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## **Sustainable growth under pollution quotas: optimal R&D, investment and replacement policies**

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# Sustainable growth under pollution quotas: optimal R&D, investment and replacement policies<sup>1</sup>

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## Abstract

We consider an optimal growth model of an economy facing an exogenous pollution quota. In the absence of an international market of pollution permits, the economy has three instruments to reach sustainable growth: R&D to develop cleaner technologies, investment in new clean capital goods, and scrapping of the old dirty capital. The R&D technology depends negatively on a complexity component and positively on investment in this sector at constant elasticity. First, we characterize possible balanced growth paths for different parameterizations of the R&D technology. It is shown that countries with an under-performing R&D sector would need an increasing pollution quota over time to ensure balanced growth while countries with a highly efficient R&D sector would supply part of their assigned pollution permits in an international market without harming their long-term growth. Second, we study transitional dynamics to balanced growth. We prove that regardless of how large the regulation quota is, the transition dynamics leads to the balanced growth with binding quota in a finite time. In particular, we discover two optimal transition regimes: an intensive growth (sustained investment in new capital and R&D with scrapping the oldest capital goods), and an extensive growth (sustained investment in new capital and R&D without scrapping the oldest capital).

**Keywords:** Sustainable growth, vintage capital, endogenous growth, R&D, pollution quotas

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## 1. Introduction

Identifying sustainable growth paths is becoming a central question in economic theory. The issue has many challenging normative, demographic, and technological aspects as pointed out by Arrow et al. (2004). On the technological side, many research avenues have been taken so far resulting in a quite dense literature. In particular, several researchers have studied the design of research and development (R&D) programs to meet the environmental constraints necessary for sustainability (like the pollution reduction targets formalized in the Kyoto Protocol). Early interesting contributions to the issue of pollution control and growth can be found in van der Ploeg and Withagen (1991), Withagen (1995) or Selden and Daqing (1995). Merging R&D-based endogenous growth models and environmental sustainability concerns has been the object of an equally important literature: Bovenberg and Smulders (1995) who explored the link between environmental quality and economic growth in an endogenous growth model that incorporates pollution-augmenting technological change, and Grimaud (1999) who studied a decentralized model of Schumpeterian growth with environmental pollution, are among the earliest contributors to this topic.

A central question of the latter literature turns out to be whether the environmental regulation can ultimately deliver a win-win situation as economies facing this regulation will have strong incentives to innovate resulting in new and “clean” growth regimes. This mechanism, often referred to as the Porter hypothesis, has been studied in numerous papers, some empirical (like the seminal paper by Newell et al., 1999) and others more theoretically-oriented (see Acemoglu et al., 2011, for one of the most recent contribution on the induced-innovation hypothesis under environmental constraints).

This paper is a contribution to the latter line of research. More precisely, we consider an optimal growth model with R&D expenditures and pollution quotas. Technological progress is therefore endogenous; it is also specified as embodied in capital goods: thanks to R&D efforts, new capital goods use less and less resources (say, energy), that is, they get cleaner over time. Such a view of technological progress is documented and commented in a substantial literature.

In modern economic theory, the technological change is usually described as an embodied, endogenous, and energy-saving phenomenon. More specifically, substantial

economic evidence supports the direct impact of the *R&D spending* on the *industry-level capital-embodied technological change*. In particular, Wilson (2002) used industrial data to confirm that “the technological change, or innovation, embodied in an industry’s capital is proportional to the R&D that is done (upstream) by the economy as a whole”. Wilson also found that the cross-industry variation in estimates of embodied technological change matches the cross-industry variation in embodied R&D. Finally, Ayres (2005) argued that “technical progress is essentially equivalent to increasing efficiency of converting raw resources, such as coal, into useful work...” (p. 142).

This said, our paper has three salient and distinctive features. First of all, it explicitly uses a vintage capital framework in the tradition of Solow et al. (1966): capital and energy are complementary (Leontief technology), and new vintages consume less energy over time (energy-saving technical progress). Second, we explicitly account for pollution quotas: because of international agreements of the Kyoto protocol style, national economies are assigned pollution reduction targets, which can be formalized as **exogenous** upper-bounds on total pollution emissions. The exogeneity of the quota is certainly a shortcoming: we don’t assume an international market of pollution permits.<sup>5</sup> Here we consider an optimal growth problem of a single economy, and as one can see, the problem is already extremely complex analytically because of the vintage structure adopted and the size of the associated optimal control problem. Extensions to two-country symmetric or asymmetric settings seem computationally manageable but certainly not analytically tractable. So we stick to a single economy under the exogenous pollution quota to bring a clear-cut analytical insight into the problem. We shall study how varying the quota dynamics can affect the optimal short-term and asymptotic properties of the model. Here, it is important to notice that the emission quota and capital-energy complementarity induce an obsolescence mechanism, which in turn opens the door to endogenous scrapping: as in Boucekkine et al. (1997) and Hritonenko and Yatsenko (1996, 2005), the oldest vintages will be removed from the workplace and replaced by less energy consuming new vintages.

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<sup>5</sup> The literature of pollution permits markets, initiated by Dales (1968) and Montgomery (1972) is huge, specially following the Kyoto protocol which stimulated an impressive conceptual and practical literature on efficient pollution control at all levels, international (see Godal and Klassen, 2006) or regional (see Boucekkine et al., 2010).

Third, after characterizing possible balanced growth paths, we shall study transitional dynamics. In particular, we aim at identifying clearly the different routes to sustainable and balanced growth, which is not so frequent in the endogenous growth literature, usually restricted to a balanced growth analysis. The value-added of studying the transitional impact of environmental policy has been already stressed by Bovenberg and Smulders (1996) in an endogenous growth set-up. In our paper, the role of the historical pollution conditions will be shown to be decisive in the shape of the optimal transition generated for given pollution quotas.

#### *Relation to the literature*

Our framework is closely related to Boucekine, Hritonenko and Yatsenko (2011), referred as BHY hereafter. The two papers share the same vintage capital production function and the same R&D technology. But while BHY address a firm problem, here we will solve an optimal growth model in the Ramsey sense. On one hand, the optimal growth setting is simpler because we get rid of the (exogenous) series of capital goods' and energy's prices. But on the other, it can be algebraically much more involved if a general utility function were to be used. To simplify, we consider a linear utility function: strictly concave utility functions render the analytical work intractable, even at the stage of a balanced growth path (BGP hereafter) computation. Within this simpler framework, we are able to derive, as in BHY, an analytical characterization of BGPs. Additionally, we are able to characterize the pollution quota paths which are compatible with the existence of BGPs for any parameterization of the R&D technology, which is a useful step to take before introducing an international market for pollution permits. This issue is not treated in BHY. Last but not least, we display the optimal transition dynamics to BGPs, an issue not covered in BHY.

To our knowledge, and with the exception of BHY, no other paper has considered R&D decisions and vintage capital with endogenous scrapping at the same time. Feichtinger et al. (2005, 2006) have developed an alternative framework balancing the efficiency gains of running new vintages with the learning costs associated, which opens the door to optimal investment in old vintages, in contrast to our modelling where such a possibility does not exist (no learning costs). Having said this, Feichtinger et al. (2005, 2006) have

not integrated R&D decisions in their setting, nor have they endogenized scrapping. Hart (2004) has constructed a multi-sector endogenous growth model with an explicit vintage structure. But his paper differs from ours in at least two aspects: it builds on two types of R&D, one output-augmenting and the other, say, environmental-friendly, while in our model only resource-saving adoptive and/or innovative R&D is allowed. More importantly, the vintage structure considered in Hart (2004) is rather short: the number of vintages is fixed, and therefore, there is no way to uncover a comprehensive modernization policy optimally combining the scrapping of the dirtiest technologies and the development of new and cleaner technologies.

Concerning the type of environmental regulation and/or constraint, a large body of the related literature uses environmental taxation, specifically emission taxation as in Feichtinger et al. (2005) or Grimaud (1999). Another stream of the literature models the pollution permits market as an alternative to taxation: Jouvét et al. (2005) in an optimal growth overlapping generations setting with exogenous technical progress, or more recently, Krysiak (2011) in an endogenous technological progress framework, have studied the specific implications of allowing for trading in pollution permits. In our paper, we consider the case of a national economy which has to deal with an optimal growth problem subject to a fixed pollution quota in the absence of an international market for pollution permits (in this sense, our economy is autarkic from the environmental point of view). As acknowledged above, this limitation is entirely due to the extreme mathematical sophistication of the problem once proper vintage structure are considered together with endogenous technical change.

### *Main findings*

This paper has essentially two contributions. In first place, it extends significantly the BGP analysis of BHY. In the latter, the R&D technology is taken “balanced” in the sense that the standard (negative) complexity component *à la* Segerstrom (2000) compensates the (positive) return to R&D investment component in the parameterization considered, which is a common assumption in the literature. In this paper, we shall explore all the cases: when the negative component dominates and when it is dominated. We believe that this extension is worth doing having in mind an extension of the model to a two-

country case in presence of an international market of pollution permits. The R&D technology may not be the same across countries: some countries (like certain Scandinavian countries) are historically more sensitive to the development of energy-saving technologies than others, and are likely to be more efficient at this. Others are lagging clearly behind. In the absence of international pollution permits, we show that they should experience different balanced growth paths if any. The countries with underperforming R&D sector would need an increasing pollution quota over time (in a very accurate sense to be given) to ensure balanced growth, while countries with a highly efficient R&D sector would supply part of their assigned pollution quotas in an international market without harming their long-term growth.

The second important new contribution of this paper is the complete dynamic analysis of the problem, which demonstrates the convergence of optimal trajectories to a long-run balanced regime. Namely, regardless how large the regulation quota is, the transition dynamics in the model leads in a finite time to the balanced growth with an active regulation. The derived transition dynamics indicates several possible short-term scenarios. In particular, we demonstrate two optimal regimes: an intensive growth (sustained investment in new capital and R&D with scrapping the oldest capital goods), and an extensive growth (sustained investment in new capital and R&D without scrapping the oldest capital). Our paper is the first one to disentangle the latter regime as a short-term optimal transition regime. Namely, if the country is not initially a large polluter (the energy pollution is lower than the quota limit), then it should initially use more new capital without scrapping the old one, so the country experiences an extensive economic growth. In other words, our model predicts that historically poor countries may find it optimal to massively invest and therefore to massively pollute during their development process, consistently with the increasing part of the environmental Kuznets curve. Such an initial growth regime comes to end when the quota limit is reached and is followed by an intensive balanced growth with scrapping of dirty capital and active energy regulation. After the transition dynamics ends, the optimal capital investment possesses everlasting replacement echoes that repeat the investment dynamics on a prehistory interval.

The paper is organized as follows. Section 2 displays the optimal growth problem and extracts the corresponding optimality conditions. Section 3 is devoted to the analysis of long-term dynamics, that is, to the existence and properties of BGPs. Section 4 characterizes optimal transitional dynamics of the model. Section 5 concludes.

