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On the Likelihood of Dummy Players in Weighted Majority Games

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Abstract

When the number of players is small in a weighted majority voting game, it can occur that one of the players has no influence on the result of the vote, in spite of a strictly positive weight. Such a player is called a "dummy" player in game theory. The purpose of this paper is to investigate the conditions that give rise to such a phenomenon and to compute its likelihood. It is shown that the probability of having a dummy player is surprisingly high and some paradoxical results are observed.

JEL classification: C7, D7

Keywords: Cooperative game theory, weighted voting games, dummy player, likelihood of voting paradoxes.

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1 Introduction

The main teaching of the literature on power indices is that, in a collective choice process, voting power or influence need not to be proportional to the relative number of votes (*weight*) an individual or a group (*player*) is entitled to. An extreme and striking consequence of this non proportionality is that a player can have a positive weight but never be a member of a minimal winning coalition (a coalition that wins and the removal of a single player does not allow the coalition to win any longer). Such players have no voting power and are known as *dummies*.

The most famous example of this somewhat paradoxical phenomenon is offered by Luxembourg in the Council of Ministers of the EU between 1958 and 1973. Luxembourg held one vote, whereas the quota for a proposition to be approved was 12 out of 17. Since other member states held an even number of votes (4 for Germany, France and Italy, 2 for Belgium and The Netherlands), Luxembourg formally was never able to make any difference in the voting process and was a dummy.

Another well known case of dummies involves Nassau County, New York. Nassau County's government took the form of a Board of Supervisors, one representative for each of various municipalities, who cast a block of votes. Here are the weighted voting systems used at various times by Nassau County. The passing quota shown reflects the number of votes needed to pass "ordinary legislation".

	1958	1964
Hempstead 1	9	31
Hempstead 2	9	31
North Hempstead	7	28
Oyster Bay	3	21
Long Beach	1	2
Glen Cove	1	2
Total votes	30	115
Quota	16	58

The numerical weights were chosen to try to take into account the populations of the different municipalities, which were quite disparate. It is easy to see that in 1958, Oyster Bay, Long Beach and Glen Cove were dummies. It can also be checked that, in 1964, there were two dummies (Glen Cove and Long Beach). After 1964, the quota was raised to guarantee that no municipality was a dummy.

A third example of dummy has recently been discovered by one of the authors (see Blancard and Lepelley, 2010) in a community of municipalities in La R?union (France). This community, called CIVIS (**C**ommunaut? **I**ntercommunale des **VI**lles **S**olidaires), gathers five municipalities: Saint-Pierre (15 representatives in the community council), Saint-Louis (10 representatives), L'Etang-Sal? (4), Petite-?le (4) and Cilaos (3), the number of representatives being roughly proportional to the municipality population . In the community Council, 19 votes are necessary for a proposition to be accepted. If we suppose that the representatives of a municipality vote as a block, it can be seen that Cilaos is a dummy: all the winning coalitions containing Cilaos remain winning when this municipality is removed.

The possibility of dummy players is clearly problematic from a democratic point of view and the diversity of the examples given above suggests that the occurrence of dummies in voting games is of practical concern and could be less rare that expected in first analysis. What is the likelihood of such an undesired phenomenon ? How the distribution of weights should be arranged in order to avoid the occurrence of dummies in voting games?

We propose in this paper a theoretical investigation of these issues in the context of weighted *majority* games, where the quota is equal to the half of the total number of votes, plus one. Our framework and our main assumptions are introduced in Section 2. We propose some analytical results in Section 3 for weighted voting games with 4, 5 and 6 players: in each case, we characterize the distributions of weights giving rise to the occurrence of the "dummy paradox" and deduce from these characterizations some representations for the likelihood of the paradox as a function of the total number of votes. Section 4 proposes exact numerical results for the likelihood of dummy players for more than 6 players and for some specified values of the total number of votes. Our results are discussed in Section 5, where we study the impact of a reduction of the weight scattering on the probability of having some dummies. Section 6 concludes the paper.

2 Framework and assumptions

We will adopt the following notation:

m is the number of players (or voters). The players are denoted by $J1$, $J2$, ..., Jm. N is the set of all players and a subset of N is called a *coalition*.

n_i is the *weight* of player i and $n = \sum_i n_i$. Hence, n_i can be interpreted as the number of votes assigned to a member Ji of a voting body. Notice however that, when the players are parties in a political assembly, the n_i's correspond to the number of representatives of each party and n is the total number of votes in the assembly.

As mentioned above, we only consider in the present study Weighted *Majority* Games (WMG): a proposition is adopted if and only if the total weight of the players in favor of this proposition is greater or equal to $n/2 + 1$ if n is even and to $(n + 1)/2$ if n is odd. In what follows, this majority quota will be denoted by $Q = [n/2]^+$, where $[x]^+$ is the smallest integer strictly higher than x. So, a coalition S is *winning* if and only if $\sum_{i \in S} n_i \geq Q$; otherwise, the coalition is said to be *loosing*.[1] A player Ji is a *dummy* if and only if, for each winning coalition S including Ji, $S - \{Ji\}$ is still winning.

Our main assumptions are the following:

[1]Notice that, when n is even, the complementary coalition of a loosing coalition is not always winning. In order to take into account this peculiarity, we will make use of the following notation: $Q^* = Q$ if n is odd and $Q^* = Q - 1$ if n is even.

