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# TESTING CONDITIONAL FACTOR MODELS

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**ABSTRACT**

Using nonparametric techniques, we develop a methodology for estimating conditional alphas and betas and long-run alphas and betas, which are the averages of conditional alphas and betas, respectively, across time. The tests can be performed for a single asset or jointly across portfolios. The traditional Gibbons, Ross, and Shanken (1989) test arises as a special case of no time variation in the alphas and factor loadings and homoskedasticity. As applications of the methodology, we estimate conditional CAPM and multifactor models on book-to-market and momentum decile portfolios. We reject the null that long-run alphas are equal to zero even though there is substantial variation in the conditional factor loadings of these portfolios.



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# 1 Introduction

Under the null of a factor model, an asset's expected excess return should be zero after controlling for that asset's systematic factor exposure. Traditional regression tests of whether an alpha is equal to zero, like the widely used Gibbons, Ross, and Shanken (1989) test, assume that the factor loadings are constant. However, there is overwhelming evidence that factor loadings, especially for the standard CAPM and Fama and French (1993) models, vary substantially over time. Factor loadings exhibit variation even at the portfolio level (see, among others, Fama and French, 1997; Lewellen and Nagel, 2006; Ang and Chen, 2007). Time-varying factor loadings can distort standard factor model tests for whether the alphas are equal to zero and, thus, render traditional statistical inference for the validity of a factor model to be possibly misleading.

We introduce a methodology to estimate time-varying alphas and betas in conditional factor models. Conditional on the realized alphas and betas, our factor specification can be regarded as a regression model with changing regression coefficients. We impose no parametric assumptions on the nature of the realized time variation of the alphas and betas and estimate them nonparametrically based on techniques similar to those found in the literature on realized volatility (see, e.g., Foster and Nelson, 1996; Andersen et al., 2006).[1] We also develop estimators of the long-run alphas and betas, defined as the averages of the conditional alphas or factor loadings, respectively, across time. Our estimators are highly robust due to their nonparametric nature.

Based on the conditional and long-run estimators, we propose short- and long-run tests for the asset pricing model. Major advantages of our estimators and tests are that they are straightforward to apply, powerful, and involve no more than running a series of kernel-weighted OLS regressions for each asset. The tests can be applied to a single asset or jointly across a system of assets. In the special case where betas are constant and there is no heteroskedasticity, our long-run tests for whether the long-run alphas equal zero are asymptotically equivalent to Gibbons, Ross, and Shanken (1989).

We analyze the estimators and tests in a continuous-time setting where the conditional alphas and betas can be thought of as the instantaneous drift of the assets and the covariance between asset and factor returns, respectively. All our estimations, however, are of discrete-

[1] Other papers in finance developing nonparametric estimators include Stanton (1997), Aït-Sahalia (1996), and Bandi (2002), who estimate drift and diffusion functions of the short rate. Bansal and Viswanathan (1993), Aït-Sahalia and Lo (1998), and Wang (2003) characterize the pricing kernel by nonparametric estimation. Brandt (1999) and Aït-Sahalia and Brandt (2007) present applications of nonparametric estimators to portfolio choice and consumption problems.

time models and so our methodology is widely applicable to the majority of empirical asset studies which estimate factor models in discrete time. As is well known from the literature on drift and volatility estimation in continuous time, one can learn about volatilities or covariances from data for any fixed time span with increasingly dense observations, while pinning down the drift requires a long span of data (see e.g. Merton, 1980). We obtain similar results in our setting: the conditional beta estimators are consistent under very weak restrictions on the data-generating process for fixed time spans, while the conditional alpha estimators are, in general, inconsistent.

Whereas a large number of applied papers implement rolling-window estimators of conditional alphas and use them in statistical inference (see the summary by Ferson and Qian, 2004), under a continuous-time model we show that conditional alphas cannot be estimated consistently. However, we demonstrate that under additional restrictions on the model involving certain time normalizations of the model parameters, a formal asymptotic theory of the conditional alpha estimators can in fact be developed. These additional assumptions supply conditions under which the popular rolling-window conditional alphas have well-defined (asymptotic) distributions, but the required time normalization is economically counterintuitive when interpreted in the context of a continuous-time factor model. In contrast to the conditional alphas, the long-run alphas are identified from data without any time normalizations. We develop an asymptotic theory for our long-run alpha and beta estimators, which converge at standard rates as found in parametric diffusion models.

Our approach builds on a literature advocating the use of short windows with high-frequency data to estimate time-varying second moments or betas, such as French, Schwert, and Stambaugh (1987) and Lewellen and Nagel (2006). In particular, Lewellen and Nagel estimate time-varying factor loadings and infer conditional alphas. In the same spirit of Lewellen and Nagel, we use local information to obtain estimates of conditional alphas and betas without having to instrument time-varying factor loadings with macroeconomic and firm-specific variables.[2] Our work extends this literature in several important ways.

First, we provide a formal distribution theory for conditional and long-run estimators which the earlier literature did not provide. For example, Lewellen and Nagel's (2006) procedure identifies the time variation of conditional betas and provides period-by-period estimates of

[2] The instrumental variables approach is taken by Shanken (1990) and Ferson and Harvey (1991), among others. As Ghysels (1998) and Harvey (2001) note, the estimates of the factor loadings obtained using instrumental variables are very sensitive to the variables included in the information set. Furthermore, many conditioning variables, especially macro and accounting variables, are only available at coarse frequencies.

conditional alphas on short, fixed windows equally weighting all observations in that window. We show this is a special case (a one-sided filter) of our general estimator and so our theoretical results apply. Lewellen and Nagel further test whether the average conditional alpha is equal to zero using a Fama and MacBeth (1973) procedure. Since this is nested as a special case of our methodology, we provide formal arguments for the validity of this procedure. We also develop data-driven methods for choosing optimal window widths used in estimation.

Second, by using kernel methods to estimate time-varying betas we are able to use all the data efficiently in the estimation of conditional alphas and betas at any moment in time. Naturally, our methodology allows for any valid kernel and so nests the one-sided, equal-weighted filters used by French, Schwert, and Stambaugh (1987), Andersen et al. (2006), Lewellen and Nagel (2006), and others, as special cases. All of these studies use truncated, backward-looking windows to estimate second moments which have larger mean square errors compared to estimates based on two-sided kernels.[3]

Third, we develop tests for the significance of conditional and long-run alphas respectively jointly across assets in the presence of time-varying betas. Earlier work incorporating time-varying factor loadings restricts attention to only single assets whereas our methodology can incorporate a large number of assets. Our procedure can be viewed as the conditional analogue of Gibbons, Ross, and Shanken (1989), who jointly test whether alphas are equal to zero across assets, where we now permit the alphas and betas to vary over time. Joint tests are useful for investigating whether a relation between conditional alphas and firm characteristics strongly exists across many portfolios and have been extensively used by Fama and French (1993) and many others.

Our work is most similar to tests of conditional factor models contemporaneously examined by Li and Yang (2011). Li and Yang also use nonparametric methods to estimate conditional parameters and formulate a test statistic based on average conditional alphas. However, they do this in a discrete-time setting, do not investigate conditional or long-run betas, and do not develop tests of constancy of conditional alphas or betas. One important issue is the bandwidth selection procedure, which requires different bandwidths for conditional or long-run estimates.

[3] Foster and Nelson (1996) derive optimal two-sided filters to estimate covariance matrices under the null of a GARCH data generating process. Foster and Nelson's exponentially declining weights can be replicated by special choice kernel weights. An advantage of using a nonparametric procedure is that we obtain efficient estimates of betas without having to specify a particular data generating process, whether this is GARCH (see for example, Bekaert and Wu, 2000) or a stochastic volatility model (see for example, Jostova and Philipov, 2005; Ang and Chen, 2007).

Li and Yang do not provide an optimal bandwidth selection procedure. They also do not derive specification tests jointly across assets as in Gibbons, Ross, and Shanken (1989), which we nest as a special case, or present a complete distribution theory for their estimators.

The rest of this paper is organized as follows. Section 2 lays out our empirical methodology. Section 3 discusses our data. In Sections 4 and 5 we investigate tests of conditional CAPM and Fama-French models on the book-to-market and momentum portfolios, respectively. Section 6 concludes. We relegate all technical proofs to the appendix.

# 2 Statistical Methodology

## 2.1 Conditional Factor Model

Let $R = (R_1, ..., R_M)'$ denote a vector of excess returns of $M$ assets observed at $n$ time points, $0 < t_1 < t_2 < ... < t_n < T$, within a time span $T > 0$. We wish to explain the returns through a set of $J$ common tradeable factors, $f = (f_1, ..., f_J)'$ which are observed at the same time points. We assume the following conditional factor model explains the returns of stock $k$ ($k = 1, ..., M$) at time $t_i$ ($i = 1, ..., n$):

$$R_{k,i} = \alpha_k (t_i) + \beta_k (t_i)' f_i + \omega_{kk} (t_i) z_{k,i}, \tag{1}$$

where $R_{k,i}$ and $f_i$ are the observed return and factors respectively at time $t_i$. This can be rewritten in matrix notation:

$$R_i = \alpha (t_i) + \beta (t_i)' f_i + \Omega^{1/2} (t_i) z_i, \tag{2}$$

where $\alpha (t) = (\alpha_1 (t), ..., \alpha_M (t))' \in \mathbb{R}^M$ is the vector of conditional alphas across stocks $k = 1, ..., M$ and $\beta (t) = (\beta_1 (t), ..., \beta_M (t))' \in \mathbb{R}^{J\times M}$ is the corresponding matrix of conditional betas. The alphas and betas can take on any sample path in the data, subject to the (weak) restrictions in Appendix A, including non-stationary and discontinuous cases, and time-varying dependence of conditional betas and factors. The vector $z_i = (z_{1,i}, ..., z_{M,i})' \in \mathbb{R}^M$ contains the errors and the covariance matrix $\Omega (t) = [\omega_{jk}^2 (t)]_{j,k} \in \mathbb{R}^{M\times M}$ allows for both heteroskedasticity and time-varying cross-sectional correlations.

Letting $\mathcal{F}_i = \mathcal{F} \{R_j, f_j, \alpha (t_j), \beta (t_j) : j \leq i\}$ denote the filtration up to time $t_i$, we assume the error term satisfies

$$\mathrm{E} [z_i|\mathcal{F}_i] = 0 \quad \text{and} \quad \mathrm{E} [z_i z_i'|\mathcal{F}_i] = I_M, \tag{3}$$

where $I_M$ denotes the $M$-dimensional identity matrix. Equation (3) is the identifying assumption of the model and rules out non-zero correlations between the factor and the errors. This orthogonality assumption is an extension of standard OLS which specifies that errors and factors are orthogonal.[4] Importantly, this condition does not rule out the alphas and betas being correlated with the factors. That is, the conditional factor loadings can be random processes in their own right and exhibit (potentially time-varying) dependence with the factors. Thus, we allow for a rich set of dynamic trading strategies of the factor portfolios.

We are interested in time-series estimates of the realized conditional alphas, $\alpha(t)$, and the conditional factor loadings, $\beta(t)$, along with their standard errors. Under the null of a factor model, the conditional alphas are equal to zero, or $\alpha(t) = 0$. As Jagannathan and Wang (1996) point out, if the correlation of the factor loadings, $\beta(t_i)$, with factors, $f_i$, is zero, then the unconditional pricing errors of a conditional factor model are mean zero and an unconditional OLS methodology could be used to test the conditional factor model. When the betas are correlated with the factors then the unconditional alpha reflects both the true conditional alpha and the covariance between the betas and the factor (see Jagannathan and Wang, 1996; Lewellen and Nagel, 2006).

Given the realized alphas and betas at each point in time, we define the long-run alphas and betas for asset $k = 1, ..., M$ as

$$\begin{aligned} \alpha_{\mathrm{LR},k} &\equiv \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \alpha_k(t_i) \in \mathbb{R}, \\ \beta_{\mathrm{LR},k} &\equiv \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \beta_k(t_i) \in \mathbb{R}^J, \end{aligned} \tag{4}$$

We use the terminology "long run" (LR) to distinguish the conditional alpha at a point in time, $\alpha_k(t)$, from the conditional alpha averaged over the sample, $\alpha_{\mathrm{LR},k}$. When the factors are correlated with the betas, the long-run alphas are potentially different from OLS alphas.

We test the hypothesis that the long-run alphas are jointly equal to zero across $M$ assets:

$$H_0 : \alpha_{\mathrm{LR},k} = 0, \quad k = 1, ..., M. \tag{5}$$

In a setting with constant alphas and betas, Gibbons, Ross, and Shanken (1989) develop a test of the null $H_0$. Our methodology can be considered to be the conditional version of the

[4] The strict factor structure rules out leverage effects and other nonlinear relationships between asset and factor returns which Boguth et al. (2011) argue may lead to additional biases in the estimators. We expect that our theoretical results are still applicable under weaker assumptions, but this requires specification of the appropriate correlation structure between the alphas, betas, and error terms. Appendix A details our technical assumptions and Appendix B contains proofs. We leave these extensions to further research. Simulation results show that our long-run alpha estimators perform well under mild misspecification of modestly correlated betas and error terms.

Gibbons-Ross-Shanken test when both conditional alphas and betas potentially vary over time. Additionally, we test the stronger hypothesis of the conditional alphas being zero at any given point in time,

$$H_{0,k} : \alpha_k(t) = 0 \text{ for all } t. \tag{6}$$

## 2.2 Conditional Estimators

Our analysis of the model and estimators is done conditional on the particular realization of alphas and betas that generated data. That is, our analysis relies on the following conditional relation between the observations and the parameters of interest which holds under the orthogonality condition in equation (3):

$$\left[\alpha\left(t_i\right), \beta\left(t_i\right)'\right]' = \Lambda^{-1}\left(t_i\right) E\left[X_i R_i' | \mathcal{F}_i\right], \quad X_i = \left(1, f_i'\right)', \tag{7}$$

where $\Lambda\left(t_i\right)$ denotes the conditional second moment of the regressors:

$$\Lambda(t_i) \equiv \mathrm{E}\left[X_i X_i' | \mathcal{F}_i\right]. \tag{8}$$

Equation (7) identifies the particular realization of alphas and betas that generated data.

The time variation in $\Lambda\left(t\right)$ reflects potential correlation between factors and betas. If there is zero correlation (and the factors are stationary), then $\Lambda\left(t\right) = \Lambda$ is constant over time, but in general $\Lambda\left(t\right)$ varies over time. One advantage of conducting the analysis conditional on the sample is that we can tailor our estimates of the particular realization of alphas and betas.[5]

A natural way to estimate $\alpha\left(t\right)$ and $\beta\left(t\right)$ is by replacing the population moments in equation (7) by their sample versions. Given observations of returns and factors, we propose the following local least-squares estimators of $\alpha_k\left(t\right)$ and $\beta_k\left(t\right)$ for asset $k$ in equation (1) at any point in time $0 \leq t \leq T$:

$$[\hat{\alpha}_k\left(t\right), \hat{\beta}_k\left(t\right)']' = \arg\min_{(\alpha,\beta)} \sum_{i=1}^{n} K_{h_k T}\left(t_i - t\right)\left(R_{k,i} - \alpha - \beta' f_i\right)^2, \tag{9}$$

[5] An alternative approach taken in the analysis of parametric conditional factor models focuses on unconditional estimates of equation (7):

$$\mathrm{E}\left[R_i\right] = \alpha_u + \beta_u' f_u + \text{cov}\left(\beta(t_i), \bar{f}_i\right),$$

where $\bar{f}_i = \mathrm{E}\left[f_i | \gamma(\cdot), \Omega(\cdot)\right]$, $f_u = \mathrm{E}\left[f_i\right]$, $\alpha_u = \mathrm{E}\left[\alpha(t)\right]$ and $\beta_u = \mathrm{E}\left[\beta(t)\right]$. Our interest lies in pinning down the particular realization of the alphas and betas in our data sample, so this unconditional approach is not very informative as it averages over all possible realizations.

where $K_{h_kT}(z) \equiv K(z/(h_kT))/(h_kT)$ with $K(\cdot)$ being a kernel and $h_k > 0$ a bandwidth. The optimal estimators solving equation (9) are simply kernel-weighted least squares:

$$[\hat{\alpha}_k(t), \hat{\beta}_k(t)']' = \left[\sum_{i=1}^{n} K_{h_kT}(t_i - t) X_i X_i'\right]^{-1} \left[\sum_{i=1}^{n} K_{h_kT}(t_i - t) X_i R_{k,i}\right]. \quad (10)$$

The proposed estimators are sample analogues to equation (7) giving weights to the individual observations according to how close in time they are to the time point of interest, $t$. The shape of the kernel, $K$, determines how the different observations are weighted. For most of our empirical work we choose the Gaussian density as kernel,

$$K(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right),$$

but also examine one-sided and uniform kernels that have been used in the literature by Andersen et al. (2006) and Lewellen and Nagel (2006), among others. In common with other nonparametric estimation methods, as long as the kernel is symmetric, the most important choice is not so much the shape of the kernel as the bandwidth, $h_k$. The bandwidth, $h_k \in (0, 1)$, controls the proportion of data obtained in the sample span $[0, T]$ that is used in the computation of the estimated alphas and betas. A small bandwidth means only observations very close to $t$ are included in the estimation. The bandwidth controls the bias and variance of the estimator and it should, in general, be sample specific. In particular, as the sample size grows, the bandwidth should shrink towards zero at a suitable rate in order for any finite-sample biases and variances to vanish. We discuss the bandwidth choice in Section 2.8.

We run the kernel regression (9) separately stock by stock for $k = 1, ..., M$. This is a generalization of the regular OLS estimators, which are also run stock by stock in the Gibbons, Ross, and Shanken (1989) test. If the same bandwidth $h$ is used for all stocks, our estimator of alphas and betas across all stocks take the simple form of a weighted multivariate OLS estimator,

$$[\hat{\alpha}(t), \hat{\beta}(t)']' = \left[\sum_{i=1}^{n} K_{hT}(t_i - t) X_i X_i'\right]^{-1} \left[\sum_{i=1}^{n} K_{hT}(t_i - t) X_i R_i\right]. \quad (11)$$

In practice it is not advisable to use one common bandwidth across all assets. We use different bandwidths for different stocks because the variation and curvature of the conditional alphas and betas may differ widely across stocks and each stock may have a different level of heteroskedasticity. We show below that for book-to-market and momentum test assets, the patterns of conditional alphas and betas are dissimilar across portfolios. Choosing stock-specific bandwidths allows us to better adjust the estimators for these effects. However, in order to avoid cumbersome notation, we present the asymptotic results for the estimators $\hat{\alpha}(t)$ and $\hat{\beta}(t)$ assuming one common bandwidth, $h$, across all stocks. The asymptotic results are identical in the case with multiple bandwidths under the assumption that these all converge at the same rate as $n \to \infty$.