## 2. The optimal growth problem

We consider a benevolent social planner of a national economy who maximizes the discounted utility from the consumption over the infinite horizon:

$$\max_{i,R,a} I = \max_{i,R,a} \int_0^{\infty} u(y(t) - i(t) - R(t))e^{-rt} dt, \quad (1)$$

where  $u(\cdot)$  is the utility function,  $r$  is the social discount rate,  $i(t)$  is the investment into new capital,  $R(t)$  is the investment into R&D,

$$y(t) = \int_{a(t)}^t i(\tau) d\tau \quad (2)$$

is the production output at time  $t$ ,  $a(t)$  is the capital scrapping time, subject to the following constraints

$$0 \leq i(t) \leq y(t) - R(t), \quad R(t) \geq 0, \quad a'(t) \geq 0, \quad a(t) \leq t. \quad (3)$$

The total energy consumption is

$$E(t) = \int_{a(t)}^t \frac{i(\tau)}{\beta(\tau)} d\tau \quad (4)$$

To address capital modernization, the model (1)-(4) departs from the concept of homogeneous capital and assumes that newer capital vintages consume less energy (and, therefore, are environmentally friendlier). In (4), the energy consumption by one machine of vintage  $t$  (i.e., installed at time  $t$ ) is equal to  $1/\beta(t)$ . The variable  $\beta(t)$  is endogenous and reflects a broadly defined energy-saving embodied technological level, which may be implemented in new energy-efficient devices, clean technologies, alternative energy sources, etc. For clarity, our model does not involve any output-augmenting embodied or disembodied technological change: each machine (old or new) produces exactly one unit



of output. Needless to say, the output not invested (either in R&D or in new capital) is consumed, that is:  $c(t) = y(t) - i(t) - R(t)$ .

Now we assume that due to international agreements, the economy is committed to limit pollution emissions. This features the presence of a pollution quota. The quota is exogenous in the absence of international markets for pollution permits, which we postulate here. Assuming that energy consumption is the unique source of pollution in the economy, the pollution quota can be formulated as follows:

$$E(t) \leq E_{\max}(t). \quad (5)$$

Next, we assume that the level of the technological progress  $\beta(\tau)$  depends on the R&D investment  $R(t)$  as

$$\frac{\beta'(\tau)}{\beta(\tau)} = \frac{f(R(\tau))}{\beta^d(\tau)}, \quad 0 < d < 1, \quad (6)$$

$f'(R) > 0$ ,  $f''(R) < 0$ . By (6), the rate  $\beta'/\beta$  of technological progress is a concave increasing function  $f(R)$  in  $R$  and a decreasing function of the level  $\beta$  itself. The latter specification reflects a negative impact of technological complexity on R&D success (see Segerstrom, 2000, for example). The parameter  $d$  measures the impact of the R&D complexity on the technological progress rate. It is consistent with the available evidence on the role of technological complexity in the adoption of clean technologies (see, for example, BHY). Also, we restrict ourselves to the case

$$f(R) = bR^n, \quad 0 < n < 1, \quad b > 0,$$

which means that the elasticity  $n$  of the rate of technological progress with respect to R&D expenditures is constant. The R&D investment is more efficient for larger  $n$ .

As one can see, the technological assumptions and constraints of the optimal control problem are identical to those considered in the firm problem studied in BHY. Yet the optimal growth setting originating in (1) requires a different interpretation of some variables and constraints as, for example, the pollution quota constraint. Algebraically speaking, the problem (1) with linear utility looks simpler than in BHY due to the absence of exogenous energy and capital prices. Assuming a nonlinear utility would not allow for a full analytical characterization of BGPs, so, we stick to the linear utility function here. In parallel with investment  $i(t)$  in output units, we will use investment

$m(t)=i(t)/\beta(t)$  in the energy consumption units for convenience. The optimization problem (1)-(6) becomes

$$\max_{m,R,a} I = \max_{m,R,a} \int_0^{\infty} [y(t) - \beta(t)m(t) - R(t)]e^{-rt} dt, \quad 0 < r < 1^6, \quad (7)$$

$$y(t) = \int_{a(t)}^t \beta(\tau)m(\tau)d\tau, \quad (8)$$

$$\frac{\beta'(\tau)}{\beta(\tau)} = b \frac{R^n(\tau)}{\beta^d(\tau)}, \quad 0 < d < 1, \quad (9)$$

$$E(t) = \int_{a(t)}^t m(\tau)d\tau \leq E_{\max}(t), \quad (10)$$

$$0 \leq \beta(t)m(t) \leq y(t) - R(t), \quad R(t) \geq 0, \quad a'(t) \geq 0, \quad a(t) \leq t, \quad (11)$$

with the given initial conditions on the prehistory:

$$a(0) = a_0 < 0, \quad \beta(a_0) = \beta_0, \quad m(\tau) \equiv m_0(\tau), \quad R(\tau) \equiv R_0(\tau), \quad \tau \in [a_0, 0]. \quad (12)$$

The optimization problem (7)-(12) includes six unknown functions  $m(t)$ ,  $R(t)$ ,  $a(t)$ ,  $y(t)$ ,  $E(t)$ , and  $\beta(t)$ ,  $t \in [0, \infty)$ , related by three equalities (8)-(10). We choose  $R$ ,  $m$ , and  $v=a'$  as *independent* controls and consider  $y$ ,  $a$ ,  $E$  and  $\beta$  as *dependent state* variables. Let  $R$ ,  $m$ ,  $v$  belong to the space  $L^\infty_{\text{loc}}[0, \infty)$  of measurable on  $[0, \infty)$  functions bounded almost everywhere (*a.e.*) on any finite subinterval of  $[0, \infty)$  (Corduneanu 1997). We also assume a priori that the integral in (7) converges (it will be true in all subsequent theorems).

Solving the differential equation (9), we obtain the explicit formula for the productivity  $\beta(\tau)$  through the previous R&D investment  $R$  on  $[a_0, \tau]$ :

$$\beta(\tau) = \left( bd \int_0^\tau R^n(v)dv + B^d \right)^{1/d}, \quad B = \left( bd \int_{a_0}^0 R_0^n(v)dv + \beta_0^d \right)^{1/d}. \quad (13)$$

The problem (7)-(12) is an optimal control problem with state constraints. To analyze its complete dynamics, we need optimality conditions that will include all possible combinations of the state constraints-inequalities  $E(t) \leq E_{\max}(t)$  and  $\beta(t)m(t) \leq y(t) - R(t)$ .

<sup>6</sup> The condition  $r < 1$  appears later in order to allow for non-trivial solutions (otherwise the optimal dynamics is no investment because of too high discount rate).

Notice that the latter is equivalent to consumption non-negativity. Clearly, having a concave utility function satisfying Inada conditions would rule out the corner regime  $c(t)=0$  associated with this condition. For mathematical consistency, we shall consider here all the possible cases allowed by linear utility. The optimality conditions are given by Theorem 1 below. As we shall see, all combinations can appear during the long-term dynamics (Section 3) or the transition dynamics (Section 4).

**Theorem 1 (necessary condition for an extremum).** *Let  $(R^*, m^*, a^*, \beta^*, y^*, E^*)$  be a solution of the optimization problem (7)-(12). Then:*

(A) *If  $E^*(t)=E_{\max}(t)$  and  $\beta^*(t)m^*(t)<y^*(t)-R^*(t)$  at  $t\in\Delta\subset[0,\infty)$ , and  $E_{\max}'(t)\leq 0$ , then*

$$I_R'(t)\leq 0 \text{ at } R^*(t)=0, \quad I_R'(t)=0 \text{ at } R^*(t)>0, \quad (14)$$

$$I_m'(t)\leq 0 \text{ at } m^*(t)=0, \quad I_m'(t)=0 \text{ at } m^*(t)>0, \quad t\in\Delta, \quad (15)$$

where

$$I_R'(t) = bnR^{n-1}(t) \int_t^{\infty} \beta^{1-d}(\tau)m(\tau) \left[ \frac{e^{-r\tau} - e^{-ra^{-1}(\tau)}}{r} - e^{-r\tau} \right] d\tau - e^{-rt}, \quad (16)$$

$$I_m'(t) = \int_t^{a^{-1}(t)} e^{-r\tau} [\beta(t) - \beta(a(\tau))] d\tau - e^{-rt} \beta(t), \quad (17)$$

the state variable  $a(t)$  is determined from (10),  $a^{-1}(t)$  is the inverse function of  $a(t)$ , and  $\beta(t)$  is given by (13).

(B) *If  $E^*(t)<E_{\max}(t)$  and  $\beta^*(t)m^*(t)<y^*(t)-R^*(t)$  at  $t\in\Delta$ , then*

$$I_R'(t)\leq 0 \text{ at } R^*(t)=0, \quad I_R'(t)=0 \text{ at } R^*(t)>0, \\ I_m'(t)\leq 0 \text{ at } m^*(t)=0, \quad I_m'(t)=0 \text{ at } m^*(t)>0, \quad (18)$$

$$I_a'(t)\leq 0 \text{ at } da^*(t)/dt=0, \quad I_a'(t)=0 \text{ at } da^*(t)/dt>0, \quad t\in\Delta,$$

where

$$I_m'(t) = \beta(t) \left[ \int_t^{a^{-1}(t)} e^{-r\tau} d\tau - e^{-rt} \right], \quad (19)$$

$$I_a'(t) = - \int_t^{\infty} e^{-r\tau} \beta(a(\tau)) m(a(\tau)) d\tau, \quad (20)$$

$I_R'(t)$  is as in (16), and  $\beta(t)$  is as in (13).

(C) If  $E^*(t) < E_{\max}(t)$  and  $\beta^*(t)m^*(t) = y^*(t) - R^*(t)$  at  $t \in \Delta \subset [0, \infty)$ , then

$$I_R'(t) \leq 0 \text{ at } R^*(t) = 0, \quad I_R'(t) = 0 \text{ at } R^*(t) > 0, \quad (21)$$

$$I_a'(t) \leq 0 \text{ at } da^*(t)/dt = 0, \quad I_a'(t) = 0 \text{ at } da^*(t)/dt > 0, \quad t \in \Delta,$$

where  $I_a'(t)$  is as in (20),  $m^*(t)$  and  $y^*(t)$  are determined from equation (8),

$$\widehat{I}_R'(t) = bnR^{n-1}(t) \int_t^{\infty} \beta^{1-d}(\tau) m(\tau) \left[ \int_{\tau}^{a^{-1}(\tau)} \chi(\xi) d\xi - \chi(\tau) \right] d\tau - \chi(t), \quad (22)$$

$\chi(t)$  is found from the integral equation

$$\chi(t) = \int_t^{a^{-1}(t)} \chi(\tau) d\tau \quad \text{at } t \in \Delta \quad (23)$$

and  $\chi(t) = e^{-rt}$  at  $t \in [0, \infty) - \Delta$ .

(D) If  $E^*(t) = E_{\max}(t)$  and  $\beta^*(t)m^*(t) = y^*(t) - R^*(t)$  at  $t \in \Delta \subset [0, \infty)$ , then

$$I_R'(t) \leq 0 \text{ at } R^*(t) = 0, \quad I_R'(t) = 0 \text{ at } R^*(t) > 0, \quad (24)$$

where  $I_R'(t)$  is given by (22),  $\chi(t)$  is found from

$$\chi(t) = \int_{t_k}^{a^{-1}(t)} \left[ 1 - \frac{\beta(a(\tau))}{\beta(t)} \right] \chi(\tau) d\tau \quad \text{at } t \in \Delta, \quad (25)$$

and  $\chi(t) = e^{-rt}$  at  $t \in [0, \infty) - \Delta$ , and  $m^*(t)$ ,  $a^*(t)$ , and  $y^*(t)$  are determined from nonlinear equations (8) and (10).

The proof is provided in Appendix. Theorem 1 is the extension of Theorem 1 in BHY (page 190) that explored only cases A and B because BHY did not analyze transition dynamics. Even for those cases, some differences show up. An essential difference emerges in the derivation of optimal scrapping when the pollution quota is not binding. It is easy to see from (20) that  $I_a'(t) < 0$ . Hence,  $a^* \equiv a_0$  is corner and the optimal regime is

boundary in  $a$ . This might not be the case in the counterpart case in the firm problem.<sup>7</sup> Economically speaking, this outcome is natural: in our optimal growth problem, the unique reason to shorten the lifetime of capital goods is the binding pollution quota, while exogenously increasing energy prices (for example, reflecting scarcity) is an additional motive to scrap in the firm problem of BHY. So, the firm model has a room for a non-boundary control in the scrapping age even when the quota is not binding.