(1) the n_i's are integer,
(2) $n/2 \geq n_1 \geq n_2 \geq ... \geq n_m \geq 1$,
(3) m and n being given, all the distributions of the n_i's verifying (1), (2) and $n = \sum_i n_i$ are equally likely to occur.

Notice that this framework fits well with the (recent) French local entities called EPCI (Etablissement Public de Cooperation Intercommunale) where each municipality belonging to the EPCI is given a number of delegates approximately proportionate to its number of inhabitants[2]. In this context, n_1 is the number of delegates of the biggest municipality in the EPCI council, n_m the number of delegates of the smallest, and n is the total number of delegates in the EPCI council (we suppose that, in this council, the delegates of a given municipality vote as a bloc). Of course, the biggest municipality should not be a dictator ($n_1 \leq n/2$) and the smallest one should obtain at least one delegate. In the EPCI council, the current decisions are taken with a quota $Q = [n/2]^+$.

3 Some analytical results

Proposition 1 *In a m-player WMG, (i) the maximum number of possible dummies is equal to $m-3$ and (ii) the number of dummies is exactly $m-3$ if and only if $n_2 + n_3 \geq Q$.*

Proof. In order to prove (i), we have to show that $J3$ cannot be a dummy in a m-player majority game, $m \geq 3$. Suppose the contrary: $J3$ is a dummy. A first consequence is that $J4, J5, ..., Jm$ are also dummies. Furthermore, the coalition $\{J1, J3\}$ is loosing (if this coalition was winning, the fact that $J3$ is a dummy would imply that $n_1 > n/2$, contradicting our assumptions). Now, if $\{J1, J3\}$ is loosing, then $\{J1, J3, J4\}$ is also loosing since $J4$ is a dummy. Similarly, as $J5$ is a dummy, the coalition $\{J1, J3, J4, J5\}$ is loosing and we can set in the same way that $\{J1, J3, J4, ..., Jm\}$ is loosing, which implies $n_1 + n_3 + n_4 + n_5 + ... + n_m < Q$. As $\sum_i n_i = n$, we would have $n_2 \geq n/2$, which is impossible.
Consider now assertion (ii) and suppose that $n_2 + n_3 \geq Q$. Let's show this implies that $J4$ is a dummy. Consider the winning coalitions including $J4$. Observe first that $n_2 + n_3 \geq Q$ implies that the coalition $\{J4, J5, ..., Jm\}$ is loosing. Observe next that the coalition $\{J1, J4\}$ is also loosing: $n_2 + n_3 \geq Q$ implies $n_1 + n_4 + n_5 + ... + n_m < Q$, which implies $n_1 + n_4 < Q$. It follows from these observations that the only winning coalitions with $J4$ must include two players among $\{J1, J2, J3\}$. As $n_2 + n_3 \geq Q$ and $n_1 \geq n_2 \geq n_3$, we have $n_1 + n_3 \geq Q$ and $n_1 + n_2 \geq Q$. Consequently, the defection of $J4$ in these coalitions lets them winning and $J4$ is a dummy.
Finally, suppose that $J4$ is a dummy. This implies that the coalition $\{J1, J4\}$ is loosing (if not, $J4$ dummy would imply $n_1 \geq Q$, a contradiction). As $J5, J6, ..., Jm$ are (also) dummies, it follows that $\{J1, J4, J5\}$, $\{J1, J4, J5, J6\}$, ..., $\{J1, J4, J5, J6, ..., Jm\}$ are also loosing. But $\{J1, J4, J5, J6, ..., Jm\}$ loosing implies $n_1 + n_4 + n_5 + ... + n_m < Q$ and, consequently, $n_2 + n_3 \geq$

[2] The CIVIS we have mentioned in the Introduction is an example of EPCI.

Q^*. As $J4$ dummy makes $n_2 + n_3 = Q^*$ impossible in the case where n is even, we finally conclude that we must have $n_2 + n_3 \geq Q$. □

Corollary 1 *There is no dummy player in a 3-player WMG and, in a 4-player WMG, J4 is a dummy player if and only if* $n_2 + n_3 \geq Q$.

The following proposition deals with 5-player and 6-player WMG's.

Proposition 2 *(i) In a 5-player WMG, J5 is a dummy player if and only if one of the following cases holds:*
- case 1: $n_2 + n_3 + n_4 \geq Q$ *and* $n_1 + n_4 \geq Q$;
- case 2: $n_2 + n_3 \geq Q$.
In case 2 (and only in this case), both J4 and J5 are dummy players.
(ii) In a 6-player WMG, J6 is a dummy player if and only if one of the following cases holds:
- case 1: $n_2 + n_3 + n_4 + n_5 \geq Q$ *and* $n_1 + n_5 \geq Q$;
- case 2: $n_2 + n_3 + n_4 \geq Q$ *and* $n_1 + n_4 \geq Q$;
- case 3: $n_2 + n_3 + n_5 \geq Q$ *and* $n_1 + n_4 + n_5 \geq Q$ *and* $n_1 + n_3 \geq Q$;
- case 4: $n_2 + n_3 \geq Q$;
- case 5: $n_2 + n_4 + n_5 \geq Q$ *and* $n_1 + n_2 \geq Q$;
- case 6: $n_3 + n_4 + n_5 \geq Q$.
In case 2, J5 and J6 are dummy players; in case 4, J4, J5 and J6 are dummy players.

