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COMMON ASSUMPTION OF RATIONALITY *

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Abstract

In this paper, we provide an epistemic characterization of iterated admissibility (IA), i.e., iterated elimination of weakly dominated strategies. We show that *rationality and common assumption of rationality* (RCAR) in complete lexicographic type structures implies IA, and that there exist such structures in which RCAR can be satisfied. Our result is unexpected in light of a negative result in Brandenburger, Friedenberg, and Keisler (2008) (BFK) that shows the impossibility of RCAR in complete continuous structures. We also show that every complete structure with RCAR has the same types and beliefs as some complete continuous structure. This enables us to reconcile and interpret the difference between our results and BFK's. Finally, we extend BFK's framework to obtain a single structure that contains a complete structure with an RCAR state for every game. This gives a game-independent epistemic condition for IA.

KEYWORDS: Epistemic game theory, rationality, admissibility, iterated weak dominance, assumption, completeness, Borel Isomorphism Theorem, o-minimality.

JEL CLASSIFICATION: C72, D80.

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1 INTRODUCTION

Analysis of games typically begins under the premise that all players are rational. Furthermore, it is often supposed, at least implicitly, that the rationality of the players is *common knowledge* in the sense of Lewis (1969) and Aumann (1976)—that is, all players know it, all players know that all players know it, and so on. It is then natural to ask which strategic choices are consistent with common knowledge of rationality (CKR).

Bernheim (1984) and Pearce (1984) gave an influential response to this question in which they argued that their notion of *rationalizability* exactly captures the implications of CKR on behavior. The rationalizable set is essentially the iteratively undominated (IU) set—that is, the set of strategy profiles surviving iterated elimination of strongly dominated strategies—with the added virtue of being defined in a way that more starkly emphasizes its intuitive connections to CKR.[1]

Bernheim (1984) and Pearce (1984) motivated their analysis as an extension of Savage's (1954) Bayesian decision theory, in which rational actors maximize subjective expected utility (SEU) subject to probabilistic beliefs about the states of the world. Therefore, in these and subsequent papers, CKR is often used interchangeably with *rationality and common belief of rationality* (RCBR), an analogous concept that is better suited for use in Bayesian settings.[2]

More formal analyses followed in Brandenburger and Dekel (1987a) and Tan and Werlang (1988), who showed that RCBR is an epistemic condition that characterizes the IU set. In other words, RCBR implies that IU strategies are played and every IU strategy can be played in some state where RCBR holds. A key fact underpinning this relationship is that SEU maximization characterizes avoidance of strongly dominated strategies. However, it is prima facie reasonable that rationality should incorporate an admissibility requirement—that is, avoidance of *weakly* dominated strategies. A long tradition in statistical decision theory, going as far back as Wald (1939), has advocated admissibility as a minimal criterion of rationality.[3]

[1] When we refer to rationalizability in this paper, we will mean *correlated* rationalizability, which omits the independence assumptions of the original definition. The correlated rationalizable set is exactly the IU set.

[2] An event is commonly believed if all players are certain of it, all players are certain that all players are certain of it, and so on, where certainty is understood to mean belief with probability 1. In the literature, common belief is also called common certainty, common belief with probability 1, and common 1-belief. See Brandenburger and Dekel (1987b); Monderer and Samet (1989).

[3] Von Neumann and Morgenstern (1944) justify the requirement from a staunchly objectivist point of view on probability while prefacing the development of their theory of two-person zero-sum games. Fur-

In light of the preceding facts, it was intuitively appealing to conjecture that *iteratively admissible* (IA) strategies—that is, strategies surviving iterated elimination of weakly dominated strategies—could be characterized by RCBR if rationality incorporates admissibility.[4] However, Samuelson (1992) demonstrated that such a conjecture would have significant obstacles associated with the limitations of SEU theory. Admissibility is typically obtained by requiring that players consider all states of the world to be probabilistically possible. However, a player who believes that her opponents are rational would exclude their inadmissible strategies from consideration. Elegant examples in Samuelson (1992) illustrated the frustrating fact that, in many games, an inadmissible strategy may maximize her SEU under such beliefs.

Brandenburger, Friedenberg, and Keisler (2008) (BFK) solved this puzzle by adopting a model of Bayesian rationality that permits the expression of a more general set of beliefs than the set allowed by SEU theory. They defined the notions of a lexicographic type structure and of *assuming* an event, which is immune to the aforementioned shortcomings of probability 1 belief that were pointed out by Samuelson (1992). In that framework, BFK formulated a condition, *rationality and common assumption of rationality* (RCAR), that gives intuitive support for the IA set as a solution concept. Given that an admissibility requirement partially reflects the view that rational players should rule out nothing, it is reasonable to consider the consequences of RCAR in model environments, such as *complete* lexicographic type structures, which, by virtue of describing sufficiently rich state spaces, do not presume much knowledge on the players' part about what BFK called "prior history or context". BFK showed that this restriction is a meaningful one by proving that RCAR in many *incomplete* type structures yields predictions outside the IA set.

In this paper we address two crucial issues that were left unresolved by BFK.

First, BFK left open the question of whether there is a complete type structure in which RCAR is possible. More broadly speaking, this first question can be subsumed under the question of whether "RCAR in complete type structures" is an epistemic condition for IA. That is, if we look across all complete type structures, is the set of strategies

thermore, later surveys by Arrow (1951) and Luce and Raiffa (1957) are uniform in their rejection of inadmissible decision rules.

[4] It is well-known that the order of eliminating weakly dominated strategies matters, whereas the order does not matter when eliminating strongly dominated strategies. When we refer to IA strategies in this paper, we will mean the strategies obtained by simultaneously deleting every weakly dominated strategy of every player in each round.

played under RCAR exactly the IA set? We answer this question in the affirmative with our Theorems 3.2 and 3.4.

The second issue can be paraphrased as one of "game independence". While the condition above—RCAR in complete type structures—is fine from the perspective a game theorist who looks across all complete type structures, it is not fully satisfactory from the perspective of a player who considers possible only those states of the world that are described by *her* type structure. It is therefore both natural and important to ask if there is a single "context-free" model environment (e.g., a complete type structure) in which an epistemic condition for IA (e.g., RCAR) can be satisfied for *all* games. We resolve this issue by showing that enough complete type structures can be embedded in a single larger model environment such that a natural generalization of RCAR is an epistemic condition for IA across all games. A complete type structure in which common assumption of rationality, as defined with respect to a given game, is possible can then be intuitively interpreted as the set of states in which there is common knowlege of that game.

The results in this paper were unexpected in the light of an impossibility theorem in BFK that left a decidedly negative message. BFK showed that, in type structures that are both complete and continuous, no state of the world can satisfy RCAR. The issues leading to this nonexistence result are independent of those that were raised in Samuelson (1992). This result appeared to cast doubt on the existence of any complete type structure in which the RCAR set is nonempty. However, our results here show that RCAR is possible when the requirement that the type structure is continuous is dropped. In the process, we also identify some of the conceptual issues that help us to reconcile the positive results herein with the negative conclusions of BFK.

Toward that end, we prove that, given each (discontinuous) type structure, there exists a continuous type structure with the same type sets that describes the exact same sets of beliefs. Where the two type structures, despite being equivalent in the sense that they have the same types and beliefs, differ is in how they classify what beliefs assume a given event. Given a belief in a discontinuous type structure, the same belief in an equivalent continuous type structure will, in general, assume fewer events. One implication of this difference is that beliefs in the continuous type structure must meet a higher standard in order to "rule out nothing". We argue that the discussion of these differences can be conveniently subsumed under the umbrella of *topological distinguishability*.[5]

[5]We caution the reader that, despite the similarity in nomenclature, these issues are completely unre-

Additionally, we give a topological characterization of an *RCAR tower*, which is a family of finite-order rationality sets—that is, the sets in which there is rationality and m-th order assumption of rationality (RmAR)—in complete type structures with RCAR.

The proofs in this paper illustrate the virtues of two key mathematical results in constructing type structures with desirable properties: the Borel Isomorphism Theorem and Tarski's celebrated theorem that every relation that is first order definable in the field of real numbers is semi-algebraic. We anticipate that these methods will prove useful in showing various existence results in other settings.

2 THE UNDERLYING FRAMEWORK

In this section, we briefly review the concepts we will need from BFK. We fix a finite game of complete information

$$G = \left\langle S^a, S^b, \pi^a, \pi^b \right\rangle,$$

where S^a, S^b are strategy spaces and π^a, π^b are payoff functions. The indices a and b stand for Ann and Bob, respectively. *Whenever we state a definition or result involving a and/or b (Ann and/or Bob), it will be understood that we also make the analogous statement with a and b reversed.*

2.1 ADMISSIBILITY

Ann's strategy $s^a \in S^a$ is **admissible** (i.e., not weakly dominated) in the game G if and only if s^a is optimal under some full-support probability measure defined over S^b. Let S^a_1 denote the set of Ann's admissible strategies. Given nonempty subsets $X \subseteq S^a$ and $Y \subseteq S^b$, let $G(X, Y)$ denote the reduced game $\langle X, Y, \pi^a, \pi^b \rangle$. We can then inductively define Ann's m-**admissible** strategy set S^a_m as follows: To get the induction started we write $S^a_0 \equiv S^a$. For each $m \in \mathbb{N}$, let S^a_{m+1} be the set of Ann's admissible strategies in the reduced game $G(S^a_m, S^b_m)$. In other words, S^a_{m+1} is the set of Ann's strategies that are **admissible with respect to** $S^a_m \times S^b_m$.

Note that $S^a_{m+1} \subseteq S^a_m$ for all $m \in \mathbb{N}$. We put $S^a_\infty \equiv \bigcap_{m=0}^{\infty} S^a_m$. The set $S^a_\infty \times S^b_\infty$ is called the **iteratively admissible** set (henceforth IA set). Since the sets S^a, S^b are finite, we have $S^a_\infty = S^a_M$ and $S^b_\infty = S^b_M$ for some $M \in \mathbb{N}$, and hence the IA set is nonempty.

lated to those raised by the extensive literature on *strategic topology*.

2.2 LEXICOGRAPHIC PROBABILITY SYSTEMS

Recall that a **Polish space** is a separable topological space that is completely metrizable. Let Ω denote the space of uncertainty faced by the decision maker (e.g., Ann). For now, let us assume only that Ω is Polish and fix a compatible metric. In the conventional Bayesian theory of choice under uncertainty, a decision maker's beliefs are represented by a Borel probability measure on Ω. The set of all Borel probability measures on Ω is denoted by $\mathcal{M}(\Omega)$.

BFK adopted the convention that a decision maker's beliefs are represented by lexicographic probability systems (henceforth LPS's), which are generalizations of probability measures. The decision theoretic foundations of LPS's was developed in Blume, Brandenburger, and Dekel (1991a). An **LPS** on Ω is any finite sequence of probability measures on Ω, e.g.,

$$\sigma = (\mu_0, \ldots, \mu_{n-1}) \in \overbrace{\mathcal{M}(\Omega) \times \cdots \times \mathcal{M}(\Omega)}^{n \text{ times}},$$

that satisfies **mutual singularity**—that is, there exist disjoint Borel sets $U_0, \ldots, U_{n-1}$ in Ω such that $\mu_i(U_i) = 1$ and $\mu_j(U_j) = 0$ for $i \neq j$.[6] The set of all LPS's on Ω is denoted by $\mathcal{L}(\Omega)$. It is immediate that $\mathcal{M}(\Omega) \subsetneq \mathcal{L}(\Omega)$. Additional notation, which will be convenient later, follows below.

$$\begin{aligned} \mathcal{N}_n(\Omega) &\equiv \overbrace{\mathcal{M}(\Omega) \times \cdots \times \mathcal{M}(\Omega)}^{n \text{ times}}; \\ \mathcal{N}(\Omega) &\equiv \bigcup_{n \in \mathbb{N}} \mathcal{N}_n(\Omega); \\ \mathcal{L}_n(\Omega) &\equiv \mathcal{L}(\Omega) \cap \mathcal{N}_n(\Omega). \end{aligned}$$

We define a Polish topology on $\mathcal{N}(\Omega)$ by following the usual conventions. First, we give $\mathcal{M}(\Omega)$ its weak* topology, which makes it a Polish space. Second, we give $\mathcal{N}_n(\Omega) = \prod_{k=1}^{n} \mathcal{M}(\Omega)$ the product topology. Then we may view $\mathcal{N}(\Omega)$ as a countable topological union of disjoint Polish spaces $\mathcal{N}_n(\Omega)$. $\mathcal{N}(\Omega)$ with this topology is again a Polish space.

An LPS $\sigma = (\mu_0, \ldots, \mu_{n-1})$ represents an ordered sequence of mutually contradictory hypotheses. We interpret μ_0 as being infinitely more likely than μ_1, which in turn is infinitely more likely than μ_2, and so on. The primary hypothesis μ_0, being more likely than all other hypotheses, can be regarded as the prior belief. The secondary hypothesis

[6] The definition of LPS's in Blume, Brandenburger, and Dekel (1991a) did not require mutual singularity. The definition above is from BFK.

μ_1 can be regarded as the conditional belief in the a priori zero-probability (i.e., μ_0-null) event that μ_0 is false. More generally, μ_j is the conditional belief in the event that all a priori more likely hypotheses (i.e., all μ_k such that $k < j$) are false. Such an event would be μ_k-null for all $k < j$.

LPS's generalize the notion of probability measures in a straightforward manner. Not surprisingly, concepts defined with respect to probability measures often have obvious analogs that are defined with respect to LPS's.

The **support** of an n-tuple of measures $\sigma \in \mathcal{N}(\Omega)$ is the union of the supports of the measures that comprise it—that is, the support of $\sigma = (\mu_0, \ldots, \mu_{n-1})$ is simply

$$\operatorname{Supp} \sigma \equiv \bigcup_{j=0}^{n-1} \operatorname{Supp} \mu_j.$$

We say that σ has **full support** if $\operatorname{Supp} \sigma = \Omega$. Equivalently, σ has full support if, for each open U, there exists $j < n$ such that $\mu_j(U) > 0$. The set of all full-support LPS's is denoted by $\mathcal{L}^+(\Omega)$. The set $\mathcal{N}^+(\Omega)$ is defined similarly.

Similarly, Bayesian optimization under belief σ is a straightforward extension of expected utility maximization. Given an act f, for each $j < n$ let u_j be the expected utility of choosing f with respect to μ_j. Then the vector $u = (u_0, \ldots, u_{n-1})$ is called the **lexicographic expected utility** (henceforth LEU) of f under σ. The order on the hypotheses that comprise σ suggests an obvious way to compare LEU vectors. Given that, for $i < j$, μ_i is infinitely more likely than μ_j, it is natural to assign infinitely more importance to the expected utility of an act under μ_i than one would to its expected utility under μ_j. We write

$$(v_0, \ldots, v_{n-1}) = v >_{\mathsf{LEX}} u = (u_0, \ldots, u_{n-1})$$

and say that v is lexicographically greater than u if there exists $k < n$ such that $v_k > u_k$, and $v_j = u_j$ for all $j < k$. LEU maximization is simply the maximization of LEU with respect to the lexicographic order. Throughout this paper, we use the terms LEU and payoff interchangeably in appropriate contexts.

2.3 ASSUMPTION

The idea of certainty (i.e., belief with probability one) admits more than one obvious analog with respect to LPS's. If the decision maker is certain of event $E \subseteq \Omega$ then she considers E to be infinitely more likely than its complement $\Omega \setminus E$. BFK introduced a new epistemic notion, called **assumption**, to capture this property in LPS's. Intuitively speaking, a decision maker with belief $\sigma = (\mu_0, \dots, \mu_{n-1})$ **assumes** an event E if she believes **every part** of E to be infinitely more likely than its complement $\Omega \setminus E$. Formally, a Borel set E is assumed under σ at level j if the following three conditions are met (cf. Proposition 5.1 in BFK)[7]

(a) $\mu_i(E) = 1$ for each $i \le j$;

(b) $\mu_i(E) = 0$ for each $i > j$; and

(c) If U is open with $U \cap E \ne \varnothing$ then $\mu_i(U \cap E) > 0$ for some $i < n$.

Note that, even if E is assumed under σ, it need not be the case that $\Omega \setminus E$ is σ-null. In contrast, if a decision maker is certain of E then $\Omega \setminus E$ is necessarily a null event.

It is clear that if an event E is assumed under an LPS $\sigma = (\mu_0, \dots, \mu_{n-1})$ then the level at which E is assumed is unique, is less than n, and is the greatest j such that $\mu_j(E) = 1$. It is also clear that if $\mu = (\mu_0, \dots, \mu_{n-1})$ and $\nu = (\nu_0, \dots, \nu_{n-1})$ are LPS's of the same length n, and μ_j, ν_j have the same null sets for each $j < n$, then μ and ν assume the same events at each level $j < n$. Verbally, the events that an LPS μ assumes depend only on the length of μ and the null sets of the μ_j.

2.4 LEXICOGRAPHIC TYPE STRUCTURES

LPS's and associated constructs were used in BFK to build a framework in which the rationale for iterated admissibility can be expressed formally (i.e., in the language of set theory).

Recall the finite game $G = \langle S^a, S^b, \pi^a, \pi^b \rangle$. In the context of the game G, Ann is uncertain of what strategy Bob will choose, what Bob believes about Ann's strategy choice, what Bob believes about what Ann believes about Bob's strategy choice, and so on. To give a parsimonious description of Ann's beliefs about the pair consisting of Bob's strategy and Bob's beliefs while sidestepping the inherent problem of self-reference, BFK followed the convention of implicitly representing beliefs as **types**.[8] Ann's type t^a is an element of a Polish space T^a, called her **type space**. The belief that Ann's type represents is given by a Borel map $\lambda^a : T^a \to \mathscr{L}(S^b \times T^b)$, where T^b denotes Bob's type space. Similarly, Bob's types are interpreted through a Borel map $\lambda^b : T^b \to \mathscr{L}(S^a \times T^a)$. Taken together, these objects form a 6-tuple

$$\mathfrak{T} = \left\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \right\rangle,$$

[7] BFK defined assumption only when σ has full-support, but we adopt this definition for all σ.

[8] An innovation due to Harsanyi (1967).

which is called an (S^a, S^b)**-based lexicographic type structure**. Members of $S^a \times T^a \times S^b \times T^b$ are called **states of the world**.

The type structure $\mathfrak{T}$ is called **complete** if $\mathscr{L}(S^b \times T^b) = \text{range}\,\lambda^a$ and $\mathscr{L}(S^a \times T^a) = \text{range}\,\lambda^b$.[9] A complete type structure contains all beliefs about beliefs.[10]

2.5 RATIONALITY

The definition of rationality in BFK combines two requirements. The first is Bayesian optimality, which is captured by LEU maximization. The second, which might be roughly described as a form of agnosticism, is reflected in full-support beliefs. Intuitively, in a complete type structure, a player with full-support beliefs will consider all possibilities. Formally, the LEU of a strategy $s^a \in S^a$ under the LPS $\sigma = (\mu_0, \ldots, \mu_n)$ is the vector $(\pi^a(s^a, \nu_0), \ldots, \pi^a(s^a, \nu_n))$ of payoffs, where $\nu_i = \text{marg}_{S^b}\,\mu_i$, and s^a is **optimal** under σ if the LEU of s^a under σ is maximal among all strategies in S^a. A strategy-type pair (s^a, t^a) is **rational** if $\lambda^a(t^a)$ is a full-support LPS, and s^a is optimal under $\lambda^a(t^a)$. The set of all rational pairs (s^a, t^a) is denoted by R_1^a. For each $m > 0$, define R_m^a inductively by

$$R_{m+1}^a \equiv R_m^a \cap \left[S^a \times A^a(R_m^b) \right],$$

where $A^a(R_m^b)$ is the set of Ann's types in $t^a \in T^a$ such that R_m^b is assumed under $\lambda^a(t^a)$. If a state $(s^a, t^a, s^b, t^b) \in R_{m+1}^a \times R_{m+1}^b$ then we say that it satisfies **rationality and m-th order assumption of rationality** (henceforth RmAR).

We write $R_0^b \equiv S^b \times T^b$, and $R_\infty^b \equiv \bigcap_{m \in \mathbb{N}} R_m^b$. Note that $S^b \times T^b$ is trivially assumed under every full-support LPS on $S^b \times T^b$. It is shown in BFK that each of the sets R_m^a, R_m^b is Borel (so the players are able to assume these sets), and that

$$R_m^a = R_1^a \cap \left[S^a \times \bigcap_{i<m} A^a(R_i^b) \right].$$

In words, R_m^a is the set of states for which Ann is rational and assumes that Bob is i-th order rational for each $i \leq m$. If a state (s^a, t^a, s^b, t^b) belongs to $R_\infty^a \times R_\infty^b$ then it satisfies **rationality and common assumption of rationality** (henceforth RCAR). In words, each player is rational and assumes that the other player is m-th order rational for each $m \in \mathbb{N}$. It is shown in BFK that any LPS that assumes each of a countable sequence of events assumes their intersection. It follows that for any RCAR state, each player assumes that

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[9] In BFK, a type structure is called complete if $\mathscr{L}^+(S^b \times T^b) \subsetneq \text{range}\,\lambda^a$ and $\mathscr{L}^+(S^a \times T^a) \subsetneq \text{range}\,\lambda^b$. This difference is immaterial with respect to both their results and ours.

