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ABSTRACT

We explore the identification of nonseparable models without relying on the property that the model can be inverted in the econometric unobservables. In particular, we allow for infinite dimensional unobservables. In the context of a demand system, this allows each product to have multiple unobservables. We identify the distribution of demand both unconditional and conditional on market observables, which allows us to identify several quantities of economic interest such as the (conditional and unconditional) distributions of elasticities and the distribution of price effects following a merger. Our approach is based on a significant generalization of the linear in random coefficients model that only restricts the random functions to be analytic in the endogenous variables, which is satisfied by several standard demand models used in practice. We assume an (unknown) countable support for the distribution of the infinite dimensional unobservables.

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1 Introduction

In this paper we study the identification of nonseparable demand systems

$$Q = D(P, \xi), \tag{1}$$

where Q is a vector of market level quantities demanded for a set of goods, P is a vector of prices for these goods, $\xi \in \Xi$ is a demand error, and $D(\cdot, \cdot)$ is the demand system. We do not impose that ξ is independent of P, as price may be determined in market equilibrium. Rather, we assume that the demand error ξ is independent of a vector of instruments Z. The non-separable demand system contrasts with models that assume the error is additively separable, i.e.,

$$D(P, \xi) = D(P) + \xi.$$

Although additive separability has convenient econometric features, there is little underlying economic basis for demand shocks being separable. For example, when market demand is the aggregation of individual discrete choices, the quantity sold of product j in a market is a function of the demand errors for product j and all competing products in the market, which is inconsistent with the additive structure above (Berry, 1994). Our interest thus centers on identification of the non-separable model (1).

There are two major existing literatures on the identification of nonseparable models with endogenous regressors. First, the literature on the nonparametric identification of simultaneous equation models takes as its starting point the assumption that the demand system (1) is invertible in the error ξ (Brown, 1983; Roehrig, 1988; Benkard and Berry, 2006; Matzkin, 2008; Berry and Haile, 2010; Berry et al., 2011). In contrast, we do not require invertibility. Invertibility in ξ requires, at a minimum, the order restriction that there are only as many unobservables as there are products demanded; the dimensionality of ξ equals the dimensionality of Q. Instead, we allow the error to be infinite dimensional and hence do not impose the structure that demand is invertible in the unobservable ξ. This allows for considerable richness in the unobserved market level heterogeneity of demand models, as we illustrate later using the standard discrete choice setting for demand for differentiated products (see e.g., Berry, Levinsohn and Pakes (1995) or BLP).

Another literature that can be applied to demand identification is the literature on nonparametric control functions (Altonji and Matzkin, 2005; Chesher, 2003; Imbens and Newey, 2009; Blundell and Matzkin, 2010). This literature can allow for multiple unobservables in the demand equation, and in particular more unobservables than the number of goods, but cannot allow for such multidimensional unobservability in the equilibrium pricing equations. This means adapting these approaches to demand models requires that strong assumptions be placed on the reduced form pricing equation that are at odds with standard supply side models: both the demand *and* supply side unobservables should impact equilibrium prices under standard mechanisms according to which prices are set. Thus allowing for multidimensional unobservables in demand requires that we also allow for multidimensional unobservables to affect prices in order to be consistent with standard economic theory.

As far as we are aware, the identification problem of introducing multiple unobservables per product in a standard supply and demand setting has yet to be studied. As discussed above, multiple unobservables preclude inverting demand, and thus the demand unobservables themselves (for any realization of a market) cannot be identified. Instead, we will seek identification of the *distribution* of demand functions, both conditional and unconditional on the realization of the market observables. We will assume that any realization of the demand function function $D(\cdot, \xi)$ for $\xi \in \Xi$ is a real analytic function in prices, which nonparametrically generalizes a property of various well known demand systems used in practice (e.g., the AIDS model, the mixed logit BLP model). The key to our approach is then showing that under a general mechanism that governs how prices are set (which allows for multidimensional unobservables that include the demand unobservable ξ, and is consistent with standard models of price equilibrium), the reduced form of the model takes on the form of a system of random analytic functions that are indexed by an infinite dimensional unobservable consisting of both the demand unobservables ξ and the supply unobservables ω. We first show this system of random analytic functions representing the reduced form of the model can be uniquely identified from the data. We then show that, given a minimal relevance condition on the instruments, the structural demand feature of interest, namely the distribution of demand functions, can be recovered from the reduced form.

Our approach can be seen as building upon the literature on the identification of random functions in a linear in random coefficients framework in order to study nonlinear models with multiple unobservables. This literature has established nonparametric identification of the distribution of a finite dimensional vector of linear random coefficients (Beran and Millar, 1994; Beran, 1995), and has modeled endogeneity via an auxiliary instrumental variables equation that is identified along with the equation of interest (Hoderlein et al., 2010). These approaches, however, fundamentally rely upon the linear functional form in both the outcome and auxiliary equations. Linearity of demand in the unobservables is a strong restriction (i.e., inconsistent with the BLP model), and the assumption that prices are also linear in the unobservables is even stronger as this does not arise readily from equilibrium assumptions even when demand is linear. We abstract from linearity and instead exploit the deeper property of analyticity, which is compatible with a rich array of possible nonlinear demand functions and standard equilibrium assumptions.[1]

Because we identify a distribution over random demand functions indexed by the infinite dimensional unobservable ξ, we impose the additional condition that the true underlying distribution of the unobservable has some ex-ante unknown countable support. This condition provides a general class of distributions with infinite support that does not require us to impose further regularity conditions on the space of unobservables Ξ. We discuss this condition in Section 5, where we outline a possible extension that would allow us to add continuous, additive unobservables to demand and prices.