The proof of this proposition is rather tedious and is given in Appendix.
Corollary 1 and Proposition 2 allow us to enumerate the distributions of the weights that give rise to dummy players and to compute the probability of their occurrence in m-player WMG's, with $m \in \{4, 5, 6\}$. Moreover, it is possible to derive from Corollary 1 and Proposition 2 some representations for this probability as a function of n, the total number of votes. This probability is denoted by $P(m, n)$ in what follows.

Proposition 3 *For* $n \equiv 9$ *modulo* 12*, the probability of having a dummy player in a 4-player WMG is given as:*

$$P(4, n) = \frac{n^2 - 33}{2(n^2 + 3n - 12)}.$$

As a consequence, $lim_{n \to \infty} P(4, n) = \frac{1}{2}$.

Proof. Given our assumption (3) and for a given value of n, we have to divide the number of those distributions of the n_i's that give rise to the occurrence of a dummy player (denoted by $D(4, n)$) by the total number of possible distributions with 4 players (denoted by $T(4, n)$). We begin by evaluating $T(4, n)$. A vector of integers (n_1, n_2, n_3, n_4) is a possible distribution of the weights is and only if

$$n_1 \geq n_2,\ n_2 \geq n_3,\ n_3 \geq n_4,\ n_4 \geq 1,\ n_1 \leq n/2 \text{ and } n_1 + n_2 + n_3 + n_4 = n.$$

We know from Ehrhart's theory and its recent developments (the reader is referred to Lepelley *et al.* (2008) for a presentation of this theory) that the number of integer solutions of such a set

of (in)equalities is a *quasi polynomial* in n with periodic coefficients (or Ehrhart's polynomial). A periodic coefficient takes various values according to n and to a given period. For example, $c(n) = [\frac{1}{2}, \frac{3}{4}, 1]_n$ is a periodic coefficient with period 3, $c(n) = \frac{1}{2}$ if $n \equiv 0$ *modulo* 3, $c(n) = \frac{3}{4}$ if $n \equiv 1$ *modulo* 3 and $c(n) = 1$ if $n \equiv 2$ *modulo* 3. Numerous algorithms exist to derive the expression of such a quasi polynomial (see, once again, Lepelley *et al.* (2008)). Using one of these algorithms, we obtain
$T(4,n) = \frac{1}{288}n^3 + [\frac{1}{32}, \frac{1}{48}]_n n^2 + [\frac{1}{24}, -\frac{1}{96}]_n n + [0, -\frac{1}{72}, -\frac{17}{72}, -\frac{1}{4}, \frac{1}{9}, \frac{7}{72}, -\frac{1}{8}, -\frac{5}{36}, -\frac{1}{9}, -\frac{1}{8}, -\frac{1}{72}, -\frac{1}{36}]_n$.
The period of such a quasi polynomial is the least common multiple of the periods of its coefficients, here 12. Consequently, the expression of $T(4,n)$ corresponds to 12 distinct polynomials; for instance, we obtain for $n \equiv 9$ *modulo* 12:[3]

$$T(4,n) = \frac{1}{288}n^3 + \frac{1}{48}n^2 - \frac{1}{96}n - \frac{1}{8} = \frac{(n+3)(n^2-33)}{576}.$$

Now, according to Corollary 1, a dummy player exists if and only if

$$n_1 \geq n_2,\ n_2 \geq n_3,\ n_3 \geq n_4,\ n_4 \geq 1,\ n_1 \leq n/2,\ n_2 + n_3 > n/2 \text{ and } n_1 + n_2 + n_3 + n_4 = n.$$

The number of associated distributions of the n_i's is given as
$D(4,n) = \frac{1}{576}n^3 + [-\frac{1}{96}, \frac{1}{192}]_n n^2 + [-\frac{1}{24}, -\frac{11}{192}, \frac{1}{48}, \frac{1}{192}]_n n + [0, \frac{29}{576}, -\frac{7}{72}, -\frac{7}{64}, \frac{2}{9}, -\frac{35}{576}, -\frac{1}{8}, \frac{65}{576}, \frac{1}{9}, -\frac{11}{64}, \frac{7}{72}, \frac{1}{576}]_n$,
which implies, for $n \equiv 9$ *modulo* 12:

$$D(4,n) = \frac{1}{576}n^3 + \frac{1}{192}n^2 - \frac{11}{192}n - \frac{11}{64} = \frac{(n+3)(n^2+3n-12)}{288}.$$

The expression of $P(4,n) = D(4,n)/T(4,n)$ for $n \equiv 9$ *modulo* 12 directly follows, as well as the limiting value[4] $P(4,\infty) = \frac{\frac{1}{576}}{\frac{1}{288}} = \frac{1}{2}$. □
The two following Propositions are obtained along the same lines as Proposition 3 and their proofs are omitted.

Proposition 4 *For $n \equiv 15$ modulo 120, the probability of having dummy player(s) in a 5-player WMG is:*

$$P(5,n) = \frac{5(n+9)(7n^3 - 51n^2 + 165n - 801)}{6(11n^4 + 120n^3 + 350n^2 + 960n + 4815)}.$$

Consequently, the probability for P5 to be a dummy when n is large is: $lim_{n\to\infty} P(5,n) = \frac{35}{66}$. And the limiting probability of having two dummy players (J4 and J5) when n is large is given as $\frac{5}{22}$.

Proposition 5 *The limiting probability of having at least one dummy player in a 6-player WMG is given by: $lim_{n\to\infty} P(6,n) = \frac{155}{312}$. The limiting probability of having two dummies (J5 and J6) is $\frac{5}{39}$ and the limiting probability of having three (J4, J5 and J6) is $\frac{5}{52}$.*

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[3]Of course, the 11 other polynomials can be derived in the same way.