More (but less essential) differences emerge in the expression of the optimal interior investment rules depicted in Theorem 1. Let us briefly interpret the optimal interior investment rules in case A, which will turn out to be the important one in the long-run as demonstrated in Section 3. As to the optimal investment rule in new capital, it can reformulated as:

$$\int_t^{a^{-1}(t)} e^{-r\tau} \left[ 1 - \frac{\beta(a(\tau))}{\beta(t)} \right] d\tau = e^{-rt}$$

The rule is very close to the counterpart in Boucekkine et al. (1997): it equalizes the present value of marginal investment cost (the right-hand side) and the discounted value of the quasi-rents generated by one unit of capital bought at  $t$  along its lifetime (the left-hand side). Here costs and benefits are expressed in terms of marginal utility, but since utility is linear, marginal utility terms do not show up. The quasi-rent at  $\tau$  generated by a machine of vintage  $t$  is the difference between the unit of consumption forgone to buy one unit of new capital and the operation cost at  $\tau$ , which is the product of the amount of energy consumed to operate any machine of vintage  $t$ , that is  $\frac{1}{\beta(t)}$ , and the shadow price of energy  $\beta(a(\tau))$  at the date  $a(\tau)$ .

The optimal investment rule in R&D in case A may be rewritten as:

$$bnR^{n-1}(t) \int_t^{\infty} \beta^{1-d}(\tau) m(\tau) \left[ \frac{e^{-r\tau} - e^{-ra^{-1}(\tau)}}{r} - e^{-r\tau} \right] d\tau = e^{-rt}.$$

As for investment in capital, this rule equalizes the marginal cost of R&D (right-hand side) and its marginal benefit (the left-hand side). As before, the marginal utility terms do not show up due to the linear utility. Now note that in contrast to a unit of capital, which

<sup>7</sup> See equation (21) in BHY, page 190.

is necessarily scrapped at finite time, the benefit of R&D investment is everlasting through R&D cumulative technology, which explains integration from  $t$  to infinity in the left-hand side. Other than this, the left-hand side of the rule can be interpreted as in BHY: the marginal increase in  $\beta(\tau)$ ,  $\tau \geq t$ , following the marginal rise in  $R(t)$ , that is  $\frac{bnR^{n-1}(t)}{\beta^d(\tau)}$ , increases profitability by improving the efficiency of all vintages after the date  $t$ , but since machines have a finite lifetime, this effect should be computed between  $\tau$  and  $a^{-1}(\tau)$  for each vintage  $\tau$ , which explains the factor  $\frac{e^{-r\tau} - e^{-ra^{-1}(\tau)}}{r} = \int_{\tau}^{a^{-1}(\tau)} e^{-rs} ds$  in the integrand.

Next, we study the dynamics of the optimization problem (7)-(12). We analyze the long-term dynamics looking for possible exponential balanced growth paths in Section 3 and then we move to the short-term transition dynamics in Section 4.

### 3. Optimal long-term dynamics.

In this section, we identify interior optimal trajectories over a “long-term” interval  $[t_l, \infty)$  starting with some finite instant  $t_l \geq 0$ . After such interior regimes are identified, the next step will be the analysis of a short-term “transition” dynamics over the interval  $[0, t_l]$ .

Let us examine what kinds of long-term interior regimes are possible in the problem (7)-(12). The necessary extremum condition of Theorem 1 specifies four possible Cases A-D. We can immediately rule out Cases C and D in the long run because then the integrand of the objective function (7) is zero over  $[t_l, \infty)$  and it is straightforward to show that these cases cannot be optimal in the sense that they are dominated by other solution paths.

Next, Case B with non-binding environmental constraint  $E < E_{\max}$  appears to be also impossible. Indeed, then an interior solution should be found from the system

$$I_R'(t)=0, \quad I_m'(t)=0, \quad I_a'(t)=0, \quad t \in [t_l, \infty),$$

where  $I_R'(t)$ ,  $I_m'(t)$  and  $I_a'(t)$  are determined by (16), (19), and (20). As we explained before, this case implies an optimal regime which is boundary in  $a$ . Therefore, no long-run interior regime with inactive environmental regulation  $E < E_{\max}$  is possible. We shall

see in Section 4 that such a regime (extensive growth) can arise in the short-term dynamics and it leads to Case A with binding constraint  $E=E_{\max}$  in a finite time.

So, the only possible long-run case is Case A with the binding environmental constraint (10):  $E(t)=E_{\max}(t)$  at  $t \in [t_l, \infty)$ . Then the optimal long-term dynamics can involve an interior regime  $(R, m, a)$  determined by the system of three nonlinear equations

$$I_R'(t)=0, \quad I_m'(t)=0, \\ \int_{a(t)}^t m(\tau) d\tau = E_{\max}(t), \quad t \in [t_l, \infty), \quad (26)$$

where  $I_R'(t)$  and  $I_m'(t)$  are determined by (16) and (17). Let  $r < 1$  here and thereafter, otherwise,  $I_R'(t) < 0$  and  $I_m'(t) < 0$  by (16), (17). The equations  $I_R'(t) = 0$  and  $I_m'(t) = 0$  lead to

$$bnR^{n-1}(t) \int_t^\infty \left[ bd \int_0^\tau R^n(\xi) d\xi + B^d \right]^{1/d-1} m(\tau) \left[ \frac{e^{-r\tau} - e^{-ra^{-1}(\tau)}}{r} - e^{-r\tau} \right] d\tau = e^{-rt}, \quad (27)$$

$$\int_t^{a^{-1}(t)} \left[ 1 - \left[ bd \int_0^{a(\tau)} R^n(\xi) d\xi + B^d \right]^{1/d} / \left[ bd \int_0^t R^n(\xi) d\xi + B^d \right]^{1/d} \right] e^{-r\tau} d\tau = e^{-rt} \quad (28)$$

at  $t \in [t_l, \infty)$ .

So, the optimal long-term growth in our model necessarily involves the active environmental regulation (Case A of Theorem 1). We can summarize this as the following theorem.

**Theorem 2.** *Long-term interior optimal regimes are possible in the problem (7)-(12) only under the binding environmental constraint  $E=E_{\max}$ .*

We are interested in *exponential* interior solutions to the problem (7)-(12). The following lemma is helpful in this context.

**Lemma 1** (BHY). *If  $R(t) = \bar{R} e^{Ct}$  for some  $C > 0$ , then the productivity  $\beta(t)$  is almost exponential:*

$$\beta(t) \approx \bar{R}^{n/d} \left( \frac{bd}{Cn} \right)^{1/d} e^{Cnt/d} \quad \text{at large } t. \quad (29)$$

The productivity is the exact exponential function  $\beta(t) = Be^{\hat{C}nt/d} = \bar{R}^{n/d} (bd/\hat{C}n)^{1/d} e^{\hat{G}t/d}$  at the specially chosen rate  $\hat{C} = nB^d/(bd\bar{R}^n)$ .

For brevity, we will later omit the expression “at large  $t$ ” in the notation  $f(t) \approx g(t)$ . Now we can formalize the concept of a balanced growth path in problem (7)-(12).

**Definition 1.** *The Balanced Growth Path (BGP) is a solution  $(R_A, m_A, a_A)$  to the system of three nonlinear equations (26), (27) and (28), such that  $R_A(t)$  grows exponentially,  $m_A(t)$  is exponential or constant,  $t - a_A(t)$  is a positive constant, and the constraints (11) hold.*

We will explore the possibility of the BGP under the binding environmental constraint separately in the cases  $n=d$ ,  $n<d$  and  $n>d$ . We start with the inequality cases  $n<d$  and  $n>d$ , which were not treated in BHY in their firm problem. We believe that this analysis is important having in mind an extension of the model to a two-country case in presence of an international market of pollution permits. The R&D technology may not be the same across countries: some countries (like certain Scandinavian countries) are historically more sensitive to the development of energy-saving technologies than others, and are likely to be more efficient at this. Others are lagging clearly behind. In the absence of international pollution permits, we show that they should possess different balanced growth paths if any.

### 3.1. Case $n<d$ .

Let us start with the situation where the complexity parameter  $d$  is larger than the efficiency parameter,  $n$ . This is the case of national economies where the R&D technology is not likely to ensure balanced growth in the long-run on its own. We show hereafter that indeed the pollution permits assigned to such an economy should increase over time for the economy to have balanced and sustainable growth.

**Theorem 3.** *Let  $n<d$ . If the quota  $E_{max}(t)$  does not increase exponentially, then there is no interior BGP in the problem (7)-(12). However, if*



$$E_{\max}(t) = \bar{E} e^{gt}, \quad 0 < g < \min\{rd/n, r(d-n)/n\}, \quad (30)$$

then the problem (7)-(12) has an interior exponential solution

$$R_A(t) \approx \bar{R} e^{Ct}, \quad y_A(t) \sim e^{Ct}, \quad \beta_A(t) \sim e^{Cn/d}, \quad m_A(t) = \bar{M} e^{gt}, \quad a_A(t) = t - T, \quad (31)$$

where

$$C = \frac{gd}{d-n}, \quad \bar{M} = \bar{E}g / (1 - e^{-gT}), \quad (32)$$

$$\bar{R}^{d-n} = bn^{2d-1} d^{1-d} \bar{M}^d \frac{C^{d-1}}{r - C(1-n)} \left( \frac{1 - e^{-rT}}{r} - 1 \right)^d, \quad (33)$$

and the positive constant  $T$  is found from the nonlinear equation

$$\frac{1 - e^{-rT}}{r} - \frac{e^{-CnT/d} - e^{-rT}}{r - Cn/d} = 1. \quad (34)$$

The solution  $(R_A, m_A, a_A)$  is a BGP, at least, when

$$n > 1 - \frac{1 - e^{-CT}}{C}. \quad (35)$$

If  $g > \min\{rd/n, r(d-n)/n\}$  in (30), then the problem does not possess a finite solution because the quota  $E_{\max}(t)$  increases too fast.

**Proof.** Let us substitute

$$R(t) = \bar{R} e^{Ct} \quad \text{and} \quad t - a(t) = T = \text{const} > 0 \quad (36)$$

into (26), (27) and (28) and estimate the growth order of  $m(t)$  at large  $t$ . By (26),  $m(t)$  satisfies

$$m(t) = m(t - T) + E_{\max}'(t). \quad (37)$$

Applying Lemma 1 and Theorems 1 and 2, we find that  $\beta(t) \approx R_0^{n/d} \left( \frac{bd}{Cn} \right)^{1/d} e^{Cn/d}$ ,

$$bn\bar{R}^{n-1} e^{C(n-1)t} \int_t^\infty \left[ \frac{bd}{Cn} \bar{R}^n e^{Cn\tau} \right]^{(1-d)/d} m(\tau) \left[ \frac{1}{r} - \frac{e^{-r\tau}}{r} - 1 \right] e^{-r\tau} d\tau - e^{-rt} \approx 0, \quad (38)$$

$$\bar{R}^{n/d} \left( \frac{bd}{Cn} \right)^{1/d} \left\{ \int_t^{t+T} \left[ e^{Cn\tau/d} - e^{Cn(\tau-T)/d} \right] e^{-r\tau} d\tau - e^{Cn/d} e^{-rt} \right\} \approx 0 \quad (39)$$

at large  $t$ . To keep (38), we need an exponentially growing  $m(t)$  with the rate  $C(1-n/d)>0$ . By equation (37), it is possible only if  $E_{\max}(t)$  increases exponentially, i.e., (30) holds. Otherwise, no BGP exists.