2 Model

We lay out primitive assumptions concerning demand and supply. First, we define a real analytic function.

Definition 1. Let $\mathcal{X}$ be a non-empty open set in $\mathbb{R}^k$ for a given k. A function $g : \mathcal{X} \to \mathbb{R}$ is **analytic** if, given any interior point $w \in \mathcal{X}$, there is a power series in $x - w$ that converges to $g(x)$ for all x in

[1] The only other paper of which we are aware that has attempted to extend the identification of random coefficients from a linear to a nonlinear setting is Liu (1996). Liu does not consider the problem of infinite dimensional unobservables, non-finite support of the unobservables, or the effects of endogeneity, and thus the results are not applicable to our problem.

some neighborhood $U \subset \mathcal{X}$ of w.

We also define an extension of real analytic functions to vector valued functions.

Definition 2. Let $\mathcal{X}$ be a non-empty open set in $\mathbb{R}^k$ for a given k. A function $g : \mathcal{X} \to \mathbb{R}^l$ (for a given l) is **vector valued analytic** if each component function $g_i : \mathcal{X} \to \mathbb{R}$ for $i = 1, \ldots, l$ is real analytic.

We will exploit the property that two real analytic functions that are equal on an open set are equal everywhere (Krantz and Parks, 2002), which implies in a straightforward way that two vector valued real analytic functions exhibit the same property

Consider a population of markets. For simplicity we will assume that each possible market in the population has J products, but we could allow the number of products to vary across markets at the expense of complicating notation. For any market in the population, we observe the realizations of prices $P = (P_1, \ldots, P_J)$ and quantities $Q = (Q_1, \ldots, Q_J)$ and a vector of cost shifters $Z = (Z_1, \ldots, Z_J)$. Thus, the joint distribution of the observables (P, Q, Z) is identified from market data. A market is also characterized by two unobservables, a demand side unobservable $\xi \in \Xi$ and a cost side unobservable $\omega \in \Omega$, whose roles will now be explained.. All other exogenous demand shifters, such as observed consumer demographics and observed product characteristics other than price, are implicitly conditioned on in the background.

Assumption 1. *Demand is a function $D(\cdot, \xi) : \mathcal{P} \to \mathbb{R}^J$ for an open set $\mathcal{P} \subseteq \mathbb{R}^J$ that maps any possible vector of prices $p \in \mathcal{P}$ into a vector $q \in \mathbb{R}^J$ of quantities demanded. Different demand functions are indexed by different demand side unobservables $\xi \in \Xi$, and each $\xi \in \Xi$ gives a unique demand system $D(\cdot, \xi)$. For any realization $\xi \in \Xi$, the demand system $D(\cdot, \xi) = (D_1(\cdot, \xi), \ldots, D_J(\cdot, \xi))$ is such that $D(\cdot, \xi)$ is a vector valued analytic function.*

With this notation, each $D(\cdot, \xi)$ is a separate system of functions and knowing the distribution of ξ tells us the distribution of demand systems. The key restriction in the assumption is that demand is an analytic function of prices. Analyticity of market demand is a nonparametric generalization of the linear random coefficients model in Beran and Millar (1994) and Hoderlein et al. (2010) as well as other standard demand systems that are used in practice, such as the AIDS demand system of

Deaton and Muellbauer (1980) and the mixed logit demand used in BLP. The generality afforded by the analyticity assumption allows Ξ to be of arbitrary dimension. In particular, it can be infinite dimensional as opposed to the standard restriction that the demand unobservable has dimension equal to the number of products J.

Remark 1. We consider the case where there is one endogenous price for each product. Our results extend naturally to the case where demand is a function of multiple endogenous variables. Of course, we would need one excluded instrument for each endogenous regressor.

Remark 2. We assume Ξ is such that each $\xi \in \Xi$ gives rise to a unique demand system $D(\cdot, \xi)$. If it were to be the case that $D(p, \xi_1) = D(p, \xi_2)\ \forall p \in \mathcal{P}$, then we could redefine Ξ to exclude one of ξ_1 or ξ_2. In other words, each $\xi \in \Xi$ indexes a unique vector valued analytic function.

Example 1. An important example of a demand system satisfying our assumptions is the mixed logit demand system used in BLP. Abstracting away non-price product characteristics, the market demand equation in this model is

$$q_j = M \int_\beta \frac{\exp\left(p_j \beta + \tilde{\xi}_j\right)}{\sum_{j'=1}^{J} \exp\left(p_{j'} \beta + \tilde{\xi}_{j'}\right)} dH(\beta), \tag{2}$$

where M is the mass of consumers, β is a random coefficient on price, H is the distribution of random coefficients within a market and $\tilde{\xi}_j$ is a scalar random error specific to product j in a particular market. BLP assume that H is the same across markets; indeed H is their structural object of estimation. This rules out that tastes for price, or more generally product characteristics, vary across markets. It is plausible that the distribution of tastes does vary across markets. In this case, the shift in the distribution of tastes across markets will cause corresponding changes in equilibrium prices, making prices a function of the market-specific realization of the distribution H.