[10] But not necessarily all hierarchies of beliefs.

the other player is rational at order ∞, that is, Ann assumes R^b_∞ and Bob assumes R^a_∞.

3 STATEMENTS OF RESULTS

Section 3.1 states our main existence results, which 1) show that there exist complete type structures with RCAR; 2) establish that RCAR in complete type structures is an epistemic condition for IA; and 3) reconcile these facts with the negative conclusions found in the literature. Section 3.2 states some complementary results that relate beliefs about strategies to iterated admissibility. We need these results to prove our existence theorems, but they also merit independent consideration because they reveal certain structural commonalities of finite-order reasoning about rationality across complete type structures. In Section 3.3, we state a sharper form of our main existence theorem that gives a topological characterization, for a fixed game G, of the RmAR sets in complete type structures in which the RCAR set is nonempty. Finally, Section 3.4 provides a game-independent condition for IA that captures the line of reasoning that RCAR describes.

In order to easily distinguish new results from previous results from the literature, we will reserve the name "Proposition" for previous results from the literature, and use "Theorem", "Corollary", and "Lemma" for new results.

3.1 RCAR AND ITERATED ADMISSIBILITY

Consider the following infinite sequence of statements.

(a1)	Ann is rational	(b1)	Bob is rational
(a2)	(a1) and Ann assumes (b1)	(b2)	(b1) and Bob assumes (a1)
(a3)	(a2) and Ann assumes (b2)	(b3)	(b2) and Bob assumes (a2)
	...		...

For each $m > 1$, the statement "a$(m+1)$ and b$(m+1)$" corresponds to rationality and m-th order assumption of rationality. The conjunction of this infinite sequence of statements corresponds to rationality and common assumption of rationality.

A type structure $\mathfrak{T} = \langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ for G provides precise interpretations of these statements by implicitly defining the possible belief hierarchies of each player. BFK found that if the universe of beliefs implied by $\mathfrak{T}$ is rich enough—that is, $\mathfrak{T}$ is a complete structure—then the set of strategies played when RmAR holds coincides exactly with the set of $(m+1)$-admissible strategies. Proposition 3.1 below is the formal statement of this result.

Proposition 3.1 (Theorem 9.1 in BFK)**.** *Fix a finite game G and a complete lexicographic type structure $\mathfrak{T}$ for G. Then for each $m \in \mathbb{N}$,*

$$\mathrm{proj}_{S^a} R^a_m \times \mathrm{proj}_{S^b} R^b_m = S^a_m \times S^b_m.$$

It is natural to ask whether there is an analogous result that characterizes iterated admissibility using RCAR. Our main results, Theorems 3.2 and 3.4 below, establish the epistemic foundations of IA along those lines. In particular, Corollary 3.3 shows that there exists a complete type structure in which the RCAR set is nonempty, answering an open question that was asked in BFK.

Theorem 3.2 (Existence Theorem)**.** *For each finite game $G = \langle S^a, S^b, \pi^a, \pi^b \rangle$ and uncountable Polish spaces T^a, T^b, there exist Borel functions λ^a, λ^b such that $\mathfrak{T} = \langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ is a complete lexicographic type structure for G in which $R^a_\infty \times R^b_\infty$ is nonempty.*

Corollary 3.3. *Fix a finite game G. There exists a complete lexicographic type structure $\mathfrak{T}$ for G in which $R^a_\infty \times R^b_\infty$ is nonempty.*

Theorem 3.4. *Fix a finite game G and suppose $\mathfrak{T}$ is a complete lexicographic type structure for G such that $R^a_\infty \times R^b_\infty$ is nonempty. Then*

$$\mathrm{proj}_{S^a} R^a_\infty \times \mathrm{proj}_{S^b} R^b_\infty = S^a_\infty \times S^b_\infty.$$

In words, Corollary 3.3 says that there exists a complete type structure for G with at least one state that belongs to the RCAR set. Theorem 3.4 says that, in every complete type structure for G in which the RCAR set is nonempty, the set of strategies played when RCAR holds is exactly equal to the IA set. Together, Theorems 3.2 and 3.4 say that "RCAR in complete type structures" is an epistemic condition for IA since 1) every IA strategy is played under RCAR in some complete type structure; and 2) the strategies that are played under RCAR in any complete type structure must be IA strategies.

However, it is not the case that every complete type structure for G has a nonempty RCAR set. Consider the following two results from BFK.

Proposition 3.5 (Theorem 10.1 in BFK)**.** *Fix a finite game G and a complete lexicographic type structure $\mathfrak{T}$ for G such that the maps λ^a, λ^b are continuous. If there exist r^a, s^a, s^b such that $\pi^a(r^a, s^b) \neq \pi^a(s^a, s^b)$ then $R^a_\infty \times R^b_\infty = \emptyset$.*

Proposition 3.6 (Proposition 7.2 in BFK)**.** *For each finite game G there exists a complete lexicographic type structure $\mathfrak{T}$ for G such that the maps λ^a, λ^b are continuous.*[11]

These two results together show that there are complete type structures for G in which the RCAR set is empty. So the set of strategies played when RCAR holds is empty, but of course the IA set is nonempty.

How do we reconcile our results with Proposition 3.5? In particular, how should we understand the fact that complete type structures having nonempty RCAR sets cannot have continuous type-belief maps? To do so, we give an improvement of Proposition 3.6. By a **Borel refinement** of a Polish space T, we mean a Polish space U such that U has the same set of points and the same Borel σ-algebra as T, and every open set in T is open in U. Thus U has the same Borel sets but more open sets.

Theorem 3.7. *Let $\mathfrak{T} = \langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ be a complete lexicographic type structure for a finite game G. Then there exist Borel refinements U^a, U^b of T^a, T^b such that the maps*

$$\lambda^a : U^a \to \mathcal{L}(S^b \times U^b), \qquad \lambda^b : U^b \to \mathcal{L}(S^a \times U^a)$$

are continuous.

It follows that $\mathfrak{U} = \langle S^a, S^b, U^a, U^b, \lambda^a, \lambda^b \rangle$ is again a complete type structure for G, so Theorem 3.7 implies Proposition 3.6. Since the Borel σ-algebras are unchanged, the sets of LPS are unchanged, i.e.,

$$\mathcal{L}(S^a \times U^a) = \mathcal{L}(S^a \times T^a), \qquad \mathcal{L}(S^b \times U^b) = \mathcal{L}(S^b \times T^b).$$

However, more open sets have been added to their topologies.

No state satisfying RCAR exists in the type structure $\mathfrak{U}$ by Proposition 3.5. Note that while $\mathcal{L}(S^b \times T^b) = \mathcal{L}(S^b \times U^b)$, it is not the case that $\mathcal{L}^+(S^b \times T^b) = \mathcal{L}^+(S^b \times U^b)$. This is because full-support LPS's must assign positive measure to every open set and U^b contains more open sets than T^b does. Effectively, there are fewer full-support types in U^a than there are in T^a. So the operation of refining the topologies of the type spaces in this fashion shrinks the set of states in which every player is rational.

The set of full-support types shrinks as a consequence of a more basic change. Recall that, to assume an event, it is necessary to assign positive measure to every "part" of it. Any two disjoint parts of an event are topologically distinguishable[12] from each other. It

[11] Since our definition of complete is slightly different from the definition in BFK, the proof must use Theorem 13.7 instead of Theorem 7.9 in Kechris (1995).

[12] Two events are topologically indistinguishable if one approximates the other and vice versa—that is, if they have identical closures.

follows that refining the topology on the state space raises the standards of assumption for the players. From such a perspective, continuous type structures are type structures with high standards for assumption.

The description of continuous type structures in BFK as type structures in which neighboring full-support LPS's are associated with neighboring full-support types, while equivalent, does not immediately call this property to attention. Our interpretation of continuous type structures is that they describe players who are more finicky about saying that they assume something. They are more agnostic than players in discontinuous type structures in this sense.

One early interpretation of BFK's negative result was that RCAR is impossible when players know too little about each other (since complete type structures are very rich). Friedenberg (2010) had previously shown that compact, complete, and continuous *standard*[13] type structures contain all hierarchies of beliefs. While this result had not been extended to lexicographic type structures, it was reasonable to suppose that the nonexistence of RCAR in complete continuous lexicographic type structures was due to the presence of *too many* hierarchies of beliefs.

Our Theorem 3.7 says that any complete type structure—even one with nonempty RCAR—contains exactly the same set of hierarchies of beliefs as some complete continuous type structure. It permits us to isolate the issue of missing hierarchies and conclude that RCAR is impossible in complete continuous type structures because players are too cautious about assuming events in such type structures, and not because they know too little about their opponents.

The Existence Theorem 3.2 is not fully satisfactory in the sense that it only says that if we fix a game, we can find a complete type structure in which RCAR for that fixed game is nonempty. It is silent on whether, given a pair of finite strategy sets (S^a, S^b), there is a single complete type structure in which RCAR for every game G on (S^a, S^b) is nonempty.

We will remedy this shortcoming in another way in Theorem 3.25.

3.2 STRATEGIC BELIEFS AND ITERATED ADMISSIBILITY

In this section, we give an alternate definition of the IA set in the style of rationalizability via an iterative refinement of the players' *strategic beliefs*—that is, their marginal beliefs

[13] i.e., type structures with standard probabilities, not lexicographic probabilities.

over opponents' strategies.[14] We will show that this refinement process captures some of the structural properties of RmAR sets that are conveniently invariant across all complete type structures. In other words, RmAR sets can be described as having the same "shape" in a sense across all such type structures. These properties are incredibly useful in proving the existence theorems of Section 3.1 and their generalization in Section 3.3.

We suppose throughout that $G = \langle S^a, S^b, \pi^a, \pi^b \rangle$ is a finite game in strategic form. We first introduce some notation. For $r^a, s^a \in S^a$ and a sequence $\nu = (\nu_0, \dots, \nu_n) \in \mathcal{N}(S^b)$, we say that s^a is **preferred** to r^a under ν, and write $s^a \succ_\nu r^a$, if the LEU of s^a under ν is greater than that of r^a—that is, $\pi^a(s^a, \nu) >_{\mathsf{LEX}} \pi^a(r^a, \nu)$, where $\pi^a(s^a, \nu)$ is the $(n+1)$-tuple $(\pi^a(s^a, \nu_0), \dots, \pi^a(s^a, \nu_n))$. Note that the leftmost term $\pi^a(s^a, \nu_0)$ has the highest priority. Intuitively speaking, under belief $\nu = (\nu_0, \dots, \nu_n)$, for each $k \le n$ the strategies in the support of $(\nu_0, \dots, \nu_k)$ are infinitely more likely than the strategies outside the support of $(\nu_0, \dots, \nu_k)$.

Given $\mu, \nu \in \mathcal{N}(S^b)$, we write $\mu \sim^\star \nu$ if for all $r^a, s^a \in S^a$, $s^a \succ_\mu r^a$ if and only if $s^a \succ_\nu r^a$. In other words, $\mu \sim^\star \nu$ if and only if they induce the same preference ordering over *pure* strategies. It is easy to see that $\sim^\star$ is an equivalence relation. If $\mu = (\mu_0, \dots, \mu_m)$ and $\nu = (\nu_0, \dots, \nu_n)$ then the **concatenation** of μ and ν is defined as the sequence

$$\mu\nu \equiv ((\mu\nu)_0, \dots, (\mu\nu)_{m+n+1}) = (\mu_0, \dots, \mu_m, \nu_0, \dots, \nu_n).$$

We note that if $\mu \sim^\star \nu$ and $\mu' \sim^\star \nu'$ then $\mu\nu \sim^\star \mu'\nu'$. Now, let

$$P_1^a \equiv \mathcal{N}^+(S^b) = \left\{\nu : \nu \in \mathcal{N}(S^b) \wedge \operatorname{Supp} \nu = S^b\right\}$$

and define the following for each $m > 0$.

$$P_{m+1}^a \equiv \left\{\nu\nu' : \nu \in \mathcal{N}(S^b) \wedge \operatorname{Supp} \nu = S_m^b \wedge \nu' \in P_m^a\right\}.$$

The set P_1^a can be interpreted as the set of strategic beliefs held by rational types. Fix a complete type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. If Ann is rational then she has a full-support belief $\mu \in \mathcal{L}^+(S^b \times T^b)$ and her strategic belief is $\operatorname{marg}_{S^b} \mu$. Then the set of strategic beliefs that may be held by rational Anns is

$$\left\{\operatorname{marg}_{S^b} \mu : \mu \in \mathcal{L}^+(S^b \times T^b)\right\} = \mathcal{N}^+(S^b) = P_1^a.$$

We will call P_1^a the set of Ann's *rational* strategic beliefs. It readily follows that S_1^a is the set of strategies played by rational Anns.

[14]The refinement process may succinctly be described as *set-valued* lexicographic rationalizability, given its similarity to Stahl's (1995) notion of *lexicographic rationalizability*.

Following the intuitive description given above, we can say that if Ann holds a strategic belief $\nu \in P^a_2$ then she considers the event that Bob is rational to be infinitely more likely than the event that he is not. Furthermore, we can say that $\nu = \operatorname{marg}_{S^b} \mu$ for some full-support belief μ of Ann that assumes that Bob is rational. An inductive argument shows that if $\nu \in P^a_{m+1}$ then ν is the marginal on S^b of some full-support belief of Ann that m-th order assumes rationality.

For each $\nu \in \mathcal{N}(S^b)$, let $\mathbb{O}(\nu)$ denote the set of all $s^a \in S^a$ such that s^a is optimal under $\succ_\nu$ (i.e., s^a maximizes LEU under ν). Note that if $\mu \sim^\star \nu$ then $\mathbb{O}(\mu) = \mathbb{O}(\nu)$. For each $m > 0$, define

$$\mathbb{X}^a_m \equiv \left\{ \mathbb{O}(\nu) : \nu \in P^a_m \right\}.$$

We have the following characterization of m-admissible strategies as strategies that are optimal under strategic beliefs in P^a_m.

Theorem 3.8. *For each $m > 0$, $\bigcup \mathbb{X}^a_m = S^a_m$. Each $s^a \in S^a_m$ belongs to some $X^a \in \mathbb{X}^a_m$, and $\mathbb{X}^a_m$ is a set of subsets of S^a_m.*

Note that Theorem 3.8 allows us to rewrite the definition of P^a_{m+1} without reference to the m-admissible set S^b_m. In fact, all results in this section would continue to hold even if we had started with the following definition of P^a_{m+1}.

$$P^a_{m+1} \equiv \left\{ \nu\nu' : \nu \in \mathcal{N}(S^b) \wedge \operatorname{Supp} \nu = \mathbb{O}(P^b_m) \wedge \nu' \in P^a_m \right\},$$

where $\mathbb{O}(P^b_m) = \bigcup \{ \mathbb{O}(\mu) : \mu \in P^b_m \}$.

We are also able to show that for every full-support belief μ of Ann who m-th order assumes rationality, there is some $\nu \in P^a_{m+1}$ such that $\operatorname{marg}_{S^b} \mu$ and ν induce the same preferences over Ann's strategies—that is, the strategic beliefs in P^a_{m+1} are representative of Ann's preferences over her strategies in states of the world satisfying RmAR in a complete type structure. Theorem 3.9 below gives a precise statement of this relationship.

Theorem 3.9. *In a complete lexicographic type structure for a finite game, for each $m > 0$,*

(i) If $(s^a, t^a) \in R^a_m$ then $\exists \nu \in P^a_m$ such that $\operatorname{marg}_{S^b} \lambda^a(t^a) \sim^\star \nu$; and

(ii) If $\nu \in P^a_m$ then $\exists (s^a, t^a) \in R^a_m$ such that $\operatorname{marg}_{S^b} \lambda^a(t^a) = \nu$.

By definition, a strategy s^a is optimal under $\lambda^a(t^a)$ if and only if s^a is optimal under $\operatorname{marg}_{S^b} \lambda^a(t^a)$. Thus we have the following corollary.

Corollary 3.10. *In a complete lexicographic type structure for a finite game G, for each $m > 0$, a set $X^a \subseteq S^a$ belongs to $\mathbb{X}_m^a$ if and only if there is a state $(s^a, t^a) \in R_m^a$ such that X^a is the set of all optimal strategies under $\lambda^a(t^a)$. Moreover, for each m we have $\mathbb{X}_m^a \supseteq \mathbb{X}_{m+1}^a$.*

Theorems 3.8 and 3.9 tell us something about the "shape" of the RmAR sets in a complete type structure for G. To see this, we consider an *arbitrary* relation $Q \subseteq S^a \times T^a$ and subset $X^a \subseteq S^a$, and define

$$\Gamma^a(X^a, Q) \equiv \left\{t^a \in T^a : X^a = \left\{s^a : (s^a, t^a) \in Q\right\}\right\}.$$

In words, $\Gamma^a(X^a, Q)$ is the set of all $t^a \in T^a$ such that the section of Q at t^a is exactly X^a. It is clear that for each set $Q \subseteq S^a \times T^a$, the family of sets $\left\{\Gamma^a(X^a, Q) : X^a \subseteq S^a\right\}$ is pairwise disjoint, and the union of the family is T^a. Thus the nonempty sets in this family form a finite partition of T^a.

It follows that in any type structure for G, and for each nonempty set $X^a \subseteq S^a$ and $m > 0$, $\Gamma^a(X^a, R_m^a)$ is the set of all types t^a for Ann such that X^a is the set of optimal strategies for $\lambda^a(t^a)$, and Ann is open-minded and assumes k-th order rationality for Bob for all $k < m$.[15]

The next corollary shows that the RmAR sets have similar "shapes" in all complete type structures for a given finite game G.

Corollary 3.11. *In a complete lexicographic type structure for a finite game, for each nonempty $X^a \subseteq S^a$, the following statements hold.*

(i) The sequence $\left\{\Gamma^a(X^a, R_m^a) : m > 0\right\}$ is a decreasing chain of Borel sets of T^a;

(ii) For each $m > 0$, $\Gamma^a(X^a, R_m^a)$ is nonempty if and only if $X^a \in \mathbb{X}_m^a$; and

(iii) The sequence $\left\{\Gamma^a(\varnothing, R_m^a) : m > 0\right\}$ is an increasing chain of nonempty Borel sets of T^a.

Corollary 3.11 gives us the following useful formula for the RmAR sets in a complete type structure for a finite game.

$$R_m^a = \bigcup \left\{X^a \times \Gamma^a(X^a, R_m^a) : X^a \in \mathbb{X}_m^a\right\}.$$

3.3 ALL POSSIBLE RCAR SETS

In this section we state a sharper form of the Existence Theorem 3.2. Consider a complete type structure such that RCAR obtains in some state. Below, we define an *RCAR*

[15] Bob is k-th order rational if he is rational and $(k-1)$-th order assumes rationality.

tower to be the family of RmAR sets in *any* such type structure. The results of this section give a list of topological properties that characterize RCAR towers.

The sequences of RmAR sets depend on the type structure $\mathfrak{T}$ as well as the game G. To indicate this dependence, we will sometimes write $R_m^a(G,\mathfrak{T})$ instead of R_m^a, $R_\infty^a(G,\mathfrak{T})$ instead of R_∞^a. Throughout this section, we fix a finite game G and a pair of uncountable Polish spaces T^a, T^b.

Definition 3.12. *An* ***RCAR tower*** *for* (G, T^a, T^b) *is a pair of sequences* $\{Q_m^a : m > 0\}$ *and* $\{Q_m^b : m > 0\}$ *of sets such that*

(i) The intersections $Q_\infty^a \equiv \bigcap_m Q_m^a$, $Q_\infty^b \equiv \bigcap_m Q_m^b$ *are nonempty; and*

(ii) There exist maps λ^a, λ^b *for which* $\mathfrak{T} = \langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ *is a complete lexicographic type structure for G, and for all* $m > 0$, $Q_m^a = R_m^a(G,\mathfrak{T})$ *and* $Q_m^b = R_m^b(G,\mathfrak{T})$.

Thus every complete type structure with RCAR gives rise to an RCAR tower. The Existence Theorem 3.2 implies that there exists an RCAR tower. Corollary 3.11 gives a limitation on the possible RCAR towers—they must have the right "shape". Property 6.2 in BFK gives a second limitation—two events assumed at the same level must be topologically indistinguishable, so the set $Q_\infty^a \times Q_\infty^b$ must be topologically indistinguishable from $Q_M^a \times Q_M^b$ for some M. Lemma E.2 in BFK gives a third limitation—for each open-minded type t^a for Ann, there are uncountably many other open-minded types that have the same optimal strategies and assumptions as t^a, and hence each "part" of Q_m^a that is not in Q_{m+1}^a must be uncountable. The following sharp existence theorem says exactly which families of sets are RCAR towers. In words, it says that a family of sets is an RCAR tower if and only if it satisfies the three limitations above.