[4]It is worth noticing that the coefficient of the leading term of the quasi polynomials is not periodic. This peculiarity allows to easily obtain the desired probabilities for n large by considering only this coefficient in the quasi plolynomials.

Table 1

Probability $P(m, n)$ of having a dummy player as a function of n (the total number of votes) for $m = 4,\ 5,\ 6$.

n	4-player WMG	5-player WMG	6-player WMG
15	0.4375	0.2609	0.1818
18	0.2258	0.1778	0.0196
21	0.4146	0.2973	0.1978
24	0.2537	0.2302	0.0529
27	0.4578	0.3696	0.2731
30	0.3089	0.2827	0.1002
33	0.4490	0.3818	0.2905
36	0.3235	0.3087	0.1259
39	0.4684	0.4145	0.3310
42	0.3535	0.3398	0.1601
45	0.4637	0.4213	0.3407
48	0.3624	0.3553	0.1809
51	0.4747	0.4402	0.3641
54	0.3813	0.3757	0.2072
57	0.4718	0.4443	0.3711
60	0.3873	0.3859	0.2232
63	0.4789	0.4564	0.3869
66	0.4002	0.4002	0.2431
69	0.4770	0.4593	0.3918
72	0.4045	0.4075	0.2558
75	0.4820	0.4678	0.4028
78	0.4139	0.4181	0.2716
81	0.4806	0.4699	0.4066
84	0.4172	0.4235	0.2818
87	0.4842	0.4761	0.4149
90	0.4243	0.4316	0.2943
93	0.4832	0.4777	0.4177
96	0.4269	0.4359	0.3027
99	0.4860	0.4825	0.4241
.	.	.	
199	0.4928	0.5061	0.4594
202	0.4645	0.4624	0.3935
.	.	.	
limit	1/2	$\frac{35}{66} = 0.530$	$\frac{155}{312} = 0.497$

Table 2

Probability $P(m, \infty)$ of having one, two or three dummies for $m = 4,\ 5,\ 6$.

m	1 dummy	2 dummies	3 dummies	Total
4	0.5	0	0	0.5
5	0.3030	0.2273	0	0.5303
6	0.2724	0.1282	0.0962	0.4968

The next section deals with the cases with more than six players.

4 Results for more than six players

In the 5-player case, 120 different formulas are necessary to compute all the probabilities $P(5, n)$ (see Proposition 4 for one of them). The number of different formulas is exponentially increasing when m increases and it becomes practically too complicated to list all of them when m is higher than 6. In order to obtain the desired probabilities for more than 6 players, we make use of a computer. This is done in two ways: exact computations and simulations. Exact computations are done with an exhaustive list of all the possible vectors of weights for a given number n of votes in the assembly. For all these vectors $(n_1, \ldots, n_m)$, we check whether or not the last player is pivotal (decisive) (remember that $n_1 \geq n_2 \geq \ldots \geq n_m$). This is done by computing the Banzhaf powr index with the generating function method. Finally, the exact probability of having at least one dummy player is the ratio between the number of times the last player J_m is never pivotal (decisive) and the number of vectors $(n_1, \ldots, n_m)$ considered as admissible (with a uniform distribution of weights vectors, as done theoretically in the previous section).

Table 3

Probability $P(m, n)$ of having a dummy player as a function of m for $n = 45$, $n = 50$, $n = 95$ and $n = 100$.

m	$n = 45$	$n = 50$	$n = 95$	$n = 100$
4	0.4637	0.3735	0.4855	0.4297
5	0.4213	0.3020	0.4806	0.4003
6	0.3407	0.1931	0.4215	0.3091
7	0.2135	0.0858	0.3173	0.1869
8	0.1050	0.0299	0.2017	0.0862
9	0.0434	0.0091	0.0963	0.0304
10	0.0185	0.0030	0.0447	0.0108
11	0.0086	0.0012	0.0194*	0.0044*
12	0.0044	0.0005	0.0098*	0.0017*
13	0.0021	0.0002	0.0060*	0.0008*
14	0.0007	0.0000	0.0038*	0.0005*
15	0.0004	0.0000	0.0025*	0.0003*

*Simulated probabilities

Our simulations are based on random vectors of weights. The estimated probability of having at least one dummy player is then obtained by dividing the number of times the last player J_m is never pivotal (decisive) by the number of vectors $(n_1, \ldots, n_m)$ randomly generated.

Table 4

Simulated[5] probability $P(\infty, n)$ of having one, two ... or x dummies

	Number of dummy players					
m	≥ 1	1	2	3	4	5
4	0.49300	0.49300				
5	0.52490	0.29830	0.22660			
6	0.49310	0.27140	0.12750	0.09240		
7	0.43530	0.24530	0.10460	0.04570	0.03980	
8	0.34470	0.21030	0.07970	0.02930	0.01300	0.01240
9	0.25750	0.17250	0.04930	0.01960	0.00790	0.00370
10	0.17750	0.13160	0.02710	0.01000	0.00340	0.00230
11	0.11844	0.09634	0.01440	0.00414	0.00138	0.00074
12	0.07140	0.06044	0.00756	0.00184	0.00058	0.00024
13	0.04340	0.03940	0.00294	0.00062	0.00020	0.00004
14	0.02282	0.02132	0.00114	0.00024	0.00004	0^+
15	0.01226	0.01168	0.00046	0.00004	0.00002	0^+