## 2.3 Continuous-Time Factor Model

For the theoretical analysis of the proposed estimators, we introduce a continuous-time version of the discrete-time factor model. Suppose that the $M$ log-prices, $s(t) = \log S(t) \in \mathbb{R}^M$, solve the following stochastic differential equation:

$$ds(t) = \alpha(t)\,dt + \beta(t)'\,dF(t) + \Sigma^{1/2}(t)\,dB(t), \tag{12}$$

where $F(t)$ are $J$ factors and $B(t)$ is a $M$-dimensional Brownian motion. This is the ANOVA model considered in Andersen et al. (2006) and Mykland and Zhang (2006). Suppose we have observed $s(t)$ and $F(t)$ over the time span $[0, T]$ at $n$ discrete time points, $0 \leq t_0 < t_1 < ... < t_n \leq T$. We wish to estimate the spot alphas, $\alpha(t) \in \mathbb{R}^M$, and betas, $\beta(t) \in \mathbb{R}^{J\times M}$, which can be interpreted as the realized instantaneous drift of $s(t)$ and (co-)volatility of $(s(t), F(t))$, respectively. For simplicity, we assume that the observations are equidistant in time such that $\Delta \equiv t_i - t_{i-1}$ is constant; in particular, $n\Delta = T$.

To facilitate the analysis of the estimators in this diffusion setting, we introduce a discretized version of the continuous-time model,

$$\Delta s_i = \alpha(t_i)\,\Delta + \beta(t_i)'\,\Delta F_i + \Sigma^{1/2}(t_i)\sqrt{\Delta} z_i, \quad i = 1, 2, ..., n, \tag{13}$$

where $z_i \sim$ i.i.d. $(0, I_M)$, and

$$\Delta s_i = s(t_i) - s(t_{i-1}), \quad \Delta F_i = F(t_i) - F(t_{i-1}).$$

In the following we treat equation (13) as the true, data-generating model. The extension to treat (12) as the true model would require some extra effort to ensure that the discretized version (13) is an asymptotically valid approximation of (12). The analysis would involve controlling the various discretization biases that would need to vanish sufficiently fast as $\Delta \to 0$. This could be done along the lines of Bandi and Phillips (2003) and Kristensen (2010), among others.

Defining

$$R_i \equiv \Delta s_i/\Delta, \quad f_i \equiv \Delta F_i/\Delta, \quad \Omega_t \equiv \Sigma(t)/\Delta, \tag{14}$$

we can rewrite the discretized diffusion model in the form of equation (2). Natural estimators of $\alpha(t)$ and $\beta(t)$ therefore take on the same form as the discrete-time estimators in equation (11).

## 2.4 Conditional Beta Estimators

We now analyze the properties of our estimator $\hat{\beta}(t)$ under the assumption that the discretized version of the diffusion model (13) is the data-generating process. As is well known in the literature on estimation of diffusion models (see e.g. Merton, 1980; Bandi and Phillips, 2003; Kristensen, 2010), we can consistently estimate the instantaneous betas, $\beta(t)$, as $\Delta \to 0$ under weak regularity conditions. In addition to equation (13), we assume that the factors satisfy the following discretized diffusion model,

$$\Delta F_i = \mu_F(t_i)\Delta + \Lambda_{FF}^{1/2}(t_i)\sqrt{\Delta}u_i, \tag{15}$$

where $u_i \sim$ i.i.d$(0, I_J)$ and $\mu(\cdot)$ and $\Lambda_{FF}(\cdot)$ are $r$ times differentiable (possibly random) functions.

Under regularity conditions stated in Appendix A, the following bias and variance expressions can be obtained:

$$E[\hat{\beta}(t)] \simeq \beta(t) + (hT)^2 \beta^{(2)}(t), \quad \text{Var}(\hat{\beta}(t)) \simeq \frac{1}{nh} \times \kappa_2 \Sigma(t),$$

where $\beta^{(2)}(t)$ denotes the second derivative of $\beta(t)$ and $\kappa_2 = \int K^2(z)\,dz$ ($= 0.2821$ for the normal kernel).[6] These expressions show the usual trade-off between bias and variance for kernel regression estimators with $h$ needing to be chosen to balance the two. In particular, as $hT \to 0$ and $nh \to \infty$, $\hat{\beta}(t) \xrightarrow{p} \beta(t)$. Letting $h \to 0$ at a suitable rate, the bias term can be ignored and we obtain the following asymptotic result:

**Theorem 1** *Assume that A.1-A.3 given in Appendix A hold and the bandwidth is chosen such that $nh \to \infty$ and $nT^4h^5 \to 0$. Then, for any $t \in [0, T]$:*

$$\sqrt{nh}\{\hat{\beta}(t) - \beta(t)\} \sim N\left(0, \kappa_2 \Lambda_{FF}^{-1}(t) \otimes \Sigma(t)\right) \quad \textit{in large samples.} \tag{16}$$

*Moreover, the conditional estimators are asymptotically independent across any set of distinct time points.*

[6] We assume that $t \mapsto \beta(t)$ is twice differentiable. This assumption could be replaced by, for example, a Lipschitz condition, $\|\beta(s) - \beta(t)\| \leq C|s-t|^{\lambda}$ for some $\lambda > 0$, in which case the bias component would change and be of order $O((hT)^{\lambda})$.

This result is a multivariate extension of the asymptotic distribution for kernel-based estimators of spot volatilities found in Kristensen (2010, Theorem 1). Andersen et al. (2006) develop estimators of integrated factor loadings, $\int_0^T \beta(s)\, ds$, which implicitly ignore the variation of beta within each window. Our estimator is a local version of the asymptotics for the integrated beta (see also Foster and Nelson, 1996). By choosing a flat kernel and the bandwidth, $h > 0$, to match the chosen time window, our proposed estimators nest the realized beta estimators. But, while Andersen et al. (2006) develop an asymptotic theory for a fixed window width, Theorem 1 establishes results where the time window shrinks with sample size. This allows us to recover the instantaneous conditional betas.

In Theorem 1, the rate of convergence of $\hat{\beta}(t)$ is $\sqrt{nh}$, which is slower than parametric estimators since $h \to 0$. This is common to all non-parametric estimators. The asymptotic analysis and properties of the estimators are closely related to the kernel-regression type estimators of diffusion models proposed in Bandi and Phillips (2003), Kanaya and Kristensen (2010), and Kristensen (2010). Bandi and Phillips (2003) focus on univariate Markov diffusion processes and use the lagged value of the observed process as kernel regressor, while Kanaya and Kristensen (2010) and Kristensen (2010) only consider estimation of univariate stochastic volatility models. In contrast, we model time-inhomogenous, multivariate processes where the observation times, $t_1, ..., t_n$, are used as the kernel regressor. Since we only smooth over the univariate time variable $t$, increasing the number of regressors, $J$, or the number of stocks, $M$, does not affect the performance of the estimator.

Estimators of the two terms appearing in the asymptotic variance in equation (16) are obtained as follows:

$$\begin{aligned} \hat{\Lambda}_{FF}(t) &= \frac{\Delta \sum_{i=1}^{n} K_{hT}(t_i - t)\left[f_i - \hat{\mu}_F(t_i)\right]\left[f_i - \hat{\mu}_F(t_i)\right]'}{\sum_{i=1}^{n} K_{hT}(t_i - t)} \\ \hat{\Sigma}(t) &= \frac{\Delta \sum_{i=1}^{n} K_{hT}(t_i - t)\, \hat{\varepsilon}_i \hat{\varepsilon}_i'}{\sum_{i=1}^{n} K_{hT}(t_i - t)}, \end{aligned} \tag{17}$$

where $\hat{\varepsilon}_i = R_i - \hat{\alpha}(t_i) - \hat{\beta}(t_i)' f_i$ are the residuals and $\hat{\mu}_F(t)$ is an estimator of the instantaneous drift in the factors,

$$\hat{\mu}_F(t) = \frac{\sum_{i=1}^{n} K_{hT}(t_i - t) f_i}{\sum_{i=1}^{n} K_{hT}(t_i - t)}. \tag{18}$$

Due to the asymptotic independence across different values of $t$, confidence bands over a given grid of time points can easily be computed.

## 2.5 Conditional Alpha Estimators

Unlike conditional betas, conditional alphas are not identified in data without additional restrictions on the time-series variation and without increasing the data over long time spans ($T \to \infty$), which was first demonstrated by Merton (1980). Without further restrictions, the estimator of $\alpha(t)$ satisfies

$$E[\hat{\alpha}(t)] \simeq \alpha(t) + (Th)^2 \alpha^{(2)}(t), \quad \text{Var}(\hat{\alpha}(t)) \simeq \frac{1}{Th} \times \kappa_2 \Sigma(t), \tag{19}$$

as $\Delta \to 0$. Relative to $\hat{\beta}(t)$, the bias of $\hat{\alpha}(t)$ is of the same order but its variance vanishes slower, $1/(Th)$ versus $1/(nh)$. The slower rate of convergence of Var($\hat{\alpha}(t)$) is a well-known feature of nonparametric drift estimators in diffusion models, as in Bandi and Phillips (2003), and is due to the smaller amount of information regarding the drift relative to the volatility found in data.

Observe that the bias and variance of $\hat{\alpha}(t)$ are perfectly balanced. To remove the bias, we have to let $Th \to 0$, but with this bandwidth choice the variance explodes. This simply mirrors the well-known fact that in continuous time, the local variation of observed returns is too noisy to extract information about the drift. As such, we cannot state any formal results regarding the asymptotic distribution of $\hat{\alpha}(t)$. However, informally, with $h$ chosen "small enough" such that the bias is negligible, we have

$$\sqrt{Th}\{\hat{\alpha}(t) - \alpha(t)\} \sim N(0, \kappa_2 \Sigma(t)) \quad \text{in large samples.} \tag{20}$$

It should be stressed though that without further restriction on the data-generating process, the conditional alpha estimates can only be interpreted as noisy estimates of the underlying conditional alpha process. In particular, the compuation of standard errors and confidence bands for the conditional alphas based on equation (20) ignores the bias component which might be quite substantial. As such, standard errors and confidence bands for conditional alphas should be interpreted with caution.

A large empirical asset pricing literature interprets constant terms in OLS regressions estimated over different sample periods as conditional alphas, at least since Gibbons and Ferson (1985).[7] Given the wide-spread use of conditional alpha estimators, it is of interest to provide conditions under which the statement in equation (20) is formally (i.e. asymptotically) correct.

[7] See, among many others, Shanken (1990), Ferson and Schadt (1996), Christopherson, Ferson and Glassman (1998), and more recently Mamaysky, Spiegel and Zhang (2008). Ferson and Qian (2004) provide a summary of this large literature.

One such condition is to impose a recurrency restriction used in the literature on nonparametric estimation of diffusion models. In particular, Bandi and Phillips (2003) assume that the instantaneous drift (in our case, the spot alpha) is a function of a recurrent process, say $Z(t)$, that visits any given point in its domain, say $z$, infinitely often. Thus, under recurrence, there is increasing *local* information around $z$ that allows identification of the drift function at this value. A similar idea in our setting is to assume there exists functions $a : [0, 1] \mapsto \mathbb{R}^M$ and $S : [0, 1] \mapsto \mathbb{R}^{M \times M}$ such that the processes $\alpha(t)$ and $\Sigma(t)$ are generated by

$$\alpha(t) = a(t/T) \quad \text{and} \quad \Sigma(t) = S(t/T). \tag{21}$$

Then, the spot alpha would become a function of $Z(t) \equiv t/T \in [0, 1]$; in particular, $Z_i \equiv Z(t_i)$, $i = 1, ..., n$, can be thought of as i.i.d. draws from the uniform distribution on $[0, 1]$ with observations growing more and more dense in $[0, 1]$ as $T \to \infty$. Thus, with this assumption, we would accomplish the same increase in local information about $\alpha(t)$ around a given point $t$ in each successive model as $T \to \infty$. Note that in contrast the un-normalized time, $Z(t) = t$, is not a recurrent process, and so without the restriction given in equation (21), we would not be able to identify $\alpha(t)$.

Under the time-normalization assumption (21), there is a sequence of models as the sample changes ($T$ increases), and this sequence of models is constructed so that the asymptotic distribution of $\hat{\alpha}(t)$ is well defined. While the time normalization is a widely used statistical tool to construct valid asymptotic distributions, and is used extensively in the large structural change literature, the restriction imposed in (21) is counterintuitive because the underlying economic structure changes as we sample over larger time spans. The time normalization is only needed to obtain formal asymptotic results for the conditional alpha estimators and not necessary for the asymptotic analysis of the conditional beta estimators (Theorem 1), or the long-run alpha and beta estimators developed in subsequent sections. As a consequence, we relegate the asymptotic theory for the conditional alpha estimators under the time normalization (21) to Appendix C.

Finally, it is worth noting that one could alternatively analyze the conditional alpha and beta estimators in a discrete-time setting, where it is necessary to impose a time normalization. This approach is pursued in a previous working paper version of this paper (Ang and Kristensen, 2011) and in Kristensen (2011). The normalization is similar to equation (21), but in a discrete-time setting the normalization restriction has to be imposed on both the alphas and betas. This is due to the fact that, in contrast to the continuous-time setting where $\Omega(t) = \Sigma(t)/\sqrt{\Delta} \to \infty$ as we sample more frequently, the variance in the discrete-time model does not change as we collect more data over time. This mirrors the fact that in discrete time we only rely on long-span asymptotics, and so cannot nonparametrically learn about the local variation of the conditional alphas and betas without imposing some type of time normalization. However, as demonstrated in Ang and Kristensen (2011), estimators and finite-sample standard errors obtained in a discrete-time and continuous-time setting respectively are numerically identical. Thus, while the asymptotic theory is different, the empirical implementation is the same. The fact that the estimators can both be given a discrete-time and continuous-time interpretation is a convenient feature of the estimators since the vast majority of empirical studies are carried out in a discrete-time setting.

### 2.6 Long-Run Alphas and Betas

To test the null of whether the long-run (LR) alphas are equal to zero ($H_0$ in equation (5)), we construct estimators of the long-run alphas and betas in equation (4). A natural way to estimate the long-run alphas and betas for stock $k$ is to simply plug the pointwise kernel estimators into the expressions found in equation (4):

$$\hat{\alpha}_{\mathrm{LR},k} = \frac{1}{n}\sum_{i=1}^{n}\hat{\alpha}_k(t_i) \quad \text{and} \quad \hat{\beta}_{\mathrm{LR},k} = \frac{1}{n}\sum_{i=1}^{n}\hat{\beta}_k(t_i).$$

Given that we can identify the conditional spot betas, we can also identify the LR betas, and so $\hat{\beta}_{\mathrm{LR},k}$ is consistent. But more importantly, we can identify the LR alphas even if we cannot identify the instantaneous ones. In particular, without the time normalization given in equation (21), $\hat{\alpha}_k(t)$ is an inconsistent estimator of $\alpha_k(t)$, but $\hat{\alpha}_{\mathrm{LR},k}$ is a consistent estimator of $\alpha_{\mathrm{LR},k}$. Thus, we can consistenly estimate the long-run alphas without imposing the time-normalization used in the theoretical analysis of the conditional alphas. The intuition behind this feature is that our estimator of $\alpha_{\mathrm{LR},k}$ involves additional averaging over time. This averaging reduces the overall sampling error of $\hat{\alpha}_{\mathrm{LR},k}$ and enables consistency as $T \to \infty$.

The following theorem states the joint distribution of $\hat{\alpha}_{\mathrm{LR}} = (\hat{\alpha}_{\mathrm{LR},1}, ..., \hat{\alpha}_{\mathrm{LR},M})' \in \mathbb{R}^M$ and $\hat{\beta}_{\mathrm{LR}} = (\hat{\beta}_{\mathrm{LR},1}, ..., \hat{\beta}_{\mathrm{LR},M})' \in \mathbb{R}^{J\times M}$:

**Theorem 2** *Assume that assumptions A.1-A.5 given in the Appendix hold. Then the long-run estimators satisfy as $T \to \infty$:*

$$\sqrt{T}(\hat{\alpha}_{\mathrm{LR}} - \alpha_{\mathrm{LR}}) \sim N(0, \Sigma_{\mathrm{LR},\alpha\alpha}), \quad \sqrt{n}(\hat{\beta}_{\mathrm{LR}} - \beta_{\mathrm{LR}}) \sim N(0, \Sigma_{\mathrm{LR},\beta\beta}) \tag{22}$$

*in large samples, where*

$$\begin{aligned}
\alpha_{\text{LR}} &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \alpha(t)\,dt \equiv\ E[\alpha(t)] \\
\beta_{\text{LR}} &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \beta(t)\,dt \equiv \mathrm{E}[\beta(t)] \\
\Sigma_{\text{LR},\alpha\alpha} &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \Sigma(t)\,dt \equiv \mathrm{E}[\Sigma(t)] \\
\Sigma_{\text{LR},\beta\beta} &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \Lambda_{FF}^{-1}(t)\otimes\Sigma(t)\,dt \equiv \mathrm{E}\left[\Lambda_{FF}^{-1}(t)\otimes\Sigma(t)\right].
\end{aligned}$$

The long-run estimators converge at standard parametric rates $\sqrt{n}$ and $\sqrt{T}$ respectively, despite the fact that they are based on preliminary estimators that converge at slower, nonparametric rates. That is, inference of the long-run alphas and betas involves the standard Central Limit Theorem (CLT) convergence properties even though the point estimates of the conditional alphas and betas converge at slower rates. Intuitively, this is due to the additional smoothing taking place when we average over the preliminary estimates in equation (10). This occurs in other semiparametric estimators involving integrals of kernel estimators (see, for example, Newey and McFadden, 1994, Section 8; Powell, Stock, and Stoker, 1989).

Consistent estimators of the asymptotic variances are obtained by simply plugging the point estimates of $\Lambda_{FF}(t)$ and $\Sigma(t)$ given in (17) into the sample versions of $\Sigma_{\text{LR},\alpha\alpha}$ and $\Sigma_{\text{LR},\beta\beta}$:

$$\hat{\Sigma}_{\text{LR},\alpha\alpha} = \frac{1}{n}\sum_{i=1}^{n}\hat{\Sigma}(t_i), \quad \Sigma_{\text{LR},\beta\beta} = \frac{1}{n}\sum_{i=1}^{n}\hat{\Lambda}_{FF}^{-1}(t)\otimes\hat{\Sigma}(t_i).$$

We can test $H_0 : \alpha_{\text{LR}} = 0$ by the following Wald-type statistic:

$$W_{\text{LR}} = T\hat{\alpha}'_{\text{LR}}\hat{\Sigma}^{-1}_{\text{LR},\alpha\alpha}\hat{\alpha}_{\text{LR}} \sim \chi^2_M \quad \text{in large samples,} \tag{23}$$

as a direct consequence of Theorem 2. This is a conditional analogue of Gibbons, Ross, and Shanken (1989) and tests if long-run alphas are jointly equal to zero across all $k = 1, ..., M$ portfolios. A special case of Theorem 2 is Lewellen and Nagel (2006), who use a uniform kernel and the Fama-MacBeth (1973) procedure to compute standard errors of long-run estimators. Theorem 2 formally validates this procedure, extends it to a general kernel, joint tests across stocks, and long-run betas.