Theorem 3.13. *The pair of sequences* $\{Q_m^a : m > 0\}, \{Q_m^b : m > 0\}$ *is an RCAR tower for* (G, T^a, T^b) *if and only if for each nonempty* $X^a \subseteq S^a$,

(i) $\{\Gamma^a(X^a, Q_m^a) : m > 0\}$ *is a decreasing chain of Borel subsets of* T^a*;*

(ii) For each $m > 0$, $\Gamma^a(X^a, Q_m^a) \neq \emptyset \iff X^a \in \mathbb{X}_m^a$*;*

(iii) For some $M > 0$, $\overline{Q_\infty^a} = \overline{Q_M^a}$*;*

(iv) $\Gamma^a(\emptyset, Q_1^a)$ *is uncountable;*

(v) $X^a \in \mathbb{X}_m^a \implies \Gamma^a(X^a, Q_m^a) \setminus \Gamma^a(X^a, Q_{m+1}^a)$ *is uncountable; and*

(vi) $X^a \in \mathbb{X}_\infty^a \implies \Gamma^a(X^a, Q_\infty^a)$ *is uncountable;*

and similarly for b.

Theorem 3.13 shows that the property of being an RCAR tower depends only on the optimality sets $\mathbb{X}_m^a, \mathbb{X}_m^b$. As shown in Section 3.2, these sets capture properties that are universal to all complete type structures. Indeed, their definitions make no reference to type structures at all. Thus the property of being an RCAR tower is partially robust. Theorem 3.13 also shows that one has a great deal of control over the properties that the RmAR sets will have.

Furthermore, Theorem 3.14, which is stated below, shows that any family of sets that is an RCAR tower is also the family of RmAR sets in some complete one-to-one type structure for G. By a **one-to-one type structure for** G we mean a type structure $\mathfrak{T}$ for G in which the mappings λ^a, λ^b are one-to-one.

Theorem 3.14. *The pair of sequences* $\{Q^a_m : m > 0\}, \{Q^b_m : m > 0\}$ *is an RCAR tower for* (G, T^a, T^b) *if and only if*

(i) $Q^a_\infty \equiv \bigcap_m Q^a_m$ *and* $Q^b_\infty \equiv \bigcap_m Q^b_m$ *are nonempty; and*

(ii) There exist mappings λ^a, λ^b *such that* $\mathfrak{T} = \langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ *is a complete one-to-one lexicographic type structure for G, and for all* $m > 0$, $Q^a_m = R^a_m(G, \mathfrak{T})$ *and* $Q^b_m = R^b_m(G, \mathfrak{T})$.

A consequence of Theorem 3.13 is that the proof of the Existence Theorem 3.2 is reduced to the problem of finding a family of sets that fits the topological characterization of RCAR towers. The following lemma about Polish spaces, together with Theorem 3.13, implies Theorem 3.2.

Lemma 3.15. *For any finite game G and uncountable Polish spaces* T^a, T^b*, there is a family of open sets* $\{Q^a_m : m > 0\}$, $\{Q^b_m : m > 0\}$ *such that* Q^a_∞, Q^b_∞ *are open and Conditions 1–6 of Theorem 3.13 hold.*

This gives us a complete one-to-one type structure $\mathfrak{T}$ with the additional property that all the RmAR sets $R^a_m(G, \mathfrak{T}), R^b_m(G, \mathfrak{T})$ and the RCAR sets $R^a_\infty(G, \mathfrak{T}), R^b_\infty(G, \mathfrak{T})$ are open and uncountable. By choosing other RCAR towers, one can get a $\mathfrak{T}$ with various other properties.

Theorem 3.13 is a very strong result and no comparable analog is in the literature about standard type structures and RCBR. It is well-known that the sets of states that satisfy rationality and m-th order belief of rationality (RmBR) being compact for each m is sufficient for the existence of RCBR (cf. Tan and Werlang, 1988). However, there is no topological characterization of the RmBR sets in standard type structures with RCBR.

3.4 A GAME-INDEPENDENT CONDITION FOR RCAR

The results of the previous sections have shown that, given any game G, there is a large class of complete type structures such that RCAR is nonempty when rationality is defined with respect to the specific game G. Furthermore, if we fix a type structure $\mathfrak{T}_G$ in this class then the play of the game under RCAR is exactly the IA set of G. However, it will not necessarily be the case that the RCAR set is nonempty in $\mathfrak{T}_G$ when rationality is defined with respect to another game G with the same strategy sets (S^a, S^b).

In the aforementioned type structure $\mathfrak{T}_G$, players can commonly assume rationality with respect to G but perhaps not with respect to another game. Our main results of the earlier sections do not guarantee the existence of any complete type structure in which, for all games G, some state satisfies the condition "RCAR with respect to G".

This form of game-dependence raises the following question: Is there a model environment in which the set of states with common knowledge of the game G is isomorphic to a complete type structure (e.g., $\mathfrak{T}_G$) in which common assumption of rationality with respect to G is possible? In this section, we answer this question in the affirmative. To do so, we first extend the BFK framework to include moves of nature, which choose the game G.

Definition 3.16. *A* ***lexicographic type structure with nature*** *is a structure*

$$\mathfrak{V} = \left\langle \Theta, S^a, S^b, V^a, V^b, \lambda^a, \lambda^b \right\rangle$$

where S^a, S^b are nonempty finite sets, Θ is the space of games over (S^a, S^b), V^a, V^b are Polish spaces, and λ^a, λ^b are Borel functions

$$\lambda^a : V^a \to \mathscr{L}(\Theta \times S^b \times V^b), \qquad \lambda^b : V^b \to \mathscr{L}(\Theta \times S^a \times V^a).$$

We say that $\mathfrak{V}$ is ***complete*** *if the mappings λ^a, λ^b are onto, and* ***one-to-one*** *if the mappings λ^a, λ^b are one-to-one.*

Hereafter, $\mathfrak{V}$ will denote a lexicographic type structure with nature, and G will denote a game in Θ.

Definition 3.17. *We say that a type $v^a \in V^a$* ***believes an event*** *$E \subseteq \Theta \times S^b \times V^b$ and write $v^a \in C^a(E)$ if*

$$(\lambda^a(v^a))(E) = \vec{1},$$

where $\vec{1}$ denotes a finite sequence of 1s. We say that v^a ***believes a game*** *G and write $v^a \in C_1^a(G)$ if v^a believes $\{G\} \times S^b \times V^b$, that is, $v^a \in C^a(\{G\} \times S^b \times V^b)$.*

We now define common belief of G. Informally, there is common belief of G if each player believes G, believes that the other player believes G, believes that the other player believes that, and so on.

Definition 3.18. *We say that v^a has* ***common belief*** *of G if $v^a \in C^a_\infty(G)$, where*

$$\forall m > 0, \quad C^a_{m+1}(G) = C^a_m(G) \cap C^a(\Theta \times S^b \times C^b_m(G));$$
$$C^a_\infty(G) = \bigcap_{m>0} C^a_m(G).$$

If both v^a and v^b have common belief of G, we say that the pair (v^a, v^b) has common belief of G. We say that there is ***common knowledge of*** *G at state $(\theta, s^a, v^a, s^b, v^b)$ if $\theta = G$ and (v^a, v^b) has common belief of G. In other words, common knowledge is equivalent to* true *common belief. The* ***common knowledge set*** *for a at G is the set*

$$K^a(G) = \{G\} \times S^a \times C^a_\infty(G).$$

One can easily verify that the common knowledge sets $K^a(G)$ and $K^b(G)$ are Borel (but possibly empty). We say that $\mathfrak{V}$ **admits common knowledge of** G if the sets $K^a(G)$ and $K^b(G)$ are nonempty Polish spaces.[16] Clearly, $K^a(G)$ is a Polish space if and only if $C^a_\infty(G)$ is a Polish space.

Definition 3.19. *Suppose $\mathfrak{V}$ admits common knowledge of G. $\mathfrak{V}_G$ is the structure*

$$\mathfrak{V}_G = \left\langle S^a, S^b, V^a_G, V^b_G, \lambda^a_G, \lambda^b_G \right\rangle$$

such that $V^a_G = C^a_\infty(G)$, and for each $v^a \in V^a_G$ and event $E \subseteq S^b \times V^b_G$,

$$(\lambda^a_G(v^a))(E) = (\lambda^a(v^a))(\Theta \times E).$$

Lemma 3.20. *If $\mathfrak{V}$ admits common knowledge of G, then $\lambda^a_G : V^a_G \to \mathscr{L}(S^b \times V^b_G)$, so $\mathfrak{V}_G$ is an ordinary lexicographic type structure.*

We formulate an analog of RCAR with respect to a given game G in type structures with nature by extending the definitions of assumption and rationality to type structures with nature as follows.

Definition 3.21. *We say that a type $v^a \in V^a$* ***assumes an event*** *$E \subseteq \Theta \times S^b \times V^b$ in $\mathfrak{V}$, and write $v^a \in A^a(E)$, if $\lambda^a(v^a) = (\mu_1, \ldots, \mu_{n-1})$ such that*

(a) $\mu_i(E) = 1$ *for each $i \le j$;*

[16] Note that common knowledge of G is equivalent to "G and common belief of G".

(b) $\mu_i(E) = 0$ *for each* $i > j$*; and*

(c) *If* U *is open with* $U \cap E \neq \varnothing$ *then* $\mu_i(U \cap E) > 0$ *for some* $i < n$.

It is easily seen that a pair (s^a, v^a) is rational for G in $\mathfrak{V}_G$ if and only if $(s^a, v^a) \in S^a \times V^a_G$, v^a assumes $S^b \times V^b_G$ in $\mathfrak{V}_G$, and s^a maximizes LEU with respect to $\lambda^a_G(v^a)$. Our definition below of G-rationality in $\mathfrak{V}$ follows this pattern.

Definition 3.22. *We say that the triple* (θ, s^a, v^a) *is* G***-rational****, and write* $(\theta, s^a, v^a) \in R^a_1(G)$*, if*

(i) $(\theta, s^a, v^a) \in K^a(G)$*, i.e., there is common knowledge of* G*;*

(ii) v^a *assumes* $K^b(G)$ *(common knowledge of* G*) in* $\mathfrak{V}$*; and*

(iii) s^a *maximizes LEU with respect to* $\lambda^a(v^a)$.

Definition 3.23. *We say that the triple* (θ, s^a, v^a) *is* G***-rational and commonly assumes*** G***-rationality (***G***-RCAR)*** *if* $(\theta, s^a, v^a) \in R^a_\infty(G)$*, where*

$$\forall m \in \mathbb{N}, \quad R^a_{m+1}(G) = R^a_m(G) \cap (\Theta \times S^a \times A^a(R^b_m(G)))$$
$$R^a_\infty(G) = \bigcap_{m>0} R^a_m(G)$$

When $\mathfrak{V}$ admits common knowledge of G, the following result gives the relationships between assumption in $\mathfrak{V}$ and in $\mathfrak{V}_G$, and between iterated G-rationality in $\mathfrak{V}$ and iterated rationality for G in $\mathfrak{V}_G$.

Theorem 3.24. *Suppose that* $\mathfrak{V}$ *admits common knowledge of* G*, and let* $v^a \in V^a_G$*. Then*

(i) For all Borel $E \subseteq S^b \times V^b_G$*,* v^a *assumes* $\{G\} \times E$ *in* $\mathfrak{V}$ *if and only if* v^a *assumes* E *in* $\mathfrak{V}_G$*;*

(ii) $R^a_1(G) = \{G\} \times R^a_1(G, \mathfrak{V}_G)$*;*

(iii) $\forall m > 0, \quad R^a_m(G) = \{G\} \times R^a_m(G, \mathfrak{V}_G)$*;*

and similarly for b.

We can now state our result that there is a model environment in which, for every game G, G-RCAR epistemically characterizes the IA set of G. In this environment, which is a complete lexicographic type structure with nature, G-RCAR may be described as a game-independent condition in the sense that it can be satisfied for every game G.

Theorem 3.25. *For each pair* (S^a, S^b) *of finite strategy sets, there is a complete one-to-one lexicographic type structure with nature,* $\mathfrak{V} = \langle \Theta, S^a, S^b, V^a, V^b, \lambda^a, \lambda^b \rangle$*, such that for every game* $G \in \Theta$*,*

(i) $\mathfrak{V}$ *admits common knowledge of* G*;*

(ii) $\mathfrak{V}_G$ *is a complete one-to-one lexicographic type structure; and*

(iii) $\mathrm{proj}_{S^a} R^a_\infty(G) \times \mathrm{proj}_{S^b} R^b_\infty(G) = S^a_\infty(G) \times S^b_\infty(G)$.

4 DISCUSSION

Here, we give the results of this paper further context by considering their relationship with other results in the literature.

RCAR as an Epistemic Condition As explained in BFK, the RCAR concept corresponds to a "line of reasoning" where each player considers all possibilities about the beliefs of the other players. We focus on the question of whether RCAR can provide an epistemic basis for the IA solution concept. We will compare our results with the literature on the corresponding question of whether RCBR provides epistemic conditions IU.

Brandenburger and Dekel (1987a) essentially give us the following fact.

Fix a game.

(i) For each type structure, the set of strategies consistent with RCBR is a subset of the IU strategies.

(ii) There exists a finite type structure such that the set of strategies consistent with RCBR is exactly the IU set.

This says that if we look at RCBR across all type structures, we get the IU set. Taken together, our Theorems 3.2 and 3.4 imply the following analogous fact.

Fix a game.

(i) For each complete lexicographic type structure, the set of strategies consistent with RCAR is a subset of the IA strategies.

(ii) There exists a complete lexicographic type structure such that the set of strategies consistent with RCAR is exactly the IA set.

This says that if we look at RCAR across all complete lexicographic type structures, we get the IA set.[17]

It may appear that this is the end of the matter. However, under the epistemic game theory (EGT) theory approach, the beliefs that the players deem possible—and therefore the type structure that generates them—are part of the description of the strategic situation. From the perspective of the players, the type structures other than the one that describes their strategic situation are simply irrelevant. Such extraneous type structures may exclude types that the players consider possible or include types that the players consider impossible.

[17] BFK showed, if we look at RCAR across all lexicographic type structures, we do not get the IA set.

While an analyst can find a justification for each IU strategy by looking across all type structures, the player, whose perception is confined to the boundaries defined by the type structure that describes her situation, is not assured of being able to the same.

This raises a question: Can the players themselves see all the IU strategies as the result of the line of reasoning captured by RCBR? This requires a type structure that is "rich enough" so that each IU strategy is justified by some RCBR state. Indeed, many type structures fail to satisfy such a richness condition.

Brandenburger and Dekel (1987a) showed that, given a fixed game G, one can tailor the type structure so that the IU strategies of G are the output of RCBR.[18] Since this construction depends on the game G, it may be the case that this type structure is not "rich enough" to give us the IU strategies of another game G' as the output of RCBR.

However, from the perspective of EGT, a good epistemic condition should involve a line of reasoning for the players that is *game-independent* (i.e., the reasoning is the same for all games). This implies that the type structure should be rich enough so that RCBR produces the IU set for every game. Tan and Werlang (1988) identified one such richness condition: In the so-called "universal type structure", the IU set is characterized by RCBR, regardless of the game in question. Friedenberg (2010) showed that any complete, compact, and continuous type structure also has this property.

This result has no direct analog with respect to RCAR and the IA set. BFK showed that if a lexicographic type structure is complete and continuous then it contains no state that satisfies RCAR.[19] Therefore, if we fix an arbitrary complete type structure, we cannot say that RCAR is an epistemic condition for IA.

We resolve this issue by showing that there is a type structure that is rich enough in the sense that it embeds enough complete type structures so that for each game, RCAR in at least one of the embedded type structures yields the IA set as output. This structure can be viewed as a model environment in which players can look across many complete type structures like the aforementioned imaginary analyst so that "RCAR in complete type structures" is a game-independent epistemic condition for IA. Our Theorem 3.25 shows the existence of such a structure in which an embedded complete type structure with RCAR for a game G can be interpreted as describing the set of states with common knowledge of G. We make this interpretation explicit in our formal treatment, which

[18] Brandenburger and Dekel (1987a) uses finite partitions, not finite type structures. However, a finite partition structure is essentially equivalent to some finite type structure.

[19] In fact, a complete, compact, and continuous lexicographic type structure does not even exist.

introduces moves of nature that choose the game G.

Continuity The pessimism in BFK with respect to finding an epistemic condition for IA strategies was due principally to their finding that complete continuous type structures must have empty RCAR sets. Our Theorem 3.2 revived the research program by showing that there are complete type structures with nonempty RCAR sets. In these type structures, the belief maps cannot be continuous. This suggested that continuity, which had appeared to be a technical condition ex ante, changes the players' reasoning in some significant way. By contrast, the much weaker requirement that the belief mappings are Borel is a technical condition that provides the structure needed to obtain results.

Our Theorem 3.7 gives a striking way to isolate the effects of continuous belief maps by showing that, given *any* complete lexicographic type structure there is a corresponding complete and continuous structure that describes exactly the same beliefs—that is, the two type structures are equally rich in at least one sense.

The difference between a complete structure $\mathfrak{T}$ and its continuous counterpart $\mathfrak{U}$ given by Theorem 3.7 is that of topological distinguishability, which affects the classification of beliefs rather than changing them. Thus it turns out that a type structure in the BFK framework captures more information than just the players' possible hierarchies of beliefs.

In Theorem 3.7, $\mathfrak{U}$ gives a finer topologization of the state space than $\mathfrak{T}$ does. How should we interpret this difference? The topology on a state space, say Ω, is essentially the set of events that open-minded Bayesians must consider in their decision making. However, we find it more convenient to start with the interpretation that a topology separates and distinguishes hypotheses about the true state of world. Consider two hypotheses, which are respectively represented by events E and E'. If their closures are equal, as determined by a topology $\mathscr{T}$ on Ω, then it may be said that E and E' are indistinguishable in $\mathscr{T}$ because E approximates E' in an arbitrarily fine way and vice versa.

Whether an event is assumed by a given belief is sensitive to the topology on the state space. As this topology is successively refined, a given belief will be classified as assuming fewer and fewer events. We might then informally describe the difference between $\mathfrak{T}$ and $\mathfrak{U}$ as follows: The players in the environment described by the latter are more cautious about assuming things than the players in the environment described by the former. This relationship gives an intuitively appealing reconciliation of BFK's negative result with our positive result. Players described by continuous type structures are just too cautious to commonly assume rationality.

Furthermore, it is apparent that a decision maker who assigns a non-zero probability to each open set is simply giving proper consideration to all distinguishable hypotheses. Therefore, it is even the case that the rational players described by $\mathfrak{U}$ must be more open-minded than the rational players described by $\mathfrak{T}$.

Other Approaches Alternate routes to an epistemic condition for IA may also exist. The most direct path of attack would be to ask, as we did at the end of Subsection 3.1, whether there exists a single complete lexicographic type structure—perhaps an analog of the universal type structure—in which the IA set of every game is the output of RCAR.

A second option would be to weaken the criteria for assumption so that they are not sensitive to variations in topological distinguishability. However, we do not want to weaken assumption so much that we no longer get IA as an output of RCAR. Analyzing the complete lexicographic type structures constructed in this paper may provide some hints on how to achieve these goals. Roughly speaking, beliefs that manifest the so-called *Best Rationalization Principle* that was articulated in Battigalli (1996) also satisfy common assumption of rationality in our constructions. In other words, if Ann attributes each admissible choice s^b of Bob to a rational decision based on the highest order mutual assumption of rationality that is consistent with it, then her beliefs satisfy common assumption of rationality. Ann, if her beliefs reflect the Best Rationalization Principle, can be viewed as assigning ex-ante explanations of all possible actions of Bob—explanations that preserve as much higher order assumption of rationality as possible.

Yang (2010) uses lexicographic type structures with a fixed finite bound M on the length of LPS's, and introduces a notion of *weak assumption* in place of assumption. This gives an alternative epistemic characterization of IA where the players do not consider possibilities involving iterated beliefs longer than M.

A third option is to adopt an approach in which admissibility itself is not justified on an epistemic basis, thus skirting around the inclusion-exclusion problem. Barelli and Galanis (2010) gives an alternative epistemic condition for IA that is built on *event-rationality*, which, like LEU, is an extension of the standard model of Bayesian rationality. An event-rational decision maker evaluates acts based on her standard probability beliefs and break ties using her personal list of tie-breakers. In that approach, admissibility is obtained by requiring that her tie-breaker list has sufficient coverage, rather than by requiring that she has full-support beliefs (i.e., an open-minded epistemic state).

A PROOFS OF THEOREMS 3.4 AND 3.7

A **Borel refinement** of a Polish space T is a Polish space U such that U has the same points as T, and every open set in T is open in U. To prove Theorem 3.7, we need the following results about Borel refinements.

Proposition A.1 (15.4 in Kechris (1995)). *If T is a Polish space and U is a Borel refinement of T, then T and U have the same Borel sets.*

Proposition A.2 (13.11 in Kechris (1995)). *Suppose T is a Polish space, Y is a second countable space, and $f : T \to Y$ is a Borel function. Then there is a Borel refinement U of T such that $f : U \to Y$ is continuous.*[20]

Proposition A.3 (13.3 in Kechris (1995)). *Let T be a Polish space and for each $n \in \mathbb{N}$, let $T_n = (T, \mathcal{T}_n)$ be a Borel refinement of T. Let $T_\infty = (T, \mathcal{T}_\infty)$ where $\mathcal{T}_\infty$ is the topology generated by $\bigcup_{n\in\mathbb{N}} \mathcal{T}_n$. Then T_∞ is a Borel refinement of T. We say that T_∞ is the **coarsest Borel refinement of the family** $\{T_n : n \in \mathbb{N}\}$.*

Proposition A.4 (Portmanteau theorem, 17.20 in Kechris (1995)). *Let X be a Polish space, let $\mathcal{M}(X)$ be the space of Borel probability measures on X, and let $\mathcal{O}$ be an open basis for X. A sequence μ_k weakly converges to μ in $\mathcal{M}(X)$ if and only if $\liminf_k \mu_k(O) \geq \mu(O)$ for every $O \in \mathcal{O}$.*

This result is stated for all open sets in Kechris (1995), but the version stated here with an open basis follows from the proof.