0^+: the estimated probability is less than 1/50 000

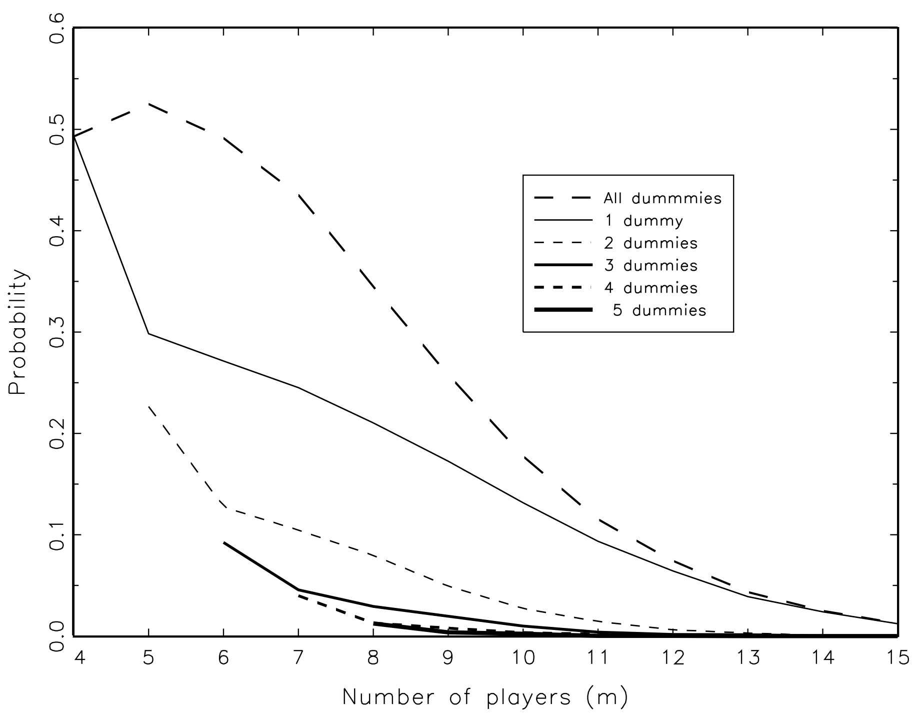

Figure 1. **Simulated Probability $P(\infty, n)$ of having exactly x dummies with m players**

[5] 10 000 simulations for $m < 11$ and 50 000 when $m \geq 11$. Moreover $P(\infty, n)$ estimating $P(99999, n)$.

5 Discussion and further results

The theoretical risk of having a dummy appears to be very high. It can be suggested that our calculations possibly overestimate this risk in the case of the french EPCI, which very often try to reduce the spread (range) of the numbers of representatives in each city. How can we introduce more realism in our analysis ?
One approach is the following. Let k be the maximal fraction of the total weight given to the "biggest" player: $n_1/n \leq k$. We wish to study the impact of parameter k on the probability of having a dummy player. Under our other assumptions, k belongs to $[\frac{1}{m}, \frac{1}{2}]$. When $k = \frac{1}{2}$, we recover the situation we have studied in the preceding sections. With $k = \frac{1}{m}$, each player obtains the same weight, hence the same power and there is no dummy. Let $P(m, n, k)$ be the probability of having a dummy when $J1$ gets $k\%$ of the total weight. It seems natural to conjecture that $P(m, n, k)$ decreases when k moves from $\frac{1}{2}$ to $\frac{1}{m}$. The following results show that this conjecture does not hold for small values of m. We will only give the proof of the first Proposition.[6]

Proposition 6 *In a 4-player WMG with n large, the probability of having a dummy player as a function of k is given by the following representation:*

$$P(4, \infty, k) = \frac{3}{4} \quad \textit{for } \frac{1}{4} \leq k < \frac{1}{3}$$

$$= \frac{240k^3 - 288k^2 + 108k - 13}{4(44k^3 - 60k^2 + 24k - 3)} \quad \textit{for } \frac{1}{3} \leq k \leq \frac{1}{2}.$$

Proof. Let K be the maximal weight of $J1$, with $k = K/n$. In order to compute the desired probability, we begin by evaluate the total number $T(4, n, K)$ of distributions on the n_i's when n_1 is constrained to be lower or equal to K. $T(4, n, K)$ is the number of integer solutions of the following inequalities :

$$n_1 \geq n_2,\ n_2 \geq n_3,\ n_3 \geq n_4,\ n_4 \geq 1,\ n_1 \leq K \quad n_1 + n_2 + n_3 + n_4 = n \text{ and } K \leq n/2.$$

We have now two parameters, n and K, and the number of integer solutions is given by bivariate quasi polynomials (see Lepelley *et al.*(2008)). Using an algorithm recently developed by Barvinok (????), we obtain for n even two distinct quasi polynomials associated with two validity domains:
For $\frac{n}{4} \leq K < \frac{n}{3}$:
$T(4, n, K) = -\frac{1}{144}n^3 + (\frac{1}{12}K + \frac{5}{48})n^2 + (-\frac{1}{3}K^2 - \frac{5}{6}K - \frac{1}{2})n + \frac{4}{9}K^3 + \frac{5}{3}K^2 + 2K + c_1$;
For $\frac{n}{3} \leq K \leq \frac{n}{2}$:
$T(4, n, K) = \frac{1}{48}n^3 + (-\frac{1}{6}K - \frac{3}{16})n^2 + (\frac{5}{12}K^2 + -\frac{11}{12}K + \frac{5}{12})n - \frac{11}{36}K^3 - \frac{23}{24}K^2 - 2K + c_2$,
where c_1 and c_2 are periodic constants the value of which depends on both n and K.
Consider the first domain. As $K = kn$, it follows that, for $\frac{n}{4} \leq K < \frac{n}{3}$, *i.e.* for $\frac{1}{4} \leq k < \frac{1}{3}$:
$T(4, n, k) = -\frac{1}{144}n^3 + (\frac{1}{12}kn + \frac{5}{48})n^2 + (-\frac{1}{3}k^2n^2 - \frac{5}{6}kn - \frac{1}{2})n + \frac{4}{9}k^3n^3 + \frac{5}{3}k^2n^2 + 2kn + c_1$

[6] Although more cumbersome, the proofs of Proposition 7 and 8 are quite similar.