Our model includes the case where the factor loadings are constant with $\beta(t) = \beta \in \mathbb{R}^{J\times M}$ for all $t$. Under the null that beta is indeed constant, $\beta(t) = \beta$, and with no heteroskedasticity, $\Sigma(t) = \Sigma$ for all $t$, the asymptotic distribution of $\hat{\alpha}_{\text{LR}}$ is identical to the standard Gibbons, Ross, and Shanken (1989) test. This is shown in Appendix D. Thus, we pay no price asymptotically for the added robustness of our estimator. Furthermore, only in a setting where the factors are uncorrelated with the betas is the Gibbons-Ross-Shanken estimator of $\alpha_{\mathrm{LR}}$ consistent. This is not surprising given the results of Jagannathan and Wang (1996) and others who show that in the presence of time-varying betas, OLS alphas do not yield estimates of conditional or long-run alphas.

## 2.7 Tests for Constancy of Alphas and Betas

We wish to test for constancy of the potentially time-varying conditional alphas and betas. The two null hypotheses of interest are formally:

$$\begin{aligned} H_k(\alpha) &: \alpha_k(t) = \alpha_k \in \mathbb{R}, \quad \text{for all } t \in [0,T], \\ H_k(\beta) &: \beta_k(t) = \beta_k \in \mathbb{R}^J, \quad \text{for all } t \in [0,T]. \end{aligned} \tag{24}$$

We propose to test each the two hypotheses through Hausman-type statistics where we compare two estimators – the first is chosen to be consistent both under the relevant null and the alternative while the second one is only consistent under the null. If the null is true, the test statistic is expected to be small and vice versa. A natural choice for the former estimator is the nonparametric estimator developed in Section 2.2. For the latter, we use the long-run estimator since under the relevant null the long-run estimator is a consistent estimator of the constant coefficient. To be more precise, we define our test stastistics for $H_k(\alpha)$ and $H_k(\beta)$, respectively, as the following two weighted least-squares statistics,

$$\begin{aligned} W_k(\alpha) &\equiv \frac{1}{n}\sum_{i=1}^{n} \hat{\sigma}_{kk}^{-2}(t_i)\left[\hat{\alpha}_k(t_i) - \hat{\alpha}_{\mathrm{LR},k}\right]^2, \\ W_k(\beta) &\equiv \frac{1}{n}\sum_{i=1}^{n} \hat{\sigma}_{kk}^{-2}(t_i)\left[\hat{\beta}_k(t_i) - \hat{\beta}_{\mathrm{LR},k}\right]' \hat{\Lambda}_{FF}(t_i)\left[\beta_k(t_i) - \hat{\beta}_{\mathrm{LR},k}\right]. \end{aligned} \tag{25}$$

The weights have been chosen to ensure that the asymptotic distributions of the statistics are nuisance parameter free. The two proposed test statistic are related to the generalized likelihood-ratio test statistics advocated in Fan, Zhang, and Zhang (2001).

The test statistic $W_k(\alpha)$ depends on $\hat{\alpha}_k(t)$ which is in general an inconsistent estimator of $\alpha_k(t)$ as discussed in Section 2.6. However, under the null of constant alphas, $\mathrm{E}[\hat{\alpha}_k(t)] \simeq \alpha_k(t)$ (c.f. equation (19)), and we are therefore able to derive a formal large-sample distribution of $W_k(\alpha)$. An important hypothesis nested within $H_k(\alpha)$ is the asset pricing hypothesis that

$\alpha_{k,t} = 0$ for all $t \in [0, T]$ ($H_{0,k}$ in equation (6)). This can be tested by simply setting $\hat{\alpha}_{\text{LR},k} = 0$ in $W_k(\alpha)$ yielding:

$$W_k(0) \equiv \frac{1}{n} \sum_{i=1}^{n} \hat{\sigma}_{kk}^{-2}(t_i) \hat{\alpha}_k^2(t_i).$$

The proposed test statistics follow normal distributions in large samples as stated in the following theorem:

**Theorem 3** *Assume that assumptions A.1-A.5 given in the Appendix hold and the bandwidth satisfies A.6. Then:*

$$\begin{aligned} \textit{Under } H_k(\alpha) &: \frac{W_k(\alpha) - m(\alpha)}{v(\alpha)} \sim N(0,1), \\ \textit{Under } H_k(\beta) &: \frac{W_k(\beta) - m(\beta)}{v(\beta)} \sim N(0,1) \end{aligned} \tag{26}$$

*in large samples, where, with* $(K * K)(z) \equiv \int K(y) K(z+y)\, dy$ *and* $J = \dim(f_i)$*,*

$$\begin{aligned} m(\alpha) &= \frac{\kappa_2}{Th} \textit{ and } v^2(\alpha) = \frac{2\int (K*K)^2(z)\, dz}{T^3 h}, \\ m(\beta) &= \frac{\kappa_2 J \Delta}{Th} \textit{ and } v^2(\beta) = \frac{2J\int (K*K)^2(z)\, dz}{n^2 T h}. \end{aligned}$$

*For Gaussian kernels,* $\int (K*K)^2(z)\, dz = 0.1995$ *and* $\kappa_2 = 0.2821$.

A convenient feature of the limiting distributions of the test statistics is that they are nuisance parameter free since $m(\alpha)$, $m(\beta)$, $v(\alpha)$, and $v(\beta)$ depend on known quantities only. Moreover, Fan, Zhang, and Zhang (2001) demonstrate in a cross-sectional setting that test statistics of the form of $W_k(\alpha)$ and $W_k(\beta)$ are, in general, asymptotically optimal and can even be adaptively optimal, and so we expect them to be able to easily detect departures from the null. As a straightforward corollary of Theorem 3, one can show that the test statistic $W_k(0)$ has the same asymptotic distribution as $W_k(\alpha)$ and so is not affected by setting $\hat{\alpha}_{\text{LR},k} = 0$.

The above test procedures can easily be adapted to construct joint tests of parameter constancy across multiple stocks. For example, to jointly test for constant alphas across all stocks, we would simply redefine the above least-squares statistic to include alpha estimates across all stocks,

$$\bar{W}(\alpha) \equiv \frac{1}{n} \sum_{i=1}^{n} [\hat{\alpha}(t_i) - \hat{\alpha}_{\text{LR}}]' \hat{\Sigma}^{-1}(t_i) [\hat{\alpha}(t_i) - \hat{\alpha}_{\text{LR}}].$$

The asymptotic distribution of this would be the same as for $W_k(\alpha)$, except that now $m(\alpha) = \kappa_2 M/(Th)$ and $v^2(\alpha) = 2M \int (K*K)^2(z)\, dz/(T^3 h)$ since we are testing $M$ hypotheses jointly.

## 2.8 Choice of Kernel and Bandwidth

As is common to all nonparametric estimators, the kernel and bandwidth need to be selected. Our theoretical results are based on using a kernel centered around zero and our main empirical results use the Gaussian kernel. Previous authors using high frequency data to estimate covariances or betas, such as Andersen et al. (2006) and Lewellen and Nagel (2006), have used one-sided filters. For example, the rolling window estimator employed by Lewellen and Nagel corresponds to a uniform kernel on $[-1, 0]$ with $K(z) = \mathbb{I}\{-1 \leq z \leq 0\}$. For the estimator to be consistent, we have to let the sequence of bandwidths shrink towards zero as the sample size grows, $h \equiv h_n \to 0$ as $n \to \infty$ in order to remove any biases of the estimator.[8] However, a given sample requires a particular choice of $h$, which we now discuss.

Since our interest lies in the in-sample estimation and testing, we advocate using two-sided symmetric kernels because in this case the bias from two-sided symmetric kernels is lower than for one-sided filters. In our data where $n$ is over 10,000 daily observations, the improvement in the integrated root mean squared error (RMSE) using a Gaussian filter over a backward-looking uniform filter can be quite substantial. For the symmetric kernel the integrated RMSE is of order $O\left(n^{-2/5}\right)$ whereas the corresponding integrated RMSE is at most of order $O\left(n^{-1/3}\right)$ for a one-sided kernel. We provide further details in Appendix E.

Bias at end points is a well-known issue common to all kernel estimators. Symmetric kernels suffer from excess bias at the beginning and end of the sample. This can be handled in a number of different ways. The easiest way, which is also the procedure we follow in the empirical work, is to simply refrain from reporting estimates close to the two boundaries. All our theoretical results are established under the assumption that our sample has been observed across (normalized) time points $t \in [-c, T + c]$ for some $c > 0$ and we then only estimate the alphas and betas for $t \in [0, T]$. In the empirical work, we do not report the time-varying

[8] For very finely sampled data, especially intra-day data, non-synchronous trading may induce bias. There is a large literature on methods to handle non-synchronous trading going back to Scholes and Williams (1977) and Dimson (1979). These methods can be employed in our setting. As an example, consider the one-factor model where $f_t = R_{m,t}$ is the market return. As an ad-hoc adjustment for non-synchronous trading, we can augment the one-factor regression to include the lagged market return, $R_t = \alpha_t + \beta_{1,t} R_{m,t} + \beta_{2,t} R_{m,t-1} + \varepsilon_t$, and add the combined betas, $\hat{\beta}_t = \hat{\beta}_{1,t} + \hat{\beta}_{2,t}$. This is done by Li and Yang (2011). More recently, there has been a growing literature on how to adjust for non-synchronous effects in the estimation of realized volatility. Again, these can be carried over to our setting. For example, it is possible to adapt the methods proposed in, for example, Hayashi and Yoshida (2005) or Barndorff-Nielsen et al. (2009) to adjust for the biases due to non-synchronous observations. In our empirical work, non-synchronous trading should not be a major issue as we work with value-weighted, not equal-weighted, portfolios at the daily frequency.

alphas and betas during the first and last year of our post-1963 sample. Alternatively, adaptive estimators which control for the boundary bias could be used.[9] Usage of these kernels does not affect the asymptotic distributions in Theorems 1-3 or the asymptotic distributions we derive for long-run alphas and betas in Section 2.6.

There are two bandwidth selection issues unique to our estimators that we now discuss, which are separate bandwidth choices for the conditional and long-run estimators. We choose one bandwidth for the point estimates of conditional alphas and betas and a different bandwidth for the long-run alphas and betas. The two different bandwidths are necessary because in our theoretical framework the conditional estimators and the long-run estimators converge at different rates. In particular, the asymptotic results suggest that for the integrated long-run estimators we need to undersmooth relative to the point-wise conditional estimates; that is, we should choose our long-run bandwidths to be smaller than the conditional bandwidths. Our strategy is to determine optimal conditional bandwidths and then adjust the conditional bandwidths for the long-run alpha and beta estimates. We propose data-driven rules for choosing the bandwidths.[10]

### 2.8.1 Bandwidth for Conditional Estimators

To estimate the conditional bandwidths, we develop a global plug-in method that is designed to mimic the optimal, infeasible bandwidth. The bandwidth selection criterion is chosen as the integrated (across all time points) mean-square error (MSE), and so the resulting bandwidth is a global one. In some situations, local bandwidth selection procedures that adapt to local features at a given point in time may be more useful; the following procedure can be adapted to this purpose by replacing all sample averages by subsample ones in the expressions below.

For a symmetric kernel with $\int K(z)\,dz = \int K(z)\,z^2 dz = 1$, the optimal global bandwidth that minimizes the (integrated over $[0,T]$) MSE of $\hat{\beta}_k(t)$ is:

$$h^*_{\beta,k} = \left(\frac{V_k(\beta)}{B_k(\beta)}\right)^{1/5} n^{-1/5}, \tag{27}$$

where $V_k(\beta) = \frac{1}{T}\int_0^T v_k(s;\beta)\,ds$ and $B_k(\beta) = \frac{1}{T}\int_0^T b_k^2(s;\beta)\,ds$ are the integrated time-varying variance and squared-bias components. Similarly, under the time normalization, the

[9] Two such estimators are boundary kernels and locally linear estimators. The former involves exchanging the fixed kernel $K$ for another adaptive kernel which adjusts to how close we are to the boundary, while the latter uses a local linear approximation of $\alpha(t)$ and $\beta(t)$ instead of a local constant one.

[10] We conducted simulation studies showing that the proposed methods work well in practice.

optimal bandwidth for the estimation of $\alpha_k(t)$ in terms of integrated MSE is

$$h^*_{\alpha,k} = \left(\frac{V_k(\alpha)}{B_k(\alpha)}\right)^{1/5} T^{-1/5}, \tag{28}$$

where $V_k(\alpha) = \frac{1}{T}\int_0^T v_k(s;\alpha)\,ds$ and $B_k(\alpha) = \frac{1}{T}\int_0^T b_k^2(s;\alpha)\,ds$ are the integrated time-varying variance and squared-bias components of $\hat{\alpha}_k(t)$. The functions appearing in the integrals are given by

$$\begin{aligned} v_k(t;\beta) &= \kappa_2 \Lambda^{-1}_{FF}(t)\,\sigma^2_{kk}(t) \quad \text{and} \quad b_k(t;\beta) = \beta^{(2)}_k(t)\,; \\ v_k(t;\alpha) &= \kappa_2 \sigma^2_{kk}(t) \quad \text{and} \quad b_k(t;\alpha) = \alpha^{(2)}_k(t)\,. \end{aligned}$$

Ideally, we would compute $v_k$ and $b_k$ in order to obtain the optimal bandwidth given in equations (27)-(28). However, these depend on unknown components, $\alpha$, $\beta$, $\Lambda_{FF}$, and $\Sigma$. In order to implement the bandwidth choice we propose a two-step method to provide preliminary estimates of these unknown quantities.[11] Since the proposed procedures for choosing the bandwidth choice for $\hat{\beta}(t)$ and $\hat{\alpha}(t)$ follow along the same lines, we only describe the one for $\hat{\beta}(t)$:

1. Choose as prior $\Lambda_{FF}(t) = \Lambda$ and $\sigma_{kk}(t) = \sigma_{kk}$ being constants, and $\beta_k(t) = b_{k0} + b_{k1}t + ... + b_{kp}t^p$ a polynomial of order $p \geq 2$. We obtain parametric least-squares estimates $\tilde{\Lambda}_{FF}$, $\tilde{\sigma}^2_{kk}$ and $\tilde{\beta}_k(t) = \tilde{b}_{k0} + \tilde{b}_{k1}t + ... + \tilde{b}_{kp}t^p$. Compute for each stock ($k = 1, ..., M$)

$$\tilde{V}_k(\beta) = \frac{\kappa_2}{T}\tilde{\Lambda}^{-1}_{FF}\tilde{\sigma}^2_{kk} \quad \text{and} \quad \tilde{B}_k(\beta) = \frac{1}{n}\sum_{i=1}^{n} ||\tilde{\beta}^{(2)}_k(t_i)||^2,$$

   where $\tilde{\beta}^{(2)}_{k,t} = 2\tilde{b}_{k2} + 6\tilde{b}_{k3}(t/n) + ... + p(p-1)\tilde{b}_{kp}(t/n)^{p-2}$. Then, using these estimates we compute the first-pass bandwidth

$$\tilde{h}_k = \left[\frac{\tilde{V}_k(\beta)}{\tilde{B}_k(\beta)}\right]^{1/5} \times n^{-1/5}. \tag{29}$$

2. Given $\tilde{h}_k$, compute the kernel estimators $\hat{\beta}_k(t)$ and the variance components given in eq. (17) with $h_k = \tilde{h}_k$. Use these to compute

$$\hat{V}_k(\beta) = \kappa_2 \frac{1}{n}\sum_{i=1}^{n} \hat{\Lambda}^{-1}_{FF}(t_i)\,\hat{\sigma}^2_{kk}(t_i) \quad \text{and} \quad \hat{B}_k(\beta) = \frac{1}{n}\sum_{i=1}^{n} ||\hat{\beta}^{(2)}_k(t_i)||^2,$$

[11] Ruppert, Sheather and Wand (1995) discuss in detail how this can done in a standard kernel regression framework. This bandwidth selection procedure takes into account the (time-varying) correlation structure between betas and factors through $\Lambda_t$ and $\Omega_t$.

with $\hat{\beta}_k^{(2)}(t)$ being the second derivative of the kernel estimator. These are in turn used to obtain the second-pass bandwidth:

$$\hat{h}_k = \left[\frac{\hat{V}_k(\beta)}{\hat{B}_k(\beta)}\right]^{1/5} \times n^{-1/5}. \tag{30}$$

We compute conditional alphas and betas using the bandwidths obtained by the described two-step procedure.

Our motivation for using a plug-in bandwidth is as follows. We believe that the betas for our portfolios vary slowly and smoothly over time as argued both in economic models such as Gomes, Kogan, and Zhang (2003) and from previous empirical estimates such as Petkova and Zhang (2005), Lewellen and Nagel (2006), and Ang and Chen (2007), and others. The plug-in bandwidth accommodates this prior information by allowing us to specify a low-level polynomial order. In our empirical work we choose a polynomial of degree $p = 6$, and find little difference in the choice of bandwidths when $p$ is below ten.[12]

One could alternatively use (generalized) cross-validation (GCV) procedures to choose the bandwidth. These procedures are completely data driven and, in general, yield consistent estimates of the optimal bandwidth. However, we find that in our data these can produce bandwidths that are extremely small, corresponding to a time window as narrow as 3-5 days with corresponding huge time variation in the estimated factor loadings. We believe these bandwidth choices are not economically sensible. The poor performance of the GCV procedures is likely due to a number of factors. First, it is well-known that cross-validated bandwidths may exhibit very inferior asymptotic and practical performance even in a cross-sectional setting (see, for example, Härdle, Hall, and Marron, 1988). This problem is further enhanced when GCV procedures are used on time-series data as found in various studies (Diggle and Hutchinson, 1989; Hart, 1991; Opsomer, Wang, and Yang, 2001).

### 2.8.2 Bandwidth for Long-Run Estimators

To estimate the long-run alphas and betas we re-estimate the conditional coefficients by undersmoothing relative to the bandwidth in equation (30). The reason for this is that the long-run estimates are themselves integrals and the integration imparts additional smoothing. Using the same bandwidth as for the conditional alphas and betas will result in over-smoothing.