If we allow the distribution H to vary across markets, it becomes a market-specific random variable, and thus H properly belongs in our potentially infinite-dimensional error term $\xi = \left(\tilde{\xi}_1, \ldots, \tilde{\xi}_J, H\right)$. If each H is restricted to take on compact support in the space of β's, the mixed logit demand system in (2) will be analytic in prices p (Stinchcombe and White, 1998, page 318).

Assumption 2. *For each realization of $\xi \in \Xi$ and the supply side unobservable $\omega \in \Omega$, there exists a unique reduced form pricing function $P(\cdot, (\omega, \xi)) : \mathcal{Z} \to \mathcal{P} \subseteq \mathbb{R}^J$ for an open set $\mathcal{Z} \subseteq \mathbb{R}^J$ that maps a vector of cost shifters $z \in \mathcal{Z}$ into prices $p \in \mathcal{P}$ such that each $P(\cdot, (\omega, \xi))$ is a vector valued analytic function in z. Furthermore, for each realization of $(\xi, \omega) \in \Xi \times \Omega$, there exists $z \in \mathcal{Z}$ such that the Jacobian of the pricing function $P(\cdot, (\omega, \xi))$ is full rank at z.*

Uniqueness of the reduced form is a standard assumption in the literature on simultaneous equations models (Brown, 1983; Roehrig, 1988; Benkard and Berry, 2006; Matzkin, 2008; Berry and Haile, 2010; Berry et al., 2011). The control function literature typically restricts the pricing equation to be invertible (see e.g., Imbens and Newey (2009)), which requires that the supply side error $\omega \in \Omega$ have the same dimension as the number of products, and does not allow the demand error ξ to separately affect prices (see Kim and Petrin (2010) for an application of these assumptions to demand estimation). These restrictions are inconsistent with standard economic theory, which predicts that both the supply and demand errors have implications for equilibrium prices. Our pricing assumption avoids this problem and is compatible with the supply and demand unobservables $\omega \in \Omega$ and $\xi \in \Xi$ both entering the mechanism that determines prices without any dimensionality restrictions. Analyticity of the reduced form pricing equation can be shown to follow given the assumed analyticity of demand in Assumption 1 and the analyticity of costs under a standard model of imperfect competition, as we now describe.

Example 2. Consider single-product firms that set prices according to Bertrand-Nash oligopoly theory. In other words, firm j sets prices to solve its first order condition

$$c_j(z) = p_j + D_j(p, \xi) \frac{\partial D_j(p, \xi)}{\partial p_j},$$

where $\omega = (c_1(\cdot), \ldots, c_J(\cdot))$ is a vector of J real analytic marginal cost functions for each market. If $D(p, \xi)$ is a vector valued analytic function of p, the derivative $\frac{\partial D_j(p,\xi)}{\partial p_j}$ is a real analytic function. Assume that for any realization of (ω, ξ) and $z \in \mathcal{Z}$, there exists a unique equilibrium price $P(z, (\omega, \xi))$ that also is the unique price that solves the above system of first order equations. Then the analytic implicit theorem (see Krantz and Parks (2002)) can be used to show that $P(z, (\omega, \xi))$ is a vector valued analytic function in $z \in \mathcal{Z}$ for any realization of the unobservables $(\xi, \omega) \in \Xi \times \Omega$, i.e., the equilibrium pricing function $P(z, (\omega, \xi))$ is everywhere analytic and hence satisfies the requirement of Assumption 2. This same argument can be adapted to multiproduct firms in a straightforward way. Further assumptions on equilibrium selection are required to address models with multiple equilibria.

The full rank part of Assumption 2 is simply a relevance assumption on the instruments for shifting prices, and unlike other nonparametric relevance conditions (the completeness condition in Newey and Powell (2003) or the measurable separability assumption in Florens et al. (2008)), its validity can be traced back directly to natural assumptions on the supply and demand model. Without this assumption on the instrument, there exist realizations of the unobservables (ξ, ω) such that the instruments could not shift prices in a full rank way anywhere, which might lead to underidentification. Our assumption on the relevance of the instrument is simply designed to avoid this pathology.

We now turn to our final assumption, which concerns the stochastic nature of the random elements of the model. Let $\Delta(\Xi \times \Omega)$ denote the set of distributions over $\Xi \times \Omega$ that have countable supports. That is, each $G \in \Delta(\Xi \times \Omega)$ has a support over some possibly infinite but countable subset $\left\{(\xi_1^G, \omega_1^G), (\xi_2^G, \omega_2^G), \ldots\right\} \subset \Xi \times \Omega$. Observe that for $G, G' \in \Delta(\Xi \times \Omega)$, G and G' can have completely non-overlapping supports – the only restriction is that each distribution's support is countable.

