and (4.14), it follows that

$$p^j(t_k, x_i) \quad = \quad -\ \mathbb{E}\ \big[\ e_j\big(\Lambda_{\tau_h(\varphi)}\big)\ \big|\ \Lambda_{t_k} = x_i\ \big].$$

And hence the sub-gradient (4.16) coincides with $\nabla u_{h,T}(\varphi)$ defined in (4.15). □

4.2.2 Projection

To compute the projection $P_{\mathrm{Quad}_{0,h}^{K,M}}$ from $B(\mathcal{N}_R)$ to $\mathrm{Quad}_{0,h}^{K,M}$, we still need to specify the norm equipped on $B(\mathcal{N}_R)$. In order to make the projection algorithm simple, we shall introduce an invertible linear map from $B(\mathcal{N}_R)$ to $\mathbb{R}^{2r+1}$, then equip on $B(\mathcal{N}_R)$ the norm induced by the classical L^2-norm on $\mathbb{R}^{2r+1}$.
Let $\mathcal{L}_R : B(\mathcal{N}_R) \to \mathbb{R}^{2r+1}$ be defined by

$$\xi_i \quad = \quad \begin{cases} \varphi(x_i)\ -\ \varphi(x_{i-1}), & \text{for } \ 0 < i \le 2m, \\ \varphi(x_0), & \text{for } \ i = 0, \\ \varphi(x_i)\ -\ \varphi(x_{i-1}), & \text{for } \ -2m \le i < 0. \end{cases} \tag{4.17}$$

We define the norm $|\cdot|_R$ on $B(\mathcal{N}_R)$ (easily be verified) by

$$|\varphi|_R \quad := \quad \big|\mathcal{L}_R(\varphi)\big|_{L^2(\mathbb{R}^{2r+1})}, \quad \forall \varphi \in B(\mathcal{N}_R).$$

Denote

$$\begin{aligned} E_0^{K,M} \quad &:= \quad \big\{\ \mathcal{L}_R\varphi \ \ :\ \ \varphi \in \mathrm{Quad}_0^{K,M}\ \big\} \\ &= \quad \Big\{\ \xi \in \mathbb{R}^{2r+1}\ :\ 0 = \xi_0 \le \xi_{\pm 1} \le \cdots \le \xi_{\pm 2m} \le 4KM\Delta x, \\ &\qquad \xi_{\pm i} = K(x_{i+1}^2 - x_i^2),\ \forall 2m < i \le r \text{ and } \ \sum_{i=1}^{2m} \xi_i = \sum_{i=-1}^{-2m} \xi_i = 4KM^2\ \Big\}. \end{aligned}$$

Then the projection $P_{\mathrm{Quad}_{0,h}^{K,M}}$ from $B(\mathcal{N}_R)$ to $\mathrm{Quad}_{0,h}^{K,M}$ under norm $|\cdot|_R$ is equivalent to the projection from $\mathbb{R}^{2r+1}$ to $E_0^{K,M}$, which consists in solving a quadratic minimization problem :

$$\xi^z \quad = \quad \arg\min_{\xi \in E_0^{K,M}} \ \sum_{i=-r}^{r} (z_i - \xi_i)^2, \quad \text{for a given } z \in \mathbb{R}^{2r+1}. \tag{4.18}$$

Clearly, for every $z \in \mathbb{R}^{2r+1}$, $\xi_0^z = 0$ and the above optimization problem (4.18) can be decomposed into two optimization problems:

$$\min_{\xi \in E_{0,+}^{K,M}} \ \sum_{i=1}^{2m} (z_i - \xi_i)^2 \quad \text{ and } \quad \min_{\xi \in E_{0,-}^{K,M}} \ \sum_{i=-1}^{-2m} (z_i - \xi_i)^2, \tag{4.19}$$

where

$$E_{0,+}^{K,M} := \Big\{\xi = (\xi_i)_{1\le i\le 2m} \;:\; 0 \le \xi_1 \le \cdots \le \xi_{2m} \le 4KM\Delta x,\ \sum_{i=1}^{2m} \xi_i = 4KM^2\Big\},$$

$$E_{0,-}^{K,M} := \Big\{\xi = (\xi_i)_{-1\ge i\ge -2m} \;:\; 0 \le \xi_{-1} \le \cdots \le \xi_{-2m} \le 4KM\Delta x,\ \sum_{i=-1}^{-2m} \xi_i = 4KM^2\Big\}.$$