[12] The order of the polynomial is an initial belief on the underlying smoothness of the process; it does not imply that a polynomial of this order fits the estimated conditional parameters.

Ideally, we would choose optimal long-run bandwidths to minimize the mean-squared errors $\mathrm{E}\left[(\hat{\alpha}_{\mathrm{LR},k}-\alpha_{\mathrm{LR},k})^2\right]$ and $\mathrm{E}[(\hat{\beta}_{\mathrm{LR},k}-\beta_{\mathrm{LR},k})^2]$ which we derive in Appendix F. As demonstrated there, the bandwidths used for the long-run estimators should be chosen to be of order $h_{\mathrm{LR},k} = O\left(n^{-1/3}\right)$ and $h_{\mathrm{LR},k} = O\left(T^{-1/3}\right)$ for the long-run betas and alphas respectively. Thus, the optimal bandwidth for the long-run estimates is required to shrink at a faster rate than the one used for pointwise estimates as we saw above.

In our empirical work, we select the bandwidth for the long-run alphas and betas by first computing the optimal second-pass conditional bandwidth $\hat{h}_k$ in equation (30) and then scaling this down by setting

$$\hat{h}_{\mathrm{LR},k} = \hat{h}_k \times n^{-2/15}. \tag{31}$$

# 3 Data

In our empirical work, we consider two specifications of conditional factor models: a conditional CAPM where there is a single factor which is the market excess return and a conditional version of the Fama and French (1993) model where the three factors are the market excess return, $MKT$, and two zero-cost mimicking portfolios, which are a size factor, $SMB$, and a value factor, $HML$.

We apply our methodology to decile portfolios sorted by book-to-market ratios and decile portfolios sorted on past returns constructed by Kenneth French. We use the Fama-French (1993) factors, $MKT$, $SMB$, and $HML$ as explanatory factors. All our data is at the daily frequency from July 1963 to December 2007, and we choose to measure time in days such that $\Delta = 1$. We use this whole span of data to compute optimal bandwidths. However, in reporting estimates of conditional factor models we truncate the first and last years of daily observations to avoid end-point bias, so our conditional estimates of alphas and factor loadings and our estimates of long-run alphas and betas span July 1964 to December 2006. Our summary statistics in Table 1 cover this truncated sample, as do all of our results in the next sections.

Panel A of Table 1 reports summary statistics of our factors. We report annualized means and standard deviations. The market premium is 5.32% compared to a small size premium for $SMB$ at 1.84% and a value premium for $HML$ at 5.24%. Both $SMB$ and $HML$ are negatively correlated with the market portfolio with correlations of -23% and -58%, respectively, but have a low correlation with each other of only -6%. In Panel B, we list summary statistics of the book-to-market and momentum decile portfolios. We also report OLS estimates of a constant alpha and constant beta in the last two columns using the market excess return factor. The book-to-market portfolios have average excess returns of 3.84% for growth stocks (decile 1) to 9.97% for value stocks (decile 10). We refer to the zero-cost strategy 10-1 that goes long value stocks and shorts growth stocks as the ”book-to-market strategy.” The book-to-market strategy has an average return of 6.13%, an OLS alpha of 7.73% and a negative OLS beta of -0.301. Similarly, for the momentum portfolios we refer to a 10-1 strategy that goes long past winners (decile 10) and goes short past losers (decile 1) as the ”momentum strategy.” The momentum strategy's returns are particularly impressive with a mean of 17.07% and an OLS alpha of 16.69%. The momentum strategy has an OLS beta close to zero of 0.072.

We first examine the conditional and long-run alphas and betas of the book-to-market portfolios and the book-to-market strategy in Section 4. Then, we test the conditional Fama and French (1993) model on the momentum portfolios in Section 5.

# 4 Portfolios Sorted on Book-to-Market Ratios

## 4.1 Tests of the Conditional CAPM

We report estimates of bandwidths, conditional alphas and betas, and long-run alphas and betas in Table 2 for the decile book-to-market portfolios. The last row contains results for the 10-1 book-to-market strategy. The columns labeled "Bandwidth" list the second-pass bandwidth $\hat{h}_{k,2}$ in equation (30). The column headed "Fraction" reports the bandwidths as a fraction of the entire sample, which is equal to one. In the column titled "Months" we transform the bandwidth to a monthly equivalent unit. For the normal distribution, 95% of the mass lies between $(-1.96, 1.96)$. If we were to use a flat uniform distribution, 95% of the mass would lie between $(-0.975, 0.975)$. Thus, to transform to a monthly equivalent unit we multiply by $533 \times 1.96/0.975$, where there are 533 months in the sample. We annualize the alphas in Table 2 by multiplying the daily estimates by 252.

For the decile 8-10 portfolios, which contain predominantly value stocks, and the value-growth strategy 10-1, the optimal bandwidth is around 20 months. For these portfolios there is significant time variation in beta and the relatively tighter windows allow this variation to be picked up with greater precision. In contrast, growth stocks in deciles 1-2 have optimal windows of 51 and 106 months, respectively. Growth portfolios do not exhibit much variation in beta so the window estimation procedure picks a much longer bandwidth. Overall, our estimated bandwidths are somewhat longer than the commonly used 12-month horizon to compute betas using daily data (see, for example, Ang, Chen, and Xing, 2006). At the same time, our 20-month window is shorter than the standard 60-month window often used at the monthly frequency (see, for example, Fama and French, 1993, 1997).

We report the standard deviation of conditional betas at the end of each month in the column labeled "Stdev of Cond Betas." Below, we further characterize the time variation of these monthly conditional estimates. The conditional betas of the book-to-market strategy a standard deviation of 0.206. The majority of this time variation comes from value stocks, as decile 1 betas have a standard deviation of only 0.056 while decile 10 betas have a standard deviation of 0.191.

Lewellen and Nagel (2006) argue that the magnitude of the time variation of conditional betas is too small for a conditional CAPM to explain the value premium. The estimates in Table 2 overwhelmingly confirm this. Lewellen and Nagel suggest that an approximate upper bound for the unconditional OLS alpha of the book-to-market strategy, which Table 1 reports as 0.644% per month or 7.73% per annum, is given by $\sigma_\beta \times \sigma_{\mathrm{E}_t[r_{m,t+1}]}$, where $\sigma_\beta$ is the standard deviation of conditional betas and $\sigma_{\mathrm{E}_t[r_{m,t+1}]}$ is the standard deviation of the conditional market risk premium. Conservatively assuming that $\sigma_{\mathrm{E}_t[r_{m,t+1}]}$ is 0.5% per month following Campbell and Cochrane (1999), we can explain at most $0.206 \times 0.5 = 0.103\%$ per month or 1.24% per annum of the annual 7.73% book-to-market OLS alpha. We now formally test for this result by computing long-run alphas and betas.

In the last two columns of Table 2, we report estimates of long-run annualized alphas and betas, along with standard errors in parentheses. The long-run alpha of the growth portfolio is $-2.26\%$ with a standard error of 0.008 and the long-run alpha of the value portfolio is $4.61\%$ with a standard error of 0.011. Thus, both growth and value portfolios overwhelmingly reject the conditional CAPM. The long-run alpha of the book-to-market portfolio is $6.81\%$ with a standard error of 0.015. Clearly, there is a significant long-run alpha after controlling for time-varying market betas. The long-run alpha of the book-to-market strategy is very similar to, but not precisely equal to, the difference in long-run alphas between the value and growth deciles because of the different smoothing parameters applied to each portfolio. There is no monotonic pattern for the long-run betas of the book-to-market portfolios, but the book-to-market strategy has a significantly negative long-run beta of -0.218 with a standard error of 0.008.

We test if the long-run alphas across all 10 book-to-market portfolios are equal to zero using the Wald test of equation (23). The Wald test statistic is 31.6 with a p-value less than 0.001. Thus, the book-to-market portfolios overwhelmingly reject the null of the conditional CAPM with time-varying betas.

Figure 1 compares the long-run alphas with OLS alphas. We plot the long-run alphas using squares with 95% confidence intervals displayed in the solid error bars. The point estimates of the OLS alphas are plotted as circles with 95% confidence intervals in dashed lines. Portfolios 1-10 on the $x$-axis represent the growth to value decile portfolios. Portfolio 11 is the book-to-market strategy. The spread in OLS alphas is greater than the spread in long-run alphas, but the standard error bands are very similar for both the long-run and OLS estimates, despite our procedure being nonparametric. For the book-to-market strategy, the OLS alpha is 7.73% compared to a long-run alpha of 6.81%. Thus accounting for time-varying betas has reduced the OLS alpha by approximately only 1.1%.

## 4.2 Time Variation of Conditional Betas

We now characterize the time variation of conditional betas from the one-factor market model.[13] We begin by testing for constant conditional alphas and betas using the Wald test of Theorem 3. Table 3 shows that for all book-to-market portfolios, we fail to reject the hypothesis that the conditional alphas are constant, with Wald statistics that are far below the 95% critical values. Note that this does not mean that the conditional alphas are equal to zero, as we estimate a highly significant long-run alpha of the book-to-market strategy and reject that the long-run alphas are jointly equal to zero across book-to-market portfolios. In contrast, we reject the null that the conditional betas are constant with p-values that are effectively zero.

Figure 2 charts the annualized estimates of conditional betas for the growth (decile 1) and value (decile 10) portfolios at a monthly frequency. Conditional factor loadings are estimated relatively precisely with tight 95% confidence bands, shown in dashed lines. Growth betas are largely constant around 1.2, except after 2000 where growth betas decline to around one. In contrast, conditional betas of value stocks are much more variable, ranging from close to 1.3 in 1965 and around 0.45 in 2000. From this low, value stock betas increase to around one at the end of the sample. We attribute the low relative returns of value stocks in the late 1990s to the low betas of value stocks at this time.

In Figure 3, we plot betas of the book-to-market strategy, which is the difference in returns between deciles 10 and 1 (value minus growth). Since the conditional betas of growth stocks are fairly flat, almost all of the time variation of the conditional betas of the book-to-

[13] Point estimates of conditional alphas cannot be (asymptotically) interpreted without the time-normalization assumption in equation (21). We analyze conditional alphas under this assumption in Appendix C.

market strategy is driven by the conditional betas of the decile 10 value stocks. Figure 3 also overlays estimates of conditional betas from a backward-looking, flat 12-month filter. Similar filters are employed by Andersen et al. (2006) and Lewellen and Nagel (2006). Not surprisingly, the 12-month uniform filter produces estimates with larger conditional variation. Some of this conditional variation is smoothed away by using the longer bandwidths of our optimal estimators.[14] However, the unconditional variation over the whole sample of the uniform filter estimates and the optimal estimates are similar. For example, the standard deviation of end-of-month conditional beta estimates from the uniform filter is 0.276, compared to 0.206 for the optimal two-sided conditional beta estimates. This implies that Lewellen and Nagel's (2006) analysis using backward-looking uniform filters is conservative. Using our optimal estimators reduces the overall volatility of the conditional betas making it even more unlikely that the value premium can be explained by time-varying market factor loadings.

Several authors like Jagannathan and Wang (1996) and Lettau and Ludvigson (2001b) argue that value stock betas increase during times when risk premia are high causing value stocks to carry a premium to compensate investors for bearing this risk. Theoretical models of risk predict that betas on value stocks should vary over time and be highest during times when marginal utility is high (see for example, Gomes, Kogan, and Zhang, 2003; Zhang, 2005). We investigate how betas move over the business cycle in Table 4 where we regress conditional betas of the value-growth strategy onto various macro factors. Kanaya and Kristensen (2010) provide theoretical justification for this two-step procedure in a continuous-time setting.

In Table 4, we find only weak evidence that the book-to-market strategy betas increase during bad times. Regressions I-IX examine the covariation of conditional betas with individual macro factors known to predict market excess returns. When dividend yields are high, the market risk premium is high, and regression I shows that conditional betas covary positively with dividend yields. However, this is the only variable that has a significant coefficient with the correct sign. When bad times are proxied by high default spreads, high short rates, or high market volatility, conditional betas of the book-to-market strategy tend to be lower. During NBER recessions conditional betas also move the wrong way and tend to be lower. The industrial production, term spread, Lettau and Ludvigson's (2001a) $cay$, and inflation regressions have insignificant coefficients. The industrial production coefficient also has the wrong predicted sign.

[14] The standard error bands of the uniform filters (not shown) are much larger than the standard error bands of the optimal estimates.

In regression X, we find that book-to-market strategy betas do have significant covariation with many macro factors. This regression has an impressive adjusted $R^2$ of 55%. Except for the positive and significant coefficient on the dividend yield, the coefficients on the other macro variables: the default spread, industrial production, short rate, term spread, market volatility, and $cay$ are either insignificant or have the wrong sign, or both. In regression XI, we perform a similar exercise to Petkova and Zhang (2005). We first estimate the market risk premium by running a first-stage regression of excess market returns over the next quarter onto the instruments in regression X measured at the beginning of the quarter. We define the market risk premium as the fitted value of this regression at the beginning of each quarter. We find that in regression XI, there is small positive covariation of conditional betas of value stocks with these fitted market risk premia with a coefficient of 0.37 and a standard error of 0.18. But, the adjusted $R^2$ of this regression is only 0.06. This is smaller than the covariation that Petkova and Zhang (2005) find because they specify betas as linear functions of the same state variables that drive the time variation of market risk premia. In summary, although conditional betas do covary with macro variables, there is little evidence that betas of value stocks are higher during times when the market risk premium is high.

## 4.3 Tests of the Conditional Fama-French (1993) Model

We now investigate the performance of a conditional version of the Fama and French (1993) model estimated on the book-to-market portfolios and the book-to-market strategy. Table 5 reports long-run alphas and factor loadings. After controlling for the Fama-French factors with time-varying factor loadings, the long-run alphas of the book-to-market portfolios are still significantly different from zero and are positive for growth stocks and negative for value stocks. The long-run alphas monotonically decline from 2.16% for decile 1 to -1.58% for decile 10. The book-to-market strategy has a long-run alpha of -3.75% with a standard error of 0.010. The joint test across all ten book-to-market portfolios for the long-run alphas equal to zero decisively rejects with a p-value of zero. Thus, the conditional Fama and French (1993) model is overwhelmingly rejected.

Table 5 shows that long-run market factor loadings have only a small spread across growth to value deciles, with the book-to-market strategy having a small long-run market loading of 0.191. In contrast, the long-run $SMB$ loading is relatively large at 0.452, and would be zero if the value effect were uniform across stocks of all sizes. Value stocks have a small size bias (see Loughran, 1997) and this is reflected in the large long-run $SMB$ loading. We expect, and find, that long-run $HML$ loadings increase from -0.672 for growth stocks to 0.792 for value stocks, with the book-to-market strategy having a long-run $HML$ loading of 1.466. The previously positive long-run alphas for value stocks under the conditional CAPM become negative under the conditional Fama-French model. The conditional Fama-French model over-compensates for the high returns for value stocks by producing $SMB$ and $HML$ factor loadings that are relatively too large, leading to a negative long-run alpha for value stocks.

In Table 6, we conduct constancy tests of the conditional alphas and factor loadings. We fail to reject that the conditional alphas are constant for all book-to-market portfolios. Whereas the conditional betas exhibited large time variation in the conditional CAPM, we now cannot reject that the conditional market factor loadings are constant for decile portfolios 3-9. However, the extreme growth and value deciles do have time-varying $MKT$ betas. Table 6 reports rejections at the 99% level that the $SMB$ loadings and $HML$ loadings are constant for the extreme growth and value deciles. For the book-to-market strategy, there is strong evidence that the $SMB$ and $HML$ loadings vary over time, especially for the $HML$ loadings. Consequently, the time variation of conditional betas in the one-factor model is now absorbed by time-varying $SMB$ and $HML$ loadings in the conditional Fama-French model.

We plot the conditional factor loadings in Figure 4. Market factor loadings range between zero and 0.5. The $SMB$ loadings generally remain above 0.5 until the mid-1970s and then decline to approximately 0.2 in the mid-1980s. During the 1990s the $SMB$ loadings strongly trended upwards, particularly during the late 1990s bull market. This is a period where value stocks performed poorly and the high $SMB$ loadings translate into larger negative conditional Fama-French alphas during this time. After 2000, the $SMB$ loadings decrease to end the sample around 0.25.

Figure 4 shows that the $HML$ loadings are well above one for the whole sample and reach a high of 1.91 in 1993 and end the sample at 1.25. Value strategies perform well coming out of the early 1990s recession and the early 2000s recession, and $HML$ loadings during these periods actually decrease for the book-to-market strategy. One may expect that the $HML$ loadings should be constant because $HML$ is constructed by Fama and French (1993) as a zero-cost mimicking portfolio to go long value stocks and short growth stocks, which is precisely what the book-to-market strategy does. However, the breakpoints of the $HML$ factor are quite different, at approximately thirds, compared to the first and last deciles of firms in the book-to-market strategy. The fact that the $HML$ loadings vary so much over time indicates that growth and value stocks in the 10% extremes covary quite differently with average growth and value stocks in the middle of the distribution. Put another way, the 10% tail value stocks are not simply levered versions of value stocks with lower and more typical book-to-market ratios.

# 5 Portfolios Sorted on Past Returns

We end by testing the conditional Fama and French (1993) model on decile portfolios sorted by past returns. These portfolios are well known to strongly reject the null of the standard Fama and French model with constant alphas and factor loadings. In Table 7, we report long-run estimates of alphas and Fama-French factor loadings for each portfolio and the 10-1 momentum strategy. The long-run alphas range from -4.68% with a standard error of 0.014 for the first loser decile to 2.97% with a standard error of 0.010 to the tenth loser decile. The momentum strategy has a long-run alpha of 8.11% with a standard error of 0.019. A joint test that the long-run alphas are equal to zero rejects with a p-value of zero. Thus, a conditional version of the Fama-French model cannot price the momentum portfolios.

Table 7 shows that there is no pattern in the long-run market factor loading across the momentum deciles and the momentum strategy is close to market neutral in the long run with a long-run beta of 0.074. The long-run $SMB$ loadings are small, except for the loser and winner deciles at 0.385 and 0.359, respectively. These effectively cancel in the momentum strategy, which is effectively $SMB$ neutral. Finally, the long-run $HML$ loading for the winner portfolio is noticeably negative at -0.175. The momentum strategy long-run $HML$ loading is -0.117 and the negative sign means that controlling for a conditional $HML$ loading exacerbates the momentum effect, as firms with negative $HML$ exposures have low returns on average.