Assumption 3.

- *The support of the cost shifters Z contains a countably dense subset of $\mathcal{Z}$.*
- $(\xi, \omega) \perp Z$ *(full independence)*
- *The true distribution G^0 of the market unobservables $(\xi, \omega) \in \Xi \times \Omega$ is such that $G^0 \in \Delta(\Xi \times \Omega)$.*

We discuss countable support of the regressors and the market unobservables in Section 5. We emphasize here that the support of the true underlying distribution is unknown to the researcher and hence learned in identification. We do not need to impose that $\Xi \times \Omega$ is compact or even choose a particular topology for $\Xi \times \Omega$. Full independence between instruments and demand and supply errors is a common assumption when there are no restrictions on how the errors enter the model. We can also extend our results to allow more excluded instruments than endogenous regressors. We do not require large support on Z because of the analyticity assumption on the demand and pricing functions.

We now state our main result.

Theorem 1. *Under Assumptions 1–3, given the joint distribution of the market observables* (P, Q, Z)*, the true distribution over demand functions* $D(\cdot, \xi)$*, both unconditional and conditional on any particular realization of the observables* $(P, Q, Z) = (p, q, z)$*, is identified.*

Much of the remainder of the paper now explains the key ideas that are needed to prove this result and show its applicability to demand analysis.

3 Identification of Random Functions

There are two key elements to how we prove Theorem 1. First, we show that the reduced form of the model gives rise to a system of random analytic functions, and that this reduced form can be identified from the distribution of the market observables (P, Q, Z). This requires us to extend the existing literature on identification in linear random coefficient models to allow for random functions that are nonlinear. The second key element is showing that the structural feature of interest, namely the distribution of demand systems, can be recovered from the identified reduced form.

To establish the first key element, let $\theta = (\xi, \omega) \in \Theta$ index the joint realization of the demand and pricing unobservables. Observe that for each realization of $\theta \in \Theta$, the model generates a reduced form mapping from $z \mapsto (p, q)$, i.e., a unique mapping from the vector of instruments to the endogenously determined prices and quantities (p, q). Let this mapping be denoted by $g(z, \theta) = (g_1(z, \theta), g_2(z, \theta))$, where

$$
\begin{aligned}
q = g_2(z, \theta) &= D\left(P(z, (\xi, \omega)), \xi\right) \\
p = g_1(z, \theta) &= P(z, (\xi, \omega)).
\end{aligned}
$$

Observe that $g : \mathcal{Z} \to \mathbb{R}^{2J}$ is a vector valued analytic function, since the composition of analytic maps is analytic. Let $\mathcal{A}$ denote the family of vector valued analytic functions from $\mathcal{Z}$ to $\mathbb{R}^{2J}$. For each realization of $\theta = (\xi, \omega)$, the reduced form $g(\cdot, \theta)$ is thus an element of $\mathcal{A}$. We can refer to each element $f \in \mathcal{A}$ as a "type", i.e., a type characterizes a particular mapping from cost shifters $z \in \mathcal{Z}$ to the endogenous prices and quantities (p, q). The true distribution of the unobservables (ξ, ω) induces a distribution μ^0 over $\mathcal{A}$. Let $\Delta(\mathcal{A})$ denote the set of probability distributions over $\mathcal{A}$ with countable support. Observe that under Assumption 3, we have that the true $\mu^0 \in \Delta(\mathcal{A})$. For a generic $\mu \in \Delta(\mathcal{A})$, let $T^\mu \subset \mathcal{A}$ denote its support. We discuss the countable support restriction further in Section 5.

Observe that the reduced form of the structural model consists of a distribution $\mu^0 \in \Delta(\mathcal{A})$ over vector valued analytic functions $\mathcal{A}$. We now show that the reduced form is identified, the objects of identification being the the identities of the types $T^0 = \{f_1^0, f_2^0, \dots\} \subset \mathcal{A}$ and their masses $\mu^0\left(f_i^0\right)$ for $i = 1, 2, \dots$ In order to identify the reduced form, we must first extend the existing literature on identification of random coefficients in linear models to allow more generally for nonlinearities. Because linear functions are analytic, our space of random functions $\mathcal{A}$ represents such a generalization. We first formally describe the meaning of identification in our setting.

By Assumption 3, the distribution of the explanatory variables Z is such that Z has support equal to a countably dense subset of $\mathcal{Z}$, and we can restrict attention to variation in Z within any closed cube $C = [a_1, b_1] \times \cdots \times [a_J, b_J]$ that is contained in $\mathcal{Z}$. Let $\mathcal{X}$ denote the support of Z within C – observe that $\mathcal{X}$ is also countably dense in C. For example, $\mathcal{X}$ could be the intersection of $\mathbb{Q}^J$ (the set of J-tuples with rational valued components) with the cube C. Observe that we only require the $\mathcal{X}$ to be a countably infinite set, and thus the cardinality of the support of Z can be the same as the cardinality of the support T^0 of the random functions. For the sake of concreteness, we will assume that $\mathcal{X}$ takes on this form.