Here in place of optimization problem (4.19), we shall consider a similar but more general optimization problem and give an algorithm for it. Let $a = (a_i)_{1\le i\le m} \in \mathbb{N}^m$ and $A \in \mathbb{R}^+$ such that $0 < A < \sum_{i=1}^m a_i$, we define

$$\mathcal{K}_m^a := \Big\{ \xi = (\xi_i)_{1\le i\le m} \in \mathbb{R}^m \;:\; \xi_1 \le \cdots \le \xi_m \Big\},$$

$$\mathcal{K}_m^A := \Big\{ \xi = (\xi_i)_{1\le i\le m} \in [0,1]^m \;:\; \sum_{i=1}^m a_i\xi_i = A \Big\}, \text{ and } \mathcal{K}_m^{a,A} := \mathcal{K}_m^a \cap \mathcal{K}_m^A.$$

The projection $P_{\mathcal{K}_m^{a,A}}(z)$ of $z \in \mathbb{R}^m$ to $\mathcal{K}_m^{a,A}$ is to solve the optimization problem

$$\xi_m^{a,A,z} := \arg\min_{\xi\in\mathcal{K}_m^{a,A}} \sum_{i=1}^m a_i(z_i - \xi_i)^2. \tag{4.20}$$

Similarly, the projection $P_{\mathcal{K}_m^a}$(resp. $P_{\mathcal{K}_m^A}$) is defined by the optimization problem (4.20), where $\mathcal{K}_m^{a,A}$ in the formula is replaced by $\mathcal{K}_m^a$ (resp. $\mathcal{K}_m^A$), and the projected element $\xi_m^{a,A,z}$ is replaced by $\xi_m^{a,z}$ (resp. $\xi_m^{A,z}$).

In the following, we shall show that

$$P_{\mathcal{K}_m^{a,A}} = P_{\mathcal{K}_m^{a,A}} \circ P_{\mathcal{K}_m^a} = P_{\mathcal{K}_m^A} \circ P_{\mathcal{K}_m^a},$$

and give the algorithms for both $P_{\mathcal{K}_m^a}$ and $P_{\mathcal{K}_m^A}$. With these algorithms, we can deduce easily an algorithm for the projection $P_{E_{K,M}^+}$. We just remark that similar algorithm to compute the convex envelope of a function is discussed in Page 143-145 of Edelsbrunner [8].
Given $a \in \mathbb{N}^m$ and $z \in \mathbb{R}^m$, we define $S^{a,z} \in \mathbb{R}^{\sum_{i=1}^m a_i}$ by $S_k^{a,z} := z_j$ for $\sum_{i=1}^{j-1} < k \le \sum_{i=1}^{j}$, and a function $F^{a,z}$ defined on the grid $\mathbb{N}\cap[0, 1 + \sum_{i=1}^m a_i]$ by

$$F^{a,z}(0) := 0 \quad \text{and} \quad F^{a,z}(k) := \sum_{i=1}^k S_i^{a,z}. \tag{4.21}$$

Lemma 4.7. *Let $z \in \mathbb{R}^m$ such that $z_k \ge z_{k+1}$, then $(\xi_m^{a,z})_k = (\xi_m^{a,z})_{k+1}$ and $(\xi_m^{a,A,z})_k = (\xi_m^{a,A,z})_{k+1}$ for $\xi_m^{a,z} = P_{\mathcal{K}_m^a}(z)$ and $\xi_m^{a,A,z} = P_{\mathcal{K}_m^{a,A}}(z)$. And therefore, in this case, the projections $P_{\mathcal{K}_m^a}(z)$ and $P_{\mathcal{K}_m^{a,A}}(z)$ are equivalent to $P_{\mathcal{K}_{m-1}^{\tilde a}}(\tilde z)$ and $P_{\mathcal{K}_{m-1}^{\tilde a,A}}(\tilde z)$ with*