We can judge the impact of allowing for conditional Fama-French loadings in Figure 5 which graphs the long-run alphas of the momentum portfolios 1-10 and the long-run alpha of the momentum strategy (portfolio 11 on the graph). We overlay the OLS alpha estimates which assume constant factor loadings. The momentum strategy has a Fama-French OLS alpha of 16.7% with a standard error of 0.026. Table 7 reports that the long-run alpha controlling for time-varying factor loadings is 8.11%. Thus, the conditional factor loadings have lowered the momentum strategy alpha by almost 9% but this still leaves a large amount of the momentum effect unexplained. Figure 5 shows that the reduction of the absolute values of OLS alphas compared to the long-run conditional alphas is particularly large for both the extreme loser and winner deciles.

Figure 6 shows that all the Fama-French conditional factor loadings vary significantly over time and their variation is larger than the case of the book-to-market portfolios. Formal constancy tests (not reported) overwhelmingly reject the null of constant Fama-French factor loadings. Whereas the standard deviation of the conditional betas is around 0.2 for the book-to-market strategy (see Table 2), the standard deviations of the conditional Fama-French betas are 0.386, 0.584, and 0.658 for $MKT$, $SMB$, and $HML$, respectively. Figure 6 also shows a marked common covariation of these factor loadings, with a correlation of 0.61 between conditional $MKT$ and $SMB$ loadings and a correlation of 0.43 between conditional $SMB$ and $HML$ loadings. During the early 1970s all factor loadings generally increased and all factor loadings also generally decrease during the late 1970s and through the 1980s. Beginning in 1990, all factor loadings experience a sharp run up and also generally trend downwards over the mid- to late 1990s. At the end of the sample the conditional $HML$ loading is still particularly high at over 1.5. Despite this very pronounced time variation, conditional Fama-French factor loadings still cannot completely price the momentum portfolios.

# 6 Conclusion

We develop a new nonparametric methodology for estimating conditional factor models. We derive asymptotic distributions for conditional alphas and factor loadings at any point in time and also for long-run alphas and factor loadings averaged over time. We also develop a test for the null hypothesis that the conditional alphas and factor loadings are constant over time. The tests can be run for a single asset and also jointly across a system of assets. In the special case of no time variation in the conditional alphas and factor loadings and homoskedasticity, our tests reduce to the well-known Gibbons, Ross, and Shanken (1989) test.

We apply our tests to decile portfolios sorted by book-to-market ratios and past returns. We find significant variation in factor loadings, but overwhelming evidence that a conditional CAPM and a conditional version of the Fama and French (1993) model cannot account for the value premium or the momentum effect. Joint tests for whether long-run alphas are equal to zero in the presence of time-varying factor loadings decisively reject for both the conditional CAPM and Fama-French models. We also find that conditional market betas for a book-to-market strategy exhibit little covariation with market risk premia. Consistent with the book-to-market and momentum portfolios rejecting the conditional models, accounting for time-varying factor loadings only slightly reduces the OLS alphas from the unconditional CAPM and Fama-French regressions which assume constant betas.

Our tests are easy to implement, powerful, and can be estimated asset-by-asset, just as in the traditional classical Gibbons, Ross, and Shanken (1989) test. There are many further empirical applications of the tests to other conditional factor models and other sets of portfolios, especially situations where betas are dynamic, such as many derivative trading, hedge fund returns, and time-varying leverage strategies. Theoretically, the tests can also be extended to incorporate adaptive estimators to take account the bias at the endpoints of the sample. Such estimators can also be adapted to yield estimates of future conditional alphas or factor loadings that do not use forward-looking information.

# Appendix

# A Technical Assumptions

Our theoretical results are derived under the assumption that the true data-generating process is given by equations (13) and (15). Throughout the appendix all assumptions and arguments are stated conditionally on the realizations of $\alpha(t)$, $\beta(t)$, $\mu_F(t)$, $\Lambda_{FF}(t)$, and $\Sigma(t)$. We assume that we have observed data at $-cT \leq t \leq (1+c)T$ for some fixed $c > 0$ chosen so that the end-point bias is negligible to avoid any boundary issues and to keep the notation simple.

Let $C^2[0,1]$ denote the space of twice continuously differentiable functions on the interval, $[0,T]$. We impose the following assumptions:

**A.1** There exists $B, L < \infty$ and $\nu > 1$ such that: $|K(u)| \leq B\|u\|^{-\nu}$ for $\|u\| \geq L$; either (i) $K(u) = 0$ for $\|u\| > L$ and $|K(u) - K(u')| \leq B\|u - u'\|$, or (ii) $K(u)$ is differentiable with $|\partial K(u)/\partial u| \leq B$ and $|\partial K(u)/\partial u| \leq B\|u\|^{-\nu}$ for $\|u\| \geq L$; $\int_{\mathbb{R}} K(z)\,dz = 1$, $\int_{\mathbb{R}} zK(z)\,dz = 0$ and $\kappa_2 := \int_{\mathbb{R}} z^2K(z)\,dz < \infty$.

**A.2** The realizations $\alpha(t)$, $\beta(t)$, $\mu_F(t)$, $\Lambda_{FF}(t)$, and $\Sigma(t)$ all lie in $C^2[0,T]$. Furthermore, $\Lambda_{FF}(t)$ is positive definite for any $t \in [0,T]$.

**A.3** Conditional on the realizations in A.2: The errors $z_i$ and $u_i$ are i.i.d. with $E[z_i] = 0$, $E[u_i] = 0$, $E[z_iz_i'] = I_M$ and $E[u_iu_i'] = I_J$.

**A.4** The processes $\alpha(t)$, $\beta(t)$, $\Lambda_{FF}(t)$, and $\Sigma(t)$ are stationary, ergodic, and bounded.

**A.5** The bandwidth satisfies $\Delta^{-1}(Th)^4 \to 0$, $\Delta/(Th)^2 \to 0$, and $\Delta^{1-\epsilon}/(Th)^{7/4} \to 0$ for some $\epsilon > 0$.

Most standard kernels, including the Gaussian and the uniform one, satisfy A.1. The smoothness conditions in A.2 rule out jumps in the coefficients; Theorem 1 and 4 remain valid at all points where no jumps have occurred, and we conjecture that Theorems 2 and 3 remain valid with a finite jump activity since this will have a minor impact as we smooth over the whole time interval. One could exchange $C^2[0,T]$ for the space of Lipschitz functions of order $\lambda > 0$, $\|f(t) - f(t')\| \leq C|t-t|^{\lambda}$ with all the theoretical results remaining correct after suitably adjustment of the requirements imposed on the bandwidth $h$. The requirement that $\Lambda_{FF}(t) > 0$ is a local version of the rank condition known from OLS. The i.i.d. assumption in A.3 can be replaced by mixing or martingale conditions; we however maintain the i.i.d. assumption for simplicity since the proofs otherwise would become more involved. Since all the moment conditions have to hold conditionally on the realizations, A.3 rules out leverage effects; we conjecture that leverage effects can be accommodated by employing arguments similar to the ones used in the realized volatility literature, see e.g. Foster and Nelson (1996). Assumption A.4 is imposed when we analyze the long-run alpha and beta estimators to ensure that the long-run alphas and betas are well-defined as the limits of sample averages. The conditions on the bandwidth in A.5 are needed for the estimators of the long-run alphas and betas and when testing the null of constant alphas or betas; the assumption entails undersmoothing (relative to the point optimal choice of $h$ in terms of MSE) in the computation of the long-run estimates and the test statistics.

# B Proofs

**Proof of Theorem 1.** Define $\bar{h} = hT$ such that $\hat{\beta}(t)$ can be rewritten as

$$\hat{\beta}(t) = \left[\Delta^2 \sum_{i=1}^{n} K_{\bar{h}}(t_i - t) f_i f_i' - \Delta \bar{f}(t)\bar{f}(t)'\right]^{-1} \left[\Delta^2 \sum_{t=1}^{n} K_{\bar{h}}(t_i - t) f_i R_i' - \Delta \bar{f}(t)\bar{R}(t)'\right],$$

where $\bar{f}(t) = \Delta\sum_{i=1}^{n} K_{\bar{h}}(t_i - t) f_i$ and $\bar{R}(t) = \Delta\sum_{i=1}^{n} K_{\bar{h}}(t_i - t) R_i$. Since

$$f_i = \Delta F_i/\Delta = \mu_F(t_i) + \frac{1}{\sqrt{\Delta}}\Lambda_{FF}^{1/2}(t_i)\,u_i,$$

we obtain that

$$\bar{f}(t) = \Delta\sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\,\mu_F(t_i) + \sqrt{\Delta}\sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\,\Lambda_{FF}^{1/2}(t_i)\,u_i =: \bar{f}_1(t) + \bar{f}_2(t).$$

Using standard results for Riemann sums and kernel estimators,

$$\bar{f}_1(t) = \Delta \sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, \mu_F(t_i) = \int_0^T K_{\bar{h}}(s-t)\, \mu_F(s)\, ds + O(\Delta) = \mu_F(t) + O\left(\bar{h}^2\right) + O(\Delta).$$

The second term has mean zero while its variance satisfies

$$\operatorname{var}\left(\bar{f}_2(\tau)\right) = \Delta \sum_{i=1}^{n} K_{\bar{h}}^2(t_i - t)\, \Lambda_{FF}(t_i) = \frac{1}{\bar{h}} \left\{\kappa_2 \Lambda_{FF}(t) + O\left(\bar{h}^2\right) + O(\Delta)\right\},$$

where we employed the same arguments as in the analysis of the first term. A similar analysis can be carried out for $\bar{R}(t)$ and we conclude that

$$\bar{f}(t) = \mu_F(t) + O_P(\bar{h}^2) + O_P(1/\sqrt{\bar{h}}), \quad \bar{R}(t) = \alpha(t) + \beta(t)' \mu_F(t) + O_P(\bar{h}^2) + O_P(1/\sqrt{\bar{h}}). \tag{B.1}$$

Similarly,

$$\begin{aligned}
\Delta^2 \sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, f_i f_i' &= \Delta \sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, \Lambda_{FF}^{1/2}(t_i)\, u_i u_i' \Lambda_{FF}^{1/2}(t_i) + \Delta^2 \sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, \mu_F(t_i)\, \mu_F(t_i)' \\
&\quad + \Delta^2 \sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, \mu_F(t_i)\, u_i' \Lambda_{FF}^{1/2}(t_i) \\
&= \Lambda_{FF}(t) + \Delta \mu_F(t)\, \mu_F(t)' + O_P\left(\bar{h}^2\right) + O_P(\sqrt{\Delta/\bar{h}}),
\end{aligned} \tag{B.2}$$

and

$$\begin{aligned}
\Delta^2 \sum_{t=1}^{n} K_{\bar{h}}(t_i - t)\, f_i R_i' &= \Delta^2 \sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, f_i f_i' \beta(t_i) + \Delta^2 \sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, f_i \alpha(t_i)' \\
&\quad + \Delta^{3/2} \sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, f_i z_i' \Sigma^{1/2}(t_i) \\
&= \Lambda_{FF}(t)\, \beta(t) + \Delta \mu_F(t)\, \alpha(t)' + O_P\left(\bar{h}^2\right) + \sqrt{\Delta/\bar{h}}\, U_n(t),
\end{aligned} \tag{B.3}$$

where, by a CLT for Martingales (see e.g. Brown, 1971),

$$U_n(t) := \Delta \sqrt{\bar{h}} \sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, f_i z_i' \Sigma^{1/2}(t_i) \xrightarrow{d} N\left(0, \kappa_2 \Lambda_{FF}(t) \otimes \Sigma(t)\right).$$

Collecting the results of equations (B.1)-(B.3), we obtain

$$\begin{aligned}
&\sqrt{\Delta^{-1}\bar{h}} \left\{\Delta^2 \sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, f_i R_i' - \Delta \bar{f}(t)\, \bar{R}(t)' - \Lambda_{FF}(t)\, \beta(t)\right\} \\
&= U_n(t) + o_P(1) \xrightarrow{d} N\left(0, \kappa_2 \Lambda_{FF}(t) \otimes \Sigma(t)\right),
\end{aligned} \tag{B.4}$$

and

$$\Delta^2 \sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, f_i f_i' - \Delta \bar{f}(t)\, \bar{f}(t)' = \Lambda_{FF}(t) + o_P(1).$$

This yields the claimed result. ■

**Proof of Theorem 2.** The proof proceeds along the same lines as in Kristensen (2010, Proof of Theorem 4) and so we only sketch the arguments. First consider $\hat{\beta}_{\mathrm{LR}}$. As demonstrated in the proof of Theorem 1, with $\bar{h} = hT$,

$$\begin{aligned}
\hat{\beta}(t) &\simeq \left[\sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, \Delta F_i \Delta F_i'\right]^{-1} \left[\sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, \Delta F_i \Delta F_i' \beta(t_i)\right] \\
&\quad + \sqrt{\Delta} \left[\sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, \Delta F_i \Delta F_i'\right]^{-1} \left[\sum_{i=1}^{n} K_{\bar{h}}(t_i - t)\, \Delta F_i z_i' \Sigma^{1/2}(t_i)\right],
\end{aligned}$$

where, uniformly over $t \in [0, T]$,

$$\sum_{i=1}^{n} K_{\bar{h}}(t_i - t) \Delta F_i \Delta F_i' = \Lambda_{FF}(t) + O_P(\bar{h}^2) + O_P\left(\sqrt{\Delta/\bar{h}}\right).$$

Thus,

$$\begin{aligned}
\hat{\beta}_{\text{LR}} &\simeq \frac{1}{n}\sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\Lambda_{FF}^{-1}(t_j) K_{\bar{h}}(t_i - t_j)\right\}\Delta F_i \Delta F_i' \beta(t_i) \\
&\quad + \frac{\sqrt{\Delta}}{n}\sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\Lambda_{FF}^{-1}(t_j) K_{\bar{h}}(t_i - t_j)\right\}\Delta F_i z_i' \Sigma^{1/2}(t_i) \\
&\simeq \frac{1}{n\Delta}\sum_{i=1}^{n}\Lambda_{FF}^{-1}(t_i)\Delta F_i \Delta F_i' \beta(t_i) + \frac{1}{n\sqrt{\Delta}}\sum_{i=1}^{n}\Lambda_{FF}^{-1}(t_i)\Delta F_i z_i' \Sigma^{1/2}(t_i) \\
&\equiv B_1 + B_2.
\end{aligned}$$

Here and in the following, "$\simeq$" is used to denote that the left and right hand side are identical up to some higher-order term which is asymptotically negiglible under our assumptions. Since $\Delta F_i \Delta F_i'/\Delta \simeq \Lambda_{FF}(t_i)$, $B_1 \simeq n^{-1}\sum_{i=1}^{n}\beta(t_i) \simeq T^{-1}\int_0^T \beta(s)\,ds$. The second term is a martingale with quadratic variation

$$\begin{aligned}
\langle B_2 \rangle &= \frac{1}{n^2\Delta}\sum_{i=1}^{n}\Lambda_{FF}^{-1}(t_i)\Delta F_i \Delta F_i' \Lambda_{FF}^{-1}(t_i) \otimes \Sigma(t_i) \simeq \frac{1}{n^2}\sum_{i=1}^{n}\Lambda_{FF}^{-1}(t_i) \otimes \Sigma(t_i) \\
&\simeq \frac{1}{nT}\int_0^T \Lambda_{FF}^{-1}(s) \otimes \Sigma(s)\,ds.
\end{aligned}$$

The result for the LR betas now follows by the CLT for martingales together with the LLN for stationary and ergodic processes.

Next, consider $\hat{\alpha}_{\text{LR}}$: With $\bar{f}(t) = \sum_{i=1}^{n} K_{\bar{h}}(t_i - t) f_i$ and $\bar{R}(t) = \sum_{i=1}^{n} K_{\bar{h}}(t_i - t) R_i$,

$$\hat{\alpha}_{\text{LR}} = \frac{1}{n}\sum_{j=1}^{n}\hat{\alpha}(t_j) = \frac{1}{n}\sum_{j=1}^{n}\frac{\bar{R}(t_j) - \hat{\beta}(t_j)\bar{f}(t_j)}{\sum_{j=1}^{n} K_{\bar{h}}(t_i - t_j)},$$

where uniformly over $t \in [0, T]$, $\Delta\sum_{i=1}^{n} K_{\bar{h}}(t_i - t) \simeq 1$ and $\hat{\beta}(t) \simeq \beta(t)$. Thus,

$$\begin{aligned}
\hat{\alpha}_{\text{LR}} &\simeq \frac{\Delta}{n}\sum_{j=1}^{n}\left\{\bar{R}(t_j) - \beta(t_j)\bar{f}(t_j)\right\} \\
&= \frac{1}{n\Delta}\sum_{i=1}^{n}\left\{\Delta\sum_{j=1}^{n} K_{\bar{h}}(t_i - t_j)\right\}\Delta s_i - \frac{1}{n}\sum_{i=1}^{n}\left\{\sum_{j=1}^{n} K_{\bar{h}}(t_i - t_j)\beta(t_j)\right\}\Delta F_i \\
&\simeq \frac{1}{n\Delta}\sum_{i=1}^{n}\left\{\Delta s_i - \beta(t_i)\Delta F_i\right\} \\
&= \frac{1}{n}\sum_{i=1}^{n}\alpha(t_i) + \frac{1}{n\sqrt{\Delta}}\sum_{i=1}^{n}\Sigma^{1/2}(t_i) z_i \\
&\equiv A_1 + A_2.
\end{aligned}$$

By the same arguments as for $\hat{\beta}_{\text{LR}}$, $A_1 \simeq T^{-1}\int_0^T \alpha(s)\,ds$, while $A_2$ is a martingale with quadratic variation

$$\langle A_2 \rangle = \frac{1}{n^2\Delta}\sum_{i=1}^{n}\Sigma(t_i) \simeq \frac{1}{T^2}\int_0^T \Sigma(s)\,ds.$$

The result for the LR alphas now follows by the CLT for martingales and the LLN for stationary and ergodic processes. ■