Let (30) hold, then  $m$  is found from (10) as  $m_A(t) = \bar{M} e^{gt}$ , where  $\bar{M} > 0$  is determined by (32). Substituting it into (38), we can have  $I_R'(t)=0$  only if  $g=C(1-n/d)$  and the level constant  $\bar{R}$  satisfies (33).

The equation (39) with respect to  $T$  has appeared before in the vintage models with exogenous technological change (Boucekkine et al 1998, and Hritonenko and Yatsenko 1996). After evaluating the integrals inside, it leads to the nonlinear (but not integral) equation (34), which has a unique positive solution  $T$  if  $C < rd/n$  (Hritonenko and Yatsenko 1996).

To prove that the path (32)-(34) is a BGP indeed, we need to show that the state constant  $\beta_A(t)m_A(t) - R_A(t) < y_A(t)$  holds, at least, at large  $t$ . By (8) and (38),

$$y_A(t) \approx \bar{R}^{n/d} \bar{M} \left( \frac{bd}{Cn} \right)^{1/d} \frac{1 - e^{-CT}}{C} e^{Ct}. \text{ Therefore,}$$

$$\begin{aligned} y_A(t) - \beta_A(t)m_A(t) - R_A(t) &\approx \bar{R}^{n/d} \left\{ \bar{M} \left( \frac{bd}{Cn} \right)^{1/d} \left[ \frac{1 - e^{-CT}}{C} - 1 \right] - \bar{R}^{1-n/d} \right\} e^{Ct}. \\ &= \bar{R}^{n/d} e^{Ct} \left\{ \bar{M} \left( \frac{bd}{Cn} \right)^{1/d} \left[ \frac{1 - e^{-CT}}{C} - 1 \right] - \frac{\bar{M}bn}{(r - C + Cn)} \left( \frac{bd}{Cn} \right)^{1/d-1} \left[ \frac{1 - e^{-rT}}{r} - 1 \right] \right\} \\ &= \bar{R}^{n/d} e^{Ct} \bar{M} \left( \frac{bd}{Cn} \right)^{1/d} \frac{Cn}{(r - C + Cn)} \left\{ \left( 1 + \frac{r - C}{Cn} \right) \left[ \frac{1 - e^{-CT}}{C} - 1 \right] - \frac{n}{d} \left[ \frac{1 - e^{-rT}}{r} - 1 \right] \right\} \end{aligned}$$

Next, substituting  $e^{-rT}$  from equation (34) into this formula and combining similar terms, we obtain

$$\begin{aligned} y_A(t) - \beta_A(t)m_A(t) - R_A(t) &\approx \left( \frac{bd}{Cn} \right)^{1/d} \frac{n\bar{R}^{n/d} \bar{M} e^{Ct}}{(r - C + Cn)} \\ &\quad \times \left\{ \frac{r - C}{Cn} \left( 1 - C(1 - n) - e^{-CT} \right) + e^{\frac{CnT}{d}} - e^{-CT} \right\}. \end{aligned} \tag{40}$$

The first term in brackets is positive at (35) and the second term is positive at  $n < d$ .

The theorem is proven.  $\square$

Some comments are in order here. At first, one has to observe that the sufficient condition (35) to ensure the existence of BGPs involves endogenous magnitudes,  $C$  and  $T$ . Unfortunately, we couldn't express it in terms of the parameters of the model. Nevertheless, it appears to be valid for all economically reasonable ranges of parameters  $n$ ,  $C$ , and  $T$ , for example, if  $C < 0.1$  and  $0.05 < n < 1$ , then (35) holds at  $T > 1$  year, which is definitely reassuring. Second, it is important to notice that balanced growth is compatible with a substantial interval (30) of  $g$  values. Small values for the growth rate of pollution quotas are enough to ensure balanced growth, which is a remarkable property. In contrast, too large values of  $g$  lead the economy to explosive growth, which is economically straightforward. Third, in this case, the growth rate of the economy  $C$  is proportional to the growth rate of pollution quotas: clearly, the R&D sector and the associated induced-innovation mechanism are too weak to ensure balanced growth in this case of underperforming R&D sector, relaxing pollution quotas over time is a necessary accompanying condition. Countries with less efficient energy-saving research program should rely on the international markets of pollution permits to build sustainable and balanced growth.

A final crucial remark is worth doing: the innovation rate is equal to  $Cn/d = gn/(d-n)$ , while the growth rate of production is  $C = gd/(d-n)$ . Consistently, if  $n=0$ , then the growth rate of innovation is zero while the growth rate of production  $C$  is  $g$ . That is to say, growth generated in this case is semi-endogenous: there are two interdependent engines of growth, one exogenous coming from the quota (and international market of pollution permits, not modeled here) and the other is endogenous reflecting the Porter mechanism. The R&D sector is not necessary for the existence of (exogenous) balanced growth paths, however, operating it allows to reach higher values of growth and welfare.

### 3.2 The case $n > d$

This case is formally symmetrical to the previous one but has an opposite interpretation. Here the R&D sector is highly efficient. While the countries studied in Section 3.2 should ask for more pollution permits to reach balanced growth, we expect just the contrary here: countries with highly efficient energy-saving technology may supply part of their assigned pollution permits in an international market without harming their growth.

Moreover, they must do that in order to achieve a sustainable growth. We shall check these claims in this section.

**Theorem 4.** *Let  $n > d$ . If the quota  $E_{\max}(t)$  does not decrease exponentially, then no BGP is possible in the problem (7)-(12). If*

$$E_{\max}(t) = \bar{E} e^{-gt}, \quad 0 < g < 1 - d/n, \quad r < 1, \quad (41)$$

then a unique BGP  $(R_\Lambda, m_\Lambda, a_\Lambda)$  exists,

$$R_\Lambda(t) \approx \bar{R} e^{Ct}, \quad y_\Lambda(t) \sim e^{Ct}, \quad \beta_\Lambda(t) \sim e^{Cn/d}, \quad m_\Lambda(t) = \bar{M} e^{-gt}, \quad a_\Lambda(t) = t - T, \quad (42)$$

where  $C = \frac{gd}{n-d}$ ,  $\bar{M} = \bar{E}g / (1 - e^{-gT})$ , and the positive constants  $\bar{R}$  and  $T$  are found from formulas (33) and (34).

**Proof.** It essentially follows the proof of Theorem 3 and leads to similar expressions with the exception that now  $m(t)$  decreases rather than increases with the rate  $g$ . Formulas (36)-(39) remain valid.

To keep  $I_R'(t) = 0$  by (38), we need an increasing  $R(t) \sim e^{Ct}$  and a decreasing  $m(t) \sim e^{-gt}$  with  $g = C(1 - d/n) > 0$ . If  $m(t)$  decreases exponentially, then by (10)  $E_{\max}(t)$  also must decrease exponentially with the same rate  $-g$  to have a BGP.

The main difference in the proof is that  $\beta_\Lambda(t)m_\Lambda(t) - R_\Lambda(t) < y_\Lambda(t)$  at large  $t$ , because the second term in brackets in (40) is negative at  $n < d$ . So, we assume that  $r$  is small,  $r \ll 1$ .

By (45),  $Cn/d < r \ll 1$  is also small. Then, as shown in (Hritonenko and Yatsenko 1996), the unique solution  $T$  of equation (34) is large such that  $T \sim (Cn/d)^{-0.5}$ . Therefore,  $nT/d \ll 1$  and  $CT \ll 1$ . Expressing the exponents in (40) as the Taylor series, we have

$$y_\Lambda(t) - \beta_\Lambda(t)m_\Lambda(t) - R_\Lambda(t) \approx \left(\frac{bd}{Cn}\right)^{1/d} \frac{n\bar{R}^{n/d}\bar{M}e^{Ct}}{(r-C+Cn)} \left\{ \frac{r-C}{n}(T-1+n) - \frac{Cn}{d}T + CT \right\}$$

Finally, because  $T$  is large, the last equality leads to

$$\begin{aligned} y_\Lambda(t) - \beta_\Lambda(t)m_\Lambda(t) - R_\Lambda(t) &\approx \left(\frac{bd}{Cn}\right)^{1/d} \frac{n\bar{R}^{n/d}\bar{M}e^{Ct}}{(r-C+Cn)} CT \left\{ \frac{r}{Cn} - \frac{1}{n} + 1 - \frac{n}{d} + o(T^{-1}) \right\} \\ &> \left(\frac{bd}{Cn}\right)^{1/d} \frac{n\bar{R}^{n/d}\bar{M}e^{Ct}}{(r-C+Cn)} CT \left\{ \left(\frac{n}{d} - 1\right) \left(\frac{1}{n} - 1\right) + o(T^{-1}) \right\} > 0 \end{aligned}$$

The theorem is proven.  $\square$

Therefore, our claims in the beginning of this section are verified. Sections 3.1 and 3.2 pave the way for the conception of multi-country extensions of our model with an international market of pollution permits. Because such extensions are not immediate and, especially, because they are computationally highly demanding, they are out of the scope of this paper. Rather, we provide with the analysis of transitional dynamics in our single country model, which is novel in some essential aspects.

### 3.3. Balanced growth at $n=d$

Let us address the situation when the parameter of “R&D efficiency”  $n$  equals the parameter of “R&D complexity”  $d$ . Then, an interior BGP regime is possible only if the quota  $E_{\max}(t)$  is constant.<sup>8</sup>

**Theorem 5.** *If  $n=d$  and the environmental quota  $E_{\max}(t)$  is not constant at large  $t$ , then no BGP with positive growth exists.*

**Proof.** By Theorems 1 and 2, any interior regime  $(R, m, a)$  has to satisfy the nonlinear system (26)-(28). Let  $R(t)=\bar{R} e^{Ct}$  and  $t-a(t)=T=\text{const}>0$ . Then, (26) leads to (37). Let us assume that  $E_{\max}(t)$  varies in time. Then,  $m(t)$  cannot be constant by (37).

On the other side, in our case  $\beta(t) \approx \bar{R} \left(\frac{b}{C}\right)^{1/n} e^{Ct}$  and equality (27) is

$$bn\bar{R}^{n-1} e^{C(n-1)t} \int_t^{\infty} \left[\frac{b}{C} \bar{R} e^{C\tau}\right]^{(1-n)} m(\tau) \left[\frac{1}{r} - \frac{e^{-rT}}{r} - 1\right] e^{-r\tau} d\tau \approx e^{-rt} \quad (43)$$

Differentiating (43), we have

$$bn \left[\frac{b}{C}\right]^{(1-n)} m(t) \left[\frac{1}{r} - \frac{e^{-rT}}{r} - 1\right] e^{-rt} e^{C(1-n)t} = d(e^{-rt} e^{C(1-n)t}) / dt$$

It means that  $m(t)$  must be constant to satisfy (43). Hence, no BGP exists.

The theorem is proven.  $\square$

<sup>8</sup> Section 3.1 is technically similar to BHY, so we expose the case  $n=d$  briefly.

We now move to the case of constant economic and institutional environment, which is the case where BGPs typically arise. The findings are summarized in Theorem 6 below. This theorem is the optimal growth counterpart of Theorem 3 established in BHY (page 193) for the firm problem.