$$\tilde a_i = \begin{cases} a_i, & 1\le i\le k-1,\\ a_k + a_{k+1}, & i = k,\\ a_{i+1}, & k+1 \le i \le m-1,\end{cases} \quad \text{and} \quad \tilde z_i = \begin{cases} z_i, & 1\le i\le k-1,\\ \frac{a_k z_k + a_{k+1}z_{k+1}}{a_k+a_{k+1}}, & i = k,\\ z_{i+1}, & k+1 \le i \le m-1,\end{cases} \tag{4.22}$$

in sense that $S^{a,\xi_m^{a,z}} = S^{\tilde a,\xi_{m-1}^{\tilde a,\tilde z}}$ and $S^{a,\xi_m^{a,A,z}} = S^{\tilde a,\xi_{m-1}^{\tilde a,A,\tilde z}}$, where $\xi_{m-1}^{\tilde a,\tilde z} = P_{\mathcal{K}_{m-1}^{\tilde a}}(\tilde z)$ and $\xi_{m-1}^{\tilde a,A,\tilde z} = P_{\mathcal{K}_{m-1}^{\tilde a,A}}(\tilde z)$.

Proof. Given $\xi \in \mathbb{R}^m$ such that $\xi_{k+1} > \xi_k$, there is $\varepsilon > 0$ satisfying that $\xi_{k+1} = \xi_k + (1+\frac{a_k}{a_{k+1}})\varepsilon$. Let $\hat{\xi}$ be defined by $\hat{\xi}_i = \begin{cases} \hat{\xi}_k + \varepsilon, & i = k, k+1, \\ \xi_i, & \text{otherwise}, \end{cases}$ we will show that

$$\sum_{i=1}^{m} a_i(\hat{\xi}_i - z_i)^2 \quad < \quad \sum_{i=1}^{m} a_i(\xi_i - z_i)^2. \tag{4.23}$$

Thus such a ξ is not optimal since $\xi \in \mathcal{K}^a_m$(resp. $\mathcal{K}^{a,A}_m$) implies that $\hat{\xi} \in \mathcal{K}^a_m$ (resp. $\mathcal{K}^{a,A}_m$) also. And therefore, $(\xi^{a,z}_m)_k = (\xi^{a,z}_m)_{k+1}$ and $(\xi^{a,A,z}_m)_k = (\xi^{a,A,z}_m)_{k+1}$.
Indeed, (4.23) holds since with the above given ξ and $\hat{\xi}$,

$$\begin{aligned} & \sum_{i=1}^{m} a_i \ (\xi_i - z_i)^2 \ - \ \sum_{i=1}^{m} a_i \ (\hat{\xi}_i - z_i)^2 \\ = \ & a_k \ (\xi_k - z_k)^2 \ + \ a_{k+1} \left(\xi_k + (1+\frac{a_k}{a_{k+1}})\varepsilon - z_{k+1}\right)^2 \\ & - \ a_k \ (\xi_k + \varepsilon - z_k)^2 \ - \ a_{k+1} \ (\xi_k + \varepsilon - z_{k+1})^2 \\ = \ & \frac{a_k}{a_{k+1}} \ (a_k + a_{k+1}) \ \varepsilon^2 \ + \ 2 \ a_k \ \varepsilon \ (z_k - z_{k+1}) \quad > \quad 0. \end{aligned}$$

Finally, the equivalence between $P_{\mathcal{K}^a_m}(z)$ (resp. $P_{\mathcal{K}^{a,A}_m}(z)$) and $P_{\mathcal{K}^{\tilde{a}}_{m-1}}(\tilde{z})$ (resp. $P_{\mathcal{K}^{\tilde{a},A}_{m-1}}(\tilde{z})$) is from the fact that for every ξ such that $\xi_k = \xi_{k+1}$,

$$\sum_{i=1}^{m} a_i(z_i - \xi_i)^2 \ = \ \sum_{i=1}^{m-1} \tilde{a}_i(\tilde{z}_i - \tilde{\xi}_i)^2 \ + a_k z_k^2 + a_{k+1} z_{k+1}^2 \ - \ (a_k + a_{k+1}) \frac{(z_k + z_{k+1})^2}{4},$$

where $\tilde{\xi}_i \ = \begin{cases} \xi_i, & i \leq k-1, \\ \xi_k, & i = k, k+1, \\ \xi_{i-1}, & k+2 \leq i \leq m-1. \end{cases}$ □

Lemma 4.7 gives an algorithm for projection $P_{\mathcal{K}^a_m}$ which finishes with less than m steps. And it simplifies the projection $P_{\mathcal{K}^{a,A}_m}$.