Proof of Theorem 3.7. Let $T_0^a = T^a$ and $T_0^b = T^b$. Using Proposition A.2 countably many times, we obtain sequences of Polish spaces T_n^a, T_n^b such that for each n, T_{n+1}^a is a Borel refinement of T_n^a (and hence of T^a), and λ^a is continuous from T_{n+1}^a to $\mathcal{L}(S^b \times T_n^b)$. Let U^a be the coarsest Borel refinement of $\{T_n : n \in \mathbb{N}\}$, and define U^b analogously. Since each open set in T_{n+1}^a is open in U^a, λ^a is continuous from U^a to $\mathcal{L}(S^b \times T_n^b)$ for each n.

Suppose that u_k^a converges to u^a in U^a. Then for each n, $\lambda^a(u_k^a)$ converges to $\lambda^a(u^a)$ in $\mathcal{L}(S^b \times T_n^b)$. For some ℓ, we have $\lambda^a(u^a) \in \mathcal{N}_\ell(S^b \times U^b)$. Then $\lambda^a(u_k^a) \in \mathcal{N}_\ell(S^b \times U^b)$ for all but finitely many k, so we may assume this holds for all k. Let $\mu_{k,m}$ be the m-th

[20] Note that each subspace of a Polish space is second countable.

coordinate of $\lambda^a(u_k^a)$, and let μ_m be the m-th coordinate of $\lambda^a(u^a)$. Hence for each n and each $m \le \ell$, $\mu_{k,m}$ weakly converges to μ_m in $\mathcal{M}(S^b \times T_n^b)$. By the Portmanteau theorem, $\liminf_k \mu_{m,k}(O) \ge \mu_m(O)$ for each m, n and each open set O in $S^b \times T_n^b$. Since the open sets in $S^b \times T_n^b$, $n \in \mathbb{N}$ form an open basis for $S^b \times U^b$, it follows from the other direction of the Portmanteau theorem A.4 that for each $m \le \ell$, $\mu_{k,m}$ weakly converges to μ_m in $\mathcal{M}(S^b \times U^b)$. Therefore, $\lambda^a(u_k^a)$ converges to $\lambda^a(u^a)$ in $\mathcal{L}(S^b \times U^b)$. This shows that λ^a is continuous from U^a to $\mathcal{L}(S^b \times U^b)$. □

To prove Theorem 3.4, we need two results from BFK about assumption.

Proposition A.5 (Property 6.2 in BFK)**.** *Let X be a Polish space, E, F be Borel subsets of X, and $\sigma = (\mu_0, \ldots, \mu_{m-1})$ a full-support LPS on X. If σ assumes both E and F the same level, then $\overline{E} = \overline{F}$.*

Proposition A.6 (Property 6.3 in BFK)**.** *Let X be a Polish space, $k \in \mathbb{N}$, and $\sigma \in \mathcal{L}^+(X)$. Suppose $E_n, n \in \mathbb{N}$ are Borel sets in X, and E_n is assumed under σ at level k for each $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} E_n$ is assumed under σ at level k.*[21]

Proof of Theorem 3.4. By the premise, $\exists (s^a, t^a, s^b, t^b) \in R_\infty^a \times R_\infty^b$. Furthermore, the LPS $\sigma = \lambda^a(t^a)$ has full-support and assumes each set in the infinite sequence $(R_1^b, R_2^b, \ldots)$.

It follows that there exists a smallest k such that σ assumes R_m^b at level k for infinitely many m, and a smallest M such that σ assumes R_M^b at level k. By Propositions A.6 and A.5, σ assumes R_∞^b at level k and $\overline{R_\infty^b} = \overline{R_M^b}$.

Since $\{s^b\} \times T^b$ is open for all $s^b \in S^b$, $\overline{R_\infty^b} = \overline{R_M^b}$ implies that $\operatorname{proj}_{S^b} R_\infty^b = \operatorname{proj}_{S^b} R_M^b = S_M^b$. Therefore $S_M^b = S_\infty^b$ since $R_\infty^b \subseteq R_m^b$ for all $m \ge M$. Analogously, $\operatorname{proj}_{S^a} R_\infty^a = S_\infty^a$. □

B PROOFS OF THEOREMS 3.8 AND 3.9

For convenience we let $S_0^a \equiv S^a$ and $R_0^a \equiv S^a \times T^a$, and similarly for b.

Lemma B.1. *For each $m > 0$,*

$$P_m^a = \left\{ \nu^{m-1} \ldots \nu^0 : \forall k < m, \quad \nu^k \in \mathcal{N}(S^b) \wedge \operatorname{Supp} \nu^k = S_k^b \right\}.$$

Proof of Lemma B.1. The proof is by induction. The base case ($m = 1$) holds trivially. Assume the result for m. Then by definition, the following are equivalent to $\mu \in P_{m+1}^a$.

[21] The proof in BFK establishes the result as stated here, but the statement in BFK did not mention the level.

- ⊳ $\mu = \nu\nu'$ for some $\nu, \nu' \in \mathcal{N}(S^b)$ such that $\operatorname{Supp}\nu = S^b_m$ and $\nu' \in P^a_m$;
- ⊳ $\mu = \nu^m\nu^{m-1}\ldots\ldots\nu^0$ for some $\nu^0, \ldots, \nu^{m-1}, \nu^m \in \mathcal{N}(S^b)$ such that $\operatorname{Supp}\nu^k = S^b_k$ for all $k \le m$.

This completes the induction. □

Lemma B.2. *For each $m > 0$ we have $P^a_{m+1} \subseteq P^a_m$.*

Proof. Suppose $\mu \in P^a_{m+1}$. By Lemma B.1, μ can be written as $\nu^m\nu^{m-1}\ldots\nu^0$ where $\nu^k \in \mathcal{N}(S^b)$ and $\operatorname{Supp}\nu^k = S^b_k$ for all $k \le m$. Then $\operatorname{Supp}\nu^m\nu^{m-1} = S^b_{m-1}$, and by Lemma B.1, we have $\mu \in P^a_m$. □

We will need the following result, which is Proposition 1 in Blume, Brandenburger, and Dekel (1991b).

Proposition B.3. *For each $\nu \in \mathcal{N}(S^b)$ there is a probability measure $\rho \in \mathcal{M}(S^b)$ such that $\operatorname{Supp}\rho = \operatorname{Supp}\nu$ and $(\rho) \sim^\star \nu$.*

Lemma B.4. *For each $m \in \mathbb{N}$ and $\sigma \in P^a_{m+1}$, there exists $\nu = (\nu_0, \ldots, \nu_m) \in \mathcal{N}_{m+1}(S^b)$ such that $\nu \sim^\star \sigma$ and $\operatorname{Supp}\nu_{m-k} = S^b_k$ for each $k \le m$ (so $\nu \in P^a_{m+1}$ by Lemma B.1).*

Proof of Lemma B.4. We argue by induction on m. The result for $m = 0$ follows from Proposition B.3. Suppose the result holds for m, and let $\sigma \in P^a_{m+2}$. Then $\sigma = \sigma'\sigma''$ where $\sigma' \in \mathcal{N}(S^b)$, $\operatorname{Supp}\sigma' = S^b_{m+1}$, and $\sigma'' \in P^a_{m+1}$. By inductive hypothesis, there exists $\nu = (\nu_1, \ldots, \nu_{m+1}) \in \mathcal{N}_{m+1}(S^b)$ such that $\nu \sim^\star \sigma''$ and $\operatorname{Supp}\nu_{m+1-k} = S^b_k$ for each $k \le m$. By Proposition B.3 there exists $\nu_0 \in \mathcal{M}(S^b)$ such that $\operatorname{Supp}\nu_0 = \operatorname{Supp}\sigma' = S^b_{m+1}$ and $\nu_0 \sim^\star \sigma'$. Then

$$\nu_0\nu = (\nu_0, \nu_1, \ldots, \nu_{m+1}) \sim^\star \sigma,$$

so the result holds for $m + 1$. □

Lemma B.5. *If $\nu, \nu' \in \mathcal{N}(S^b)$ then $\mathbb{O}(\nu\nu') \subseteq \mathbb{O}(\nu)$.*

Proof of Lemma B.5. It is easily seen that if $r^a \succ_\nu s^a$ then $r^a \succ_{\nu\nu'} s^a$. If $s^a \in \mathbb{O}(\nu\nu')$ then there is no r^a such that $r^a \succ_{\nu\nu'} s^a$, so there is no r^a such that $r^a \succ_\nu s^a$, and thus $s^a \in \mathbb{O}(\nu)$. □

Proof of Theorem 3.8. The proof is by induction on m. First, the base case: Since $P^a_1 = \mathcal{N}^+(S^b)$ and S^a_1 is the set of Ann's admissible strategies, we have

$$\bigcup \mathbb{X}^a_1 \equiv \bigcup \{\mathbb{O}(\nu) : \nu \in P^a_1\} = S^a_1.$$

Now fix an $m > 1$ and assume the induction hypothesis that $S^a_m = \bigcup \mathbb{X}^a_m$. We will show that $S^a_{m+1} = \bigcup \mathbb{X}^a_{m+1}$ in two steps.

Step 1: We want to show that $S^a_{m+1} \supseteq \bigcup \mathbb{X}^a_{m+1}$. Equivalently, we want to show that $\mathbb{O}(\mu) \subseteq S^a_{m+1}$ for any $\mu \in P^a_{m+1}$. By Lemma B.4 there exists $\nu = (\nu_0, \ldots, \nu_m) \in \mathcal{N}_{m+1}(S^b)$ such that $\nu \sim^\star \mu$ and $\operatorname{Supp} \nu_{m-k} = S^b_k$ for each $k \le m$. Then $\mathbb{O}(\nu) = \mathbb{O}(\mu)$. By Lemma B.5, $\mathbb{O}(\mu) = \mathbb{O}(\nu) \subseteq \mathbb{O}(\nu_0)$. By Lemma B.2, $\mu \in P^a_m$, so $\mathbb{O}(\mu) \in \mathbb{X}^a_m$. By the induction hypothesis $S^a_m = \bigcup \mathbb{X}^a_m$, we then have $\mathbb{O}(\mu) \subseteq S^a_m$. We note that $\operatorname{Supp} \nu_0 = S^b_m$, so by the definition of S^a_{m+1}, $\mathbb{O}(\nu_0) \cap S^a_m \subseteq S^a_{m+1}$. But $\mathbb{O}(\mu) \subseteq \mathbb{O}(\nu_0) \cap S^a_m$, so $\mathbb{O}(\mu) \subseteq S^a_{m+1}$.

Step 2: We want to show that $S^a_{m+1} \subseteq \bigcup \mathbb{X}^a_{m+1}$. Equivalently, we want to show that, for each $s^a \in S^a_{m+1}$, there exists a $\mu \in P^a_{m+1}$ such that $s^a \in \mathbb{O}(\mu)$. If $s^a \in S^a_{m+1}$, then for each $k \le m$ we have $s^a \in S^a_{k+1}$, so there exists $\nu^k \in \mathcal{N}_1(S^b)$ such that $\operatorname{Supp} \nu^k = S^b_k$ and $s^a \in \mathbb{O}(\nu^k)$. By Lemma B.5, $s^a \in \mathbb{O}(\mu)$ where $\mu = \nu^m \nu^{m-1} \ldots \nu^0$. By Lemma B.1, $\mu \in P^a_{m+1}$. □

For the proof of Theorem 3.9, we will need the following two results. The result below is an immediate consequence of Lemma E.2 in BFK.

Proposition B.6. *For each LPS $\sigma \in \mathcal{L}^+(S^b \times T^b)$ there are continuum-many $\hat{\sigma} \in \mathcal{L}^+(S^b \times T^b)$ such that*

(i) $\operatorname{marg}_{S^b} \sigma = \operatorname{marg}_{S^b} \hat{\sigma}$;

(ii) For each Borel set $E \subseteq S^b \times T^b$ and each $k \in \mathbb{N}$, E is assumed under σ at level k if and only if E is assumed under $\hat{\sigma}$ at level k.

Proposition B.7 (Lemma E.3 in BFK)**.** *In a complete lexicographic type structure for G, for each $m \in \mathbb{N}$ we have* $\operatorname{proj}_{S^b} R^b_m = \operatorname{proj}_{S^b}(R^b_m \setminus R^b_{m+1})$.

Proof of Theorem 3.9. **Proof of (i).** We want to show that $(s^a, t^a) \in R^a_{m+1} \implies \exists \nu \in P^a_{m+1}$ such that $\operatorname{marg}_{S^b} \lambda^a(t^a) \sim^\star \nu$. Let $\sigma = (\mu_0, \ldots, \mu_n) = \lambda^a(t^a)$. For each $k \le m$, let $[k]$ denote the level at which σ assumes R^b_k. Then $n = [0] \ge \cdots \ge [m]$. For the *proof of (i)* only, let $\nu^k = \operatorname{marg}_{S^b}(\mu_0, \ldots, \mu_{[k]})$ for each $k \le m$. Then $\nu^0 = \operatorname{marg}_{S^b} \sigma$. Since σ assumes R^b_k at level $[k]$, we see from Proposition 3.1 that $\operatorname{Supp} \nu^k = \operatorname{proj}_{S^b} R^b_k = S^b_k$. Note that for each $k \le m$, ν^{k+1} is an initial segment of ν^k. It is readily verified that if ν is an initial segment of ν' then $\nu\nu' \sim^\star \nu'$. It follows by induction that $\nu^m \nu^{m-1} \ldots \nu^k \sim^\star \nu^k$ for each $k < m$, and hence $\nu^m \nu^{m-1} \ldots \nu^0 \sim^\star \nu^0 = \operatorname{marg}_{S^b} \sigma$. By Lemma B.1, $\nu^m \nu^{m-1} \ldots \nu^0 \in P^a_{m+1}$.

Proof of (ii). We want to show that $\nu \in P^a_{m+1} \implies \exists (s^a, t^a) \in R^a_{m+1}$ s.t. $\operatorname{marg}_{S^b} \lambda^a(t^a) = \nu$. By Lemma B.1, we can write ν as $\nu^m \nu^{m-1} \ldots \nu^0$, where $\operatorname{Supp} \nu^k = S^b_k$ for all $k \le m$.

We first consider ν^m. We can write $\nu^m = (\nu^m_0, \ldots, \nu^m_n)$ where each ν^m_i is a probability measure on S^b. By Proposition 3.1, $\operatorname{Supp} \nu^m = S^b_m = \operatorname{proj}_{S^b} R^b_m$. By Proposition B.6, for each $s^b \in S^b_m$, the set $(\{s^b\} \times T^b) \cap R^b_m$ is uncountable. Since S^b is finite and T^b is separable, R^b_m has a countable dense subset. We may therefore pick sets $Y_0, \ldots, Y_n$ such that

- ▷ The sets Y_i are countable and pairwise disjoint;
- ▷ The union $Y_0 \cup \cdots \cup Y_n$ is a dense subset of R^b_m; and
- ▷ For each $i \le n$, $\mathrm{proj}_{S^b} Y^m_i = \mathrm{Supp}\, \nu^m_i$.

For each $s^b \in S^b$ and $i \le n$, we can assign positive measures to the points of $(\{s^b\} \times T^b) \cap Y_i$ that add up to $\nu^m_i(\{s^b\})$. This gives probability measures $\mu^m_i \in \mathcal{M}(S^b \times T^b)$ such that $\mathrm{Supp}\, \mu^m_i \supseteq Y_i$, $\mu^m_i(Y_i) = 1$, and $\mathrm{marg}_{S^b}\, \mu^m_i = \nu^m_i$. Then the measures $\mu^m_i, i \le n$ are mutually singular, so the $(n+1)$-tuple $\mu^m \equiv (\mu^m_0, \ldots, \mu^m_n)$ is an LPS such that $\mathrm{Supp}\, \mu^m \supseteq R^b_m$, $\mu^m(R^b_m) = \vec{1}$, and $\mathrm{marg}_{S^b}\, \mu^m = \nu^m$.

We now consider ν^k for $k < m$. By Proposition B.7, $\mathrm{Supp}\, \nu^k = S^b_k = \mathrm{proj}_{S^b}(R^b_k \setminus R^b_{k+1})$. By the above construction with $R^b_k \setminus R^b_{k+1}$ in place of R^b_m, we obtain an LPS μ^k such that $\mathrm{Supp}\, \mu^k \supseteq R^b_k \setminus R^b_{k+1}$, $\mu^k(R^b_k \setminus R^b_{k+1}) = \vec{1}$, and $\mathrm{marg}_{S^b}\, \mu^k = \nu^k$.

Now, let μ be the concatenation $\mu \equiv \mu^m \mu^{m-1} \ldots \mu^0$. Then $\mathrm{marg}_{S^b}\, \mu = \nu$, and μ is a full-support LPS that assumes R^b_k for all $k \le m$. By completeness, there exists a type $t^a \in T^a$ such that $\lambda^a(t^a) = \mu$. Since μ has full support, there exists an $s^a \in S^a$ such that (s^a, t^a) is a rational pair. Then (s^a, t^a) satisfies rationality and m-th order assumption of rationality, so $(s^a, t^a) \in R^a_{m+1}$. □

For each $\sigma \in \mathcal{L}(S^b \times T^b)$, let $\mathbb{O}(\sigma)$ be the set of all strategies $s^a \in S^a$ that are optimal under $\mathrm{marg}_{S^b}\, \sigma$

Proof of Corollary 3.10. Note that for each $\sigma \in \mathcal{L}(S^b \times T^b)$, we have $\mathbb{O}(\sigma) = \mathbb{O}(\mathrm{marg}_{S^b}\, \sigma)$. Then by Theorem 3.9,

$$\mathbb{X}^a_m = \{\mathbb{O}(\nu) : \nu \in P^a_m\} = \{\mathbb{O}(\lambda^a(t^a)) : (s^a, t^a) \in R^a_m\}. \qquad \square$$

Proof of Corollary 3.11. **Proof of (i).** By Lemma C.4 in BFK, each of the sets R^a_m is Borel. Since S^a is finite, each of the sets $\Gamma^a(X^a, R^a_m)$ is Borel. Let U^a_m is the set of Ann's full-support types that assume k-th order rationality of Bob for all $k < m$, i.e.,

$$U^a_m = \{t^a \in T^a : (\exists s^a)(s^a, t^a) \in R^a_m\}.$$

Then for each nonempty $X^a \subseteq S^a$,

$$\Gamma^a(X^a, R^a_m) = \{t \in U^a_m : \mathbb{O}(\lambda^a(t^a)) = X^a\}.$$

Since $R^a_{m+1} \subseteq R^a_m$, $U^a_{m+1} \subseteq U^a_m$, and therefore $\Gamma^a(X^a, R^a_{m+1}) \subseteq \Gamma^a(X^a, R^a_m)$. This proves (i).

Proof of (ii). By Corollary 3.10, the following equivalences hold.

$$X^a \in \mathbb{X}^a_m \iff (\exists (s^a, t^a) \in R^a_m)\left[X^a = \mathbb{O}(\lambda^a(t^a))\right]$$
$$\iff (\exists t^a \in U^a_m)\left[X^a = \mathbb{O}(\lambda^a(t^a))\right] \iff (\exists t^a)\left[t^a \in \Gamma^a(X^a, R^a_m)\right]$$

Proof of (iii). Since $\mathfrak{T}$ is complete, there exists $t^a \in T^a$ such that Ann is not open-minded, so there is no s^a such that $(s^a, t^a) \in R^a_1$ and hence $t^a \in \Gamma(\varnothing, R^a_1)$. $\Gamma^a(\varnothing, R^a_m)$ is the complement of the union of the sets $\Gamma^a(X^a, R^a_m)$ with X^a nonempty. Thus by **(i)**, $\left\{\Gamma^a(\varnothing, R^a_m) : m > 0\right\}$ is an increasing sequence of Borel sets. □

C POLISH SPACES AND ASSUMPTION

In this section we establish some useful properties of Polish spaces and assumption. A topological space $(X, \mathscr{T})$ is called **Polish** if it is separable and completely metrizable. It is well known that all uncountable subsets of Polish spaces have cardinality equal to $2^{\aleph_0}$ (i.e., the cardinality of the continuum). This is a consequence of Proposition C.1 below.

The **Cantor space** $\mathscr{C}$ is the set $\{0,1\}^{\mathbb{N}}$ endowed with the product topology. It is a Polish space of cardinality $2^{\aleph_0}$. A **Cantor set** C in a topological space X is a homeomorphic copy of $\mathscr{C}$ in X—that is, $(C, \mathscr{T}|C)$ is homeomorphic to $\mathscr{C}$, where $\mathscr{T}|C = \left\{U \cap C : U \in \mathscr{T}\right\}$ is the subspace topology on C. A subset of a topological space is **perfect** if it is closed and has no isolated points.