**Theorem 6.** *If  $n=d$  and  $E_{\max}(t)=\bar{E}=\text{const}$ , then an interior solution of problem (7)-(12)*

$$R_A(t) \approx \bar{R} e^{Ct}, \quad \beta_A(t) \sim e^{Ct}, \quad y_A(t) \sim e^{Ct}, \quad m_A(t) = \bar{M} = \text{const}, \quad a_A(t) = t - \bar{E} / \bar{M}, \quad (44)$$

*is possible, where constants  $C$  and  $\bar{M}$  are determined by the nonlinear system*

$$C^{1/d} [r/C + d - 1] = d\bar{M}b^{1/d} \left[ \frac{1 - e^{-r\bar{E}/\bar{M}}}{r} - 1 \right], \quad (45)$$

$$\cdot \frac{1 - e^{-r\bar{E}/\bar{M}}}{r} - \frac{e^{-C\bar{E}/\bar{M}} - e^{-r\bar{E}/\bar{M}}}{r - C} = 1, \quad (46)$$

*The solution  $(R_A, m_A, a_A)$  exists and represents a BGP, at least, in the following cases:*

(i)  $d > 0.5$  and large enough  $\bar{E}$ ; then the optimal  $C \rightarrow 0$  and  $t - a_A(t) \rightarrow \infty$  as  $\bar{E} \rightarrow \infty$ .

(ii)  $r\bar{E} \ll 1$ ,  $r^{1/d} < \bar{E}b^{1/d} [1 - \sqrt{2r}]$ , (47)

*then  $C$ ,  $0 < C < r$ , is a solution of the nonlinear equation*

$$C^{(1-d)/d} [r - C(1-d)] = d\bar{E}b^{1/d} \left[ 1 - \sqrt{\frac{1}{2} \left( \frac{r}{\sqrt{C}} + \sqrt{C} \right)} \right] + o(r) \quad (48)$$

*and  $\bar{M} = \bar{E} \sqrt{C/2} + o(r)$ . The uniqueness of the solution is guaranteed if*

$$r^{1/d-1/2} < \frac{d^2}{4(1-d)} \bar{E}b^{1/d} \sqrt{2}. \quad (49)$$

**Proof.** Formulas (44)-(49) follow from Theorem 3 in BHY, where it is also shown that the system (45)-(46) has a solution  $C > 0$  and  $\bar{M} > 0$  in the cases (i) and (ii).

To prove that the path (44) is a BGP indeed, we need to show that the state constant  $y_A(t) - \beta_A(t)m_A(t) - R_A(t) > 0$  holds along (44), at least, at large  $t$ . By (8), (29), and (32),

$$y_A(t) \approx \bar{R}\bar{M} \left( \frac{b}{C} \right)^{1/d} \frac{1 - e^{-C\bar{E}/\bar{M}}}{C} e^{Ct}. \quad \text{Therefore,}$$

$$y_{\Lambda}(t) - \beta_{\Lambda}(t)m_{\Lambda}(t) - R_{\Lambda}(t) \approx \bar{\text{Re}}^{Ct} \left\{ \bar{M} \left( \frac{b}{C} \right)^{1/d} \left[ \frac{1 - e^{-C\bar{E}/\bar{M}}}{C} - 1 \right] - 1 \right\}.$$

Expressing the exponent above as the Taylor series, we obtain

$$y_{\Lambda}(t) - \beta_{\Lambda}(t)m_{\Lambda}(t) - R_{\Lambda}(t) = \bar{R} \left\{ \bar{M} \left( \frac{b}{C} \right)^{1/d} \left[ \bar{E}/\bar{M} - \frac{C}{2} (\bar{E}/\bar{M})^2 + o(r) - 1 \right] - 1 \right\} e^{Ct}.$$

On the other side, expressing the exponent in (33) as the Taylor series, we have

$$C^{(1-d)/d} [r - C(1-d)] = d\bar{M}b^{1/d} \left[ \bar{E}/\bar{M} - \frac{r}{2} (\bar{E}/\bar{M})^2 + o(r) - 1 \right].$$

Combining the last two formulas, we obtain

$$y_{\Lambda}(t) - \beta_{\Lambda}(t)m_{\Lambda}(t) - R_{\Lambda}(t) \approx \bar{R} \left\{ \frac{r - C(1-d)}{Cd} - 1 \right\} e^{Ct} = \frac{r - C}{Cd} \bar{\text{Re}}^{Ct} > 0.$$

The theorem is proven.  $\square$

The conditions (i) and (ii) are sufficient for the existence of the BGP. Of course, the BGP can also exist when these conditions do not hold. The uniqueness condition (49) is also sufficient. The only possible case of non-uniqueness when we need condition (49) is when the optimal  $C$  is close to  $r$ .

It is clear that the BGP in case  $n=d$  is also induced by the R&D sector of the economy and illustrates a Porter-like mechanism. Indeed, as statement (i) of Theorem 6 indicates, the growth rate tends to zero when the quota disappears (increases indefinitely). The long term growth is endogenous and is determined by the model parameters  $r$  and  $d$  and the quota level  $\bar{E}$ . It can readily shown that a further decrease of  $\bar{E}$  leads to the decrease of both optimal growth rate  $C$  and optimal investment in efficiency units  $\bar{M}$ . In other words, while an induced-innovation mechanism is at work, strengthening environmental regulation by tightening quotas negatively affects the rate of innovation and growth of the economy. As in BHY's firm problem, we uncover a kind of *mild Porter-like mechanism*: quotas are necessary for R&D to get launched but too strict quotas kill the growth. In the

balanced case  $n=d$ , the innovation rate, that is the growth rate of  $\beta(t)$ , is equal to the growth rate of production (and investment variables  $i(t)$  and  $R(t)$ ).<sup>9</sup>

Finally, it is also interesting to note that two different BGPs are possible in the firm's model of BHY, one being a Porter-induced BGP and the other one caused by a monotonically increasing (exogenous) price of energy resource. Without the resource scarcity, the only sustainable regime there is the BGP analogous to our (44). When the resource becomes increasingly scarcer, this BGP ceases to exist and another scarcity generated BGP appears.

#### 4. Transition Dynamics

We can show that the short-term dynamics will remain qualitatively the same for any bounded regulation function  $E_{\max}(t)$ , provided that a long-term interior regime exists. However, as shown in Section 3, essential difficulties arise in finding such regimes. For this reason and for clarity sake, we restrict ourselves in this section with the case of  $n=d$  and a constant function  $E_{\max}(t)=\bar{E}$ . The long-term interior regime in this case is the BGP  $(R_A, m_A, a_A)$  determined by Theorem 6.

As proven in Theorem 2, the long-term dynamics necessarily involves the active environmental balance restriction (10). In this section, we will show that:

(1) All Cases A-D from Theorem 1 are possible in short-term dynamics. The optimal trajectories during the transition period appear to be qualitatively different depending on whether the environmental restriction (10) is initially active,  $E(0)=E_{\max}$  (Cases B and C), or inactive,  $E(0)<E_{\max}$  (Cases A and D).

(2) The short-term transition dynamics always leads to the long-term interior regime with the active environmental restriction.

The solution  $R^*(t)$ ,  $m^*(t)$ , and  $a^*(t)$ ,  $t \in [0, \infty)$ , of the optimization problem (7)-(12) must satisfy the initial conditions (12). The essential initial condition is  $a(0)=a_0$  because the unknown  $a(t)$  is continuous. If  $a_0 \neq a_A(0)$ , then the dynamics of  $(R^*, m^*, a^*)$  involves a

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<sup>9</sup> We don't detail here the computation of the BGP. It goes without saying that given that growth is endogenous, we also face a problem of indeterminacy in levels. This technical point is made precisely in BHY.



transition from the initial state  $a(0)=a_0$  to the long-term interior trajectory  $a_A(t)$  from Theorem 6. Also, the given model functions shall satisfy the inequality

$$R_0(0) + Bm_0(0) \leq \int_{-a_0}^0 \left[ \beta_0 + \int_{-a_0}^{\tau} R_0(v)dv \right] m_0(\tau) d\tau . \quad (50)$$

Otherwise, the constraint (11) is violated at  $t=0$  and the economic system is not possible.

#### 4.1. Optimal intensive growth at active environment regulation.

Let  $E(t)=E_{\max}$  starting from the initial time  $t=0$  (**the case of a country-polluter**). Then the optimal dynamics is subjected to Case A or D of Theorem 1 (with the active restriction  $E(t)=E_{\max}$  on  $[0, \infty)$ ). This regime is a **growth with intensive capital renovation** induced by technical progress. In order to make a new capital investment  $m(t)$  at  $t \geq t_k$ , some obsolete capital  $m(a(t))a'(t)$  should be removed, following the equality (10) under the given  $E(t)=E_{\max}$  or

$$\int_{a(t)}^t m(\tau) d\tau = E_{\max}.$$

In the long-term dynamics considered in Section 3, the optimal R&D innovation  $R^*(t)$  is the interior trajectory  $R_A(t)$  determined from  $I_R'(t)=0$ , where  $I_R'(t)$  is given by (16). The optimal  $R^*(t)$  reaches the trajectory  $R_A(t)$  immediately at  $t=0$ . The long-term dynamics has the interior turnpike trajectory  $a_A$  for the capital lifetime, determined from  $I_m'(t)=0$  or

$$\int_t^{a^{-1}(t)} e^{-r\tau} [\beta(t) - \beta(a(\tau))] d\tau = e^{-rt} \beta(t).$$

If  $a_0=a_A(0)$ , then the optimal capital lifetime  $a^* \equiv a_A$ , that is, no transition dynamics at all. If  $a_0 \neq a_A(0)$ , then we can show that the optimal  $a^*(t)$  will reach  $a_A(t)$  at some time  $t_1 > 0$ . If  $a_0 > a_A(0)$ , then the optimal investment  $m^*(t)=0$  is minimal at  $0 < t \leq t_1$  (Case A). If  $a_0 < a_A(0)$ , then the optimal investment  $m^*(t)=(y^*(t) - R^*(t))/\beta^*(t)$  is maximal at  $0 < t \leq t_1$  (Case D).

After the transition, at  $t > t_1$ , the optimal long-term trajectory  $m^*(t)$ , in a general case, possesses a repetitive pattern (Hritonenko and Yatsenko, 1996, and Boucekine, Germain and Licandro, 1997) determined by the dynamics of  $m(t)$  on the interval  $[a_0, t_1]$ . These

replacement echoes are absent at the “perfect” initial condition  $a_0=a_A(0)$ ,  $m_0(\tau)=\bar{M}$ ,  $\tau \in [a_0, 0]$ . To illustrate them, we provide a numeric example shown in Figure 1 that will be used and developed in the next section.

**Example 1.** Let

$$r=0.05, \quad d=n=0.5, \quad b=0.005, \quad E_{\max}(t)=\bar{E}=22,$$

$$a_0=-22, \quad \beta_0=1, \quad R_0(\tau)=0, \quad m_0(\tau)=1, \quad \tau \in [-22, 0].$$

Then,  $B=\beta(0)=1$  by (13) and the BGP exists:

$$R_A(t)=R_0e^{Ct}, \quad C=0.00225, \quad m_A(t)=M_0=0.55, \quad a_A(t)=t-40, \quad t \in [0, \infty).$$

The BGP is indicated by the dotted lines in Figure 1. In this case,  $E(0)=m_0a_0=22$  is equal to  $E_{\max}(0)=\bar{E}$ , hence, the environmental balance (10) is active starting  $t=0$ . Since  $a_A(0)=-40 < a_0=-22$ , then the optimal  $a^*(t)=-22$  and  $m^*(t)=0$  at  $0 < t \leq t_1=18$  (Case A). After  $t_1$ , the optimal  $a^*(t)$  coincides with  $a_A(t)$  and  $m^*(t)=m^*(t-40)$  exhibits *replacement echoes* (shown with dotted lines).