Algorithm 2. *For projection $P_{\mathcal{K}^a_m}(z)$:*

- *1, Given system parameters (m, a, z), stop if $m = 1$.*
- *2, Find k such that $z_k \geq z_{k+1}$, stop if it does not exist.*
- *3, With the found k in step 2, reduce parameters (m, a, z) to $(m-1, \tilde{a}, \tilde{z})$ as in equation (4.22).*
- *4, Go to 1.*

Proposition 4.8. $P_{\mathcal{K}^{a,A}_m} = P_{\mathcal{K}^{a,A}_m} \circ P_{\mathcal{K}^a_m}$*, and for every $z \in \mathbb{R}^m$, $F^{a,\xi}$ (with $\xi := P_{\mathcal{K}^a_m}(z)$) is the convex envelope of $F^{a,z}$, where the functions $F^{a,\xi}$ and $F^{a,z}$ are define in* (4.21)

Proof. Suppose that the entrance data of Algorithm 2 is (m_1, a_1, z_1) and exit data is (m_2, a_2, z_2), then clearly $P_{\mathcal{K}^{a_2}_{m_2}}(z_2) = z_2$. And by Lemma 4.7, we have $S^{a_1,\xi_1} = S^{a_2,z_2}$ (with $\xi_1 := P_{\mathcal{K}^{a_1}_{m_1}}(z_1)$) and $S^{a_1,\xi_1^A} = S^{a_2,\xi_2^A}$ (with $\xi_1^A := P_{\mathcal{K}^{a_1,A}_{m_1}}(z_1)$ and $\xi_2^A := P_{\mathcal{K}^{a_2,A}_{m_2}}(z_2)$), from which we deduce that, $P_{\mathcal{K}^{a,A}_m} = P_{\mathcal{K}^{a,A}_m} \circ P_{\mathcal{K}^a_m}$.

To see that $F^{a,\xi}$ (with $\xi := P_{\mathcal{K}_m^a}(z)$) is the convex envelope of $F^{a,z}$, it is enough to verify that at every step in Algorithm 2, $F^{a,\tilde{z}}$ is greater than the convex envelope of $F^{a,z}$. And at the exit, $F^{a,\xi}$ is a convex function. □

Now, we shall prove that $P_{\mathcal{K}_m^{a,A}} \circ P_{\mathcal{K}_m^a} = P_{\mathcal{K}_m^A} \circ P_{\mathcal{K}_m^a}$, in this order, we just need to show that for every $z \in \mathcal{K}_m^a$, $P_{\mathcal{K}_m^{a,A}}(z) = P_{\mathcal{K}_m^A}(z)$. In fact, we shall give an algorithm of projection $P_{\mathcal{K}_m^A}(z)$ for $z \in \mathcal{K}_m^a$, and then verify that $P_{\mathcal{K}_m^A}(z) \in \mathcal{K}_m^{a,A}$.

Given $\nu \in \mathbb{R}$, let us denote by $z - \nu$ the sequence $(z_i - \nu)_{1\le i\le m}$, and by z^ν the sequence $(z_i^\nu)_{1\le i\le m}) = (0 \vee (z_i - \nu) \wedge 1)_{1\le i\le m}$.