Proposition C.1 (The Perfect Set Theorem for Borel Sets, 13.6 in Kechris, 1995)**.** *Let X be a Polish space and $A \subseteq X$ be Borel. Then either A is countable, or else A contains a Cantor set and has cardinality $2^{\aleph_0}$.*

Proposition C.2 (Cantor-Bendixson, 6.4 in Kechris, 1995)**.** *Let X be a Polish space. Then X has a unique perfect subset P such that $X \setminus P$ is countable and open. Furthermore, every open neighborhood of every $x \in P$ is uncountable.*

Lemma C.3. *Let X be an uncountable Polish space and $n \in \mathbb{N}$. Then there exist disjoint open sets $U_1, \ldots, U_n$ in X such that*

(i) U_i is uncountable for all i; and

(ii) $X \setminus \biguplus\{U_1, \ldots, U_n\}$ is uncountable.

Proof of Lemma C.3. By Proposition C.2, X has a perfect subset P such that $X \setminus P$ is countable and open. We can choose $n+1$ distinct points $x_1, \ldots, x_{n+1} \in P$. Since X is metrizable, it is normal—that is, any two disjoint closed sets in X have disjoint open neighborhoods. It follows that there exist disjoint open sets $U_1, \ldots, U_{n+1}$ such that $x_j \in U_j$ for all j. By Proposition C.2, $U_1, \ldots, U_{n+1}$ are uncountable. Finally, $X \setminus \biguplus\{U_1, \ldots, U_n\}$ is uncountable since it contains U_{n+1}, which is itself uncountable. □

Lemma C.4. *$\mathscr{C} = \biguplus_{n\in\mathbb{N}} K_n$, where $(K_0, K_1, \ldots)$ is a sequence of disjoint uncountable compact sets.*

Proof of Lemma C.4. Define $K_0, K_1, \ldots$ as follows.

$$\begin{aligned} &K_0 = \{0\}^{\mathbb{N}} \cup \{c \in \mathscr{C} : c_0 = 1\}; \\ \forall n > 0, \quad &K_n = \{c \in \mathscr{C} : (\forall k < n) c_k = 0 \wedge c_n = 1\}. \end{aligned}$$

For each $n > 0$, K_n is a Cantor set, and therefore it is uncountable and compact. K_0 is the union of a Cantor set and a single point, therefore it is also uncountable and compact. By construction, $\mathscr{C} = \biguplus_{n\in\mathbb{N}} K_n$, and $(K_0, K_1, \ldots)$ is a sequence of disjoint sets. □

Given a Polish space $(X, \mathscr{O}(X))$, a **Borel subspace** of X is a topological space $(A, \mathscr{O}(A))$ where A is a nonempty Borel subset of X endowed with the subspace topology $\mathscr{O}(A) = \{U \cap A : U \in \mathscr{O}(X)\}$.

Proposition C.5 (Borel Isomorphism Theorem, Theorem 15.6 in (Kechris, 1995))**.** *Let A, B be Borel subspaces of Polish spaces. If* $\operatorname{card}(A) = \operatorname{card}(B)$*, then there is a one-to-one Borel mapping from A onto B.*[22]

Lemma C.6. *Let X, Y be Polish spaces, and let $\{X_n : n \in \mathbb{N}\}$ and $\{Y_n : n \in \mathbb{N}\}$ be be countable partitions of X, Y into Borel sets such that* $\operatorname{card}(X_n) = \operatorname{card}(Y_n)$ *for each $n \in \mathbb{N}$. Then there is a one-to-one Borel mapping from X onto Y that maps X_n onto Y_n for each $n \in \mathbb{N}$.*

Proof of Lemma C.6. Each of the sets X_n, Y_n with its subspace topology is a Borel subspace of a Polish space. By Proposition C.5, for each $n \in \mathbb{N}$ there is a one-to-one Borel mapping λ_n from X_n onto Y_n. Then the union $\lambda = \bigcup_{n\in\mathbb{N}} \lambda_n$ is a one-to-one Borel mapping from X onto Y that sends X_n onto Y_n for each $n \in \mathbb{N}$, as required. □

We will need the following facts from BFK about assumption.

Proposition C.7 (Lemma C.3 in BFK)**.** *For each Polish space X and Borel set E in X, the set of $\sigma \in \mathscr{L}^+(X)$ such that E is assumed under σ is Borel.*

[22] In Kechris, 1995, this result is stated in terms of standard Borel spaces, which are the measure spaces associated with Borel subspaces of Polish spaces.

Proposition C.8 (Lemma B.1 in BFK[23])**.** *Let X be a Polish space, E be a Borel subset of X, $\sigma = (\mu_0, \ldots, \mu_{n-1})$ be a full-support LPS on X, and $k < n$. Then σ assumes E at level k if and only if the following conditions are met.*

(i) $\mu_i(E) = 1$ *for each* $i \le k$*;*

(ii) $\mu_i(E) = 0$ *for each* $i > k$*; and*

(iii) $E \subseteq \bigcup_{i \le k} \operatorname{Supp} \mu_i$*.*

In a topological space, a set D is said to be **dense in** a set E if $D \subseteq E$ and $\overline{D} = \overline{E}$. Note that if D_1 is dense in E_1 and D_2 is dense in E_2, then $D_1 \cup D_2$ is dense in $E_1 \cup E_2$. Also, if D is dense in E and $D \subseteq F \subseteq E$, then D is dense in F and F is dense in E.

Lemma C.9. *Let X be a Polish space, and E an uncountable Borel set in X. Then there exists a Cantor set $C \subseteq E$ such that $E \setminus C$ is uncountable and $E \setminus C$ is dense in E.*

Proof of Lemma C.9. The Cantor space $\mathscr{C}$ contains the Cantor set $\{(0,0),(1,1)\}^{\mathbb{N}}$ and the complement of this set is uncountable and dense in $\mathscr{C}$. By Proposition C.1, E contains a Cantor set D. It follows that D contains a Cantor set C such that $D \setminus C$ is uncountable and dense in D. But $D \subseteq E$, so $C \subseteq E$, and $E \setminus C$ is uncountable and dense in E. □

Lemma C.10. *Let X be a Polish space, and U_0 an uncountable open set in X. Then there exists a decreasing sequence of open sets $(U_0, U_1, U_2, \ldots)$ such that*

(i) For all $n \in \mathbb{N}$, $U_n \setminus U_{n+1}$ is uncountable;

(ii) $U_\infty \equiv \bigcap_{n \in \mathbb{N}} U_n$ is an uncountable open set; and

(iii) U_∞ is dense in U_0.

Proof of Lemma C.10. By Lemma C.9, there exists a Cantor set $C \subseteq U_0$ such that $U_0 \setminus C$ is uncountable and dense in U_0. By Lemma C.4, there exists a sequence $(K_0, K_1, \ldots)$ of disjoint uncountable compact sets such that $\biguplus_{n \in \mathbb{N}} K_n = C$. For each $n > 0$, define $U_n \equiv U_0 \setminus \biguplus_{j<n} K_j$. Then U_n is open, and $U_n \setminus U_{n+1} = K_n$, which is uncountable. Moreover, $U_\infty = U_0 \setminus C$, so U_∞ is uncountable, open, and dense in U_0. □

Lemma C.11. *Let X, Y be Polish spaces, X finite, and let $Z_0 = X \times Y$. Let $\nu = (\nu_0, \ldots, \nu_m) \in \mathscr{N}_{m+1}(X)$. If $(Z_1, Z_2, \ldots, Z_{m+1})$ is a decreasing sequence of nonempty Borel subsets of Z_0 such that*

$$\forall k \le m, \quad \operatorname{proj}_X Z_k = \operatorname{proj}_X (Z_k \setminus Z_{k+1}) \text{ and } \operatorname{Supp} \nu_{m-k} = \operatorname{proj}_X Z_k$$

then there exists $\mu = (\mu_0, \ldots, \mu_m) \in \mathscr{L}^+_{m+1}(Z)$ such that

[23] The proof in BFK establishes this fact, but statement of Lemma B.1 was garbled in BFK.

(i) $\text{marg}_X \mu = \nu$;

(ii) $\forall k \le m$, μ *assumes* Z_k *at level* $m-k$;

(iii) μ *does not assume* Z_{m+1}; *and*

(iv) $\forall x \in \text{proj}_X Z_{m+1}$, $\mu_0(Z_{m+1} \cap (\{x\} \times Y)) > 0$.

Proof of Lemma C.11. Using the fact that Polish spaces are separable, there is a countable subset U of Z_0 such that $U \cap Z_{m+1}$ is dense in Z_{m+1}, and $U \cap (Z_k \setminus Z_{k+1})$ is dense in $Z_k \setminus Z_{k+1}$ for each $k \le m$. It follows that $U \cap Z_k$ is dense in Z_k for each $k \le m$.

Choose any $\rho \in \mathcal{M}(Z_0)$ such that $\rho(U) = 1$ and $\rho(\{u\}) > 0$ for each $u \in U$. Since U is dense in Z_0, $\rho \in \mathcal{M}^+(Z_0)$. For all $k \in \mathbb{N}$, let $X_k \equiv \text{proj}_X Z_k$. For all $x \in X$ and $k \in \mathbb{N}$, let $Z_k(x) = Z_k \cap (\{x\} \times Y)$. This set is clearly Borel. Since X is finite, it readily follows that

$$\forall x \in X_k, k \ge 0, \quad Z_k(x) \cap U \neq \varnothing \text{ and dense in } Z_k(x);$$

$$\forall x \in X_k, k \le m, \quad (Z_k(x) \setminus Z_{k+1}(x)) \cap U \neq \varnothing \text{ and dense in } Z_k(x) \setminus Z_{k+1}(x).$$

Note that, for every Borel set V of Z_0 such that $V \cap U$ is nonempty, the conditional measure $\rho(\cdot|V)$ is well-defined. We define $\mu_0, \ldots, \mu_m$ as follows.

$$\forall k < m, \quad \mu_{m-k}(E) \equiv \sum_{x \in X_k} \nu_{m-k}(x) \rho(E | Z_k(x) \setminus Z_{k+1}(x)); \text{ and}$$

$$\mu_0(E) \equiv \sum_{x \in X_m} \nu_0(x) \rho(E | Z_m(x)).$$

It is clear from these definitions that $\sum_{k=0}^{m} \mu_k$ and ρ are mutually absolutely continuous. Therefore $\mu = (\mu_0, \ldots, \mu_m)$ is a full-support LPS on Z_0. It is also clear that $\mu_0(Z_{m+1}(x)) > 0$ for each $x \in X_{m+1}$, and that $\text{marg}_X \mu_k = \nu_k$ for all $k \le m$.

For each $k \le m$, $Z_k \subseteq \text{Supp}(\mu_0, \ldots, \mu_{m-l})$, because $Z_k \cap S$ is dense in Z_k. Using Proposition C.8, we can easily verify that for all $k \le m$, μ assumes Z_k at level $m-k$. $Z_m \setminus Z_{m+1}$ has a nonempty intersection with U, so μ_0 gives the set positive probability. However, since $\mu_0(Z_m) = 1$, it follows that $\mu_0(Z_{m+1}) < 1$. Proposition C.8 makes it clear that μ does not assume Z_k when $k > m$. □

Lemma C.12. *Let* X, Y *be Polish spaces with* X *be finite, and let* $Z_0 = X \times Y$. *Let* $\nu = (\nu_0, \ldots, \nu_m) \in \mathcal{N}_{m+1}(X)$, *and let* $(Z_1, Z_2, \ldots)$ *a strictly decreasing sequence of nonempty Borel subsets of* Z_0, *such that*

$$\forall k \ge 0, \quad \text{proj}_X Z_k = \text{proj}_X (Z_k \setminus Z_{k+1});$$

$$\forall k \le m, \quad \text{Supp}\, \nu_{m-k} = \text{proj}_X Z_k;$$

$$Z_\infty \equiv \bigcap \{Z_k : k \in \mathbb{N}\} \text{ is dense in } Z_m.$$

Then there exists $\mu = (\mu_0, \dots, \mu_m) \in \mathscr{L}^+_{m+1}(Z)$ *such that*

(i) *For all* $k \le m$, μ *assumes* Z_k *at level* $m - k$;

(ii) *For all* $k > m$, μ *assumes* Z_k *at level* 0;

(iii) $\operatorname{marg}_X \mu = \nu$.

Proof of Lemma C.12. For each $k \in \mathbb{N} \cup \{\infty\}$, let $X_k \equiv \operatorname{proj}_X Z_k$, and for each $x \in X$, let $Z_k(x) = Z_k \cap (\{x\} \times Y)$. Since Z_∞ is dense in Z_m and X is finite, we have $X_\infty = X_m$, and for each $x \in X_m$, $Z_\infty(x)$ is dense in $Z_m(x)$. By Lemma C.11, there exists $\phi = (\phi_0, \dots, \phi_m) \in \mathscr{L}^+_{m+1}(Z)$ such that

- ▷ $\operatorname{marg}_X \phi = \nu$;
- ▷ for each $k \le m$, ϕ assumes Z_k at level $m - k$;
- ▷ for each $x \in X_m$, $\mu_0(Z_\infty(x)) > 0$.

Then the conditional probability $\phi_0(\cdot | Z_\infty(x))$ is well-defined for each $x \in X_m$. For each $0 < k \le m$, let $\mu_k = \phi_k$. Define μ_0 in the following way.

$$\mu_0(E) \equiv \sum_{x \in X_m} \nu_0(x) \phi_0(E | Z_\infty(x)).$$

By construction, $\operatorname{Supp} \mu_0 = \operatorname{Supp} \phi_0(\cdot | Z_\infty) = \overline{Z_\infty} = \overline{Z_m} = \operatorname{Supp} \phi_0$. Therefore, it is readily apparent that $\operatorname{Supp}(\mu_0, \dots, \mu_m) = \operatorname{Supp}(\phi_0, \dots, \phi_m) = Z_0$. We have $\operatorname{Supp} \nu_0 = X_m$, so $\mu_0(Z_\infty) = \mu_0(Z_m) = \nu_0(X_m) = 1$.

By Proposition C.8, we can easily verify that μ assumes Z_k at level 0 for all $k \ge m$. □

D PROOFS OF THEOREMS 3.2, 3.13, AND 3.14

Theorem 3.13 says that a necessary and sufficient condition for a family of sets to be an RCAR tower is that the sets have the right "shape", that the intersections are topologically indistinguishable from the sets at some finite level, and that each "part" is uncountable. The ingredients for the proof are given in the preceding two subsections. To prove necessity, we must show for that in every complete type structure for G, the RmAR sets must satisfy conditions **(i)–(vi)** of Theorem 3.13. To prove sufficiency, one must start with a given family of sets that satisfy conditions **(i)–(vi)**, and construct a pair of Borel mappings λ^a, λ^b such that the resulting type structure has the given sets as its RmAR sets. Our construction will produce mappings that are one-to-one, so we will get a proof of Theorem 3.14 as well. The idea is to construct these mappings by gluing together countably many Borel mappings from pieces of T^a to corresponding sets of beliefs about $S^b \times T^b$.

As we have done throughout the paper, we fix the underlying game G. We also fix T^a and T^b and assume that they are uncountable Polish spaces.

Lemma D.1 (Necessity half of Theorem 3.13)**.** *Let T^a, T^b be uncountable Polish spaces. Every RCAR tower for (G, T^a, T^b) satisfies* **(i)–(vi)** *of Theorem 3.13.*

Proof of Lemma D.1. Let $\{Q^a_m : m > 0\}, \{Q^b_m : m > 0\}$ form an RCAR tower for (G, T^a, T^b). Therefore, there is a complete type structure $\mathfrak{T}$ for G with RCAR such that for each $m > 0$, $Q^a_m = R^a_m$ and $Q^b_m = R^b_m$. Conditions **(i)** and **(ii)** follow from Corollary 3.11.

Proof of (iii). Since $\mathfrak{T}$ has RCAR, there is a state $(s^a, t^a) \in R^a_\infty$. Then $\sigma \equiv \lambda^a(t^a)$ has full support, and R^b_m is assumed under σ for each $m > 0$. Then by the same argument that was used in the proof of Theorem 3.4, we see that $\overline{R^b_\infty} = \overline{R^b_M}$ for some $M > 0$, as required.

Proof of (iv.) Since T^b is uncountable, there are uncountably many LPS's on $S^b \times T^b$ that do not have full support. $\mathfrak{T}$ is complete, so there are uncountably many $t^a \in T^a$ such that $\lambda^a(t^a)$ do not have full support, and hence $t^a \in \Gamma(\emptyset, Q^a_1)$.

Proof of (v). Let $X^a \in \mathbb{X}^a_m$. This means that there exists $\nu \in P^a_m$ such that $\mathbb{O}(\nu) = X^a$. By Lemma B.4 we may take ν to be of the form $\nu = (\nu_0, \ldots, \nu_{m-1})$ where each ν_k is a measure on S^b. By Lemma B.1, for each $k < m$ we have $\operatorname{Supp} \nu_{m-1-k} = S^b_k$. By Proposition 3.1, each of the sets R^b_k is nonempty, and $\operatorname{proj}_{S^b} R^b_k = S^b_k$. By Lemma C.11, there exists $\mu \in \mathscr{L}^+(S^b \times T^b)$ such that

- ▹ $\operatorname{marg}_{S^b} \mu = \nu$;
- ▹ for all $k < m$, μ assumes R^b_k at level $m - 1 - k$;
- ▹ μ does not assume R^b_m.

Because $\mathfrak{T}$ is complete, there exists $t^a \in T^a$ such that $\lambda^a(t^a) = \mu$. It follows that $t^a \in \Gamma^a(X^a, R^a_m) \setminus \Gamma^a(X^a, R^a_{m+1})$, so this set is nonempty. Finally, Proposition B.6 implies that $\Gamma^a(X^a, R^a_m) \setminus \Gamma^a(X^a, R^a_{m+1})$ is uncountable.

Proof of (vi). Let $X^a \in \mathbb{X}^a_\infty$, and let M be large enough so that $X^a \in \mathbb{X}^a_M$ and **(iii)** holds for M. As in the preceding paragraph, there exists $\nu \in P^a_M$ of the form $\nu = (\nu_0, \ldots, \nu_{M-1})$ such that $\mathbb{O}(\nu) = X^a$. By Lemma C.12, there exists $\mu \in \mathscr{L}^+(S^b \times T^b)$ such that

- ▹ $\operatorname{marg}_{S^b} \mu = \nu$;
- ▹ for all $k < M$, μ assumes R^b_k at level $M - 1 - k$;
- ▹ for all $k \geq M$, μ assumes R^b_k at level 0.

By Proposition A.6, μ assumes R^b_∞ at level 0. As before, because $\mathfrak{T}$ is complete, there exists $t^a \in T^a$ such that $\lambda^a(t^a) = \mu$. It follows that $t^a \in \Gamma^a(X^a, R^a_\infty)$, so this set is nonempty, and by Proposition B.6, $\Gamma^a(X^a, R^a_\infty)$ is uncountable. □

Lemma D.2 (Sufficiency half of Theorems 3.13 and 3.14)**.** *Let G be a finite game and T^a, T^b be uncountable Polish spaces. For every family of sets*

$$\mathscr{Q} = (\{Q^a_m : m > 0\}, \{Q^b_m : m > 0\})$$

*that satisfies **(i)–(vi)** of Theorem 3.13, there is a complete one-to-one lexicographic type structure $\mathfrak{T}$ for G such that $R^a_m = Q^a_m$ and $R^b_m = Q^b_m$ for all $m > 0$, and the RCAR set is nonempty.*

Proof of Lemma D.2. We must find a pair of one-to-one Borel mappings λ^a, λ^b such that $\mathfrak{T}$ is a complete one-to-one type structure for G, and $R^a_m = Q^a_m, R^b_m = Q^b_m$ for all $m > 0$. **(vi)** will guarantee that $\mathfrak{T}$ has an RCAR state.

By **(i)–(vi)**, the following family of sets is a partition of T^a into countably many uncountable Borel sets.

(a) $\Gamma^a(\varnothing, Q^a_1)$;

(b) $\forall m > 0, \forall X^a \in \mathbb{X}^a_{m+1}, \quad \Gamma^a(X^a, Q^a_m) \setminus \Gamma^a(X^a, Q^a_{m+1})$;

(c) $\forall m > 0, \forall X^a \in \mathbb{X}^a_m \setminus \mathbb{X}^a_{m+1}, \quad \Gamma^a(X^a, Q^a_m)$;

(d) $\forall X^a \in \mathbb{X}^a_\infty, \quad \Gamma^a(X^a, Q^a_\infty)$.

We now introduce notation for the sets of beliefs that correspond to the sets of types $\Gamma^a(X^a, Q^a_m)$. For each $m > 0$ and each $X^a \in \mathbb{X}^a_m$, let $\Lambda^a_m(X^a, \mathscr{Q})$ be the set of all μ such that

- ▷ $\mu \in \mathscr{L}^+(S^b \times T^b)$;
- ▷ $\mathbb{O}(\mu) = X^a$;
- ▷ For all $k < m$, Q^b_k is assumed under μ.