Also by assumption, the exogenous variables Z and the random coefficients $f \in \mathcal{A}$ are independent of one another. Thus for any distributions $\mathbb{P}_Z$ of Z and and $\mu \in \Delta(\mathcal{A})$, let $\mathcal{L}[\mathbb{P}_Z, \mu]$ denote the joint distribution of the endogenous and exogenous variables (P, Q, Z) implied by the model. In particular, for measurable subsets $A \subseteq \mathbb{R}^{2J}$ and $B \subseteq \mathcal{X}$,

$$\mathcal{L}[\mathbb{P}_Z, \mu](A, B) = \sum_{z \in B} \sum_{f \in T^\mu} \mathbf{1}[f(z) \in A]\, \mu(f) \mathbb{P}_Z(z). \tag{3}$$

We follow Beran (1995) and define identification as follows.

Definition 3. The true distribution μ^0 is *identified* in a class of measures $\Delta(\mathcal{A})$ if both $\mu^0 \in \Delta(\mathcal{A})$, and for $\mu, \mu' \in \Delta(\mathcal{A})$, $\mathcal{L}[\mathbb{P}_Z, \mu] \neq \mathcal{L}[\mathbb{P}_Z, \mu']$, meaning there exists measurable subsets $A \subseteq \mathbb{R}^{2J}$ and $B \subseteq \mathcal{X}$ such that $\mathcal{L}[P_X, \mu](A, B) \neq \mathcal{L}[P_X, \mu'](A, B)$.

Intuitively, identification states that any distribution $\mu \neq \mu^0$ implies a different distribution over the observables (P, Q, Z) than the truth μ^0. Because the distribution over the observables (P, Q, Z) is identified in the data, the true $\mu^0 \in \Delta(\mathcal{A})$ can be uniquely inferred from the data.

Theorem 2. *The true probability measure μ^0 is identified in the set $\Delta(\mathcal{A})$.*

Consider any two $\mu, \nu \in \Delta(\mathcal{A})$. Let $T \subset \mathcal{A}$ be the union of T^μ and T^ν, which is countable because the union of two countable sets is countable. Letting $T = \{f_1, f_2, \dots\}$ and $\alpha_i = \mu(f_i) - \nu(f_i)$, identification according to Definition 3 is equivalent to finding the existence of an $A \subseteq \mathbb{R}^{2J}$ and $B \subseteq \mathcal{X}$ such that

$$\sum_{x \in B} \sum_{i=1}^{\infty} \alpha_i \mathbf{1}[f_i(x) \in A]\ \mathbb{P}_Z(z) \neq 0. \tag{4}$$

Because μ and ν are assumed to be distinct, then $\alpha_i \neq 0$ for at least one $i \geq 1$, and thus without loss of generality we let $\alpha_1 \neq 0$. The key to showing the existence of subsets A and B that play the role of the "identifying sets" in (4) is the following lemma concerning the class of functions $\mathcal{A}$.

Lemma 1. *For any countable subset of (vector valued analytic) functions $S = \{f_1, f_2, \dots\} \subset \mathcal{A}$, there exists a $z \in C = [a_1, b_1] \times \cdots \times [a_J, b_J]$ such that $f_1(z) \neq f_j(z)$ for all $j > 1$.*

Proof. The first step in the proof is to show that for distinct $f, g \in \mathcal{A}$, the set $D = \{z \in C : f(z) = g(z)\}$ is a nowhere dense set. Observe that since both f and g are continuous, the set D is closed. Now suppose by way of contradiction that D is somewhere dense. Then D has a non-empty interior, and hence there is a non-empty open set of the form $U = \times_{i=1}^{J}(a_i, b_i) \subset C$ on which f and g coincide. But because $\mathcal{A}$ is comprised of vector valued analytic functions, this would imply that $f = g$, thus contradicting the fact that they are distinct.

Now let $E = \cup_{i \geq 2}\{z \in C : f_1(z) = f_i(z)\}$. The Baire category theorem implies that its complement in C, E^c, is non-empty. Any point $z \in E^c$ satisfies the condition of the lemma. $\square$

Now we can prove Theorem 2.

Proof of Theorem 2

According to the previous lemma, there exists a $\hat{z} \in C$ such that $f_1(\hat{z}) \neq f_j(\hat{z})$ for any $j > 1$. Let $\hat{y} = f_1(\hat{z})$. Let A_ϵ denote an $\epsilon \geq 0$ open ball centered at $\hat{y}$. Now define the functions

$$h_i(z, \epsilon) = \mathbf{1}\left[f_i(z) \in A_\epsilon\right],$$

which are defined for any $i \geq 1$ and $z \in \mathcal{Z}$ and $\epsilon \geq 0$. Also define

$$H(z, \epsilon) = \sum_{i=1}^{\infty} \alpha_i h_i(z, \epsilon).$$

Because $\sum_{i \geq 1} \alpha_i$ is an absolutely convergent series, it is straightforward to show (via the Weierstrass M-test) that $H(z, \epsilon)$ is a uniformly convergent series over all possible values (z, ϵ). By the uniform limit theorem, if there is any point (z, ϵ) at which all the h_i are continuous, then H will also be continuous at this point (z, ϵ).