Lemma 4.9. *Given $\nu \in \mathbb{R}$, $z \in \mathbb{R}^m$, then $P_{\mathcal{K}_m^{a,A}}(z) = P_{\mathcal{K}_m^{a,A}}(z-\nu)$ and $P_{\mathcal{K}_m^A}(z) = P_{\mathcal{K}_m^A}(z-\nu)$. In addition, if $z \in \mathcal{K}_m^a$, then there is $\hat\nu \in \mathbb{R}$ such that $\sum_{i=1}^m a_i z_i^{\hat\nu} = A$ and $P_{\mathcal{K}_m^A}(z) = P_{\mathcal{K}_m^{a,A}}(z) = z^{\hat\nu}$. And it follows that $P_{\mathcal{K}_m^{a,A}} = P_{\mathcal{K}_m^{a,A}} \circ P_{\mathcal{K}_m^a} = P_{\mathcal{K}_m^A} \circ P_{\mathcal{K}_m^a}$.*

Proof. To prove that $P_{\mathcal{K}_m^{a,A}}(z) = P_{\mathcal{K}_m^{a,A}}(z-\nu)$ or $P_{\mathcal{K}_m^A}(z) = P_{\mathcal{K}_m^A}(z-\nu)$, it is enough to see that for every $\xi \in \mathbb{R}^m$ such that $\sum_{i=1}^m a_i\xi_i = A$,

$$\sum_{i=1}^m a_i(z_i - \nu - \xi_i)^2 \quad = \quad \sum_{i=1}^m a_i(z_i - \xi_i)^2 \ + \ \nu^2 \sum_{i=1}^m a_i \ - \ 2\nu\Big(\sum_{i=1}^m a_i z_i - A\Big).$$

For the existence of $\hat\nu$, we remark that $\nu \mapsto \sum_{i=1}^m a_i z_i^\nu$ is continuous, and that $0 < A < \sum_{i=1}^m a_i$ is supposed at the beginning of the section. Clearly, by its definition, z^ν is the projected element of $z - \nu$ to $[0,1]^m$ in sense that $\xi_0 = z^\nu$ minimizes $\sum_{i=1}^m a_i(z_i - \nu - \xi_i)^2$ among all $\xi \in [0,1]^m$. Then for $z \in \mathcal{K}_m^a$, it is easy to verify that $z^{\hat\nu} \in \mathcal{K}_m^{a,A} \subset \mathcal{K}_m^A \subset [0,1]^m$ with the found $\hat\nu$. Therefore $P_{\mathcal{K}_m^A}(z) = P_{\mathcal{K}_m^{a,A}}(z) = P_{\mathcal{K}_m^A}(z-\hat\nu) = P_{\mathcal{K}_m^{a,A}}(z-\hat\nu) = z^{\hat\nu}$. □

Algorithm 3. *To find $\hat\nu$ such that $\sum_{i=1}^m a_i z_i^{\hat\nu} = A$:*

- *1, Set $z_0 = -\infty$ and $z_{m+1} = \infty$.*
- *2, Find the maximum k such that $\sum_{i=1}^m a_i z_i^{z_{k-1}} \ge A$ and $\sum_{i=1}^m a_i z_i^{z_k} \le A$, then $z_{k-1} \le \hat\nu \le z_k$.*
- *3, Find the minimum j such that $\sum_{i=1}^m a_i z_i^{z_{j+1}-1} \le A$ and $\sum_{i=1}^m a_i z_i^{u_j-1} \ge A$, then $z_j - 1 \le \hat\nu \le z_{j+1} - 1$.*
- *4, Set $\hat\nu = \frac{\sum_{i=j+1}^m a_i + \sum_{i=k}^j a_i z_i - A}{\sum_{i=k}^j a_i}$ when $k \le j$, and $\hat\nu = z_{k-1}$ when $k = j+1$.*

By the way how to find k and j, we can easily have $k \le j+1$, then step 4 of Algorithm 3 gives the right $\hat\nu$ since $z_i^{\hat\nu} = \begin{cases} 0, & \text{if } i \le k-1, \\ 1, & \text{if } i \ge j+1, \\ z_i - \hat\nu, & \text{otherwise.} \end{cases}$ for k, j found in step 2 and 3, and hence for $k \le j$,

$$\sum_{i=k}^j a_i(z_i - \hat\nu) \ + \ \sum_{i=j+1}^m a_i = A \quad \Longrightarrow \quad \hat\nu \ = \ \frac{\sum_{i=j+1}^m a_i + \sum_{i=k}^j a_i z_i - A}{\sum_{i=k}^j a_i}.$$

Finally, we propose the following algorithm for projection $P_{\mathrm{Quad}_{0,h}^{K,M}}$:

Algorithm 4. *For projection $P_{\mathrm{Quad}_{0,h}^{K,M}}$ in* (4.18)*:*

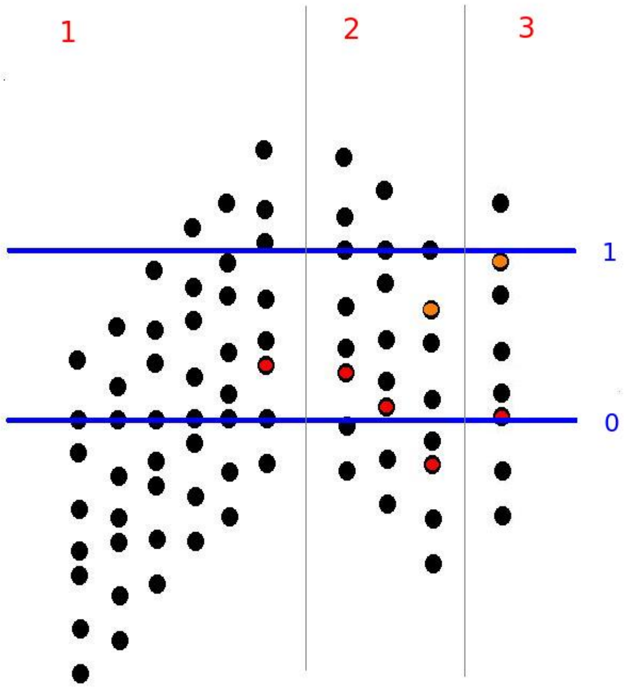

Figure 2: An illustration of Algorithm 3.

- *1, Compute the convex envelope $\hat{\varphi}$ of φ on $[0, 2M]$ and on $[-2M, 0]$.*
- *2, Set $z = \mathcal{L}_R(\hat{\varphi}|_{\mathcal{N}_R})$, use Algorithm 3 to compute $P_{E_0^{K,M}}(u)$.*
- *3, Let $P_{\mathrm{Quad}_{0,h}^{K,M}}(\varphi) = \mathcal{L}_R^{-1} P_{E_0^{K,M}}(z)$.*

4.2.3 Convergence rate

We shall give a convergence rate for the gradient projection algorithm. In preparation, let us first give a bound for the sub-gradients $\nabla u_{h,T}$.

Proposition 4.10. *Let $\varphi_1, \varphi_2 \in B(\mathcal{N}_R)$, then under the CFL condition* (4.3)*,*

$$\big|\ u_{h,T}(\varphi_1)\ -\ u_{h,T}(\varphi_2)\ \big|\ \leq\ 2\ |\varphi_1 - \varphi_2|_\infty, \tag{4.24}$$

and it follows that

$$\big|\nabla u_{h,T}(\varphi)\big|_R\ \leq\ 2\sqrt{2m+1}\ =\ 2\sqrt{\frac{2M}{\Delta x}+1}, \quad \forall \varphi \in B(\mathcal{N}_R). \tag{4.25}$$

Proof. Under the CFL condition (4.3), the $\theta-$scheme is monotone, which implies that $|\lambda_h^{\varphi,T,R,\varphi_1} - \lambda_h^{\varphi,T,R,\varphi_2}|_\infty \leq |\varphi_1 - \varphi_2|_\infty$, and hence by the definition of $u_{h,T}$ in (4.4), (4.24) holds true. Next, denote $\xi^i := \mathcal{L}_R(\varphi_i)$, $i = 1, 2$, then by Cauchy-Schwarz inequality,

$$|\varphi_1 - \varphi_2|_\infty \leq \max\Big(\sum_{i=0}^{2m} \big|\xi_i^1 - \xi_i^2\big|, \sum_{i=0}^{-2m} \big|\xi_i^1 - \xi_i^2\big|\Big) \leq \sqrt{2m+1}\cdot \big\|\xi^1 - \xi^2\big\|_{\mathbb{L}^2},$$

which implies immediately (4.25). □

Finally, let us finish this section by providing a convergence rate of the proposed gradient projection algorithm. Denote