$= (-\frac{1}{144} + \frac{1}{12}k - \frac{1}{3}k^2 + \frac{4}{9}k^3)n^3 + (\frac{5}{48} - \frac{5}{6}k + \frac{5}{3}k^2)n + (-\frac{1}{2} + 2k)n + c_1$.

Observe that, in order to compute the limiting probability $P(4, \infty, k)$, only the coefficient of the leading term in n^3 matters. For this reason, we will only give the coefficient of n_3 of the quasi polynomials we exhibit in the remaining of this proof.

Proceeding as above, we obtain for the second domain, $\frac{1}{3} \leq k \leq \frac{1}{2}$:

$T(4, n, k) = (\frac{1}{48} - \frac{1}{6}k + \frac{5}{12}k^2 - \frac{11}{36}k^3)n^3 + ...$

Consider now the number $D(4, n, K)$ of distributions with a dummy player with $n_1 \leq K$. All we have to do is to add to the above set of inequalities $n_2 + n_3 > n/2$. Replacing K by kn in the quasi polynomials associated with this new set on inequalities gives:

For $\frac{1}{4} \leq k < \frac{1}{3}$: $D(4, n, k) = (-\frac{1}{192} + \frac{1}{16}k - \frac{1}{4}k^2 + \frac{1}{3}k^3)n^3 + ...$

For $\frac{1}{3} \leq k \leq \frac{1}{2}$: $D(4, n, k) = (\frac{13}{576} - \frac{3}{16}k + \frac{1}{2}k^2 - \frac{5}{12}k^3)n^3 + ...$

We finally obtain:

For $\frac{1}{4} \leq k < \frac{1}{3}$:

$$P(4, \infty, k) = \frac{D(4, \infty, k)}{T(4, \infty, k)} = \frac{-\frac{1}{192} + \frac{1}{16}k - \frac{1}{4}k^2 + \frac{1}{3}k^3}{-\frac{1}{144} + \frac{1}{12}k - \frac{1}{3}k^2 + \frac{4}{9}k^3} = \frac{\frac{(4k-1)^3}{192}}{\frac{(4k-1)^3}{144}} = \frac{3}{4},$$

and for $\frac{1}{4} \leq k < \frac{1}{3}$:

$$P(4, \infty, k) = \frac{D(4, \infty, k)}{T(4, \infty, k)} = \frac{\frac{13}{576} - \frac{3}{16}k + \frac{1}{2}k^2 - \frac{5}{12}k^3}{\frac{1}{48} - \frac{1}{6}k + \frac{5}{12}k^2 - \frac{11}{36}k^3} = \frac{240k^3 - 288k^2 + 108k - 13}{4(44k^3 - 60k^2 + 24k - 3)}. \quad \square$$

Proposition 7 *In a 5-player WMG with n large, the probability of having at least one dummy player depends on k as shown in the following representation:*

$$P(5, \infty, k) = 0 \quad \textit{for } \frac{1}{5} < k < \frac{1}{4}$$
$$= \frac{5(4k-1)^3(44k-23)}{32(655k^4 - 780k^3 + 330k^2 - 60k + 43)} \quad \textit{for } \frac{1}{4} < k < \frac{1}{3}$$
$$= \frac{-5(3264k^4 - 3840k^3 + 1440k^2 - 192k + 5)}{96(155k^4 - 300k^3 + 210k^2 - 60k + 6)} \quad \textit{for } \frac{1}{3} < k < \frac{1}{2}.$$

Proposition 8 *In a 6-player WMG with n large, the probability of having at least one dummy player depends on k as shown in the following representation:*

$$P(6, \infty, k) = \frac{5}{12} \quad \textit{for } \frac{1}{6} < k < \frac{1}{5}$$
$$= \frac{186120k^5 - 192600k^4 + 79200k^3 - 16200k^2 + 1650k - 67}{12(10974k^5 - 12270k^4 + 5340k^3 - 1140k^2 + 120k - 5)} \quad \textit{for } \frac{1}{5} < k < \frac{1}{4}$$
$$= -\frac{5034240k^5 - 7027200k^4 + 3916800k^3 - 1094400k^2 + 153600k - 8669}{768(2193k^5 - 3465k^4 + 2130k^3 - 630k^2 + 90k - 5)} \quad \textit{for } \frac{1}{4} < k < \frac{3}{10}$$
$$= \frac{5(1153152k^5 - 1834560k^4 + 1160640k^3 - 364320k^2 + 56760k - 3515)}{768(2193k^5 - 3465k^4 + 2130k^3 - 630k^2 + 90k - 5)} \quad \textit{for } \frac{3}{10} < k < \frac{1}{3}$$
$$= -\frac{5(282240k^5 - 486720k^4 + 331200k^3 - 110880k^2 + 18360k - 1211)}{768(237k^5 - 585k^4 + 570k^3 - 270k^2 + 60k - 5)} \quad \textit{for } \frac{1}{3} < k < \frac{3}{8}$$
$$= \frac{5(307584k^5 - 619200k^4 + 498240k^3 - 200160k^2 + 39960k - 3163)}{768(237k^5 - 585k^4 + 570k^3 - 270k^2 + 60k - 5)} \quad \textit{for } \frac{3}{8} < k < \frac{1}{2}.$$