## 4.2. Optimal extensive growth

Let the energy pollution  $E(t)$  be lower than the limit  $E_{\max}$  at time  $t=0$ , which means that the country is not initially a big polluter.

We assume that  $E(t) < E_{\max}$  over a finite interval  $0 \leq t < t_k$ , where the moment  $t_k$  is to be determined. Then, we have Case B or C of Theorem 1, at least, at the beginning of the planning horizon. Since  $I_a'(t) < 0$  by (20), the boundary regime  $a^*(t) \equiv a_0$  is always optimal while  $E(t) < E_{\max}$ .

First, let  $m(t) < (y(t) - R(t))/\beta(t)$  (Case B), then  $I_m'(t) \leq 0$ , otherwise the optimal investment  $m^*$  is maximal possible and we immediately switch to Case C. By (19), Case B is highly unlikely in economic practice. It means an extremely underfunded initial capital (determined by the length  $a_0$  of prehistory) combined with a high impatience (high discount rate  $r$ ). Indeed, simple calculations show that for the discount rates  $10\% < r < 50\%$ , Case B happens if the initial prehistory length  $a_0$  is less than 1.05 - 1.4 years. For such values of  $a_0$ , the constraint (50) imposes extremely severe restrictions on the initial functions  $m_0$  and  $R_0$  and value  $\beta$ . In Case B, the optimal investment  $m^*$  is zero and no

capital scrapping occurs, which corresponds to the trivial solution  $R^0 \equiv 0$ ,  $m^0 \equiv 0$ ,  $a^0$  of the problem (7)-(12) described in Remark 1. In this case, the non-trivial long-run solution with investing into new capital and R&D is not possible.

For economically reasonable values of the discount  $r < 10\%$ /year and the initial capital lifetime  $a_0$  more than one year,  $I_m'(t) > 0$  by (19). Hence, the optimal investment  $m^*$  is maximal possible and we have Case C. Then, the country can use more new capital and there is no need to remove the old one, which can be classified as ***an extensive economic growth***. The upper bound for  $m(t)$  is given by the constraint (11) and the optimal  $m^*(t)$  jumps to this bound immediately after  $t=0$ . In this case, the inequality-constraint  $m(t) \leq (y(t) - R(t))/\beta(t)$  limits both optimal controls  $R^*$  and  $m^*$ . Therefore, the transition dynamics on some initial period  $[0, t_k]$  is determined by the restriction

$$R^*(t) + \beta^*(t)m^*(t) = y^*(t) \quad (51)$$

until  $E(t_k) = E_{\max}$ . The energy pollution amount  $E(t) = \int_{a_0}^t m^*(\tau) d\tau$  is accumulated fast and the energy regulation limit  $E_{\max}$  will be reached soon, which will mean the end of the extensive growth phase. Following Case C of Theorem 1, the optimal  $R^*(t)$ ,  $m^*(t)$  and  $y^*(t)$  over  $[0, t_k]$  are determined from the system of three nonlinear equations (10), (51), and  $I_R'(t) = 0$ .

The end  $t_k$  of the “extensive-growth” transition period  $[0, t_k]$  is determined from the condition  $E(t_k) = E_{\max}$ . After the transition period  $[0, t_k]$ , the optimal dynamics will switch to the scenario of Section 4.1 with the active constraint (10).

If  $a^*(t_k) \neq a_A(t_k)$ , then the “extensive-growth” transition on  $[0, t_k]$  is followed by one of the intensive growth transition scenarios on  $[t_k, t_1]$ ,  $t_1 > t_k$ , described above in Section 4.1. If  $a^*(t_k) > a_A(t_k)$ , then the optimal investment  $m^*(t) = 0$  is minimal on  $[t_k, t_1]$  (Case A). If  $a^*(t_k) < a_A(t_k)$ , then the optimal investment  $m^*(t) = (y^*(t) - R^*(t))/\beta^*(t)$  is maximal on  $[t_k, t_1]$  (Case D).

**Example 2.** Let all given parameters be as in Example 1 except  $m_0(\vartheta) = 0.5$ ,  $\varpi \in [-22, 0]$ . Then the BGP is the same as in Example 1 but the transition dynamics is different and is shown in Figure 2.

In this case,  $E(0)=m_0a_0=0.5*22=11$  is less than  $E_{\max}(0)=22$ , hence, the environmental balance (10) is inactive on an initial interval  $[0, t_k]$  at the beginning of the planning horizon. The dynamics of the optimal  $m^*(t)$  and  $R^*(t)$  on  $[0, t_k]$  follows the restriction  $R^*(t)+\beta^*(t)m^*(t)=y^*(t)$  (Case C of Theorem 1). The optimal  $R^*(t)$  over  $[0, t_k]$  is found from (27) as

$$R^{1/2}(t) = 2b\chi^{-1}(t) \int_{t_k}^{\infty} \beta^{1-d}(\tau) m(\tau) \left[ \frac{e^{-r\tau} - e^{-ra^{-1}(\tau)}}{r} - e^{-r\tau} \right] d\tau$$

where  $\chi(t) = \int_t^{t_k} \chi(\tau) d\tau + \int_{t_k}^{a^{-1}(t)} e^{-r\tau} d\tau$  over  $[0, t_k]$  is found from (23) and  $\chi(t) = e^{-rt}$  on  $[t_k, \infty)$ .

Finding an approximate solution of the arising equations, we obtain that  $R^*(t) \approx 0.2$  at  $0 \leq t \leq t_k$ . Then, the optimal  $m^*(t) \approx 10.8$  at  $t=0$  and  $m^*(t) \approx 21.8$  at  $t=t_k$ . The corresponding  $E^*(t)$  increases fast and reaches the limit value  $E_{\max}=22$  at  $t_k \approx 0.75$ . The corresponding  $y^*(t)$  also increases fast from  $y^*(0) \approx 11$  to  $y^*(t_k) \approx 22$ .

The further optimal dynamics on  $[t_k, \infty)$  is similar to Example 1 and follows Case A. It is shown in Figure 2 with black curves.

As opposed to the “intensive-growth” scenario of Example 2, the optimal trajectory  $m^*(t)$  always possesses the replacement echoes after the transition. Indeed, no “perfect” initial condition is possible in this case. If  $m_0(\tau) = \bar{M}$  on  $[a_0, 0]$ , then  $a_0 > a_A(0)$  by  $E(0) < E_{\max}$ . Alternatively, if  $a_0 = a_A(0)$ , then  $\int_{a_0}^0 m_0(\tau) d\tau < \int_{a_0}^0 \bar{M} d\tau$ . The optimal short-term trajectory  $m^*(t)$  is different from  $\bar{M}$  on the “extensive-growth” transition period  $[0, t_k]$ , and the optimal trajectory  $m^*(t)$  will repeat the dynamics of  $m(t)$  on  $[a_0, t_k]$ .

We can summarize the above reasoning in the following statement.

**Theorem 7.** *In the case  $n=d$  and a constant regulation quota  $E_{\max}$ , the transition dynamics of the problem (7)-(12) leads to the BGP with active energy regulation (described by Theorem 6) in a finite time  $t_k \geq 0$ , regardless how large the value  $E_{\max}$  is. The transition dynamics is absent ( $t_k=0$ ) only if*

$$a_0 = a_A(0) \quad \text{and} \quad E(0) = E_{\max}. \quad (52)$$

If (52) holds and  $m_0 \equiv \bar{M}$ , then the optimal  $m^* \equiv \bar{M}$ , otherwise, the optimal trajectory  $m^*$  possesses everlasting replacement echoes that repeat the dynamics of  $m^*$  on the interval  $[a_0, t_k]$ .

This theorem describes the complete dynamics of the central planner problem (7)-(12) in case  $d=n$ . The dynamics will be qualitatively similar for any values of  $n$  and  $d$  and any bounded regulation function  $E_{\max}(t)$ . As it is already shown in Section 3, the presence of the environmental quota constraint is essential for getting a meaningful optimal dynamics in the central planner problem (7)-(12). Another justification is that the model (8)-(12) with no quota ( $E_{\max}=\infty$ ) has only a *blow-up solution* that strives to  $\infty$  in a finite time for any  $n$  and  $d$  and the corresponding objective functional (7) is always infinite (Yatsenko, Boucekkine and Hritonenko, 2009).

The optimal dynamics highlighted in this scenario are quite new in the related economic literature (see for example, Boucekkine, Germain and Licandro, 1997). They deserve some comments:

- i) At first, note that the modernization policy chosen by the firm consists in increasing investment in new equipment and R&D **without** scrapping the older and more resource consuming machines. In Hritonenko and Yatsenko (1996) and Boucekkine et al. (1997), the modernization policy also encompasses scrapping part of the older capital goods in a way similar to the intensive growth scenario described above. The reason behind this difference is quite elementary: a firm with a low enough initial capital stock (and so, with low enough initial resource consumption) has no incentive to scrap its old machines as long as its resource quota constraint is not binding. In contrary, at a binding quota, investing in new machines is not possible without scrapping some obsolete older machines because of market clearing conditions or binding regulation constraints.
- ii) Note that in our case firms which are historically “small” polluters are precisely those which are historically “small” producers. Extended to a country level, our exercise predicts that historically poor countries will find it optimal to massively

invest and therefore to massively pollute during their development process. During such a transition, new and clean machines will co-exist with old and dirty machines in the productive sectors, implying an unambiguously dirty transition, mimicking the increasing part of the environmental Kuznets curve.

## 5. Concluding remarks

In this paper, we have studied the optimal R&D, investment and replacement policies in an economy subject to a fixed pollution quota with no access to international pollution permits markets. In the first place, we have significantly extended previous work by characterizing all possible balanced growth paths for any parameterizations of the R&D technology. In particular, we have shown that countries with under-performing R&D sector would need an increasing pollution quota over time to ensure balanced growth while countries with a highly efficient R&D sector should supply part of their assigned pollution permits to an international market without harming their long-term growth. In second place, we have studied transitional dynamics to balanced growth, a task not undertaken so far. We have uncovered two optimal transition regimes: an intensive growth (sustained investment in new capital and R&D **with** scrapping the oldest capital goods), and an extensive growth (sustained investment in new capital and R&D **without** scrapping the oldest capital).

As mentioned repeatedly along the text, a natural extension of the present work is to consider multi-country extensions of the model and to incorporate an international market of pollution permits. Such extensions would allow for asymmetric countries (due to different R&D technologies in the sense of Sections 3.1 and 3.2) and study optimal allocations of permits and its implications for long-term in each country. If the vintage structures and associated scrapping decisions are kept, it is likely that the required solution approach would be less analytical than the one followed in the single country case studied in this paper. Another possible extension is to sacrifice the endogenous scrapping of capital in order to obtain an analytically tractable multi-country vintage model with international pollution market.

## Appendix

**Proof of Theorem 1:** The proof is based on perturbation techniques of the optimization theory. It extends the approach that has been earlier applied by Hritonenko and Yatsenko (2005, 2008) to vintage models with exogenous technological change and state constraints.

**Case B.** Let the restrictions (10),(11) be inactive on a certain subset  $\Delta$  of the interval  $[0, \infty)$ :  $E^*(t) < E_{\max}(t)$  and  $R(t) + \beta(t)m(t) < y(t)$  at  $t \in \Delta \subset [0, \infty)$ . We choose  $R$ ,  $m$ , and  $v = a'$  to be the *independent* unknown variables of the OP (7)-(12). Then, the differential restriction  $a'(t) \geq 0$  in (11) takes the standard form  $v(t) \geq 0$ . The dependent variables  $y(t)$ ,  $E(t)$  and  $\beta(t)$  can be found from (8), (10), and (13). We assume that  $R, m, v \in L^\infty_{\text{loc}}[0, \infty)$ .