$$\Phi \;:=\; \max_{\varphi_1,\varphi_2\in \text{Quad}_{0,h}^{K,M}} |\varphi_1-\varphi_2|_R^2 \;\le\; 4m\,(4KM\Delta x)^2 \;\le\; 64K^2M^3\Delta x,$$

then from Section 5.3.1 of Ben-Tal and Nemirovski[3], we have the convergence rate:

$$\begin{aligned}\min_{n\le N} u_{h,T}(\varphi_n) \;-\; U_h^{K,M,T} \;&\le\; \frac{\Phi \;+\; \sum_{i=n}^N \gamma_n^2 \big|\nabla u_{h,T}(\varphi_n)\big|_R^2}{2\sum_{n=1}^N \gamma_n} \\ &=\; \frac{32K^2M^3\Delta x \;+\; \big(4\frac{M}{\Delta x}+2\big)\;\sum_{i=n}^N \gamma_n^2}{\sum_{n=1}^N \gamma_n}. \end{aligned} \tag{4.26}$$

For the sequence $\gamma = (\gamma_n)_{n\ge1}$, there are several choices:

- Divergent Series : $\gamma_n \ge 0$, $\sum_{n=1}^\infty \gamma_n = +\infty$ and $\sum_{n=1}^\infty \gamma_n^2 < +\infty$. We get convergence as $N\to\infty$.
- Optimal stepsizes : $\gamma_n = \frac{\sqrt{\Phi}}{\left|\nabla u_{h,T}(\varphi_n)\right|_R \sqrt{n}}$, we have by [3] that

$$\min_{n\le N} u_{h,T}(\varphi_n) \;-\; U_h^{K,M,T} \;\le\; O(1)\cdot \frac{16KM\sqrt{2M^2+M\Delta x}}{\sqrt{N}}.$$

5 Numerical example

As shown in Corollary 3.9, the model-free price upper bound of variance swap is C_0 defined in (2.2). Let $(S_t)_{t\ge0}$ follow the Black-Scholes dynamics $dS_t = \sigma S_t dW_t$, where $(W_t)_{t\ge0}$ is a standard Brownian motion, and $\mu_0 \sim S_{\frac12}$ and $\mu_1 \sim S_1$. Then

$$C_0 \;=\; \mathbb{E}\left(\,S_1^2 - S_{\frac12}^2\,\right) \;=\; \mathbb{E}\int_{\frac12}^1 \sigma^2 S_t^2\,dt \;=\; \frac12\sigma^2 S_0^2.$$

We set $\sigma = 0.2$, $S_0 = 1$, it follows that $C_0 = 0.02$. In our implemented example, with a 2.40GHz CPU computer, it takes 57.24 seconds to finish 4×10^4 iterations, and we get the numerical upper bound 0.2019, i.e. the relative error is less than 1 %, see also Figure 3.

6 Appendix

We give a proof for Theorem 2.5, where we use the weak dyanmic progrmming technique proposed in Bouchard and Touzi [4].

Proof of Theorem 2.5. We first introduce

$$\overline{U}^K \;:=\; \inf_{\phi\in\text{Quad}_0^K} \bar u(\phi) \qquad\text{and}\qquad \overline{U}^{K,M} \;:=\; \inf_{\phi\in\text{Quad}_0^{K,M}} \bar u(\phi),$$

and we claim that

$$\bar u(\phi) \;=\; u(\phi), \qquad \forall \phi\in \text{Quad}_0^{K,M}, \tag{6.1}$$