Table 5

Exact probability $P(m,\infty,k)$ of having a dummy player as a function of k for large n and $m = 4,\ 5,\ 6$.

k	4-player WMG	5-player WMG	6-player WMG
0.23	-	0	0.381
0.25	0.750	0	0.282
0.27	0.750	0.060	0.269
0.29	0.750	0.174	0.311
0.31	0.750	0.270	0.354
0.33	0.750	0.344	0.391
0.35	0.748	0.403	0.418
0.37	0.737	0.452	0.439
0.39	0.718	0.492	0.455
0.41	0.693	0.525	0.469
0.43	0.662	0.549	0.482
0.45	0.624	0.563	0.495
0.47	0.580	0.563	0.505
0.49	0.529	0.547	0.505
0.50	0.500	0.530	0.497

Table 6

Simulated[7] probability $P(m, 9999, k)$ of having a dummy player as a function of k and m.

k	7-player WMG	8-player WMG	9-player WMG	10-player WMG
0.23	0.199	0.175	0.146	0.099
0.25	0.235	0.198	0.163	0.117
0.27	0.259	0.221	0.182	0.125
0.29	0.290	0.238	0.192	0.133
0.31	0.319	0.255	0.205	0.143
0.33	0.346	0.271	0.211	0.152
0.35	0.365	0.283	0.219	0.157
0.37	0.376	0.297	0.223	0.160
0.39	0.387	0.302	0.228	0.163
0.41	0.393	0.313	0.234	0.165
0.43	0.402	0.322	0.238	0.167
0.45	0.413	0.333	0.243	0.174
0.47	0.422	0.340	0.247	0.177
0.49	0.428	0.347	0.252	0.180
0.50	0.435	0.435	0.258	0.1775

[7] 10 000 simulations for $m < 11$ and 50 000 when $m \geq 11$.

6 Concluding remark

We have shown in this paper that the probability of having at least one dummy player in Weighted Majority Games with a small number of player is very high. This probability can reach about 50% for 4, 5 or 6 players ; for more than 6 players, the probability decreases but we have to consider more than 15 players for obtaining results lower than 1%. Of course, it can be suspected that our probabilistic assumption (all admissible weight distributions are supposed to be equally likely to occur) could tend to exaggerate the probability of having a dummy. We have proved however that, for a very small number of players, the introduction of some degree of homogeneity in the distribution of the weights has a weak impact on this probability.
Finally, it is worth to emphasize that our results are limited to *majority* games, in which the quota for a proposition to be approved is equal to 50% of the total weight. It should be of interest to consider the impact of the quota value on the probability of having a dummy player. We plan to study this question in another paper.

7 Appendix: Proof of Proposition 2

(i) In order to characterize the distributions of the n_i's for which $J5$ is a dummy player, we consider the set of coalitions to which $J5$ is susceptible to belong: $\{J1, J5\}$, $\{J2, J5\}$, $\{J3, J5\}$, $\{J4, J5\}$, $\{J1, J2, J5\}$, $\{J1, J3, J5\}$, $\{J1, J4, J5\}$, $\{J2, J3, J5\}$, $\{J2, J4, J5\}$, $\{J3, J4, J5\}$, $\{J1, J2, J3, J5\}$, $\{J1, J2, J4, J5\}$, $\{J1, J3, J4, J5\}$, $\{J2, J3, J4, J5\}$ and $\{J1, J2, J3, J4, J5\}$. Consider first the two-player coalitions; $J5$ is a dummy player if these coalitions are loosing (if not, a zero power for $J5$ would imply that a coalition with only one player is winning, contradicting our assumptions) and these two-player coalitions will be loosing if $n_1 + n_5 < Q$. Consider now the coalitions $\{J1, J2, J5\}$ and $\{J1, J3, J5\}$; these coalitions are necessarily winning (recall that $n_1 \geq n_2 \geq n_3 \geq n_4 \geq n_5 \geq 1$, by assumption (2)). $J5$ is a dummy if we have $n_1 + n_3 \geq Q$ (which implies $n_1 + n_2 \geq Q$). The next three-player coalitions $\{J1, J4, J5\}$, $\{J2, J3, J5\}$, $\{J2, J4, J5\}$ and $\{J3, J4, J5\}$ can be winning or loosing. Hence, $J5$ is a dummy if: ($n_1+n_4+n_5 < Q$ or $n_1+n_4 \geq Q$) and ($n_2+n_3+n_5 < Q$ or $n_2+n_3 \geq Q$) and ($n_2+n_4+n_5 < Q$ or $n_2 + n_4 \geq Q$) and ($n_3 + n_4 + n_5 < Q$ or $n_3 + n_4 \geq Q$). Finally, consider coalitions with four or five players, which are winning coalitions. $J5$ is a dummy player is $n_2 + n_3 + n_4 \geq Q$. To summing up, $J5$ is a dummy player if and only if we have:
$n_1 + n_5 < Q$ and $n_1 + n_3 \geq Q$ and ($n_1 + n_4 + n_5 < Q$ or $n_1 + n_4 \geq Q$) and ($n_2 + n_3 + n_5 < Q$ or $n_2 + n_3 \geq Q$) and ($n_2 + n_4 + n_5 < Q$ or $n_2 + n_4 \geq Q$) and ($n_3 + n_4 + n_5 < Q$ or $n_3 + n_4 \geq Q$) and $n_2 + n_3 + n_4 \geq Q$.
Since $\sum_i n_i = n$, these inequalities can be written in the following way:
$n_1 + n_5 < Q$ and $n_1 + n_3 \geq Q$ and ($n_2 + n_3 \geq Q^*$ or $n_1 + n_4 \geq Q$) and ($n_1 + n_4 \geq Q^*$ or $n_2 + n_3 \geq Q$) and ($n_1 + n_3 \geq Q^*$ or $n_2 + n_4 \geq Q$) and ($n_1 + n_2 \geq Q^*$ or $n_3 + n_4 \geq Q$) and $n_1 + n_5 < Q^*$,
Eliminating redundant inequalities, we obtain:
$n_1 + n_5 < Q^*$ and $n_1 + n_3 \geq Q$ and ($n_2 + n_3 \geq Q$ or $n_1 + n_4 \geq Q$) and $n_1 + n_3 \geq Q$ and