We also let $\Lambda^a_1(\varnothing, \mathscr{Q}) = \mathscr{L}(S^b \times T^b) \setminus \mathscr{L}^+(S^b \times T^b)$, and let

$$\forall X^a \in \mathbb{X}^a_\infty, \quad \Lambda^a_\infty(X^a, \mathscr{Q}) = \bigcap_{m>0} \Lambda^a_m(X^a, \mathscr{Q}).$$

It follows from Proposition C.7 that for each $m > 0$ and $X^a \in \mathbb{X}^a_m$, the set $\Lambda^a_m(X^a, \mathscr{Q})$ is Borel. Therefore, the following family of sets is a countable partition of $\mathscr{L}(S^b \times T^b)$ into Borel sets.

(a') $\Lambda^a_1(\varnothing, \mathscr{Q})$;

(b') $\forall m > 0, \forall X^a \in \mathbb{X}^a_{m+1}, \quad \Lambda^a_m(X^a, \mathscr{Q}) \setminus \Lambda^a_{m+1}(X^a, \mathscr{Q})$;

(c') $\forall m > 0, \forall X^a \in \mathbb{X}^a_m \setminus \mathbb{X}^a_{m+1}, \quad \Lambda^a_m(X^a, \mathscr{Q})$;

(d') $\forall X^a \in \mathbb{X}^a_\infty, \quad \Lambda^a_\infty(X^a, \mathscr{Q})$.

We show that each of the sets listed in **(a')–(d')** is uncountable. The case **(c')** is listed separately because in that case $\Lambda^a_{m+1}(X^a, \mathscr{Q})$ is empty. By Proposition B.6, it is enough to show that each of these sets is nonempty. It will be convenient to put $Q^a_0 = S^a \times T^a, Q^b_0 =$

$S^b \times T^b$.

(a') Since T^b is infinite, there are probability measures on $S^b \times T^b$ which do not have full support, so the set $\Lambda_1^a(\emptyset, \mathcal{Q})$ is nonempty.

(b') Let $m > 0$ and $X^a \in \mathbb{X}_{m+1}^a$. By **(i)** and **(ii)**, $Q_1^b, \ldots, Q_m^b$ is a strictly decreasing sequence of nonempty Borel subsets of $S^b \times T^b$. By **(v)**, we have for each $k < m$, $\mathrm{proj}_{S^b} Q_k^b = \mathrm{proj}_{S^b}(Q_k^b \setminus Q_{k+1}^b)$.

By Lemma C.11 there exists $\mu \in \mathcal{L}^+(S^b \times T^b)$ such that $\mathbb{O}(\mu) = X^a$, and Q_k^b is assumed under μ for all $k < m$, but Q_m^b is not assumed under μ. This shows that the set $\Lambda_m^a(X^a, \mathcal{Q}) \setminus \Lambda_{m+1}^a(X^a, \mathcal{Q})$ is nonempty.

(c') Let $m > 0$ and $X^a \in \mathbb{X}_m^a \setminus \mathbb{X}_{m+1}^a$. By the same argument as above, the set $\Lambda_m^a(X^a, \mathcal{Q})$ is nonempty.

(d') Let $X^a \in \mathbb{X}_\infty^a$. $Q_1^b, Q_2^b, \ldots$ is a strictly decreasing sequence of nonempty Borel subsets of $S^b \times T^b$ by **(i)** and **(ii)**; and by **(v)**, we have for each $k \in \mathbb{N}$, $\mathrm{proj}_{S^b} Q_k^b = \mathrm{proj}_{S^b}(Q_k^b \setminus Q_{k+1}^b)$. By **(iii)**, there exists $M > 0$ such that Q_∞^b is dense in Q_M^b. Then by Lemma C.12, there exists $\mu \in \mathcal{L}^+(S^b \times T^b)$ such that $\mathbb{O}(\mu) = X^a$, and Q_k^b is assumed under μ for all $k \in \mathbb{N}$. Therefore, the set $\Lambda_\infty^a(X^a, \mathcal{Q})$ is nonempty.

We can now apply Lemma C.6 to obtain a bijective Borel map λ^a from T^a onto $\mathcal{L}(S^b \times T^b)$ such that λ^a maps

(a'') $\Gamma^a(\emptyset, Q_1^a)$ onto $\Lambda_1^a(\emptyset, \mathcal{Q})$;

(b'') $\forall m > 0$, $\forall X^a \in \mathbb{X}_{m+1}^a$, $\Gamma^a(X^a, Q_m^a) \setminus \Gamma^a(X^a, Q_{m+1}^a)$ onto $\Lambda_m^a(X^a, \mathcal{Q}) \setminus \Lambda_{m+1}^a(X^a, \mathcal{Q})$;

(c'') $\forall m > 0$, $\forall X^a \in \mathbb{X}_m^a \setminus \mathbb{X}_{m+1}^a$, $\Gamma^a(X^a, Q_m^a)$ onto $\Lambda_m^a(X^a, \mathcal{Q})$;

(d'') $\forall X^a \in \mathbb{X}_\infty^a$, $\Gamma^a(X^a, Q_\infty^a)$ onto $\Lambda_\infty^a(X^a, \mathcal{Q})$.

A mapping $\lambda^b : T^b \to \mathcal{L}(S^a \times T^a)$ can be constructed similarly. The resulting type structure $\mathfrak{T}$ is a complete one-to-one type structure for G. Using the definition of R_m^a, it follows by induction that $Q_m^a = R_m^a$ for all $m > 0$. Therefore, $\mathcal{Q}$ is an RCAR tower. □

Theorems 3.13 and 3.14 both follow immediately from Lemmas D.1 and D.2.

Proof of Lemma 3.15. By Lemma C.3, we may choose a finite family of disjoint uncountable open sets

$$\Gamma_1(X^a) \subseteq T^a, X^a \in \mathbb{X}_1^a$$

such that the complement of their union is also uncountable.

Let M be large enough so that $\mathbb{X}_M^a = \mathbb{X}_\infty^a$. Consider an $X^a \in \mathbb{X}_1^a \setminus \mathbb{X}_M^a$. There is a unique $m < M$ such that $X^a \in \mathbb{X}_m^a \setminus \mathbb{X}_{m+1}^a$. By Lemma C.10, there is a finite decreasing chain of uncountable open sets

$$\Gamma_1(X^a) \supseteq \Gamma_2(X^a) \supseteq \ldots \Gamma_m(X^a)$$

such that the difference $\Gamma_k(X^a) \setminus \Gamma_{k+1}(X^a)$ is uncountable whenever $0 < k < m$. Now, consider an $X^a \in \mathbb{X}^a_M$. By Lemma C.10 again, there is an infinite decreasing chain of uncountable open sets $\Gamma_1(X^a) \supseteq \Gamma_2(X^a) \supseteq \ldots$ such that

- ▷ $\Gamma_k(X^a) \setminus \Gamma_{k+1}(X^a)$ is uncountable whenever $k > 0$;
- ▷ $\Gamma_\infty(X^a) \equiv \bigcap_{k>0} \Gamma_k(X^a)$ is an uncountable open set;
- ▷ $\Gamma_\infty(X^a)$ is dense in $\Gamma_M(X^a)$.

For each $m > 0$, define $Q^a_m = \bigcup_{X^a \in \mathbb{X}^a_m} X^a \times \Gamma_m(X^a)$. Then Q^a_m is open for each $m > 0$, and Q^a_∞ is open. It follows from our construction that $\Gamma_m(X^a) = \Gamma^a(X^a, Q^a_m)$ for each $m > 0$ and $X^a \in \mathbb{X}^a_m$, and that **(i)–(vi)** of Theorem 3.13 hold.

Theorem 3.2 now follows at once from Theorem 3.13 and Lemma 3.15. □

E PROOF OF THEOREM 3.24

Proof of Lemma 3.20. Let $v^a \in V^a_G$ and let $\lambda^a(v^a) = \sigma = (\sigma_0, \ldots, \sigma_k)$. Then $\lambda^a_G(v^a)$ is the marginal $\rho = (\rho_0, \ldots, \rho_k)$ of σ on $S^b \times V^b$, so λ^a_G is a Borel map from V^a into $\mathcal{N}(S^b \times V^b)$. σ is mutually singular, so there are pairwise disjoint Borel sets $U_i \subseteq \Theta \times S^b \times V^b$ such that $\sigma_i(U_i) = 1$ for each $i \le k$. The G-sections $W_i = \{(s^b, v^b) : (G, s^b, v^b) \in U_i\}$ are Borel and pairwise disjoint. Since $v^a \in V^a_G \subseteq C^a_1(G)$, $\sigma(\{G\} \times S^b \times V^b) = \vec{1}$. Therefore $\rho_i(W_i) = 1$ for each $i \le k$, and hence $\rho \in \mathcal{L}(S^b \times V^b)$. □

Lemma E.1. *Let Θ, X, Y be Polish spaces, where $Y \subseteq X$, and let $G \in \Theta$. Then*

(i) For each open U in the topology of $\{G\} \times X$, there exists an open W in the topology of $\Theta \times Y$ such that $W = U \cap (\{G\} \times Y)$; and

(ii) For each open W in the topology of $\Theta \times Y$, there exists an open U in the topology of $\{G\} \times X$ such that $W = U \cap (\{G\} \times Y)$.

Proof of Lemma E.1. Both **(i)** and **(ii)** follow immediately from the definition of subspace topology. □

Proof of Theorem 3.24. **Proof of (i).** Since $v^a \in V^a_G = C^a_1(G)$, $\lambda^a(v^a)m(\{G\} \times S^b \times V^b) = 1$. Therefore $(\lambda^a_G(v^a))(E) = \lambda^a(v^a)(\Theta \times E) = (\lambda^a(v^a))(\{G\} \times E)$. So if v^a assumes E in $\mathfrak{V}$ or

$\{G\} \times E$ in $\mathfrak{V}_G$, then

$$(\lambda^a_G(v^a))(E) = (\underbrace{1,\ldots,1}_{\text{1 or more}}, \underbrace{0,\ldots,0}_{\text{0 or more}}) = (\lambda^a(v^a))(\{G\} \times E).$$

Therefore conditions (a) and (b) hold for assuming E in $\mathfrak{V}$ if and only if they hold for assuming $\{G\} \times E$ in $\mathfrak{V}_G$. Furthermore, Lemma E.1 implies that for every Borel $F \subseteq S^b \times V^b_G$, $F = E \cap U \neq \varnothing$ for some open $U \subseteq S^b \times V^b_G$ if and only if $\{G\} \times F = W \cap (\{G\} \times E) \neq \varnothing$ for some open $W \subseteq \Theta \times S^b \times V^b$. So condition (c) for assuming E in $\mathfrak{V}$ is equivalent to condition (c) for assuming $\{G\} \times E$ in $\mathfrak{V}_G$.

Proof of (ii). It is quite trivial that s^a maximizes LEU with respect to $\lambda^a(v^a)$ if and only if s^a maximizes LEU with respect to $\lambda^a_G(v^a)$. $\lambda^a_G(v^a)$ has full support in $S^b \times V^b_G$ if and only if v^a assumes $S^b \times V^b_G$ in $\mathfrak{V}_G$. By **(i)**, this holds if and only if $\lambda^a(v^a)$ assumes $K^b(G) = \{G\} \times S^b \times V^b_G$ in $\mathfrak{V}$. This proves **(ii)**.

Proof of (iii). The base case is handled by **(ii)**. Assume the induction hypothesis for $M > 1$:

$$\forall m \leq M, \quad R^a_m(G) = \{G\} \times R^a_m(G, \mathfrak{V}_G)$$

By **(i)** and **(ii)**, for all $v^a \in \operatorname{proj}_{V^a} R^a_1(G) = \operatorname{proj}_{V^a_G} R^a_1(G, \mathfrak{V}_G)$, v^a assumes $\{G\} \times R^b_M(G, \mathfrak{V}_G)$ in $\mathfrak{V}$ if and only if v^a assumes $R^b_M(G, \mathfrak{V}_G)$ in $\mathfrak{V}_G$. Therefore,

$$\begin{aligned} R^a_{M+1}(G) &= R^a_M(G) \cap (\Theta \times S^a \times A^a(R^b_M(G))) \\ &= \left[\{G\} \times R^a_M(G, \mathfrak{V}_G)\right] \cap \left[\Theta \times S^a \times A^a(\{G\} \times R^b_M(G, \mathfrak{V}_G))\right] \\ &= \{G\} \times \left[R^a_M(G, \mathfrak{V}_G) \cap (S^a \times A^a(R^b_M(G, \mathfrak{V}_G)))\right] \\ &= \{G\} \times R^a_{M+1}(G, \mathfrak{V}_G). \end{aligned}$$

□

F PROOF OF THEOREM 3.25

To prove Theorem 3.25, we will first show that it is a consequence of Theorem F.1 below. The proof of Theorem F.1 is much longer, and is given in the next section.

Fix the finite strategy sets (S^a, S^b). Recall from Section 3.4 that we identify a game G with strategy sets (S^a, S^b) with the N-tuple of real numbers that represents the pair (π^a, π^b) of payoff functions, where $N = 2 \cdot |S^a \times S^b|$. Therefore, we let the space of all games on (S^a, S^b) be $\Theta = \mathbb{R}^N$. We maintain this definition of Θ throughout this paper. Theorem F.1 says there is a "Borel family" of type structures $\mathfrak{T}_G$ indexed by $G \in \Theta$ such that each $\mathfrak{T}_G$ has RCAR for G.

Given Polish spaces X, Y, Z and a Borel function $f : X \times Y \to Z$, we let $f_y : X \to Z$ be the Borel function defined by $f_y(x) = f(x, y)$.

Theorem F.1. *Let T^a, T^b be uncountable Polish spaces. There exist Borel maps*

$$\kappa^a : T^a \times \Theta \to \mathscr{L}(S^b \times T^b), \qquad \kappa^b : T^b \times \Theta \to \mathscr{L}(S^a \times T^a)$$

such that for every $G \in \Theta$, $\mathfrak{T}_G = \langle S^a, S^b, T^a, T^b, \kappa^a_G, \kappa^b_G \rangle$ is a complete one-to-one lexicographic type structure such that

$$\mathrm{proj}_{S^a} R^a_\infty(G, \mathfrak{T}_G) \times \mathrm{proj}_{S^b} R^b_\infty(G, \mathfrak{T}_G) = S^a_\infty(G) \times S^b_\infty(G).$$

This result would follow immediately from Theorems 3.2 and 3.4 if we only required that the map κ^a_G is Borel in t^a for each fixed G, and similarly for b. The extra difficulty lies in finding maps κ^a and κ^b that are Borel in both variables. Intuitively, $\{\mathfrak{T}_G : G \in \Theta\}$ is a Borel family of type structures indexed by $G \in \Theta$. Ann's beliefs depend on both the game G and a type t^a, and Bob's beliefs depend on both a game G and a type t^b.

To prepare for the proof of Theorem 3.25, we first prove an easier intermediate result, in which the requirement that $\mathfrak{V}$ is complete is omitted. Theorem 3.25 can then be proved by carefully embedding this $\mathfrak{V}$ into a complete type structure so that, for each game $G \in \Theta$, the set of states in which there is common knowledge of G remains unaltered.

Theorem F.2. *There is a one-to-one lexicographic type structure with nature, $\mathfrak{V} = \langle \Theta, S^a, S^b, V^a, V^b, \lambda^a, \lambda^b \rangle$, such that for every game $G \in \Theta$,*

(i) $\mathfrak{V}$ admits common knowledge of G;

(ii) For every game $G \in \Theta$, $\mathfrak{V}_G$ is a complete one-to-one lexicographic type structure such that

$$\mathrm{proj}_{S^a} R^a_\infty(G, \mathfrak{V}_G) \times \mathrm{proj}_{S^b} R^b_\infty(G, \mathfrak{V}_G) = S^a_\infty(G) \times S^b_\infty(G); \text{ and}$$

(iii) Any pair of types (v^a, v^b) that believes G has common belief of G.

Proof of Theorem F.2 from Theorem F.1. Let $S^a, S^b, T^a, T^b, \kappa^a, \kappa^b$ be as they were in Theorem F.1. Let $V^a = T^a \times \Theta$ and $V^b = T^b \times \Theta$. We will define Borel maps λ^a, λ^b so that $\mathfrak{V} = \langle \Theta, S^a, S^b, V^a, V^b, \lambda^a, \lambda^b \rangle$ has the required properties. The plan will be to make $T^a \times \{G\}$ be the set of types that have common belief of G.

Define the function

$$\alpha^b : \mathscr{L}(S^b \times T^b) \times \Theta \to \mathscr{L}(\Theta \times S^b \times V^b)$$

as follows. For each $(\sigma, G) \in \mathscr{L}(S^b \times T^b) \times \Theta$, let $\alpha^b(\sigma, G)$ be the unique $\mu \in \mathscr{L}(\Theta \times S^b \times V^b)$ such that

- $\mu(\{G\} \times S^b \times (T^b \times \{G\})) = \vec{1}$; and
- For each Borel set $E \subseteq S^b \times T^b$, $\mu(\{G\} \times E \times \{G\}) = \sigma(E)$.

Note that $E \times \{G\} \subseteq S^b \times V^b$, so $\{G\} \times E \times \{G\} \subseteq \Theta \times S^b \times V^b$.

Claim. α^b is a continuous map.

Proof of Claim: Suppose $\sigma_n \to \sigma$ in $\mathscr{L}(S^b \times T^b)$, and $G_n \to G$ in Θ, where $\to$ indicates weak convergence. We must prove that $\alpha^b(\sigma_n, G_n) \to \alpha^b(\sigma, G)$. It suffices to prove this in the case that each σ_n and σ have length one, because it would then follow that each coordinate of $\alpha^b(\sigma_n, G_n)$ converges to the corresponding coordinate of $\alpha^b(\sigma, G)$.

Let $\beta : \Theta \to \mathscr{M}(\Theta)$ be the map $H \mapsto \delta_H$, where $\delta_H(\{H\}) = 1$. We have $\beta(G_n) \to \beta(G)$, because for every continuous $f : \Theta \to \mathbb{R}$, $\int f \, d\delta_{G_n} = f(G_n)$ converges to $\int f \, d\delta_G = f(G)$. We note that

$$\alpha^b(\sigma_n, G_n) = \beta(G_n) \otimes \sigma_n \otimes \beta(G_n), \quad \alpha^b(\sigma, G) = \beta(G) \otimes \sigma \otimes \beta(G).$$

Therefore, we have $\alpha^b(\sigma_n, G_n) \to \alpha^b(\sigma, G)$, which proves the claim.

Now, define $\lambda^a(t^a, G) = \alpha^b(\kappa^a(t^a, G), G)$. Since κ^a is Borel and α^b is continuous, λ^a is a Borel map, and hence $\mathfrak{V}$ is a type structure with nature. Let $G \in \Theta$. We see from the definition of λ^a that a type $v^a \in V^a$ believes G if and only if $v^a = (t^a, G)$ for some $t^a \in T^a$. Thus $C_1^a(G) = T^a \times \{G\}$. Moreover, $C_2^a(G) = C_1^a(G)$, and hence by induction, $C_m^a(G) = C_1^a(G)$. Therefore, $\mathfrak{V}$ has the property that $C_1^a(G) = C_\infty^a(G) = V_G^a$, that is, every v^a that believes G has common belief of G. It follows that $\mathfrak{V}$ admits common knowledge of G. Finally, the mappings $t^a \mapsto (t^a, G), t^b \mapsto (t^b, G)$ are topological homeomorphisms from T^a to V_G^a and T^b to V_G^b that give an isomorphism from the type structure $\mathfrak{T}_G$ of Theorem F.1 to $\mathfrak{V}_G$. Therefore, $\mathfrak{V}_G$ has the same properties as $\mathfrak{T}_G$. In particular, $\mathfrak{V}_G$ is a complete one-to-one type structure such that $\text{proj}_{S^a} R_\infty^a(G, \mathfrak{V}_G) \times \text{proj}_{S^b} R_\infty^b(G, \mathfrak{V}_G) = S_\infty^a(G) \times S_\infty^b(G)$. □

Proof of Theorem 3.25 from Theorem F.1. Let $S^a, S^b, T^a, T^b, \kappa^a, \kappa^b$ be as in Theorem F.1. Let V^a be the topological union

$$V^a = [0,1) \uplus ([1,\infty) \times \Theta) \uplus (T^a \times \Theta),$$

where the three parts of the union are disjoint and clopen in V^a. Note that $V^a = [0,1) \uplus (([1,\infty) \uplus T^a) \times \Theta)$. Define V^b analogously. We will define Borel mappings λ^a, λ^b so that $\mathfrak{V} = \langle \Theta, S^a, S^b, V^a, V^b, \lambda^a, \lambda^b \rangle$ has the required properties.

For each $G \in \Theta$, our plan will be to let $T^a \times \{G\}$ be the set of types having common belief of G; let $[m, m+1) \times \{G\}$ be the set of types having m-th order, but not $(m+1)$-th order, belief of G; and let $[0,1)$ be the set of types not having belief of G. We will use the Borel Isomorphism Theorem (Proposition C.5), as we did in the proof of Theorem 3.13.