By construction we have that

$$H(\hat{z}, 0) = \alpha_1. \tag{5}$$

Without loss of generality let $\alpha_1 > 0$. Observe also that each $h_i(\hat{z}, \epsilon)$ for $i \geq 1$ is continuous at $\epsilon = 0$, and hence $H(\hat{z}, \epsilon)$ is continuous at $\epsilon = 0$.[2] Hence there exists $\bar{\epsilon} > 0$ such that $H(\hat{z}, \epsilon) > 0$ for all $\epsilon < \bar{\epsilon}$. Letting $d(y_1, y_2)$ denote the denote the distance between two points in $\mathbb{R}^{2J}$, we can choose a $\tilde{\epsilon} \leq \bar{\epsilon}$ such that $\tilde{\epsilon} \neq d\left(f_1(\hat{z}), f_j(\hat{z})\right)$ for any $j \geq 1$.[3] Finally, observe that each $h_i(z, \tilde{\epsilon})$ for $i \geq 1$ is continuous at $\hat{z}$ and hence $H(z, \tilde{\epsilon})$ is continuous at $z = \hat{z}$. Thus for all $z \in B$ where B is a sufficiently small open neighborhood of $\hat{z}$ we have that $H(z, \tilde{\epsilon}) > 0$. By the definition of H, we have that

$$\sum_{i=1}^{\infty} \alpha_i \mathbf{1}\left[f_i(z) \in A_{\tilde{\epsilon}}\right] > 0$$

[2] To show continuity of $H(\hat{z}, \epsilon)$, consider that $h_1(\hat{z}, \epsilon)$ is always 1 around $\epsilon = 0$ and $h_i(\hat{z}, \epsilon)$ is always 0 around $\epsilon = 0$ for $i > 1$.

[3] This follows because $\{d\left(f_1(\hat{z}), f_j(\hat{z})\right) \mid j \geq 1\}$ is at most a countable set and the interval $[0, \bar{\epsilon}]$ is uncountable.

for each $z \in B$. Because the support $\mathcal{X}$ of Z is dense in the cube C, the intersection of the neighborhood B and the support $\mathcal{X}$ is non-empty, and thus

$$\sum_{z \in B \cap \mathcal{X}} \sum_{i=1}^{\infty} \alpha_i \mathbf{1}\left[f_i(z) \in A_{\hat{\epsilon}}\right] \mathbb{P}_Z(z) > 0,$$

and (4) is satisfied.

Proof of Theorem 1

We have now established that the reduced form of the model, i.e., a system of random analytic functions, is identified from the data. We now wish to show that the structural features of interest, i.e., the distribution of demand, is identified from the reduced form. Identification of the reduced form gives identities of the types $T^0 = \{f_1^0, f_2^0, \dots\} \subset \mathcal{A}$ and their masses $\mu^0\left(f_i^0\right)$ for $i = 1, 2, \dots$, where each type $f_i^0 : \mathcal{Z} \to \mathbb{R}^{2J}$ represents a mapping from the cost shifters $z \in \mathcal{Z}$ to prices and quantitates $(p, q) \in \mathbb{R}^{2J}$. By assumption, each $f_i^0 = g\left(\cdot, (\xi_i, \omega_i)\right)$ for some realization of $(\xi_i, \omega_i) \in \Xi \times \Omega$. The key to recovering the structural features of interest from the reduced form is the following lemma, which shows that each realization of the random function $f_i^0 \in \mathcal{A}$ is consistent with at most one demand unobservable $\xi_i \in \Xi$, i.e. the demand unobservable can be inverted $\xi\left(f_i^0\right) \in \Xi$ as a function of any realization f_i^0 of the reduced form random function.

Lemma 2. *For any realization of $f_i^0 \in \mathcal{A}$, there is at most one $\xi_i \in \Xi$ such that $f_i^0 = g\left(\cdot, (\xi_i, \omega_i)\right)$ for some $\omega_i \in \Omega$.*

Proof. To see this, partition $f_i^0 = \left(f_{i,1}^0, f_{i,2}^0\right)$ where $q = f_{i,1}^0(z) \in \mathbb{R}^J$ predicts quantities and $p = f_{i,2}(z) \in \mathbb{R}^J$ predicts price for any $z \in \mathcal{Z}$. By Assumption 2 and the fact each $f_{i,2}^0$ is smooth, there exists an open subset $U \subseteq \mathcal{Z}$ such that $f_{i,2}^0$ has a full rank Jacobian for each $z \in U$. Therefore $f_{i,2}^0(W) \subseteq \mathcal{P}$ is an open set of prices by the open mapping theorem. Now suppose there exists two $\xi, \xi' \in \Xi$ such that $f_{i1}^0(z) = D\left(f_{i,2}(z), \xi\right) = D\left(f_{i,2}(z), \xi'\right)$ for all $z \in U$. Then it would be the case that $D(\cdot, \xi)$ and $D(\cdot, \xi')$ coincide on an open set. Because ξ and ξ' index different vector valued analytic functions, by the key property of analytic functions, we must thus have that $\xi = \xi'$. $\square$