$n_1 + n_2 \geq Q$,
which can be reduced to:
$(n_1 + n_5 < Q^*$ and $n_2 + n_3 \geq Q)$ or $(n_1 + n_5 < Q^*$ and $n_1 + n_4 \geq Q)$.
As $n_2 + n_3 \geq Q$ implies $n_1 + n_5 < Q^*$ and $n_1 + n_5 < Q^*$ is equivalent to $n_2 + n_3 + n_4 \geq Q$, we finally obtain:
$n_2 + n_3 \geq Q$ or $(n_2 + n_3 + n_4 \geq Q$ and $n_1 + n_4 \geq Q)$, in accordance with Proposition 2 (i). Furthermore, it follows from Proposition 1 that $J4$ and $J5$ are both dummy players if and only if $n_2 + n_3 \geq Q$.
(ii) Proceeding as above, it is easily checked that $J6$ is a dummy player if and only if a) all the two-player coalitions including $J6$ are loosing, b) either $\{J1, J2, J6\}$, $\{J1, J3, J6\}$, $\{J1, J4, J6\}$, $\{J1, J5, J6\}$, $\{J2, J3, J6\}$ are loosing or (respectively) $\{J1, J2\}$, $\{J1, J3\}$, $\{J1, J4\}$, $\{J1, J5\}$, $\{J2, J3\}$ are winning, c) either $\{J1, J4, J5, J6\}$, $\{J2, J3, J4, J6\}$, $\{J2, J3, J5, J6\}$, $\{J2, J4, J5, J6\}$, $\{J3, J4, J5, J6\}$ are loosing or (respectively) $\{J1, J4, J5\}$, $\{J2, J3, J4\}$, $\{J2, J3, J5\}$, $\{J2, J4, J5\}$, $\{J3, J4, J5\}$ are winning, d) $\{J1, J2, J3\}$, $\{J1, J2, J4\}$, $\{J1, J2, J5\}$, $\{J1, J3, J4\}$, $\{J1, J3, J5\}$ are winning, e) $\{J1, J2, J3, J4\}$, $\{J1, J2, J3, J5\}$, $\{J1, J2, J4, J5\}$, $\{J1, J3, J4, J5\}$, $\{J2, J3, J4, J5\}$ are winning and f) $\{J1, J2, J3, J4, J5\}$ is winning. This implies a) $n_1+n_6 < Q$, b) $(n_1+n_2+n_6 < Q$ or $n_1+n_2 \geq Q)$ and $(n_1+n_3+n_6 < Q$ or $n_1+n_3 \geq Q)$ and $(n_1+n_4+n_6 < Q$ or $n_1+n_4 \geq Q)$ and $(n_1+n_5+n_6 < Q$ or $n_1+n_5 \geq Q)$ and $(n_2+n_3+n_6 < Q$ or $n_2+n_3 \geq Q)$, c) $(n_1+n_4+n_5+n_6 < Q$ or $n_1+n_4+n_5 \geq Q)$ and $(n_2+n_3+n_4+n_6 < Q$ or $n_2+n_3+n_4 \geq Q)$ and $(n_2+n_3+n_5+n_6 < Q$ or $n_2+n_3+n_5 \geq Q)$ and $(n_2+n_4+n_5+n_6 < Q$ or $n_2+n_4+n_5 \geq Q)$ and $(n_3+n_4+n_5+n_6 < Q$ or $n_3+n_4+n_5 \geq Q)$, d) $n_1+n_3+n_5 \geq Q$, e) $n_2+n_3+n_4+n_5 \geq Q$ and f) $n_1+n_2+n_3+n_4+n_5 \geq Q$. The reduction of this set of inequalities leads to the six cases given in Proposition 2 (ii). To complete the proof, it remains to observe that, in case 4, $J4$, $J5$ and $J6$ are dummy players (by Proposition 1); and if $J6$ is a dummy and $J4$ is not $(n_2 + n_3 < Q)$, then it results from part (i) of Proposition 2 that $J5$ is also a dummy player if and only if $n_1+n_4 \geq Q$ and $n_2+n_3+n_4 \geq Q$.□