For each $m > 0$, let $T^a_m = [m, \infty) \uplus T^a$. Then $T^a_1 \supseteq T^a_2 \supseteq \cdots$, and $T^a = \bigcap_{m>0} T^a_m$. According to our plan, $T^a_m \times \{G\}$ will be the set of types having m-th order belief of G. These types will be mapped to the beliefs in set $J^b_m(G)$, which we define inductively as follows.

$$J^b_1(G) = \left\{\sigma \in \mathcal{L}(\Theta \times S^b \times V^b) : \sigma(\{G\} \times S^b \times V^b) = \vec{1}\right\}$$
$$J^b_{m+1}(G) = \left\{\sigma \in J^b_m(G) : \sigma(\{G\} \times S^b \times (T^b_m \times \{G\})) = \vec{1}\right\}$$

Intuitively, $J^b_m(G)$ is the set of LPS's that have m-th order belief of G. We also write

$$J^b_0 = \mathcal{L}(\Theta \times S^b \times V^b) \setminus \bigcup_{G \in \Theta} J^b_1(G).$$

Now, let $\alpha^b : \mathcal{L}(S^b \times T^b) \times \Theta \to \mathcal{L}(\Theta \times S^b \times V^b)$ be as in the proof of Theorem F.2. We will construct a one-to-one Borel function $\lambda^a : V^a \to \mathcal{L}(\Theta \times S^b \times V^b)$ such that

(I) $\lambda^a([0,1)) = J^b_0$;

(II) For each $m > 1$ and $G \in \Theta$, $\lambda^a(T^a_m \times \{G\}) = J^b_m(G)$; and

(III) For each $G \in \Theta$ and $t^a \in T^a$, $\lambda^a(t^a, G) = \alpha^b(\kappa^a(t^a, G), G)$.

Note that **(I)** and **(II)** imply that the map λ^a is onto. Since λ^a will be one-to-one, **(I)** and **(II)** will also imply that for each $G \in \Theta$, $\lambda^a(T^a \times \{G\}) = \bigcap_{m>0} J^b_m(G)$.

It is clear that the set J^b_0 has the cardinality of the continuum. We show that J^b_0 is also Borel. To see this, let

$$\gamma^b : \mathcal{L}(S^b \times V^b) \times \Theta \to \mathcal{L}(\Theta \times S^b \times V^b)$$

be the function such that $\gamma^b(\sigma, G)$ is the unique $\mu \in \mathcal{L}(\Theta \times S^b \times V^b)$ such that $\mu(\{G\} \times S^b \times V^b) = \vec{1}$ and for each Borel set $E \subseteq S^b \times V^b$, $\mu(\{G\} \times E) = \sigma(E)$. Note that

$$\forall G \in \Theta, \quad \gamma^b(\mathcal{L}(S^b \times V^b) \times \{G\}) = J^b_1(G)$$

Therefore, the range of γ^b is $\bigcup_{G \in \Theta} J^b_1(G)$ and the complement of the range of γ^b is J^b_0. Arguing as in the proof of the Claim in Theorem F.2, we see that γ^b is a continuous map. It is also clear that γ^b is one-to-one. By Corollary 15.2 in Kechris (1995), images of Borels sets under such maps are Borel sets themselves. Therefore, for every Borel $F \subseteq \mathcal{L}(S^b \times V^b) \times \Theta$, $\gamma^b(F)$ is Borel. In particular, the range of γ^b is Borel, and therefore its complement J^b_0 is Borel.

By the Borel Isomorphism Theorem, there is a one-to-one and onto Borel map $\lambda_0^a : [0,1) \to J_0^b$. This will take care of **(I)** since we will eventually let λ^a coincide with λ_0^a on $[0,1)$.

For each $G \in \Theta$ and each $m > 0$, the difference $J_m^b(G) \setminus J_{m+1}^b(G)$ clearly has the cardinality of the continuum. Moreover, since $J_m^b(G)$ is the image under γ^b of a Borel set, $J_m^b(G)$ is Borel. Hence the difference sets $J_m^b(G) \setminus J_{m+1}^b(G)$ are Borel as well. By the Borel Isomorphism Theorem, there is a one-to-one Borel function from $[m, m+1) \times \{G\}$ onto this difference. However, since there are uncountably many G's, we cannot in general glue these functions together into a single Borel function.

To get around this problem, we introduce mappings that translate the games and keep everything else unchanged. Let G_0 be the particular game whose payoff functions are everywhere zero. Given two games $G, H \in \Theta$, let $G + H$ be the game obtained by adding the payoff functions of G and H pointwise at each strategy profile. Note that $G_0 + H = H$ for each $H \in \Theta$. For each $H \in \Theta$, the map $G \mapsto G + H$ is a homeomorphism from Θ to itself that sends G_0 to H.

For $H \in \Theta$, let $\psi_H^a : V^a \to V^a$ be the map defined by

$$\psi_H^a(r) = \begin{cases} (t^a, G+H) & \text{if } r = (t^a, G) \in T_1^a \times \Theta; \\ r & \text{if } r \in [0,1). \end{cases}$$

Then ψ_H^a is a homeomorphism from V^a to V^a and we have

$$\forall m > 0, \quad \psi_H^a(T_m^a \times \{G_0\}) = T_m^a \times \{H\}.$$

Moreover, $(v^a, H) \mapsto \psi_H^a(v^a)$ is a continuous map from $V^a \times \Theta$ onto V^a.

Let ϕ_H^b be a function from $\mathscr{L}(\Theta \times S^b \times V^b)$ to itself such that for each $\sigma \in \mathscr{L}(\Theta \times S^b \times V^b)$ and Borel set $E \subseteq \Theta \times S^b \times V^b$,

$$(\phi_H^b(\sigma))\left(\left\{(G+H, s^b, \psi_H^b(v^b)) : (s^b, v^b, G) \in E\right\}\right) = \sigma(E).$$

Then ϕ_H^b is a homeomorphism from $\mathscr{L}(\Theta \times S^b \times V^b)$ onto itself such that for each $m > 1$, $\phi_H^b(J_m^b(G_0)) = J_m^b(H)$.

By the Borel Isomorphism Theorem, for each $m > 0$, there is a one-to-one Borel function ρ_m from $[m, m+1) \times \{G_0\}$ onto $J_m^b(G_0) \setminus J_{m+1}^b(G_0)$. Let

$$\lambda_m^a : [m, m+1) \times \Theta \to \mathscr{L}(\Theta \times S^b \times V^b)$$

be the mapping given by $\lambda_m^a(r, H) = \phi_H^b(\rho_m(\sigma, G_0))$. It follows that λ_m^a is one-to-one and

Borel, and for each $H \in \Theta$,

$$\lambda_m^a([m,m+1) \times \{H\}) = J_m^b(H) \setminus J_{m+1}^b(H).$$

Let $\lambda_\infty^a : T^a \times \Theta \to \mathscr{L}(\Theta \times S^b \times V^b)$ be the mapping given by

$$\lambda_\infty^a(t^a, G) = \alpha^b(\kappa^a(t^a, G), G).$$

It is clear that λ_∞^a is one-to-one. Since α^a is continuous and κ^a is Borel, λ_∞^a is Borel. Also, for each $H \in \Theta$, $\lambda_\infty^a(T^a \times \{H\}) = \bigcap_m J_m^b(H)$.

It follows that the union $\lambda^a = \lambda_0^a \cup \left(\bigcup_m \lambda_m^a\right) \cup \lambda_\infty^a$ is a one-to-one Borel mapping from V^a onto $\mathscr{L}(\Theta \times S^b \times V^b)$ that satisfies **(I)–(III)**. Therefore $\mathfrak{V}$ is a complete one-to-one type structure with nature.

It follows from **(II)** that for each $H \in \Theta$, $C_1^a(H) = T_1^a \times \{H\}$. We then see by induction that for each $m > 0$ and $H \in \Theta$, $C_m^a(H) = T_m^a \times \{H\}$. Therefore, the set of $v^a \in V^a$ with common belief of H is

$$V_H^a = C_\infty^a(H) = \bigcap_m T_m^a \times \{H\} = T^a \times \{H\}.$$

So $\mathfrak{V}$ admits common knowledge of every game $H \in \Theta$. As in the proof of Theorem F.2, for each $G \in \Theta$, the type structure $\mathfrak{V}_G$ is isomorphic to $\mathfrak{T}_G$. Therefore $\mathfrak{V}_G$ is a complete one-to-one type structure such that

$$\text{proj}_{S^a} R_\infty^a(G, \mathfrak{V}_G) \times \text{proj}_{S^b} R_\infty^b(G, \mathfrak{V}_G) = S_\infty^a(G) \times S_\infty^b(G).$$

By applying Theorem 3.24, we get

$$\text{proj}_{S^a} R_\infty^a(G) \times \text{proj}_{S^b} R_\infty^b(G) = S_\infty^a(G) \times S_\infty^b(G). \qquad \square$$

G PROOF OF THEOREM F.1

Note that each of the objects P_m^a, $\mathbb{O}(v)$, $\mathbb{X}_m^a$ defined in Section 3.2 depends on a game $G \in \Theta$. In this section, we will let $P_m^a(G)$, $\mathbb{O}(G, v)$, $\mathbb{X}_m^a(G)$ denote these objects to indicate the dependence on G.

By the Existence Theorem 3.2 and Theorem 3.4, for each game $G \in \Theta$ there is a complete type structure

$$\mathfrak{T} = \left\langle S^a, S^b, T^a, T^b, \lambda_G^a, \lambda_G^b \right\rangle$$

such that $R_\infty^a(G, \mathfrak{T})$ and $R_\infty^b(G, \mathfrak{T})$ are nonempty. Our task will be to choose such maps λ_G^a, λ_G^b for each $G \in \Theta$ so that $(G, u^a) \mapsto \lambda_G^a(u^a)$ is a Borel map from $\Theta \times T^a$ into $\mathscr{L}(S^b \times$

T^b), and similarly with a and b reversed. If the set of games Θ were countable, then we could directly appeal to the Borel Isomorphism Theorem and glue the maps λ_G together. However, we will need to choose the maps λ_G more carefully since Θ is uncountable.

The following lemma improves Theorem 3.2 by specifying in advance the length of $\lambda^a(t^a)$ for each type $t^a \in T^a$. For the remainder of this section, let $M = |S^a| + |S^b|$.

Lemma G.1. *Let T^a, T^b be uncountable Polish spaces and let $\{T^a_n : n > 0\}$ and $\{T^b_n : n > 0\}$ be countable partitions of T^a, T^b. For each game $G \in \Theta$, there exists a complete one-to-one lexicographic type structure $\mathfrak{T} = \langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ such that for each $k \geq M+1$,*

$$R^a_\infty(G, \mathfrak{T}) \cap (S^a \times T^a_k) \neq \varnothing, \qquad R^b_\infty(G, \mathfrak{T}) \cap (S^b \times T^b_k) \neq \varnothing;$$

and for each $k > 0$, $t^a \in T^a_k$, $t^b \in T^b_k$, $\lambda^a(t^a)$ and $\lambda^b(t^b)$ have length k.

Proof of Lemma G.1. The proof is a routine modification of the proofs of Lemma 3.15 and Theorem 3.13 so that for each $k > 0$, types in T^a_k are mapped to LPS's of length k. By the method of Lemma 3.15, one can build a family of sets

$$\{Q^a_m : m > 0\}, \qquad \{Q^b_m : m > 0\}$$

such that **(i)–(vi)** of Theorem 3.13 hold within T^a_k for each $k \geq \min(m, M)$. That is, we have the following for each nonempty $X^a \subseteq S^a$ and each $k > 0$, and similarly for b.

(i) $\{\Gamma^a(X^a, Q^a_m) \cap T^a_k : m > 0\}$ is a decreasing chain of Borel subsets of T^a_k;

(ii) For each $m > 0$, $\Gamma^a(X^a, Q^a_m) \cap T^a_k \neq \varnothing \iff (X^a \in \mathbb{X}^a_m \wedge k \geq \min(m, M))$;

(iii) $\Gamma^a(X^a, Q^a_\infty) \cap T^a_k$ is dense in $\Gamma^a(X^a, Q^a_{M+1}) \cap T^a_k$,

(iv) $\Gamma^a(\varnothing, Q^a_1) \cap T^a_k$ is uncountable;

(v) If $X^a \in \mathbb{X}^a_m$ and $k \geq \min(m, M)$ then

$$(\Gamma^a(X^a, Q^a_m) \setminus \Gamma^a(X^a, Q^a_{m+1})) \cap T^a_k$$

is uncountable, and if $m < M$ then $\Gamma^a(X^a, Q^a_{m+1})$ is not even dense in $\Gamma^a(X^a, Q^a_m)$;

(vi) If $X^a \in \mathbb{X}^a_\infty$ and $n \geq M$ then $\Gamma^a(X^a, Q^a_\infty) \cap X^a_n$ is uncountable;

Condition **(v)** is upgraded to insure that for $k \leq M$, no LPS in $\mathcal{L}(S^b \times T^b)$ of length k can assume all of $Q^a_0, \ldots, Q^a_k$. Then each piece of T^a_k will have the same cardinality of the corresponding piece of $\mathcal{L}_k(S^b \times T^b)$. The Borel Isomorphism Theorem can now be used as in the proof of Theorem 3.13 to construct the required mappings λ^a and λ^b. □

Next, we show that the games $G \in \Theta$ can be classified into finitely many shapes. We say that two games $G, H \in \Theta$ have the **same shape** if $\mathbb{X}^a_m(G) = \mathbb{X}^a_m(H)$ and $\mathbb{X}^b_m(G) = \mathbb{X}^b_m(H)$

for all m. By Theorem 3.8, if G and H have the same shape, then $S^a_m(G) = S^a_m(H)$ and $S^b_m(G) = S^b_m(H)$ for each m.

The next lemma shows that the sequences $S^a_m(G)$ and $\mathbb{X}^a_m(G)$ stabilize at $M = |S^a| + |S^b|$, and hence there are only finitely many possible shapes of games in Θ.

Lemma G.2. *For each $G \in \Theta$ and $m \geq M$ we have*

(i) $S^a_m(G) = S^a_M(G) = S^a_\infty(G)$ and $S^b_m(G) = S^b_M(G) = S^b_\infty(G)$;

(ii) $\mathbb{X}^a_m(G) = \mathbb{X}^a_{M+1}(G) = \mathbb{X}^a_\infty(G)$ and $\mathbb{X}^b_m(G) = \mathbb{X}^b_{M+1}(G) = \mathbb{X}^b_\infty(G)$.

Hence there are only finitely many shapes of games in Θ.

Proof of Lemma G.2. **Proof of (i).** If $S^a_m(G) = S^a_{m+1}(G)$ and $S^b_m(G) = S^b_{m+1}(G)$, then we see from the definition of $S^a_m(G)$ that $S^a_m(G) = S^a_n(G)$ and $S^b_m(G) = S^b_n(G)$ for all $n \geq m$. Moreover, $S^a_0(G) = S^a$ and $S^b_0(G) = S^b$, and the sets $S^a_m(G), S^b_m(G)$ decrease with m. Therefore, the pair of sets $(S^a_m(G), S^b_m(G))$ can change at most M times, and **(i)** follows.

Proof of (ii). Let $m > M$ and $X^a \in \mathbb{X}^a_m(G)$. Then $X^a = \mathbb{O}(G, \mu)$ for some $\mu \in P^a_m(G)$. We have $\mu = \nu\nu'$ for some $\nu \in \mathcal{N}(S^b)$ with $\mathrm{Supp}(\nu) = S^b_{m-1}(G)$ and some $\nu' \in P^a_{m-1}(G)$. By 1., $S^b_m(G) = S^b_{m-1}(G)$, so $\mu' = \nu\mu \in P^a_m(G)$. It is clear that $\mathbb{O}(G, \mu') = \mathbb{O}(G, \mu)$, so $X^a \in \mathbb{X}^a_{m+1}(G)$. This proves **(ii)**. □

Lemma G.2 shows that the shape of G depends only on $\mathbb{X}^a_m(G), \mathbb{X}^b_m(G)$ for $m \leq M+1$. We may therefore define the **shape of** G as follows. Given a sequence

$$\mathbb{S} = (\mathbb{X}^a_1, \ldots, \mathbb{X}^a_{M+1}, \mathbb{X}^b_1, \ldots, \mathbb{X}^b_{M+1}),$$

we say that G has shape $\mathbb{S}$, and write $\mathbb{S}(G) = \mathbb{S}$, if $\mathbb{X}^a_m = \mathbb{X}^a_m(G)$ and $\mathbb{X}^b_m = \mathbb{X}^b_m(G)$ for $m = 1, \ldots, M+1$. And we say that $\mathbb{S}$ is a **game shape** if there exists a game $G \in \Theta$ such that $\mathbb{S} = \mathbb{S}(G)$.

The intuitive idea of our proof of Theorem F.1 will be to build the type structures $\mathfrak{T}_G$ in such a way that they can be glued together by an inductive construction on the length of LPS's. For each fixed length $k > 0$, we will see that the set Θ of games can be partitioned into finitely many classes such that within each class, the length k parts of the type structures $\mathfrak{T}_G$ can be chosen to be the same up to a Borel transformation, and thus can be combined into a single type structure.

To do this, we will need some results from the literature about definable sets in the ordered field of real numbers. We let $\mathbb{F} = \langle \mathbb{R}, 0, 1, +, \cdot, < \rangle$ be the **ordered field** of real numbers. A set of n-tuples $A \subseteq \mathbb{R}^n$ is said to be **definable** (in $\mathbb{F}$) if A is the set of all n-tuples that satisfy a first order formula $\varphi(x_1, \ldots, x_n, \vec{c})$ in $\mathbb{F}$ that has the variables $x_0, \ldots, x_n$ and a finite tuple $\vec{c}$ of parameters in $\mathbb{R}$. Given two definable sets $A \subseteq \mathbb{R}^m, B \subseteq \mathbb{R}^n$ in $\mathbb{F}$, a function $f : B \to A$ is said to be **definable** (in $\mathbb{F}$) if its graph $\{(\vec{x}, \vec{y}) : f(\vec{x}) = \vec{y}\}$ is definable.

The celebrated classical result of Tarski (1951) shows that a set is definable in $\mathbb{F}$ if and only if it is semi-algebraic (i.e., definable by finite collections of equations and inequalities between polynomials). Tarski's theorem has the following easy consequence.

Proposition G.3.

(i) $\mathbb{F}$ *is* ***o-minimal****, that is, every set* $A \subseteq \mathbb{R}$ *that is definable in* $\mathbb{F}$ *is the union of finitely many open intervals and singletons.*

(ii) Every set $A \subseteq \mathbb{R}^k$ *that is definable in* $\mathbb{F}$ *is Borel.*

We refer to the monograph van den Dries (1998) for an exposition of o-minimal structures, but we will only need the particular o-minimal structure $\mathbb{F}$. We will need a result of Hardt (1980), which says that every definable function can be partitioned into finitely many definable pieces that each look like the projection of a product of two sets onto one of the factors. This result was generalized to o-minimal structures (see van den Dries, 1998, chap. 9, Theorem 1.2).

Definition G.4. *Suppose* $A \subseteq \mathbb{R}^m, B \subseteq \mathbb{R}^n$, $g : B \to A$ *is definable, and* g *maps* B *onto* A. *We say that* g *is* ***definably trivial*** *if there exists a definable set* $C \subseteq \mathbb{R}^k$ *for some* k *and a definable function* $h : B \to C$ *such that the function* $(g, h) : B \to A \times C$ *is a homeomorphism.*

Proposition G.5 (Hardt (1980))**.** *Let* $A \subseteq \mathbb{R}^m, B \subseteq \mathbb{R}^n$*, and* $g : B \to A$ *be definable, and suppose* g *maps* B *onto* A*. Then there exists a finite partition* $\{A_1, \ldots, A_p\}$ *of* A *into definable sets such that for each* $i \le p$*, the restriction of* g *to* $g^{-1}(A_i)$ *is definably trivial.*

In the above proposition, note that for each i, the set $B_i = g^{-1}(A_i)$ and the restriction g_i of g to B_i are also definable. The result says that there is a definable set C_i and a definable function $h_i : B_i \to C_i$ such that $(g_i, h_i) : B_i \to A_i \times C_i$ is a homeomorphism, i.e., one-to-one, onto, and bi-continuous.

We now look at definable properties of games and tuples of probability measures. Recall that $\mathcal{M}(S^b)$ is the set of all probability measures on S^b. We may identify a probability measure $\nu \in \mathcal{M}(S^b)$ with the real vector $\langle \nu(s^b) : s^b \in S^b \rangle \in \mathbb{R}^{|S^b|}$ and note that this tuple satisfies the first order formulas $0 \le \nu(s^b) \le 1$ and $\sum_{s^b \in S^b} \nu(s^b) = 1$. Similarly, for each fixed k, the k-tuple of probability measures $\nu \in \mathcal{N}_k(S^b)$ is identified with a $k \cdot |S^b|$-tuple of reals in the obvious way.

By a **k-fold support in S^b** we mean a k-tuple $Y = (Y_0, \dots, Y_{k-1})$ of nonempty sets $Y_i \subseteq S^b$ such that $\bigcup_{j<k} Y_j = S^b$. The **k-fold support** of a k-tuple

$$\nu = (\nu_0, \dots, \nu_{k-1}) \in \mathcal{N}_k(S^b)$$

is the k-tuple $\mathrm{Supp}_k(\nu) = \big(\mathrm{Supp}(\nu_0), \dots, \mathrm{Supp}(\nu_{k-1})\big)$.

The next lemma shows that for each fixed k, certain relations involving games and k-tuples of probability measures on S^b are definable.