We refer to measurable functions  $\delta R$ ,  $\delta m$ , and  $\delta v$  as *admissible variations*, if  $R, m, v, R + \delta R, m + \delta m$ , and  $v + \delta v$ , satisfy constraints (8)-(11). Let us give small admissible variations  $\delta R(t)$ ,  $\delta m(t)$ , and  $\delta v(t)$ ,  $t \in (0, \infty)$ , to  $R, a, m$ , and find the corresponding variation  $\delta I = I(R + \delta R, m + \delta m, v + \delta v) - I(R, m, v)$  of the objective functional  $I$ . Using (7)-(10) and (13), we obtain that

$$\begin{aligned} \delta I = & \int_0^\infty e^{-rt} \left[ \int_{a(t) + \delta a(t)}^t \left( db \int_0^\tau (R(\xi) + \delta R(\xi))^n d\xi + B^d \right)^{\frac{1}{d}} (m(\tau) + \delta m(\tau)) d\tau \right. \\ & \left. - (R(t) + \delta R(t)) - \left( db \int_0^t (R(\xi) + \delta R(\xi))^n d\xi + B^d \right)^{\frac{1}{d}} (m(t) + \delta m(t)) \right] dt \\ & - \int_0^\infty e^{-rt} \left[ \int_{a(t)}^t \left( db \int_0^\tau R^n(\xi) d\xi + B^d \right)^{\frac{1}{d}} m(\tau) d\tau - m(t) \left( db \int_0^t R^n(\xi) d\xi + B^d \right)^{\frac{1}{d}} + R(t) \right] dt \end{aligned} \quad (\text{A1})$$

where  $\delta a(t) = \int_0^t \delta v(\xi) d\xi$ . To prove the theorem, we shall transform the expression (A1) to the form

$$\delta I = \int_0^\infty (I'_R(t) \cdot \delta R(t) + I'_m(t) \cdot \delta m(t) + I'_v(t) \cdot \delta v(t)) dt + o(\|\delta R\|, \|\delta m\|, \|\delta v\|), \quad (\text{A2})$$

where the norm is  $\|f\| = \text{ess sup}_{[0, \infty)} |e^{-rt} f(t)|$ . This transformation involves several steps. First,

applying the Taylor expansion  $f(x + \delta x) = f(x) + f'(x)\delta x + o(\delta x)$  twice, we have

$$\begin{aligned}
& \left( db \int_0^\tau (R(\xi) + \delta R(\xi))^n d\xi + B^d \right)^{\frac{1}{d}} \\
&= \left( db \int_0^\tau (R^n(\xi) + nR^{n-1}(\xi)\delta R(\xi) + o(\delta R(\xi))) d\xi + B^d \right)^{\frac{1}{d}} \\
&= \beta(\tau) + bn\beta^{1-d}(\tau) \int_0^\tau R^{n-1}(\xi)\delta R(\xi) d\xi + \int_0^\tau o(\delta R(\xi)) d\xi.
\end{aligned} \tag{A3}$$

Next, using (A3) and the elementary property  $\int_{a(t)+\delta a(t)}^t f(\tau) d\tau = \int_{a(t)}^t f(\tau) d\tau - \int_{a(t)}^{a(t)+\delta a(t)} f(\tau) d\tau$  of

integrals, we transform (A1) to

$$\begin{aligned}
\delta I &= \int_0^\infty e^{-rt} [bn \int_{\max\{a(t),0\}}^t m(\tau) \beta^{1-d}(\tau) \int_0^\tau R^{n-1}(\xi) \delta R(\xi) d\xi d\tau dt \\
&\quad - \int_0^\infty e^{-rt} \int_{\max\{a(t),0\}}^t \beta(\tau) \delta m(\tau) d\tau dt \\
&\quad - \int_0^\infty e^{-rt} \int_{a(t)}^{a(t)+\delta a(t)} \beta(\tau) m(\tau) d\tau dt - \int_0^\infty e^{-rt} [\delta R(t) + \beta(t) \delta m(t)] dt \\
&\quad - \int_0^\infty e^{-rt} m(t) \beta^{1-d}(t) \int_0^t bn R^{n-1}(\xi) \delta R(\xi) d\xi dt + \int_0^\infty e^{-rt} o(\delta R(t), \delta m(t)) dt,
\end{aligned} \tag{A4}$$

where  $\max\{a(t),0\}$  appears because the variations  $\delta R(t)$ ,  $\delta m(t)$  are zero on the interval  $[a_0,0]$ .

Next, we interchange the limits of integration in the second term of (A4) as

$$\int_0^\infty e^{-rt} \left[ \int_{a(t)}^t \beta(\tau) \delta m(\tau) d\tau dt \right] = \int_0^\infty \int_t^{a^{-1}(t)} e^{-r\tau} d\tau \cdot \beta(t) \delta m(t) dt,$$

in the first term as

$$\begin{aligned}
& \int_0^\infty e^{-rt} \left[ \int_{\max\{a(t),0\}}^t m(\tau) \beta^{1-d}(\tau) \int_0^\tau bn R^{n-1}(\xi) \delta R(\xi) d\xi d\tau \right] dt \\
&= bn \int_0^\infty \int_t^\infty \int_\tau^{a^{-1}(\tau)} e^{-r\xi} d\xi \cdot m(\tau) \beta(\tau)^{1-d} d\tau \cdot R^{n-1}(t) \delta R(t) dt,
\end{aligned}$$

and similarly in the fifth term. To transform the third term, we use the Taylor expansion

$$\int_{a(t)}^{a(t)+\delta a(t)} f(t, \tau) d\tau = f(t, a(t)) + o(\delta a(t)).$$

Then, collecting coefficients of  $\delta R$ ,  $\delta m$ , and  $\delta a$ , we

rewrite (A4) as:



$$\begin{aligned}
\delta I &= \int_0^\infty [-e^{-rt} + bn] \int_t^{a^{-1}(\tau)} (e^{-r\xi} d\xi - e^{-r\tau}) \cdot m(\tau) \beta^{1-d}(\tau) d\tau \cdot R^{n-1}(t) \cdot \delta R(t) dt \\
&+ \int_0^\infty \left[ \int_t^{a^{-1}(t)} e^{-r\tau} d\tau - e^{-rt} \right] \cdot \beta(t) \delta m(t) dt \\
&+ - \int_0^\infty e^{-rt} \beta(a(t)) m(a(t)) \cdot \delta a(t) dt + \int_0^\infty e^{-rt} o(\delta R(t), \delta m(t), \delta a(t)) dt.
\end{aligned}$$

Finally, recalling  $\delta a(t) = \int_0^t \delta v(\xi) d\xi$ , we convert the last expression to

$$\begin{aligned}
\delta I &= \int_0^\infty [-e^{-rt} + bn] \int_t^{a^{-1}(\tau)} (e^{-r\xi} d\xi - e^{-r\tau}) \cdot m(\tau) \beta^{1-d}(\tau) d\tau \cdot R^{n-1}(t) \cdot \delta R(t) dt \\
&+ \int_0^\infty \left[ \int_t^{a^{-1}(t)} e^{-r\tau} d\tau - e^{-rt} \right] \cdot \beta(t) \delta m(t) dt \\
&- \int_0^\infty \int_t^\infty e^{-r\tau} \beta(a(\tau)) m(a(\tau)) d\tau \cdot \delta v(t) dt + \int_0^\infty e^{-rt} o(\delta R(t), \delta m(t), \delta v(t)) dt
\end{aligned} \tag{A5}$$

Formula (A5) in notations (17), (19), (20) provides the required expression (A2). The domain (11) of admissible controls  $R, m, v$  has the simple standard form  $R \geq 0, m \geq 0, v \geq 0$ . So, the optimality condition (18) follows from the obvious necessary condition that the variation  $\delta I$  of functional  $I$  can not be positive for any admissible variations  $\delta R(t), \delta m(t), \delta v(t), t \in [0, \infty)$ .

**Case A.** If the constraint  $R(t) + \beta(t)m(t) < y(t)$  is inactive and the restriction of (10) is active:  $E(t) = E_{\max}(t)$  at  $t \in \Delta \subset [0, \infty)$ , then we choose  $R$  and  $m$  to be *independent unknowns* of the OP. The dependent (*state*) variable  $a$  is uniquely determined from the initial problem

$$m(a(t))a'(t) = m(t) - E_{\max}'(t), \quad a(0) = a_0,$$

obtained after differentiating (10). As shown in Hritonenko and Yatsenko (2008), if  $E_{\max}'(t) \leq 0$ , then for any measurable  $m(t) \geq 0$ , a unique a.e. continuous function  $a(t) < t$  exists and a.e. has  $a'(t) \geq 0$ <sup>10</sup>. Therefore, the state restrictions  $a'(t) \geq 0$  and  $a(t) < t$  in (11) are satisfied automatically, so we can exclude the dependent variable  $a$  from the optimality condition.

Similarly to the previous case, let us give small admissible variations  $\delta R(t)$  and  $\delta m(t), t \in [0, \infty)$ , to  $R$  and  $m$  and find the corresponding variation  $\delta I = I(R + \delta R, m + \delta m) - I(R, m)$  of the functional  $I$ . In this case, the variation  $\delta a$  is determined by  $\delta m$ . To find their connection, let us present (10) as

<sup>10</sup> For brevity, the theorem omits the possible case  $E_{\max}'(t) > 0$  that is also treated in Hritonenko and Yatsenko (2005, 2008).

$$E_{\max}(t) = \int_{a(t)}^t m(\tau) d\tau = \int_{a(t)+\delta a(t)}^t (m(\tau) + \delta m(\tau)) d\tau$$

then

$$\int_{\max\{a(t),0\}}^t \delta m(\tau) d\tau = \int_{a(t)}^{a(t)+\delta a(t)} m(\tau) d\tau + o(\|\delta m\|, \|\delta a\|). \quad (\text{A6})$$

Next, we use the above formula (A4) for the variation  $\delta I$  as a function of  $\delta R$ ,  $\delta m$ , and  $\delta a$  and eliminate  $\delta a$  from (A4) using (A6). To do that, we rewrite the third term of (A4) by adding

$$\pm \int_0^{\infty} e^{-rt} \beta(a(t)) \int_{a(t)}^{a(t)+\delta a(t)} m(\tau) d\tau dt \text{ and applying (A6) as}$$

$$\begin{aligned} & - \int_0^{\infty} e^{-rt} \int_{a(t)}^{a(t)+\delta a(t)} \beta(\tau) m(\tau) d\tau dt \\ & = - \int_0^{\infty} e^{-rt} \beta(a(t)) \int_{a(t)}^{a(t)+\delta a(t)} m(\tau) d\tau dt + \int_0^{\infty} e^{-rt} \int_{a(t)}^{a(t)+\delta a(t)} (\beta(a(t)) - \beta(\tau)) m(\tau) d\tau dt \\ & = - \int_0^{\infty} e^{-rt} \beta(a(t)) \int_{\max\{a(t),0\}}^t \delta m(\tau) d\tau dt + \int_0^{\infty} e^{-rt} o(\delta a(t), \delta m(t)) dt \\ & = - \int_0^{\infty} \int_t^{a^{-1}(\tau)} e^{-r\tau} \beta(a(\tau)) d\tau \cdot \delta m(t) dt + \int_0^{\infty} e^{-rt} o(\delta a(t), \delta m(t)) dt \end{aligned} \quad (\text{A7})$$

The integral  $\int_a^{a+\delta a} (\beta(a) - \beta(\tau)) m(\tau) d\tau$  in (A7) has the order  $o(\delta a)$  because  $\beta(\tau)$  is continuous.