It is now possible, in light of Theorem 2, to prove Theorem 1. For each identified type f_i^0 from the reduced form, let $\xi\left(f_i^0\right) \in \Xi$ be the unique demand unobservable consistent with it as implied by Lemma 2. Now observe that

$$\begin{aligned}
\Pr\left(\xi = \xi^*\right) &= \sum_{i \geq 1} \mu^0\left(f_i^0\right) \mathbf{1}\left[\xi\left(f_i^0\right) = \xi^*\right] \\
\Pr\left(\xi = \xi^* \mid (P, Q, Z) = (p, q, z)\right) &= \frac{\sum_{i \geq 1} \mu^0\left(f_i^0\right) \mathbf{1}\left[\xi\left(f_i^0\right) = \xi^*\right] \mathbf{1}\left[(p, q) = f_i^0(z)\right]}{\sum_{i \geq 1} \mu^0\left(f_i^0\right) \mathbf{1}\left[(p, q) = f_i^0(z)\right]},
\end{aligned}$$

where $\Pr\left(\xi = \xi^*\right)$ is the unconditional probability that the demand unobservable ξ equals a particular value ξ^*, and $\Pr\left(\xi = \xi^* \mid (P, Q, Z) = (p, q, z)\right)$ is the probability ξ equals ξ^* conditional on a particular realization of the market observables (p, q, z). For any possible realization of demand $\xi^* \in \Xi$, Theorem 1 allows us to identify both the unconditional and conditional (for any realization of the market observables (p, q, z)) probability of the demand unobservable being ξ^*.

4 Uses of Demand Functions

Some of the major structural uses of demand functions are to measure price elasticities for antitrust purposes and to predict the sales from a good with changed characteristics. We show how our identification result applies to these purposes. We also discuss uses of demand functions that require more supply side information, such as predicting prices and quantities after a merger.

Consider a situation where we want to measure price elasticities for product j in a particular geographic market with observables (p, q, z). The mean (across the econometrician's uncertainty) own-price demand derivative for product j in this market is

$$\int_{\xi^\star} \frac{\partial D_j\left(p, \xi^\star\right)}{\partial p_j} \Pr\left(\xi = \xi^* \mid p, q, z\right) d\xi^\star.$$

One can similarly calculate the variance and indeed the entire distribution of the own-price demand derivative (or elasticity) for product j. In demand estimation approaches involving inverting an error term, identification would give $\Pr\left(\xi = \xi^* \mid p, q, z\right) = 1$ for some ξ^*, and thus there would be no uncertainty over the own-price demand derivative for each market. The difference is that one cannot learn ξ^* when ξ has dimension more than the number of products. It is quite natural that we cannot ascertain exactly whether the price and sales of product j are high due to a low dislike of price, a high unobserved quality, or possibly other explanations. Our approach captures the econometrician's uncertainty about the true demand derivative in the market.

Now consider an example where non-price product characteristics x enter demand and supply. Let $x = (x_1, \ldots, x_J)$ be the vector of product characteristics for all J choices, where each x_j is itself a vector. Let $D(p, x, \xi)$ be the demand function. Say that we want to predict demand when the characteristics of product 1 change from x_1 to x_1'. Let $x' = (x_1', x_2, \ldots, x_J)$. Then $D(p, x', \xi)$ is the demand for all products at the new product characteristics. Across the population of markets,

$$\int_{\xi^\star} D_1(p, x', \xi^\star) \Pr(\xi = \xi^*) \, d\xi^\star$$

is the unconditional mean of predicted sales for product 1. This mean can also be computed conditional on a particular market's characteristics (p, q, z, x).

Our approach does not recover the distribution of supply unobservables ω without further assumptions on the supply side. We can make such assumptions. Say we assume that each firm j sets prices to maximize profits

$$\pi_j = D_j(p, \xi)(p_j - c_j(z)),$$

where $c_j(z)$ is the marginal cost of product j, which isa part of ω and a function of the instruments z. We do not focus on counterfactuals involving changing z here, so we let $c_j = c_j(z)$. Profit maximization leads to the first order condition that

$$\frac{\partial D_j(p, \xi)}{\partial p_j}(p_j - c_j) + D_j(p, \xi) = 0,$$

which implies

$$c_j = p_j + D_j(p, \xi)\left(\frac{\partial D_j(p, \xi)}{\partial p_j}\right)^{-1}.$$

Therefore, each realization of ξ leads to a value of c_j for each firm j. So, with this additional structure, we identify a joint distribution of marginal costs $c = (c_1, \ldots, c_J)$ and demand errors ξ. Now say we are interested in a merger between firms 1 and 2. We can use the theory of Bertrand-Nash pricing by multiproduct firms to predict prices under each combination of ξ and c, at least if the pricing game has a unique equilibrium. We can then integrate using the distribution of (ξ, c) to calculate the mean or variance of counterfactual prices and quantities after the merger.

5 Countable Support

A key condition that we use is the (unknown) countably infinite support of the distribution G of the market unobservables $(\xi, \omega) \in \Xi \times \Omega$. This class of distributions is among the most general that one can use without imposing further structure on the space of the unobservables $\Xi \times \Omega$. Thus we allow for a countably infinite support of the distribution of unobservables in a possibly infinite dimensional space that does not impose that $\Xi \times \Omega$ has a particular topology or that $\Xi \times \Omega$ is compact. We only require variation of the instrument z in a countable set (the rationals) as well, and thus the cardinalities of the support of the data and the support of the unobservables are the same.