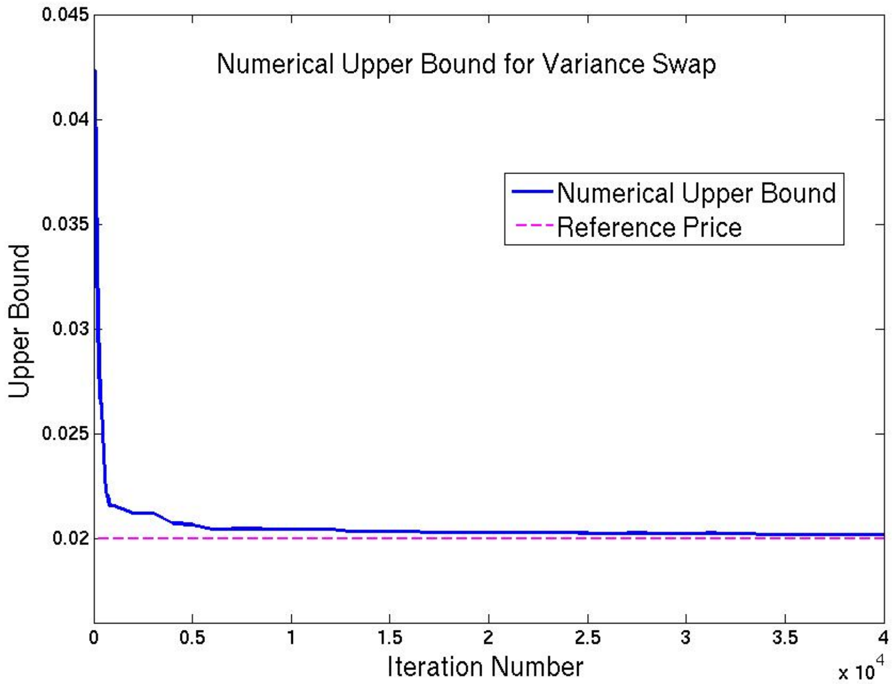

Figure 3: Numerical result for variance swap with approximation parameters: $T = 0.1$, $K = 1$, $M = 1$, $R = 2$, $\Delta t = 0.002$, $\Delta x = 0.1$ and $\gamma_n = \sqrt{n}$.

which implies that $\overline{U}^{K,M} = U^{K,M}$. Clearly, by the same arguments as in (3.3) and Proposition 3.2, we have $\overline{U}^K \to \overline{U}$ and $\overline{U}^{K,M} \to \overline{U}^K$ as $(K, M) \to \infty$. It follows that $\overline{U} = U$.

Therefore, it is enough to prove (6.1) to conclude, which is in fact a dynamic programming principle for $\bar{u}$ defined in (1.5). Moreover, by the dominated convergence theorem, λ^{ϕ,τ_R} defined in (3.8) is a continuous function for every $\phi \in$ Quad. Hence λ^ϕ is continuous for every $\phi \in \mathrm{Quad}_0^{K,M}$ by Lemma 3.3. Therefore, it is enough to derive a weak dynamic programming principle following Bouchard and Touzi [4].

Let $\phi \in \mathrm{Quad}_0^{K,M}$, $\tau \in \mathcal{T}$ which is defined in (1.6), since the stopping time τ conditioned on W_0 belongs to $\mathcal{T}^\infty$, then by a simple conditioning argument, $\mathbb{E}\big[g^\phi(\tau, W_\tau)\big] \geq \mu_0(\lambda_0^\phi)$, which implies that $u(\phi) \leq \bar{u}(\phi)$. On the other hand, as in the proof of Theorem 4.1 in [4], for every $\varepsilon > 0$, there is a countable subdivision $\Delta = (\Delta_n)_{n\geq 1}$ of $\mathbb{R}$, a sequence of stopping times $(\tau_n^\varepsilon)_{n\geq 1}$ in $\mathcal{T}^\infty$ such that $\mathbb{E}\big[(g^\phi(\tau_n^\varepsilon, x + B_{\tau_n^\varepsilon})\big] \leq \lambda_0^\phi(x) + \varepsilon$, $\forall x \in \Delta_n$. We then construct $\tau^\varepsilon \in \mathcal{T}$ by $\tau^\varepsilon(W) := \sum_{n=1}^\infty \tau_n^\varepsilon(W - W_0)\mathbf{1}_{W_0\in\Delta_n}$, so that $\mathbb{E}\big[g^\phi(\tau^\varepsilon, W_{\tau^\varepsilon})\big] \leq \mu_0(\lambda_0^\phi) + \varepsilon$. By the arbitrariness of $\varepsilon > 0$, we then get $\bar{u}(\phi) \leq \mu_0(\lambda_0^\phi) + \mu_1(\phi) = u(\phi)$, and hence establish (6.1) which concludes the proof of Proposition. □

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