Substituting (A7) into (A4) and collecting the coefficients of  $\delta m$  and  $\delta R$ , we obtain the expression

$$\delta I = \int_0^{\infty} (I'_R(t) \cdot \delta R(t) + I'_m(t) \cdot \delta m(t)) dt + o(\|\delta R\|, \|\delta m\|) \quad (\text{A8})$$

in the notations (16) and (17). The rest of the proof is identical to Case A.

**Case C.** Now the active constraint  $R(t) + \beta(t)m(t) = y(t)$  on  $\Delta$  involves four unknown variables. So, we cannot handle this constraint as easy as the constraint  $E(t) = E_{\max}(t)$  in Case B. We will apply the method of Lagrange multipliers to take into account the equality-constraint  $R(t) + \beta(t)m(t) = y(t)$ ,  $t \in \Delta$ .

Let us introduce the Lagrange multiplier  $\lambda(t)$ ,  $t \in [0, \infty)$ , for the equality  $R(t) + \beta(t)m(t) = y(t)$  on  $\Delta$  and make the usual assumption that  $\lambda(t) = 0$  at  $t \in [0, \infty) - \Delta$  because of the complementary slackness condition. Now we minimize the Lagrangian

$$L = I + \int_0^{\infty} (y(t) - R(t) - \beta(t)m(t))\lambda(t)dt \quad (\text{A9})$$

instead of the functional  $I$  (7). As in previous cases, we give small admissible variations  $\delta R(t)$ ,  $\delta m(t)$ , and  $\delta v(t)$ ,  $t \in (0, \infty)$ , to  $R$ ,  $m$ , and  $a$  and find the corresponding variation  $\delta L = L(R + \delta R, m + \delta m, v + \delta v) - L(R, m, v)$  of the objective functional (A9). Providing all necessary transformations as above, we arrive to the following expression

$$\delta L = \int_0^{\infty} (\hat{I}'_R(t) \cdot \delta R(t) + \hat{I}'_m(t) \cdot \delta m(t) + I'_v(t) \cdot \delta v(t))dt,$$

where

$$\hat{I}'_R(t) = bnR^{n-1}(t) \int_t^{\infty} \beta^{1-d}(\tau)m(\tau) \left[ \int_{\tau}^{a^{-1}(\tau)} e^{-r\xi} [1 - \lambda(\xi)] d\xi - e^{-r\tau} [1 - \lambda(\tau)]k(\tau) \right] d\tau - e^{-rt} [1 - \lambda(t)]$$

$$\hat{I}'_m(t) = \beta(t) \int_t^{a^{-1}(t)} e^{-r\tau} [1 - \lambda(\tau)] d\tau - e^{-rt} [1 - \lambda(t)]\beta(t),$$

and  $I'_v(t)$  is given by the same formula (20).

As usually in the method of Lagrange multipliers, we choose  $\lambda(t)$  from the condition  $\hat{I}'_m(t) = 0$  at  $t \in \Delta$ . Introducing the new variable  $\chi(t) = [1 - \lambda(t)]e^{-rt}$ , it leads to the formula (23). The expression for  $\hat{I}'_R(t)$  in the variable  $\chi$  is (22).

**Case D** is the combination of Case C and Case A. It is proven by combining reasoning and transformations of Cases A and C. The theorem is proven.  $\square$

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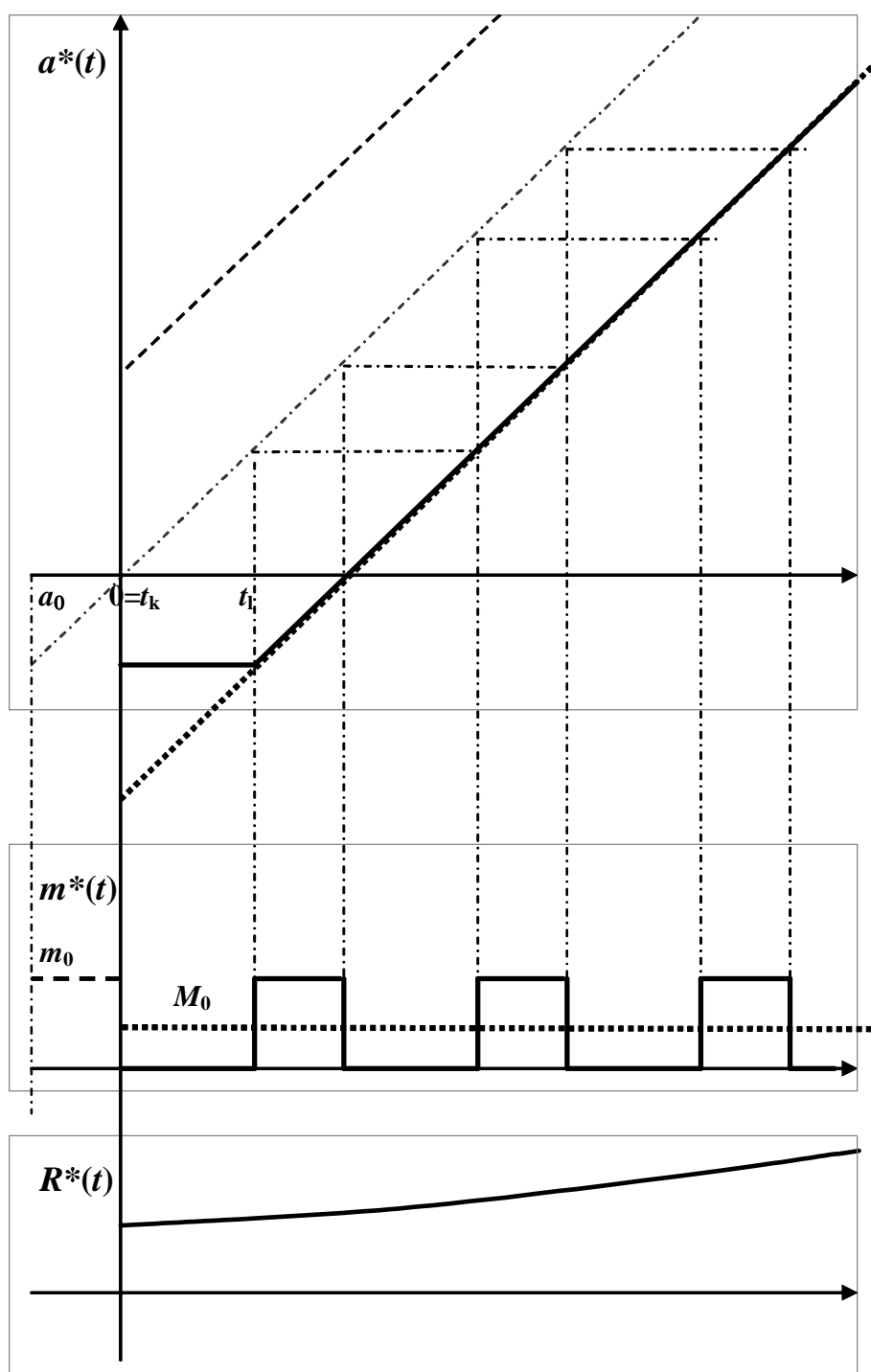


Figure 1. Transition and long-term dynamics under active environment regulation from Example 1. The dashed line shows the inverse function  $a^{-1}$ . The dotted lines indicate the BGP regime.

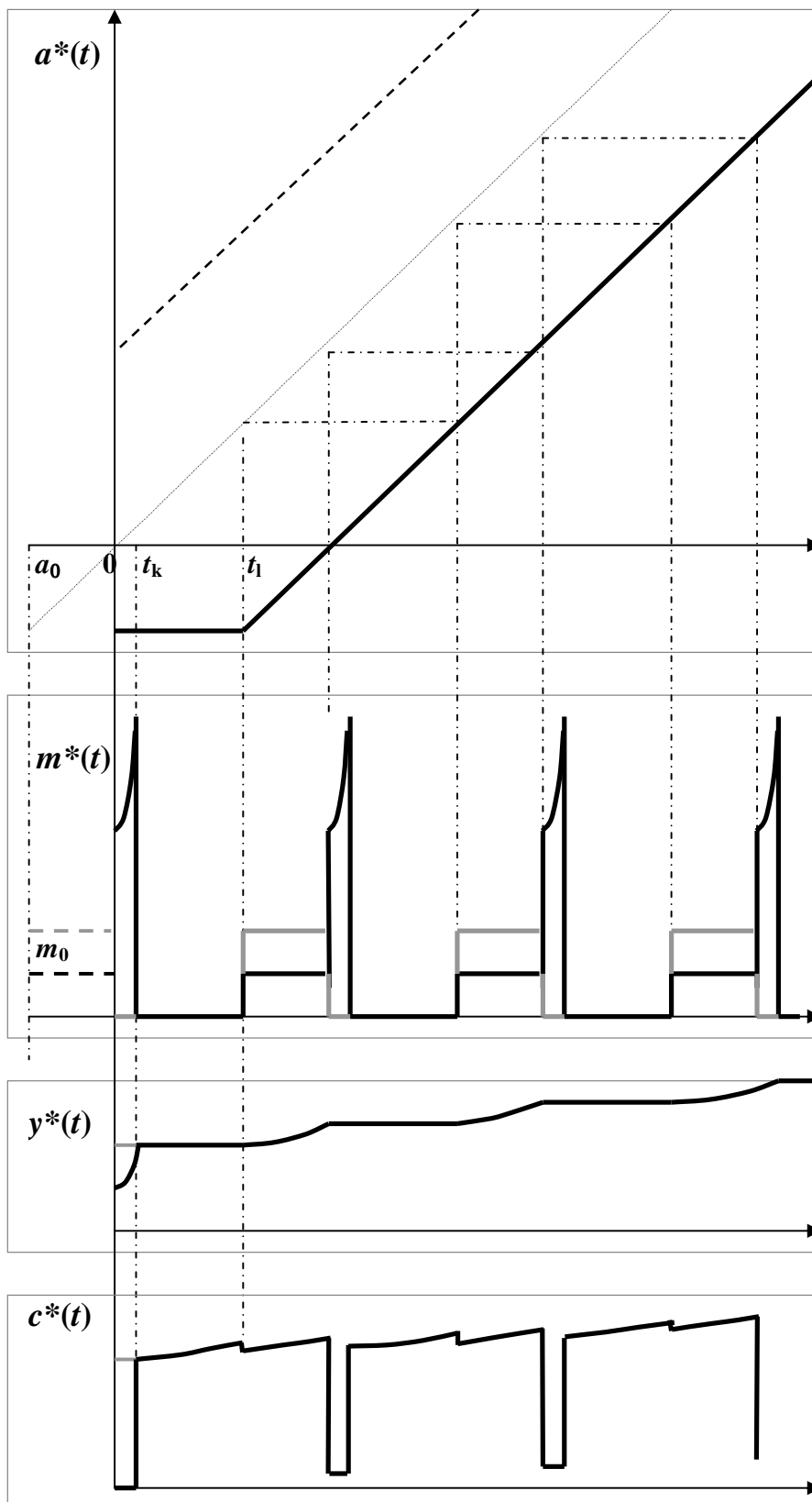


Figure 2. Transition and long-term dynamics under initial inactive environment regulation from Example 3. The optimal dynamics at active regulation from Example 2 is shown in grey color.