Lemma G.6. *For each k, the following sets are definable.*

(i) For each n, the set $\big\{(G,\nu) \in \Theta \times \mathcal{N}_k(S^b) : \nu \in P^a_n(G)\big\}$;

(ii) For each k-fold support Y in S^b, the set $\big\{\nu \in \mathcal{N}_k(S^b) : \mathrm{Supp}_k(\nu) = Y\big\}$;

(iii) For each $X^a \subseteq S^a$, the set $\big\{(G,\nu) \in \Theta \times \mathcal{N}_k(S^b) : \mathbb{O}(G,\nu) = X^a\big\}$; and

(iv) For each game shape $\mathbb{S} = \big(\mathbb{X}^a_1, \dots, \mathbb{X}^a_{M+1}, \mathbb{X}^b_1, \dots, \mathbb{X}^b_{M+1}\big)$, the set $\{G \in \Theta : \mathbb{S}(G) = \mathbb{S}\}$.

Proof of Lemma G.6. **Proof of (i–iii).** These can be seen by writing the definitions formally in first order logic.

Proof of (iv). By Lemma B.4, for each n and each set $X^a \subseteq S^a$, we have $X^a \in \mathbb{X}^a_n(G)$ if and only if there exists an $\nu \in \mathcal{N}_{M+1}(S^b) \cap P^a_n(G)$ such that $\mathbb{O}(G,\nu) = X^a$. The point is that we need only consider ν's of length $M+1$. The result now follows from **(i)** and **(iii)**. □

Definition G.7. *A k-**good partition** of Θ is a finite partition $\{A_1, \dots, A_p\}$ of Θ such that for each $i \le p$,*

(i) A_i is definable;

(ii) $\exists \mathbb{S}_i \forall G \in A_i, \quad \mathbb{S}(G) = \mathbb{S}_i$, i.e., all the games in A_i have the same shape;

(iii) For each set $X \subseteq S^a$ and each k-fold support Y in S^b, the projection function from the set

$$B_{i,X,Y} = \{(G,\nu) \in A_i \times \mathcal{N}_k(S^b) : \mathbb{O}(G,\nu) = X \wedge Y = \mathrm{Supp}_k(\nu)\}$$

to A_i is definably trivial.

Remark G.8. *For each k-good partition of Θ, the family of sets*

$$\{B_{i,X,Y} : i \le p \wedge X \subseteq S^a \wedge Y = \mathrm{Supp}_k(\nu)\}$$

*indexed by (i,X,Y) in **(iii)** is a finite partition of $\Theta \times \mathcal{N}_k(S^b)$ into definable sets. The set $B_{i,X,Y}$ may be empty for some values of (i,X,Y).*

Lemma G.9. *Suppose $\{A_1, \dots, A_p\}$ is a k-good partition of Θ. Let $i \le p$; $G_i \in A_i$, $X \subseteq S^a$; Y be a k-fold support in S^b; g be the projection function from $B_{i,X,Y}$ to A_i; and let $C_{i,X,Y} =$*

$\{v \in \mathcal{N}_k(S^b) : (G_i, v) \in B_{i,X,Y}\}$.[24] *Then there is a definable function* $h : B_{i,X,Y} \to C_{i,X,Y}$ *such that the function* $(g, h) : B_{i,X,Y} \to A_i \times C_{i,X,Y}$ *is a homeomorphism.*

Proof of Lemma G.9. By **(iii)** in Definition G.7, the projection function g from $B_{i,X,Y}$ to A_i is definably trivial, so there is a set D and a definable function $f : B_{i,X,Y} \to D$ such that the function $(g, f) : B_{i,X,Y} \to A_i \times D$ is a homeomorphism. Then the restriction of f to $\{G_i\} \times C_{i,X,Y}$ is a homeomorphism from $\{G_i\} \times C_{i,X,Y}$ to D. Therefore, there is a definable homeomorphism ℓ from D to $C_{i,X,Y}$, and hence the composition $h = \ell \circ f$ has the required properties. □

Proposition G.5 gives us the following lemma about game-LPS pairs.

Lemma G.10. *For each* $k > 0$*, there exists a* k*-good partition of* Θ.

Proof of Lemma G.10. By Lemma G.6, for each game shape $\mathbb{S}$ and $k > 0$, the sets

$$A_{\mathbb{S}} = \{G \in \Theta : \mathbb{S}(G) = \mathbb{S}\}, \qquad B_{\mathbb{S}} = A_{\mathbb{S}} \times \mathcal{N}_k(S^b)$$

are definable. It suffices to prove that for each game shape $\mathbb{S}$, the set $A_{\mathbb{S}}$ admits a finite partition into definable sets $\{A_1, \ldots, A_p\}$ such that **(iii)** holds for each $i \le p$. If so then the union of these partitions will be a k-good partition of Θ.

Now fix a game shape $\mathbb{S}$, and let g be the projection function from $B_{\mathbb{S}}$ onto $A_{\mathbb{S}}$. Since the sets S^a and S^b are finite, there are only finitely many (X, Y) such that $X \subseteq S^a$ and Y is a k-fold support in S^b. For such (X, Y), let

$$B_{X,Y} = \{(G, v) \in B_{\mathbb{S}} : \mathbb{O}(G, v) = X \text{ and } \mathrm{Supp}_k(v) = Y\}.$$

By Lemma G.6, each set $B_{X,Y}$ is definable, and hence the restriction of g to $B_{X,Y}$ is definable. By Proposition G.5, there is a finite partition $\{A_{1,X,Y}, \ldots, A_{q,X,Y}\}$ of $A_{\mathbb{S}}$ into definable sets such that for each $j \le q$, the restriction of g to $(g^{-1}(A_{j,X,Y})) \cap B_{X,Y}$ is definably trivial. Let us say that two games $G, H \in A$ are **equivalent** if

$$\{(j, X, Y) : G \in A_{j,X,Y}\} = \{(j, X, Y) : H \in A_{j,X,Y}\}.$$

There are finitely many equivalence classes in $A_{\mathbb{S}}$ and each equivalence class is definable. Therefore, this equivalence relation implicitly defines a finite partition of $A_{\mathbb{S}}$ into definable sets $\{A_1, \ldots, A_p\}$ and this partition satisfies **(iii)** as required. □

We are now ready to prove Theorem F.1.

[24] i.e., $C_{i,X,Y}$ is the fiber in $B_{i,X,Y}$ above G_i with respect to g.

Proof of Theorem F.1. We will construct a pair of Borel functions

$$\kappa^a : \Theta \times T^a \to \mathscr{L}(S^b \times T^b), \qquad\qquad \kappa^b : \Theta \times T^b \to \mathscr{L}(S^a \times T^a)$$

with the required properties in several steps. Steps 3 and 5 will require additional proof.

1. First, choose partitions $\{T^a_k : k > 0\}$ and $\{T^b_k : k > 0\}$ of T^a and T^b into continuum-large Borel sets so that $T^a = \biguplus_{k>0} T^a_k$ and $T^b = \biguplus_{k>0} T^b_k$.
2. By Lemma G.10, we can choose a k-good partition $\{A_{1,k}, \ldots, A_{p(k),k}\}$ of Θ for each k. Recall that for all i, games in $A_{i,k}$ have the same shape.
3. Next, for each (i,k), we construct a Borel map $\kappa^a : A_{i,k} \times T^a_k \to \mathscr{L}_k(S^b \times T^b)$ such that for all $G \in A_{i,k}$, $\kappa^a(G, T^a_k) = \mathscr{L}_k(S^b \times T^b)$. We will subdivide the domain even further in this step.
4. By joining such maps for all $i \le p(k)$, we will get a Borel map $\kappa^a : \Theta \times T^a_k \to \mathscr{L}_k(S^b \times T^b)$ since $p(k)$ is finite. Finally, we will join such maps for all $k \in \mathbb{N}$, to get a Borel map $\kappa^a : \Theta \times T^a \to \mathscr{L}(S^b \times T^b)$.
5. Lastly, we will verify that κ^a and κ^b satisfy the desired properties.

Step 3. We begin by fixing k, a k-good partition $\{A_{1,k}, \ldots, A_{p(k),k}\}$ of Θ, and $i \le p(k)$.

By Lemma G.1, for each game $G \in \Theta$, we can choose a complete one-to-one type structure $\mathfrak{U}_G = \langle S^a, S^b, T^a, T^b, \lambda^a_G, \lambda^b_G \rangle$ such that for each $j \ge M+1$,

$$R^a_\infty(G, \mathfrak{U}_G) \cap (S^a \times T^a_j) \neq \varnothing, \qquad\qquad R^b_\infty(G, \mathfrak{U}_G) \cap (S^b \times T^b_j) \neq \varnothing;$$

and for each $j > 0$, $t^a \in T^a_j$, $t^b \in T^b_j$, $\lambda^a_G(t^a)$ and $\lambda^b_G(t^b)$ have length j.

If we could glue together the map λ^a_G for each $G \in A_{i,k}$ to define a Borel map $(G, t^a) \mapsto \lambda_G(t^a)$ then we would be done. However, there are uncountably many G's in $A_{i,k}$, so we cannot appeal to the Borel Isomorphism Theorem.

In order to get around this problem, we fix some $G_{i,k} \in A_{i,k}$ and the associated type structure $\mathfrak{U}_{G_{i,k}}$. For the sake of avoiding subscripts of subscripts, we will let $\mathfrak{U}_{i,k} = \mathfrak{U}_{G_{i,k}}$, $\lambda^a_{i,k} = \lambda^a_{G_{i,k}}$, and $\lambda^b_{i,k} = \lambda^b_{G_{i,k}}$. We will soon show that the structural properties shared by the games in $A_{i,k}$ allow us to define a Borel map κ^a on $A_{i,k} \times T^a_k$ from the mapping $\lambda^a_{i,k}$ so that κ^a has the desired properties.

For each $X \subseteq S^a$ and each k-fold support Y in S^b, let

$$B_{i,X,Y,k} = \Big\{(G, \nu) \in A_{i,k} \times \mathcal{N}_k(S^b) : \mathbb{O}(G, \nu) = X \wedge \operatorname{Supp}_k(\nu) = Y\Big\}; \text{ and}$$

$$C_{i,X,Y,k} = \Big\{\nu \in \mathcal{N}_k(S^b) : (G_{i,k}, \nu) \in B_{i,X,Y,k}\Big\} \text{ as in Lemma G.9; and}$$

$$D_{i,X,Y,k} = \Big\{t^a \in T^a_k : \operatorname{marg}_{S^b}(\lambda^a_{i,k}(t^a)) \in C_{i,X,Y,k}\Big\}.$$

Note that the sets $B_{i,X,Y,k}$ and $C_{i,X,Y,k}$ are definable. By Proposition G.3, $B_{i,X,Y,k}$ is Borel, and hence $D_{i,X,Y,k} \subseteq T_k^a$ is Borel as well. We note that for each $i \le p(k)$, the family of sets

$$\left\{D_{i,X,Y,k} : X \subseteq S^a \text{ and } Y \text{ is a } k\text{-fold support in } S^b\right\}$$

is a partition of T_k^a into finitely many Borel sets, some of which may be empty.

We will define the restriction of κ^a to $A_{i,k} \times D_{i,X,Y,k}$. We fix $X \subseteq S^a$, and a k-fold support Y in S^b. Let g be the projection function from $B_{i,X,Y,k}$ to $A_{i,k}$. By Lemma G.9, there is a definable function $h : B_{i,X,Y,k} \to C_{i,X,Y,k}$ such that the function $(g,h) : B_{i,X,Y,k} \to A_{i,k} \times C_{i,X,Y,k}$ is a homeomorphism.

Since Y is a k-fold support in S^b, we can write $Y = (Y_0, \ldots, Y_{k-1})$. Now, let $L_Y = \left\{\mu \in \mathscr{M}(S^b \times T^b) : \mathrm{Supp}_k \,\mathrm{marg}_{S^b} \mu = Y\right\}$ and $M_Y = \left\{\nu \in \mathscr{M}(S^b) : \mathrm{Supp}_k \nu = Y\right\}$. Let $\phi_Y : L_Y \times M_Y \to L_Y$ be the function that maps (μ, ν) to $\phi_Y(\mu,\nu)$ such that the j-th component of $[\phi_Y]_j$ is defined as follows for each $j < k$.

$$[\phi_Y(\mu,\nu)]_j(E) = \sum_{s^b \in Y_j} \mu_j(E \mid \{s^b\} \times T^b) \cdot \nu_j(s^b) \quad \text{for each Borel set } E \subseteq S^b \times T^b.$$

That is, $\phi_Y(\mu,\nu)$ is the measure such that its marginal on S^b is equal to ν; and for each $s^b \in Y_j$, its beliefs conditional on $\left\{s^b\right\} \times T^b$ is the same as those of μ. It is clear that ϕ_Y is Borel and that μ and $\phi_Y(\mu,\nu)$ have idential null sets. Furthermore, $\phi_Y(\mu,\cdot)$ is a one-to-one map.[25] Lastly, note that for all $G, G' \in A_{i,k}$,

$$\phi_Y\left(\left\{\mu \in L_Y : \mathbb{O}(G, \mathrm{marg}_{S^b} \mu)\right\} \times \{\nu \in M_Y : \mathbb{O}(G,\nu)\}\right) = \left\{\mu \in L_Y : \mathbb{O}(G', \mathrm{marg}_{S^b} \mu)\right\}.$$

We define $\kappa^a : A_{i,k} \times D_{i,X,Y,k} \to \mathscr{L}_k(S^b \times T^b)$ as the following composition of Borel maps, where $(g,h)^{-1}$ denotes the inverse function of (g,h) and $\pi_{\mathscr{N}_k(S^b)}$ is the projection function onto $\mathscr{N}_k(S^b)$.

$$\kappa^a(G, t^a) = \phi_Y\left(\lambda_{i,k}^a(t^a), \pi_{\mathscr{N}_k(S^b)}\left[(g,h)^{-1}(G, \mathrm{marg}_{S^b} \lambda_{i,k}^a(t^a))\right]\right)$$

Therefore κ^a is Borel. We also let $\kappa_G^a(t^a) = \kappa^a(G, t^a)$. $\kappa^a(G, t^a)$ has marginal beliefs such that X is the optimal set under it in game G; and it also has the same null sets as $\lambda_{i,k}^a(t^a)$ at *every* level.[26] An important implication of this is that $\kappa_G^a(t^a) \in \mathscr{L}^+(S^b \times T^b) \iff \lambda_{i,k}^a(t^a) \in \mathscr{L}^+(S^b \times T^b)$ and they assume the same events at each level.

Claim. For each $G \in A_{i,k}$, κ_G^a is a one-to-one map.

Proof of claim. Let $t^a, r^a \in D_{i,X,Y,k}$, and $t^a \neq r^a$. Now, consider the case when $\mathrm{marg}_{S^b} \lambda_{i,k}^a(t^a) \neq \mathrm{marg}_{S^b} \lambda_{i,k}^a(r^a)$. Then $\mathrm{marg}_{S^b} \kappa_G^a(t^a) \neq \mathrm{marg}_{S^b} \kappa_G^a(r^a)$ since it is clear

[25] $\phi_Y(\cdot,\nu)$ is not one-to-one.

[26] In comparison, $\lambda_{i,k}^a(t^a)$ has marginal beliefs such that X is the optimal set under it in game $G_{i,k}$.

that the map $\pi_{\mathcal{N}_k(S^b)}\left[(g,h)^{-1}(G,\cdot)\right]$ is one-to-one from the properties of (g,h). Therefore, $\kappa^a_G(t^a) \neq \kappa^a_G(r^a)$. Now, consider the case when $\operatorname{marg}_{S^b}\lambda^a_{i,k}(t^a) = \operatorname{marg}_{S^b}\lambda^a_{i,k}(r^a)$. If $\kappa^a_G(t^a) = \kappa^a_G(r^a)$ then $\lambda^a_{i,k}(t^a)$ and $\lambda^a_{i,k}(r^a)$ induce the same conditional beliefs on sets of the form $\{s^b\} \times T^b$ for each $s^b \in S^b$. However, if that is so, then $\operatorname{marg}_{S^b}\lambda^a_{i,k}(t^a) = \operatorname{marg}_{S^b}\lambda^a_{i,k}(r^a) \implies \lambda^a_{i,k}(t^a) = \lambda^a_{i,k}(r^a) \iff t^a = r^a$. Therefore, $\kappa^a_G(t^a) \neq \kappa^a_G(r^a)$.

Claim. κ^a_G maps $D_{i,X,Y,k}$ onto the set $\{\sigma \in \mathscr{L}_k(S^b \times T^b) : (G, \operatorname{marg}_{S^b}(\sigma)) \in B_{i,X,Y,k}\}$.

Proof of claim. This follows immediately from the previous two claims.

Step 4. Since the sets $A_{i,k} \times D_{i,X,Y,k}$ ranging over (i,X,Y) partition $\Theta \times T^a_k$ into finitely many Borel sets, the union of the parts of κ^a on each of these sets is a Borel function from $\Theta \times T^a_k$ into $\mathscr{L}_k(S^b \times T^b)$. Since each of the functions $\lambda^a_{i,k}$ maps T^a_k onto $\mathscr{L}_k(S^b \times T^b)$, κ^a_G maps to T^a_k onto $\mathscr{L}_k(S^b \times T^b)$ for each $G \in A_{i,k}$. Moreover, since the sets X and Y can be recovered from each game G and LPS $\sigma \in \mathscr{L}_k(S^b \times T^b)$, and κ^a_G is one-to-one on each $D_{i,X,Y,k}$, it follows that κ^a_G is one-to-one on T^a_k.

We now define the full function κ^a on $\Theta \times T^a$ by taking the union of the pieces we have defined on each $\Theta \times T^a_k$. Since the sets T^a_k are disjoint Borel sets, this union is a Borel function from $\Theta \times T^a$ onto $\mathscr{L}(S^b \times T^b)$, and for each $g \in \Theta$, κ^a_G is a one-to-one Borel mapping from T^a onto $\mathscr{L}(S^b \times T^b)$. Thus for each $G \in \Theta$, $\mathfrak{T}_G = \langle S^a, S^b, T^a, T^b, \kappa^a_G, \kappa^b_G \rangle$ is a complete one-to-one type structure.

Step 5. We now complete the proof by showing that for every game $G \in \Theta$,

$$\operatorname{proj}_{S^a} R^a_\infty(G, \mathfrak{T}_G) \times \operatorname{proj}_{S^b} R^b_\infty(G, \mathfrak{T}_G) = S^a_\infty(G) \times S^b_\infty(G).$$

We do this by proving two claims.

Claim. Let $k > 0$, $i \leq p(k)$, and $G \in A_{i,k}$. Then

$$R^a_1(G_{i,k}, \mathfrak{U}_{i,k}) \cap (S^a \times T^a_k) = R^a_1(G, \mathfrak{T}_G) \cap (S^a \times T^a_k).$$

Proof of claim. Let $s^a \in S^a$ and $t^a \in T^a_k$. For some X, Y we have $t^a \in D_{i,X,Y,k}$. Then the following are equivalent to $(s^a, t^a) \in R^a_1(G_{i,k}, \mathfrak{U}_{i,k})$.

$$s^a \in \mathbb{O}(G_{i,k}, \lambda^a_{i,k}(t^a)) \text{ and } \lambda^a_{i,k}(t^a) \in \mathscr{L}^+_k(S^b \times T^b);$$
$$s^a \in \mathbb{O}(G, \kappa^a_G(t^a)) \text{ and } \kappa^a_G(t^a) \in \mathscr{L}^+_k(S^b \times T^b);$$
$$(s^a, t^a) \in R^a_1(G, \mathfrak{T}_G).$$

Claim. Let $k > 0$, $i \leq p(k)$, and $G \in A_{i,k}$. Then

$$R^a_m(G_{i,k}, \mathfrak{U}_{i,k}) \cap (S^a \times T^a_k) = R^a_m(G, \mathfrak{T}_G) \cap (S^a \times T^a_k).$$

Proof of claim. We prove the result for a and b together by induction on m. The case $m = 1$ is proved in the preceding claim. We suppose the claim holds for m with b instead of a. Let $s^a \in S^a$ and $t^a \in D_{i,X,Y,k}$. Then the following are equivalent to $(s^a, t^a) \in R^a_{m+1}(G_{i,k}, \mathfrak{U}_{i,k})$.

$(s^a, t^a) \in R^a_m(G_{i,k}, \mathfrak{U}_{i,k})$ and $\lambda^b_{i,k}(t^a)$ assumes $R^b_m(G_{i,k}, \mathfrak{U}_{i,k})$;

$(s^a, t^a) \in R^a_m(G, \mathfrak{T}_G)$ and $\kappa^a_G(t^a)$ assumes $R^b_m(G, \mathfrak{T}_G)$;

$(s^a, t^a) \in R^a_{m+1}(G, \mathfrak{T}_G)$.

This proves our claim.

Now, take $k \geq M + 1$ and let $G \in \Theta$. Then $G \in A_{i,k}$ for some $i \leq p(k)$. We chose the type structure $\mathfrak{U}_{i,k}$ so that $R^a_\infty(G_{i,k}, \mathfrak{U}_{i,k}) \cap (S^a \times T^a_k) \neq \emptyset$. By the preceding claim, we have $R^a_\infty(G, \mathfrak{T}_G) \cap (S^a \times T^a_k) \neq \emptyset$, and similarly for b. The result now follows by Theorem 3.4. □

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