**Proof of Theorem 3.** The proof follows along the same lines as the proof of Kristensen (2011, Theorem 3.3) and so we only sketch it. We suppress the index $k$ in the following, and first consider the test statistic for constant alphas, $W(\alpha)$: Let $\alpha(t) = \alpha$ denote the true, constant alpha realization, and for simplicity suppose that $F_t := 0$ such that the DGP under $H(\alpha)$ is $\Delta s_i = \alpha\Delta + \sqrt{\Delta}\sigma(t_i) z_i$; the extension to $F_t \neq 0$ is straightforward but tedious. Using that $\Delta \sum_{j=1}^{n} K_{\bar{h}}(t_j - t) \simeq 1$ uniformly over $t \in [0, T]$, the nonparametric estimators of $\alpha$ can therefore be written as

$$\hat{\alpha}(t) = \frac{\sum_{j=1}^{n} K_{\bar{h}}(t_j - t)\Delta s_j}{\Delta \sum_{j=1}^{n} K_{\bar{h}}(t_j - t)} = \alpha + \frac{\sum_{j=1}^{n} K_{\bar{h}}(t_j - t)\sigma(t_j) z_j}{\sqrt{\Delta}\sum_{j=1}^{n} K_{\bar{h}}(t_j - t)} \simeq \alpha + \sqrt{\Delta}\sum_{j=1}^{n} K_{\bar{h}}(t_j - t)\sigma(t_j) z_j, \quad \text{(B.5)}$$

where $\bar{h} = hT$. Since $\hat{\alpha}_{\text{LR}}$ is $\sqrt{T}$-consistent, we are allowed to set $\hat{\alpha}_{\text{LR}} = \alpha$. Also, by the same arguments as in Kristensen (2011, Proof of Theorem 3.3), we can treat $\sigma(t)$ as known. Thus,

$$\begin{aligned}
W(\alpha) &\simeq \frac{1}{n}\sum_{i=1}^{n} \sigma^{-2}(t_i)\left[\hat{\alpha}(t) - \alpha\right]^2 \\
&\simeq \frac{1}{n}\sum_{i=1}^{n} \sigma^{-2}(t_i)\left[\sqrt{\Delta}\sum_{j=1}^{n} K_{\bar{h}}(t_j - t)\sigma(t_j) z_j\right]^2 \\
&\simeq \frac{\Delta}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \sigma^{-2}(t_i) K_{\bar{h}}(t_j - t_i) K_{\bar{h}}(t_k - t_i)\sigma(t_j)\sigma(t_k) z_j z_k \\
&= \frac{1}{n}\sum_{j=1}^{n}\left\{\Delta\sum_{i=1}^{n} K_{\bar{h}}^2(t_j - t_i)\sigma^{-2}(t_i)\right\}\sigma^2(t_j) z_j^2 \\
&\quad + \frac{1}{n}\sum_{j\neq k}\left\{\Delta\sum_{i=1}^{n} K_{\bar{h}}(t_j - t_i) K_{\bar{h}}(t_k - t_i)\sigma^{-2}(t_i)\right\}\sigma(t_j)\sigma(t_k) z_j z_k,
\end{aligned}$$

where, uniformly over $t_j, t_k \in [0, T]$,

$$\begin{aligned}
\Delta\sum_{i=1}^{n} K_{\bar{h}}(t_j - t_i) K_{\bar{h}}(t_k - t_i)\sigma^{-2}(t_i) &\simeq \int_0^T K_{\bar{h}}(t_j - s) K_{\bar{h}}(t_k - s)\sigma^{-2}(s)\, ds \\
&\simeq (K * K)_{\bar{h}}(t_j - t_k)\sigma^{-2}(t_j).
\end{aligned}$$

Thus, since $\kappa_2 = (K * K)(0)$,

$$W(\alpha) \simeq \frac{\kappa_2}{n\bar{h}}\sum_{j=1}^{n} z_j^2 + \frac{1}{n}\sum_{j\neq k}\phi_{j,k} \simeq \frac{\kappa_2}{\bar{h}} + \frac{1}{n}\sum_{j\neq k}\phi_{j,k},$$

where $\phi_{j,k} := (K * K)_{\bar{h}}(t_j - t_k)\sigma^{-1}(t_j)\sigma(t_k) z_j z_k$. We recognize $\sum_{j\neq k}\phi_{j,k}$ as a degenerate $U$-statistic. Since

$$\begin{aligned}
\text{Var}\Big(\sum_{j\neq k}\phi_{j,k}\Big) &= 2\sum_{j\neq k}(K * K)_{\bar{h}}^2(t_j - t_k)\sigma^{-2}(t_j)\sigma^2(t_k) E\left[z_j^2\right] E\left[z_k^2\right] \\
&= 2\sum_{j\neq k}(K * K)_{\bar{h}}^2(t_j - t_k)\sigma^{-2}(t_j)\sigma^2(t_k) \\
&\simeq \frac{2}{\Delta^2}\int_0^T\int_0^T (K * K)_{\bar{h}}^2(s - t)\sigma^{-2}(s)\sigma^2(t)\, ds dt \\
&\simeq \frac{2}{\Delta^2\bar{h}}\int (K * K)^2(z)\, dz,
\end{aligned}$$

it follows by standard arguments for degenerate $U$-statistics that $\sqrt{\bar{h}}\Delta \sum_{j\neq k}\phi_{j,k} \to^d N\left(0, 2\int (K*K)^2(z)\,dz\right)$. In total, with $m(\alpha) = \kappa_2/\bar{h}$ and $v^2(\alpha) = 2\int (K*K)^2(z)\,dz/\left(T^2\bar{h}\right)$, $[W(\alpha) - m(\alpha)]/v(\alpha) \xrightarrow{d} N(0,1)$. This shows the result.

Next, consider $W(\beta)$: We focus on the simple case where $\alpha(t) = 0$ such that, under $H(\beta)$, $\Delta s_i = \beta'\Delta F_i + \sqrt{\Delta}\sigma(t_i) z_i$. The nonparametric estimators of $\beta$ can therefore be written as

$$\hat{\beta}(t) \simeq \beta + \Lambda_{FF}^{-1}(t)\sqrt{\Delta}\sum_{j=1}^{n} K_{\bar{h}}(t_j - t)\,\Delta F_j \sigma(t_j) z_j,$$

and we obtain, using that $\Delta F_j \Delta F_j' \simeq \Delta \Lambda_{FF}^{-1}(t_j)$,

$$\begin{aligned}
W(\beta) &\simeq \frac{1}{n}\sum_{i=1}^{n}\sigma^{-2}(t_i)\left[\hat{\beta}(t_i) - \beta\right]'\Lambda_{FF}(t_i)\left[\hat{\beta}(t_i) - \beta\right] \\
&\simeq \frac{1}{n}\sum_{j=1}^{n}\sigma^2(t_j) z_j^2 \Delta F_j' \left\{\Delta\sum_{i=1}^{n}\Lambda_{FF}^{-1}(t_i)\sigma^{-2}(t_i) K_{\bar{h}}^2(t_j - t_i)\right\}\Delta F_j \\
&\quad + \frac{1}{n}\sum_{j\neq k}^{n}\sigma(t_j) z_j \Delta F_j' \left\{\Delta\sum_{i=1}^{n}\Lambda_{FF}^{-1}(t_i)\sigma^{-2}(t_i) K_{\bar{h}}(t_k - t_i) K_{\bar{h}}(t_j - t_i)\right\}\Delta F_k \sigma(t_k) z_k \\
&\simeq \frac{\kappa_2}{n\bar{h}}\sum_{j=1}^{n} z_j^2 \Delta F_j' \Lambda_{FF}^{-1}(t_j)\Delta F_j \\
&\quad + \frac{1}{n}\sum_{j\neq k}^{n}\sigma^{-1}(t_j) z_j \Delta F_j' \Lambda_{FF}^{-1}(t_j)(K*K)_{\bar{h}}(t_j - t_k)\Delta F_k \sigma(t_k) z_k \\
&\simeq \frac{\kappa_2 J\Delta}{\bar{h}} + \frac{1}{n}\sum_{j\neq k}^{n}\phi_{j,k},
\end{aligned}$$

where $\phi_{j,k} = (K*K)_h(t_j - t_k)\,\Delta F_j' \Lambda_{FF}^{-1}(t_j)\Delta F_k \sigma^{-1}(t_j)\sigma(t_k) z_j z_k$. $\sum_{j\neq k}^{n}\phi_{j,k}$ is a degenerate $U$-stastistic satisfying

$$\begin{aligned}
\mathrm{Var}\Big(\sum_{j\neq k}\phi_{j,k}\Big) &= 2\Delta^2 \sum_{j\neq k}(K*K)_{\bar{h}}^2(t_j - t_k)\Lambda_{FF}^{-1}(t_j)\Lambda_{FF}(t_k)\sigma^{-2}(t_j)\sigma^2(t_k) \\
&\simeq \frac{2J}{\bar{h}}\int (K*K)^2(z)\,dz,
\end{aligned}$$

and so $\sqrt{\bar{h}}\sum_{j\neq k}^{n}\phi_{j,k} \to^d N\left(0, 2J\int (K*K)^2(z)\,dz\right)$. In conclusion, with $m(\beta) = \kappa_2 J\Delta/\bar{h}$ and $v^2(\beta) = 2J\int (K*K)^2(z)\,dz/\left(n^2\bar{h}\right)$, $[W(\beta) - m(\beta)]/v(\beta) \xrightarrow{d} N(0,1)$ as desired. ■

# C Conditional Alphas

In this section we develop an asymptotic theory for the conditional alpha estimators under the following additional assumption:

**A.6** The realizations satisfy $\alpha(t) = a(t/T)$, $\beta(t) = b(t/T)$, $\mu_F(t) = m_F(t/T)$, $\Lambda_{FF}(t) = L_{FF}(t/T)$, and $\Sigma(t) = S(t/T)$ for some functions $a(\tau)$, $b(\tau)$, $m_F(\tau)$, $L_{FF}(\tau)$, and $S(\tau)$ that all lie in $C^2[0,1]$.

The normalization imposed in Assumption A.6 is similar to the one used in the structural change (or break points) literature. These models are originally developed by, among others, Andrews (1993) and Bai and Perron (1998). Bekaert, Harvey, and Lumsdaine (2002), Paye and Timmermann (2006), and Lettau and Van Nieuwerburgh (2007), among others, apply these models in finance. In testing for structural breaks with one fixed model, the number of observations before the break remains fixed as the sample size increases, $n \to \infty$, and so identification of the break point is not possible. Instead, the literature normalizes time $t$ by sample size $n$ such that the number of observations both before and after the break increases as $n \to \infty$. This allows the break point and the parameters describing the coefficients before the break to be identified as $n$ increases. The same time normalization is also used in nonparametric methods for detecting structural change as initally proposed in Robinson (1989) and further extended by Cai (2007) and Kristensen (2011). The structural break specifications, where the coefficients take on different (constant) values before and after a break point, are special cases of our more general model which allows for a (finite) number of jumps (see Appendix A).

The same idea of employing a sequence of models to construct valid asymptotic inference is also used in the literature on local-to-unity tests introduced by Chan and Wei (1987) and Phillips (1987). The local-to-unity models employ a sequence of models indexed by sample size to ensure that the same asymptotics apply for both stationary and non-stationary cases. Analogously, our sequence of models scales the underlying parameters by time span in order to construct well-defined asymptotic distributions.

With the time normalization in Assumption A.6, the bias of the conditional alpha estimator changes from the one stated in equation (19) and now becomes $\mathrm{E}[\hat{\alpha}(t)] \simeq \alpha(t) + h^2\alpha^{(2)}(t)$ while the variance in equation (19) remains unchanged. Thus, we can now choose the bandwidth such that the bias vanishes faster than the variance component, and so obtain the following formal result:

**Theorem 4** *Assume that Assumption A.1-A.3 and A.6 hold and the bandwidth is chosen such that $Th \to \infty$ and $Th^5 \to 0$. Then, for any $t \in [0, T]$:*

$$\sqrt{Th}\,\{\hat{\alpha}(t) - \alpha(t)\} \sim N(0, \kappa_2\Sigma(t)) \quad \textit{in large samples.} \tag{C.1}$$

*Moreover, the conditional estimators are asymptotically independent across any set of distinct time points.*

**Proof.** With $\tau := t/T$ and $\tau_i := t_i/T$, we can write the estimator as $\hat{\alpha}\,(t) = \hat{a}\,(\tau T)$ where

$$\hat{a}\,(\tau) = \frac{\sum_{i=1}^{n} K_h\,(\tau_i - \tau)\left\{R_i - \hat{b}\,(\tau)' f_i\right\}}{\sum_{i=1}^{n} K_h\,(\tau_i - \tau)},$$

and $\hat{b}\,(\tau) = \hat{\beta}\,(\tau T)$. Given the normalizations in Assumption A.6, we obtain by the same arguments used in the proof of Theorem 1 that $\hat{b}\,(\tau) = b\,(\tau) + O_P\left(h^2\right) + O_P\left(1/\sqrt{nh}\right)$ such that $R_i - \hat{b}\,(\tau_i)' f_i \simeq a\,(\tau_i) + S^{1/2}\,(\tau_i)\, z_i/\sqrt{\Delta}$. This in turn implies that

$$\sqrt{Th}(\hat{a}\,(\tau) - a\,(\tau)) = O_P\left(\sqrt{Th^5}\right) + \frac{\sqrt{h}}{\sqrt{n}}\sum_{t=1}^{n} K_h\,(\tau_i - \tau)\, S^{1/2}\,(\tau_i)\, z_i, \tag{C.2}$$

where, as $Th \to \infty$,

$$\frac{\sqrt{h}}{\sqrt{n}}\sum_{t=1}^{n} K_{\bar{h}}\,(\tau_i - \tau)\, S^{1/2}\,(\tau_i)\, z_i \xrightarrow{d} N\left(0, \kappa_2 S\,(\tau)\right).$$

■

The crucial assumption in Theorem 4 is the time-normalization in assumption A.6, without which we cannot estimate conditional alphas as discussed in Section 2.5. Given the large literature which identifies the constant terms in rolling-window regression estimates as conditional alphas, Theorem 4 can be used to identify these as estimates of conditional alphas only with the time-normalization assumption in equation (21).

Using Theorem 4, we can test the hypothesis that $\alpha(t) = 0$ jointly across $M$ stocks for a given value of $t \in [0, T]$ by the following Wald statistic:

$$W(t) \equiv Th\hat{\alpha}(t)'\left[\kappa_2\hat{\Sigma}(t)\right]^{-1}\hat{\alpha}(t) \sim \chi^2_M, \tag{C.3}$$

in large samples, where $\hat{\Sigma}(t)$ is given in equation (17). Given independence across distinct time points, we can also construct tests across any given finite set of time points. However, this test is not able to detect all departures from the null, since we only test for departures at a finite number of time points. To test the conditional alphas being equal to zero uniformly over time, we advocate using the test for constancy of the conditional alphas in Section 2.7.

In Figure A-1 we plot conditional alphas from the growth and value portfolios. Again we emphasize that without the time normalization assumption A.6, the estimates of the conditional alphas cannot consistently estimate the true conditional alphas. In Figure A-1 the conditional alphas of both growth and value stocks have fairly wide standard errors, which often encompass zero. These results are similar to Ang and Chen (2007) who cannot reject that conditional alphas of value stocks is equal to zero over the post-1926 sample. Conditional alphas of growth stocks are significantly negative during 1975-1985 and reach a low of -7.09% in 1984. Growth stock conditional alphas are again significantly negative from 2003 to the end of our sample. The conditional alphas of value stocks are much more variable than the conditional alphas of growth stocks, but their standard errors are wider and so we cannot reject that the conditional alphas of value stocks are equal to zero except for the mid-1970s, the early 1980s, and the early 1990s. During the mid-1970s and the early 1980s, estimates of the conditional alphas of value stocks reach approximately 15%. During 1991, value stock conditional alphas decline to below -10%. Interestingly, the poor performance of value stocks during the late 1990s does not correspond to negative conditional alphas for value stocks during this time.

The contrast between the wide standard errors for the conditional alphas in Figure A-1 compared to the tight confidence bands for the long-run alphas in Table 2 is due to the fact that the conditional alpha estimators converge at the nonparametric rate $\sqrt{Th}$ which is less than the classical rate $\sqrt{T}$, and thus the conditional standard error bands are quite wide. This is exactly what Figure A-1 shows and what Ang and Chen (2007) pick up in an alternative parametric procedure. In comparison, the long-run alpha estimators converge at the standard parametric rate $\sqrt{T}$ (see Theorem 2) causing the long-run alphas to have much tighter standard error bounds than the conditional alphas.

# D Gibbons, Ross, and Shanken (1989) as a Special Case

First, we derive the asymptotic distribution of the Gibbons, Ross, and Shanken (1989) [GRS] estimators within the setting of the continuous-time diffusion model. The GRS estimator, which we denote $\tilde{\gamma}_{\text{LR}} = (\tilde{\alpha}_{\text{LR}}, \tilde{\beta}_{\text{LR}})$, is a standard least squares estimator of the form $\tilde{\gamma}_{\text{LR}} = \left[\sum_{i=1}^{n} X_i X_i'\right]^{-1} \left[\sum_{i=1}^{n} X_i R_i'\right]$. Under assumptions A.1-A.2 and A.5,

$$\begin{aligned}
\tilde{\beta}_{\text{LR}} &\simeq \left[\frac{1}{T}\sum_{i=1}^{n} \Delta F_i \Delta F_i'\right]^{-1} \left[\frac{1}{T}\sum_{i=1}^{n} \Delta F_i \Delta F_i' \beta(t_i)\right] + \left[\frac{1}{T}\sum_{i=1}^{n} \Delta F_i \Delta F_i'\right]^{-1} \left[\frac{\Delta^{1/2}}{T}\sum_{i=1}^{n} \Delta F_i z_i' \Sigma^{1/2}(t_i)\right] \\
&\simeq \bar{\beta}_{\text{LR}} + \mathrm{E}\left[\Lambda_{FF}(t)\right]^{-1} \frac{\Delta^{1/2}}{T} \sum_{i=1}^{n} \Delta F_i z_i' \Sigma^{1/2}(t_i)
\end{aligned}$$

where $\sqrt{n}\frac{\Delta^{1/2}}{T}\sum_{i=1}^{n} \Delta F_i z_i' \Sigma^{1/2}(t_i) \xrightarrow{d} N\left(0, \mathrm{E}\left[\Lambda_{FF}(t) \otimes \Sigma(t)\right]\right)$ and

$$\bar{\beta}_{\text{LR}} = \mathrm{E}\left[\Lambda_{FF}(t)\right]^{-1} \mathrm{E}\left[\Lambda_{FF}(t)\beta(t)\right]$$

Thus, $\sqrt{n}(\tilde{\beta}_{\text{LR}} - \bar{\beta}_{\text{LR}}) \to^d N\left(0, \mathrm{E}\left[\Lambda_{FF}(t)\right]^{-1} \mathrm{E}\left[\Lambda_{FF}^2(t) \otimes \Sigma(t)\right] \mathrm{E}\left[\Lambda_{FF}(t)\right]^{-1}\right)$. Next,

$$\begin{aligned}
\tilde{\alpha}_{\text{LR}} &\simeq \frac{1}{T}\sum_{i=1}^{n} \Delta s_i - \hat{\beta}_{\text{LR}}' \frac{1}{T}\sum_{t=1}^{n} \Delta F_i \simeq \frac{1}{T}\sum_{i=1}^{n} \left\{\Delta s_i - \bar{\beta}_{\text{LR}}' \Delta F_i\right\} \\
&= \frac{1}{T}\sum_{i=1}^{n} \left\{\alpha(t_i)\Delta + \left[\beta(t_i) - \bar{\beta}_{\text{LR}}\right]' \Delta F_i\right\} + \frac{\sqrt{\Delta}}{T}\sum_{i=1}^{n} \Sigma^{1/2}(t_i) z_i \\
&\simeq \bar{\alpha}_{\text{LR}} + \frac{\sqrt{\Delta}}{T}\sum_{i=1}^{n} \Sigma^{1/2}(t_i) z_i,
\end{aligned}$$

where $\frac{\sqrt{\Delta}}{\sqrt{T}}\sum_{i=1}^{n} \Sigma^{1/2}(t_i) z_i \xrightarrow{d} N\left(0, \mathrm{E}\left[\Sigma(t)\right]\right)$ and

$$\bar{\alpha}_{\text{LR}} = \mathrm{E}\left[\alpha(t)\right] + \mathrm{E}\left[\left[\beta(t) - \bar{\beta}_{\text{LR}}\right]' \mu_F(t)\right].$$

We conclude that$\sqrt{T}(\tilde{\alpha}_{\text{LR}} - \bar{\alpha}_{\text{LR}}) \xrightarrow{d} N\left(0, \mathrm{E}\left[\Sigma(t)\right]\right)$.