Our main interest lies in identification that imposes minimal structure on the space of unobservables $\Xi \times \Omega$, which motivates the use of distributions with countably infinite support as the class of distributions over which we seek identification. Considering more general classes of distributions introduces measurability problems that would require putting more structure on $\Xi \times \Omega$ to resolve.[4] Working with distributions that admit countable support avoids these measurability issues in a general way given the economic question, while not putting any structure on $\Xi \times \Omega$.

An alternative research question is to restrict the space of the unobservables $\Xi \times \Omega$ to address these measurability issues and consider instead some other class of distributions that is perhaps non-nested with the space of countable distributions. While we fully expect future work to move in this direction, Ackerberg, Hahn and Ridder (2010) present an example that suggests such an alternative inquiry

[4] For example, defining the class of all probability measures on a real space introduces paradoxes (e.g., the Banach-Tarski paradox). Lebesgue measure resolves this paradox, but it is well known that there is no analog of Lebesgue measure on an infinite dimensional Banach space.

could prove difficult. The example in Ackerberg et al. presents a function $y = h_1(x, \epsilon_1, \epsilon_2)$, where h is known and analytic in x and ϵ_1 and ϵ_2 are scalar, real-valued unobservables. This function h and a particular, continuous distribution for ϵ_1 and ϵ_2 give rise to a particular cumulative distribution function $F(y \mid x)$. By inverting this cumulative distribution function, Ackerberg et al. show that another analytic function $h_2(x, \nu)$ that is non-nested with h_1 and a uniform distribution over the scalar ν give rise to the same distribution for the data $F(y \mid x)$. Thus, the same data $F(y \mid x)$ can be explained by two continuous distributions over analytic functions, one putting support only on functions $h_1(x, \epsilon_1, \epsilon_2)$, indexed by ϵ_1 and ϵ_2, and the other putting support only on $h_2(x, \nu)$, indexed by ν. Positive results in this direction would thus require further substantive restrictions on $\Xi \times \Omega$, for example restrictions on the space of demand functions.

Some researchers may be uncomfortable with the assumption of countable support, because it implies that $F(p, q \mid z)$ has countable support, where F is the distribution function of prices and quantities. Uncirculated results by Kitamura (2011) explore a finite mixture model where each regression component i corresponds to a $f_i(x) + \epsilon_i$, where $\epsilon_i \sim F_i$, where $f_i(x)$ is a regression function and F_i is a cumulative distribution function with continuous support. Letting d_i be the weight on each function, Kitamura identifies $(d_i, f_i, F_i)_{i=1}^{N}$, where N is a finite number of mixture components. Kitamura requires assumptions that can be showed to be implied by analytic functions. Thus another possible extension of our results is to apply Kitamura's mixtures theorem for random functions to our supply and demand setup to identify demand with finite support for the unobservables ξ (as opposed to the possibly infinite support we consider) while allowing an extra, additively separable continuous error in prices and demand. Then $F(p, q \mid z)$ will have continuous support.

6 Conclusions

We explore the identification of distributions of demand functions with endogenous prices. Our approach does not involve the inversion of one error ξ_j for each product. Thus, we can allow the market-level errors ξ and ω to be of high finite dimension or even of infinite dimension. In the context of BLP demand systems, we can allow the distribution of preferences to vary across markets in addition to allowing unobserved product characteristics to vary across products within a market. We can allow multiple unobserved characteristics per product. Unlike the control function literature, we require conditions on pricing functions that can be established by primitive assumptions on the supply side of the market.

Our two main technical assumptions are 1) our demand and pricing functions are analytic and 2) the true distribution of unobserved demand and pricing errors has an (unknown) countable support. Analytic demand arises from some functional forms, such as the mixed logit with a compact support for random coefficients. Given analytic demand, analytic pricing functions arise from the analytic implicit function theorem if the pricing equilibrium, for example, is the unique solution to the first order conditions.

We recover a distribution of the demand unobservables ξ and the pricing unobservables η. These can be used to find the distribution of elasticities, across the population of markets or conditional on the observables in a given market. We can also extend our model to forecast the sales of a good with changed product characteristics. Finally, one can impose a particular supply side model and recover supply side unobservables for each demand side unobservable. The model can then be used to predict prices and sales after counterfactual changes to the supply side, such as a merger.

We have not discussed estimation. Our model gives rise to a likelihood for price and quantity as a function of the unknown distribution of demand and pricing functions. Therefore, we could apply mixtures estimators found in the literature, such as NPMLE, the EM algorithm, MCMC, and simulated maximum likelihood. For one computationally simple mixtures estimator, Fox and Kim (2011) present a consistency theorem that shows that the estimated distribution function converges to the true distribution function in the Lévy-Prokhorov metric on the space of multivariate distributions that take on positive support on a compact real space of random coefficients. Using this nonparametric consistency theorem requires showing that the distribution of random coefficients is identified, which our paper has done for the more general infinite dimensional case in Theorem 1.

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