From the above expressions, we see that the GRS estimator $\tilde{\alpha}_{\text{LR}}$ of the long-run alphas in general is inconsistent since it is centered around $\bar{\alpha}_{\text{LR}} \neq \mathrm{E}\left[\alpha(t)\right]$. It is only consistent if the factors are uncorrelated with the loadings such that $\mu_F(t) = \mu$ and $\Lambda_{FF}(t) = \Lambda_{FF}$ are constant, in which case $\bar{\beta}_{\text{LR}} = \beta_{\text{LR}}$ and $\bar{\alpha}_{\text{LR}} = \alpha_{\text{LR}}$.

Finally, we note that in the case of constant alphas and betas and homoskedastic errors, $\Sigma(s) = \Sigma$, the variance of our proposed estimator of $\gamma_{\text{LR}}$ is identical to the one of the GRS estimator.

# E Two-Sided versus One-Sided Filters

We focus on the estimator of $\beta(t)$; the same arguments apply to the alpha estimator. When using a two-sided symmetric kernel where $\mu_1 = \int K(z)\, z dz = 0$ and $\mu_2 = \int K(z)\, z^2 dz < \infty$, the finite-sample variance is, with $\bar{h} = h/T$,

$$\operatorname{var}\left(\hat{\beta}_k(t)\right) \simeq \frac{\Delta}{\bar{h}} v_k(t;\beta) \quad \text{with} \quad v_k(t;\beta) = \kappa_2 \Lambda_{FF}^{-1}(t)\, \sigma_{kk}^2(t), \tag{E.1}$$

while the bias is given by

$$\operatorname{Bias}\left(\hat{\beta}_k(t)\right) \simeq \bar{h}^2 b_k^{\mathrm{sym}}(t) \quad \text{with} \quad b_k^{\mathrm{sym}}(t) = \frac{\mu_2}{2} \beta_k^{(2)}(t), \tag{E.2}$$

where we have assumed that $\beta_k(t)$ is twice differentiable with second order derivative $\beta_k^{(2)}(t)$. In this case the bias is of order $O\left(h^2\right)$. When a one-sided kernel is used, the variance remains unchanged, but since $\mu_1 = \int K(z)\, z dz \neq 0$ the bias now takes the form

$$\operatorname{Bias}\left(\hat{\beta}_k(t)\right) \simeq h b_k^{\mathrm{one}}(t) \quad \text{with} \quad b_k^{\mathrm{one}}(t) = \mu_1 \beta_k^{(1)}(t). \tag{E.3}$$

The bias is in this case of order $O(h)$ and is therefore larger compared to when a two-sided kernel is employed.

As a consequence, for the symmetric kernel the optimal global bandwidth minimizing the integrated MSE, $IMSE = \frac{1}{T}\int_0^T \mathrm{E}[\|\hat{\gamma}_{j,\tau} - \gamma_{j,\tau}\|^2] d\tau$, is given by:

$$h_k^* = \left(\frac{V_k}{B_k^{\mathrm{sym}}}\right)^{1/5} n^{-1/5}, \tag{E.4}$$

where $B_k^{\mathrm{sym}} = \lim_{n\to\infty} n^{-1}\sum_{i=1}^n (b_k^{\mathrm{sym}}(t_i))^2$ and $V_k = \lim_{n\to\infty} n^{-1}\sum_{k=1}^n v(t_i)$ are the integrated versions of the time-varying (squared) bias and variance components. With this bandwidth choice, $\sqrt{IMSE}$ is of order $O\left(n^{-2/5}\right)$. If on the other hand a one-sided kernel is used, the optimal bandwidth is

$$h_k^* = \left(\frac{V_k}{B_k^{\mathrm{one}}}\right)^{1/3} n^{-1/3}, \tag{E.5}$$

where $B_k^{\mathrm{one}} = \lim_{n\to\infty} n^{-1}\sum_{i=1}^n (b_k^{\mathrm{one}}(t_i))^2$, with the corresponding $\sqrt{IMSE}$ being of order $O\left(n^{-1/3}\right)$. Thus, the symmetric kernel estimator's root-MSE is generally smaller and substantially smaller if $n$ is large.[15]

# F Bandwidth Choice for Long-Run Estimators

We sketch the outline of the derivation of an optimal bandwidth for estimating the integrated or long-run betas in a discrete-time setting.[16] We follow the same strategy as in Cattaneo, Crump, and Jansson (2010) and Ichimura and Linton (2005), among others. With $\hat{\beta}_{\mathrm{LR},k}$ denoting the long-run estimators of the alphas and betas for the $k$th asset, first note that by a third order Taylor expansion with respect to $m(t) = \Lambda_{FF}(t)\,\beta(t)$ and $\Lambda_{FF}(t)$,

$$\hat{\beta}_{\mathrm{LR},k} - \beta_{\mathrm{LR},k} = U_{1,n} + U_{2,n} + R_n, \tag{F.1}$$

where $R_n = O\left(\sup_{1\le i\le n} ||\hat{m}_k(t_i) - m_k(t_i)||^3\right) + O\left(\sup_{1\le i\le n} ||\hat{\Lambda}_{FF}(t_i) - \Lambda_{FF}(t_i)||^3\right)$ and

$$U_{1,n} = \Delta \sum_{i=1}^n \left\{\Lambda_{FF}^{-1}(t_i)\left[\hat{m}_k(t_i) - m_k(t_i)\right] - \Lambda_{FF}^{-1}(t_i)\left[\hat{\Lambda}_{FF}(t_i) - \Lambda_{FF}(t_i)\right]\beta_k(t_i)\right\};$$

[15] The two exceptions are if one wishes to estimate alphas and betas at time $t = 0$ and $t = T$. In these cases, the symmetric kernel suffers from boundary bias while a forward- and backward-looking kernel estimator, respectively, remain asymptotically unbiased. We avoid this case in our empirical work by omitting the first and last years in our sample when estimating conditional alphas and betas.

[16] We wish to thank Matias Cattaneo for helping us with this part.

$$
\begin{aligned}
U_{2,n} &= \Delta \sum_{i=1}^{n} \Lambda_{FF}^{-1}(t_i) \left[\hat{\Lambda}_{FF}(t_i) - \Lambda_{FF}(t_i)\right] \Lambda_{FF}^{-1}(t_i) \left[\hat{\Lambda}_{FF}(t_i) - \Lambda_{FF}(t_i)\right] \beta_k(t_i) \\
&\quad - \Delta \sum_{i=1}^{n} \Lambda_{FF}^{-1}(t_i) \left[\hat{\Lambda}_{FF}(t_i) - \Lambda_{FF}(t_i)\right] \Lambda_{FF}^{-1}(t_i) \left[\hat{m}_k(t_i) - m_k(t_i)\right].
\end{aligned}
$$

Thus, our estimator is (approximately) the sum of a second and third order $U$-statistic. We proceed to compute the mean and variance of each of these to obtain a MSE expansion of the estimator as a function of the bandwidth $h$.

To compute the variance of $U_{1,n}$, define $\phi(W_i, W_j) = a(W_i, W_j) + a(W_i, W_j)$, where $W_i = (\Delta F_i, u_i)$,

$$
a(W_i, W_j) = K_{i,j} \Lambda_{FF}^{-1}(t_i) \Delta F_j \varepsilon_j' + K_{i,j} \Lambda_{FF}^{-1}(t_i) \Delta F_j \Delta F_j' \left[\beta_k(t_j) - \beta_k(t_i)\right], \tag{F.2}
$$

and $\varepsilon_{k,i} \equiv \sigma_{kk}(t_i) z_i$. Observe that

$$
\mathrm{E}\left[\phi(w, W_j) \phi(w, W_j)'\right] = \mathrm{E}\left[a(w, W_j) a(w, W_j)'\right] + \mathrm{E}\left[a(W_j, w) a(W_j, w)'\right] + 2\mathrm{E}\left[a(w, W_j) a(W_j, w)'\right],
$$

Here,

$$
\begin{aligned}
\mathrm{E}\left[a(w, W_j) a(w, W_j)'\right] &= \Delta \sum_{j=1}^{n} K_{i,j}^2 \Lambda_{FF}^{-1}(t_i) \sigma_{kk}^2(t_j) \Lambda_{FF}(t_j) \Lambda^{-1}(t_i) \\
&\quad + \Delta \sum_{j=1}^{n} K_{i,j}^2 \Lambda_{FF}^{-1}(t_i) \Lambda_{FF}(t_j) \left[\beta_k(t_j) - \beta_k(t_i)\right] \left[\beta_k(t_j) - \beta_k(t_i)\right]' \Lambda_{FF}(t_j) \Lambda_{FF}^{-1}(t_i) \\
&\simeq \frac{1}{h} \times q_1(w),
\end{aligned}
$$

where $q_1(w) = \kappa_2 \Lambda_{FF}^{-1}(t_i) \sigma_{kk}^2(t_i)$. Similarly,

$$
\begin{aligned}
\mathrm{E}\left[a(W_t, w) a(W_t, w)'\right] &= \Delta \sum_{j=1}^{n} K_{i,j}^2 \Lambda_{FF}^{-1}(t_j) x x' e_k^2 \Lambda_{FF}^{-1}(t_j) \\
&\quad + \Delta \sum_{j=1}^{n} K_{i,j}^2 \Lambda_{FF}^{-1}(t_i) x e \left[\beta_k(t_j) - \beta_k(t_i)\right]' x x' \Lambda_{FF}^{-1}(t_j) \\
&\quad + \frac{1}{n} \sum_{j=1}^{n} K_{i,j}^2 \Lambda_{FF}^{-1}(t_j) x x' \left[\beta_k(t_j) - \beta_k(t_i)\right] \left[\beta_k(t_j) - \beta_k(t_i)\right]' x x' \Lambda_{FF}^{-1}(t_j) \\
&\simeq \frac{1}{h} \times q_2(w),
\end{aligned}
$$

where $q_1(w) = \kappa_2 \Lambda_{FF}^{-1}(t_i) x x' e_k^2 \Lambda_{FF}^{-1}(t_i)$, while the cross-product term is of smaller order. Employing the same arguments as in Powell and Stoker (1996), it therefore holds that $\mathrm{var}[U_{1,n}] \simeq n^{-1} V_{\mathrm{LR},kk} + \left(n^2 h\right)^{-1} \times \Sigma_{\mathrm{LR},kk}$, where

$$
\Sigma_{\mathrm{LR},kk} = \mathrm{E}\left[q_1(W_t)\right] + \mathrm{E}\left[q_2(W_t)\right] = 2\kappa_2 \frac{1}{n} \sum_{i=1}^{n} \Lambda_{FF}^{-1}(t_i) \sigma_{kk}^2(t_i) = 2\kappa_2 \frac{1}{T} \int_0^T \Lambda_{FF}^{-1}(t) \sigma_{kk}^2(t)\, dt,
$$

while $\mathrm{var}(U_{2,n})$ is of higher order and so can be ignored. In total,

$$
MSE(\hat{\gamma}_{\mathrm{LR},k}) \simeq \left\| B_{\mathrm{LR},k}^{(1)} \times h^2 + B_{\mathrm{LR},k}^{(2)} \times \frac{1}{nh} \right\|^2 + \frac{1}{n} \times V_{\mathrm{LR},kk} + \frac{1}{n^2 h} \times \Sigma_{\mathrm{LR},kk}. \tag{F.3}
$$

When minimizing this expression with respect to $h$, we can ignore the two last terms in the above expression since they are of higher order, and so the optimal bandwidth minimizing the squared component is

$$
h_{\mathrm{LR}}^* = \begin{cases} \left[-B_{\mathrm{LR},k}^{(1)\prime} B_{\mathrm{LR},k}^{(2)} / ||B_{\mathrm{LR},k}^{(1)}||^2\right]^{1/3} \times n^{-1/3}, & B_{\mathrm{LR},k}^{(1)\prime} B_{\mathrm{LR},k}^{(2)} < 0 \\ \left[\frac{1}{2} B_{\mathrm{LR},k}^{(1)\prime} B_{\mathrm{LR},k}^{(2)} / ||B_{\mathrm{LR},k}^{(1)}||^2\right]^{1/3} \times n^{-1/3} & B_{\mathrm{LR},k}^{(1)\prime} B_{\mathrm{LR},k}^{(2)} > 0 \end{cases},
$$

since in general $B_{\mathrm{LR},k}^{(1)} \neq -B_{\mathrm{LR},k}^{(2)}$.

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Table 1: Summary Statistics of Factors and Portfolios

**Panel A: Factors**

| | | | Correlations | | |
|---|---|---|---|---|---|
| | Mean | Stdev | $MKT$ | $SMB$ | $HML$ |
| $MKT$ | 0.0532 | 0.1414 | 1.0000 | -0.2264 | -0.5821 |
| $SMB$ | 0.0184 | 0.0787 | -0.2264 | 1.0000 | -0.0631 |
| $HML$ | 0.0524 | 0.0721 | -0.5812 | -0.0631 | 1.0000 |

**Panel B: Portfolios**

| | | | OLS Estimates | |
|---|---|---|---|---|
| | Mean | Stdev | $\hat{\alpha}_{OLS}$ | $\hat{\beta}_{OLS}$ |
| Book-to-Market Portfolios | | | | |
| 1 Growth | 0.0384 | 0.1729 | -0.0235 | 1.1641 |
| 2 | 0.0525 | 0.1554 | -0.0033 | 1.0486 |
| 3 | 0.0551 | 0.1465 | 0.0032 | 0.9764 |
| 4 | 0.0581 | 0.1433 | 0.0082 | 0.9386 |
| 5 | 0.0589 | 0.1369 | 0.0121 | 0.8782 |
| 6 | 0.0697 | 0.1331 | 0.0243 | 0.8534 |
| 7 | 0.0795 | 0.1315 | 0.0355 | 0.8271 |
| 8 | 0.0799 | 0.1264 | 0.0380 | 0.7878 |
| 9 | 0.0908 | 0.1367 | 0.0462 | 0.8367 |
| 10 Value | 0.0997 | 0.1470 | 0.0537 | 0.8633 |
| 10-1 Book-to-Market Strategy | 0.0613 | 0.1193 | 0.0773 | -0.3007 |
| Momentum Portfolios | | | | |
| 1 Losers | -0.0393 | 0.2027 | -0.1015 | 1.1686 |
| 2 | 0.0226 | 0.1687 | -0.0320 | 1.0261 |
| 3 | 0.0515 | 0.1494 | 0.0016 | 0.9375 |
| 4 | 0.0492 | 0.1449 | -0.0001 | 0.9258 |
| 5 | 0.0355 | 0.1394 | -0.0120 | 0.8934 |
| 6 | 0.0521 | 0.1385 | 0.0044 | 0.8962 |
| 7 | 0.0492 | 0.1407 | 0.0005 | 0.9158 |
| 8 | 0.0808 | 0.1461 | 0.0304 | 0.9480 |
| 9 | 0.0798 | 0.1571 | 0.0256 | 1.0195 |
| 10 Winners | 0.1314 | 0.1984 | 0.0654 | 1.2404 |
| 10-1 Momentum Strategy | 0.1707 | 0.1694 | 0.1669 | 0.0718 |

**Note to Table 1**

We report summary statistics of Fama and French (1993) factors in Panel A and book-to-market and momentum portfolios in Panel B. Data is at a daily frequency and spans July 1964 to December 2006 and are from http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html. We annualize means and standard deviations by multiplying the daily estimates by $252$ and $\sqrt{252}$, respectively. The portfolio returns are in excess of the daily Ibbotson risk-free rate except for the 10-1 book-to-market and momentum strategies which are simply differences between portfolio 10 and portfolio 1. The last two columns in Panel B report OLS estimates of constant alphas ($\hat{\alpha}_{OLS}$) and betas ($\hat{\beta}_{OLS}$). These are obtained by regressing the daily portfolio excess returns onto daily market excess returns.

Table 2: Alphas and Betas of Book-to-Market Portfolios

| | Bandwidth | | Stdev of | Long-Run Estimates | |
|---|---|---|---|---|---|
| | Fraction | Months | Cond Betas | Alpha | Beta |
| 1 Growth | 0.0474 | 50.8 | 0.0558 | -0.0226 | 1.1705 |
| | | | | (0.0078) | (0.0039) |
| 2 | 0.0989 | 105.9 | 0.0410 | -0.0037 | 1.0547 |
| | | | | (0.0069) | (0.0034) |
| 3 | 0.0349 | 37.4 | 0.0701 | 0.0006 | 0.9935 |
| | | | | (0.0072) | (0.0034) |
| 4 | 0.0294 | 31.5 | 0.0727 | 0.0043 | 0.9467 |
| | | | | (0.0077) | (0.0035) |
| 5 | 0.0379 | 40.6 | 0.0842 | 0.0077 | 0.8993 |
| | | | | (0.0083) | (0.0039) |
| 6 | 0.0213 | 22.8 | 0.0871 | 0.0187 | 0.8858 |
| | | | | (0.0080) | (0.0038) |
| 7 | 0.0188 | 20.1 | 0.1144 | 0.0275 | 0.8767 |
| | | | | (0.0084) | (0.0038) |
| 8 | 0.0213 | 22.8 | 0.1316 | 0.0313 | 0.8444 |
| | | | | (0.0082) | (0.0039) |
| 9 | 0.0160 | 17.2 | 0.1497 | 0.0373 | 0.8961 |
| | | | | (0.0094) | (0.0046) |
| 10 Value | 0.0182 | 19.5 | 0.1911 | 0.0461 | 0.9556 |
| | | | | (0.0112) | (0.0055) |
| 10-1 Book-to-Market Strategy | 0.0217 | 23.3 | 0.2059 | 0.0681 | -0.2180 |
| | | | | (0.0155) | (0.0076) |

Joint test for $\alpha_{LR,i} = 0$, $i = 1, ..., 10$
Wald statistic $W = 31.6$, p-value = 0.0005

The table reports conditional bandwidths ($\hat{h}_k$ in equation (30)) and various statistics of conditional and long-run alphas and betas from a conditional CAPM of the book-to-market portfolios. The bandwidths are reported in fractions of the entire sample, which corresponds to 1, and in monthly equivalent units. We transform the fraction to a monthly equivalent unit by multiplying by $533 \times 1.96/0.975$, where there are 533 months in the sample, and the intervals $(-1.96,\ 1.96)$ and $(-0.975,\ 0.975)$ correspond to cumulative probabilities of $95\%$ for the unscaled normal and uniform kernel, respectively. The conditional alphas and betas are computed at the end of each calendar month, and we report the standard deviations of the monthly conditional beta estimates in the column labeled "Stdev of Cond Betas" following Theorem 1 using the conditional bandwidths in the columns labeled "Bandwidth." The long-run estimates, with standard errors in parentheses, are computed following Theorem 2 and average daily estimates of conditional alphas and betas. The long-run bandwidths apply the transformation in equation (31) with $n = 11202$ days. Both the conditional and the long-run alphas are annualized by multiplying by 252. The joint test for long-run alphas equal to zero is given by the Wald test statistic in equation (23). The full data sample is from July 1963 to December 2007, but the conditional and long-run estimates span July 1964 to December 2006 to avoid the bias at the endpoints.

Table 3: Tests of Constant Conditional Alphas and Betas of Book-to-Market Portfolios

| | $W$ | | Critical Values | |
|---|---|---|---|---|
| | Alpha | Beta | 95% | 99% |
| 1 Growth | 49 | 424** | 129 | 136 |
| 2 | 9 | 331** | 65 | 71 |
| 3 | 26 | 425** | 172 | 180 |
| 4 | 47 | 426** | 202 | 211 |
| 5 | 30 | 585** | 159 | 167 |
| 6 | 50 | 610** | 276 | 286 |
| 7 | 75 | 678** | 311 | 322 |
| 8 | 70 | 756** | 276 | 286 |
| 9 | 84 | 949** | 361 | 373 |
| 10 Value | 116 | 1028** | 320 | 331 |
| 10-1 Book-to-Market Strategy | 114 | 830** | 270 | 280 |

We test for constancy of the conditional alphas and betas in a conditional CAPM using the Wald test of Theorem 3. In the columns labeled "Alpha" ("Beta") we test the null that the conditional alphas (betas) are constant. We report the test statistic $W$ in Theorem 3 and 95% and 99% critical values of the asymptotic distribution. We mark rejections at the 99% level with $^{**}$.

Table 4: Characterizing Conditional Betas of the Value-Growth Strategy

| | I | II | III | IV | V | VI | VII | VIII | IX | X | XI |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Dividend yield | 4.55 | | | | | | | | | 16.5 | |
| | (2.01)* | | | | | | | | | (2.95)** | |
| Default spread | | -9.65 | | | | | | | | -1.86 | |
| | | (2.14)** | | | | | | | | (3.68) | |
| Industrial production | | | 0.50 | | | | | | | 0.18 | |
| | | | (0.21) | | | | | | | (0.33) | |
| Short rate | | | | -1.83 | | | | | | -7.33 | |
| | | | | (0.50)** | | | | | | (1.22)** | |
| Term spread | | | | | 1.08 | | | | | -3.96 | |
| | | | | | (1.20) | | | | | (2.10) | |
| Market volatility | | | | | | -1.38 | | | | -0.96 | |
| | | | | | | (0.38)** | | | | (0.40)* | |
| $cay$ | | | | | | | 0.97 | | | -0.74 | |
| | | | | | | | (1.12) | | | (1.31) | |
| Inflation | | | | | | | | 1.01 | | | |
| | | | | | | | | (0.55) | | | |
| NBER recession | | | | | | | | | -0.07 | | |
| | | | | | | | | | (0.03)* | | |
| Market Risk Premium | | | | | | | | | | | 0.37 |
| | | | | | | | | | | | (0.18)* |
| Adjusted $R^2$ | 0.06 | 0.09 | 0.01 | 0.06 | 0.01 | 0.15 | 0.02 | 0.02 | 0.01 | 0.55 | 0.06 |

We regress conditional betas of the value-growth strategy onto various macro variables. The betas are computed from a conditional CAPM and are plotted in Figure 3. The dividend yield is the sum of past 12-month dividends divided by current market capitalization of the CRSP value-weighted market portfolio. The default spread is the difference between BAA and 10-year Treasury yields. Industrial production is the log year-on-year change in the industrial production index. The short rate is the three-month T-bill yield. The term spread is the difference between 10-year Treasury yields and three-month T-bill yields. Market volatility is defined as the standard deviation of daily CRSP value-weighted market returns over the past month. We denote the Lettau-Ludvigson (2001a) cointegrating residuals of consumption, wealth, and labor from their long-term trend as $cay$. Inflation is the log year-on-year change of the CPI index. The NBER recession variable is a zero/one indicator which takes on the variable one if the NBER defines a recession that month. All RHS variables are expressed in annualized units. All regressions are at the monthly frequency except regressions VII and XI which are at the quarterly frequency. The market risk premium is constructed in a regression of excess market returns over the next quarter on dividend yields, default spreads, industrial production, short rates, industrial production, short rates, term spreads, market volatility, and $cay$. The instruments are measured at the beginning of the quarter. We define the market risk premium as the fitted value of this regression at the beginning of each quarter. Robust standard errors are reported in parentheses and we denote 95% and 99% significance levels with $^*$ and $^{**}$, respectively. The data sample is from July 1964 to December 2006.

Table 5: Long-Run Fama-French (1993) Alphas and Factor Loadings of Book-to-Market Portfolios

| | Alpha | $MKT$ | $SMB$ | $HML$ |
|---|---|---|---|---|
| 1 Growth | 0.0216 | 0.9763 | -0.1794 | -0.6721 |
| | (0.0056) | (0.0041) | (0.0060) | (0.0074) |
| 2 | 0.0123 | 0.9726 | -0.0634 | -0.2724 |
| | (0.0060) | (0.0043) | (0.0063) | (0.0079) |
| 3 | 0.0072 | 0.9682 | -0.0228 | -0.1129 |
| | (0.0067) | (0.0048) | (0.0072) | (0.0087) |
| 4 | -0.0057 | 0.9995 | 0.0163 | 0.1584 |
| | (0.0072) | (0.0050) | (0.0073) | (0.0093) |
| 5 | -0.0032 | 0.9668 | 0.0005 | 0.2567 |
| | (0.0075) | (0.0053) | (0.0079) | (0.0099) |
| 6 | -0.0025 | 0.9821 | 0.0640 | 0.3022 |
| | (0.0072) | (0.0051) | (0.0077) | (0.0094) |
| 7 | -0.0113 | 1.0043 | 0.0846 | 0.4294 |
| | (0.0071) | (0.0050) | (0.0074) | (0.0091) |
| 8 | -0.0153 | 1.0352 | 0.1061 | 0.7053 |
| | (0.0057) | (0.0041) | (0.0061) | (0.0075) |
| 9 | -0.0153 | 1.1011 | 0.1353 | 0.7705 |
| | (0.0069) | (0.0049) | (0.0074) | (0.0090) |
| 10 Value | -0.0158 | 1.1667 | 0.2729 | 0.7925 |
| | (0.0090) | (0.0064) | (0.0095) | (0.0118) |
| 10-1 Book-to-Market Strategy | -0.0375 | 0.1911 | 0.4521 | 1.4660 |
| | (0.0103) | (0.0073) | (0.0108) | (0.0133) |

Joint test for $\alpha_{LR,i} = 0, i = 1, ..., 10$
Wald statistic $W_0 = 77.5$, p-value = 0.0000

The table reports estimates of long-run alphas and factor loadings from a conditional Fama and French (1993) model applied to decile book-to-market portfolios and the 10-1 book-to-market strategy. The long-run estimates, with standard errors in parentheses, are computed following Theorem 2 and average daily estimates of conditional alphas and betas. The long-run alphas are annualized by multiplying by 252. The joint test for long-run alphas equal to zero is given by the Wald test statistic $W_0$ in equation (23). The full data sample is from July 1963 to December 2007, but the long-run estimates span July 1964 to December 2006 to avoid the bias at the endpoints.

Table 6: Tests of Constant Conditional Fama-French (1993) Alphas and Factor Loadings of Book-to-Market Portfolios

| | $W$ | | | | Critical Values | |
|---|---|---|---|---|---|---|
| | Alpha | $MKT$ | $SMB$ | $HML$ | 95% | 99% |
| 1 Growth | 59 | 441** | 729** | 3259** | 285 | 295 |
| 2 | 64 | 369** | 429** | 1035** | 348 | 360 |
| 3 | 44 | 184 | 198 | 450** | 284 | 294 |
| 4 | 40 | 219 | 209 | 421** | 236 | 246 |
| 5 | 31 | 212 | 182 | 675** | 216 | 225 |
| 6 | 29 | 201 | 316** | 773** | 240 | 436 |
| 7 | 49 | 195 | 440** | 1190** | 353 | 365 |
| 8 | 60 | 221 | 483** | 3406** | 369 | 381 |
| 9 | 37 | 192 | 520** | 3126** | 270 | 281 |
| 10 Value | 42 | 367** | 612** | 2194** | 242 | 440 |
| 10-1 Book-to-Market Strategy | 46 | 283** | 1075** | 4307** | 257 | 267 |

The table reports $W$ test statistics in Theorem 3 of tests of constancy of conditional alphas and factor loadings from a conditional Fama and French (1993) model applied to decile book-to-market portfolios and the 10 - 1 book-to-market strategy. Constancy tests are done separately for each alpha or factor loading on each portfolio. We report the test statistic $W$ and 95% critical values of the asymptotic distribution. We mark rejections at the 99% level with **. The full data sample is from July 1963 to December 2007, but the conditional estimates span July 1964 to December 2006 to avoid the bias at the endpoints.

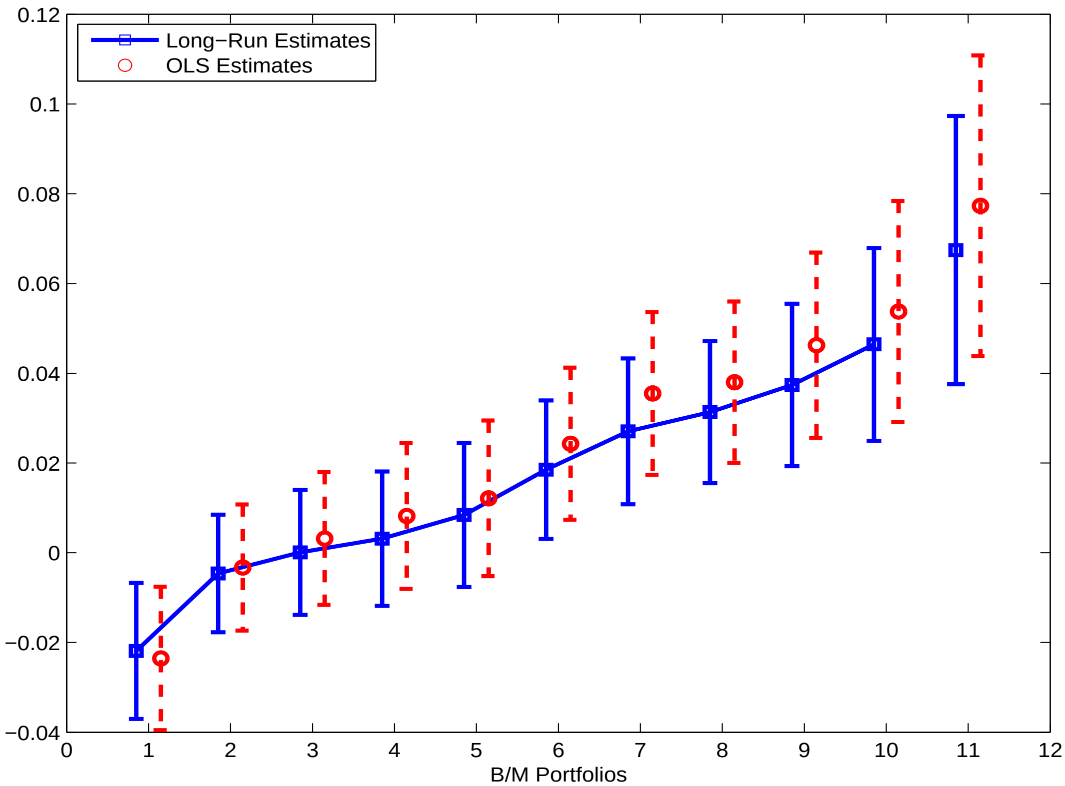
Table 7: Long-Run Fama-French (1993) Alphas and Factor Loadings of Momentum Portfolios

| | Alpha | $MKT$ | $SMB$ | $HML$ |
|---|---|---|---|---|
| 1 Losers | -0.0468 | 1.1762 | 0.3848 | -0.0590 |
| | (0.0138) | (0.0091) | (0.0132) | (0.0164) |
| 2 | 0.0074 | 1.0436 | 0.0911 | 0.0280 |
| | (0.0103) | (0.0072) | (0.0102) | (0.0128) |
| 3 | 0.0226 | 0.9723 | -0.0263 | 0.0491 |
| | (0.0087) | (0.0061) | (0.0088) | (0.0111) |
| 4 | 0.0162 | 0.9604 | -0.0531 | 0.0723 |
| | (0.0083) | (0.0058) | (0.0084) | (0.0104) |
| 5 | -0.0056 | 0.9343 | -0.0566 | 0.0559 |
| | (0.0082) | (0.0056) | (0.0082) | (0.0101) |
| 6 | -0.0043 | 0.9586 | -0.0363 | 0.1111 |
| | (0.0076) | (0.0053) | (0.0079) | (0.0097) |
| 7 | -0.0176 | 0.9837 | -0.0270 | 0.0979 |
| | (0.0074) | (0.0052) | (0.0077) | (0.0094) |
| 8 | 0.0082 | 1.0258 | -0.0238 | 0.1027 |
| | (0.0073) | (0.0052) | (0.0077) | (0.0095) |
| 9 | -0.0078 | 1.0894 | 0.0815 | 0.0337 |
| | (0.0078) | (0.0055) | (0.0081) | (0.0101) |
| 10 Winners | 0.0297 | 1.2523 | 0.3592 | -0.1753 |
| | (0.0103) | (0.0074) | (0.0183) | (0.0135) |
| 10-1 Momentum Strategy | 0.0811 | 0.0739 | -0.0286 | -0.1165 |
| | (0.0189) | (0.0127) | (0.0183) | (0.0230) |

Joint test for $\alpha_{LR,i} = 0, i = 1, ..., 10$
Wald statistic $W_0 = 91.0$, p-value = 0.0000

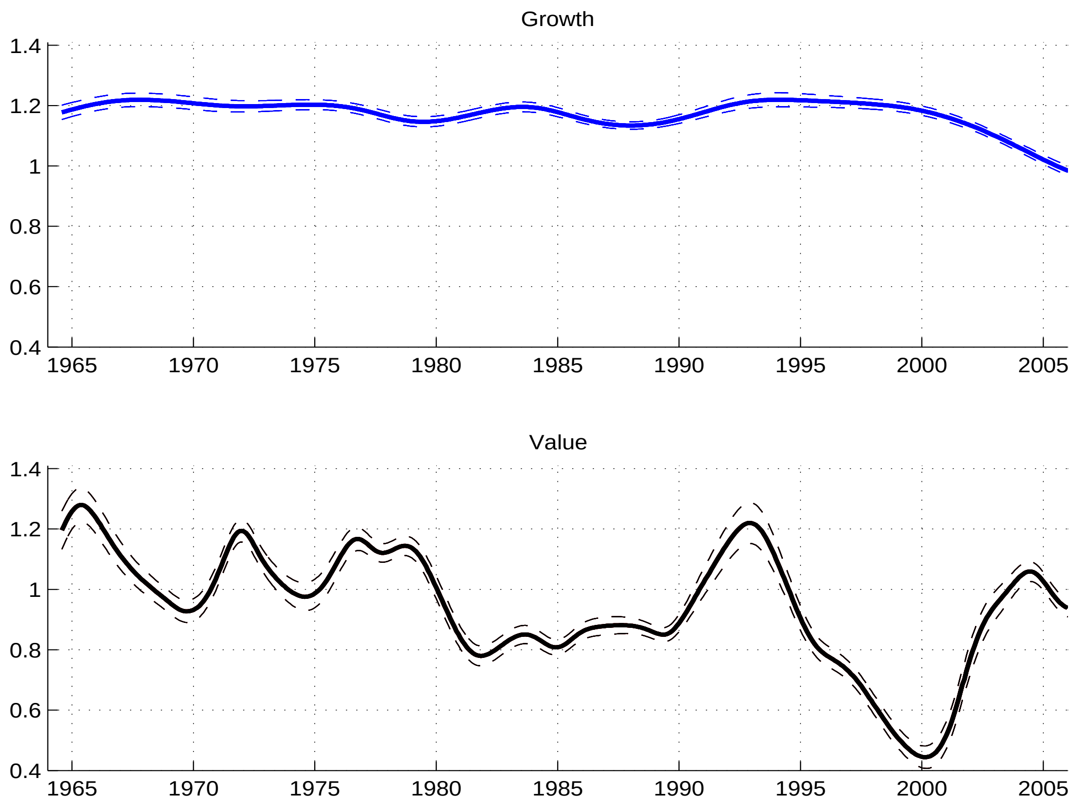
The table reports estimates of long-run alphas and factor loadings from a conditional Fama and French (1993) model applied to decile momentum portfolios and the 10-1 momentum strategy. The long-run estimates, with standard errors in parentheses, are computed following Theorem 2 and average daily estimates of conditional alphas and betas. The long-run alphas are annualized by multiplying by 252. The joint test for long-run alphas equal to zero is given by the Wald test statistic $W_0$ in equation (23). The full data sample is from July 1963 to December 2007, but the long-run estimates span July 1964 to December 2006 to avoid the bias at the endpoints.

Figure 1: Long-Run Conditional CAPM Alphas versus OLS Alphas for the Book-to-Market Portfolios



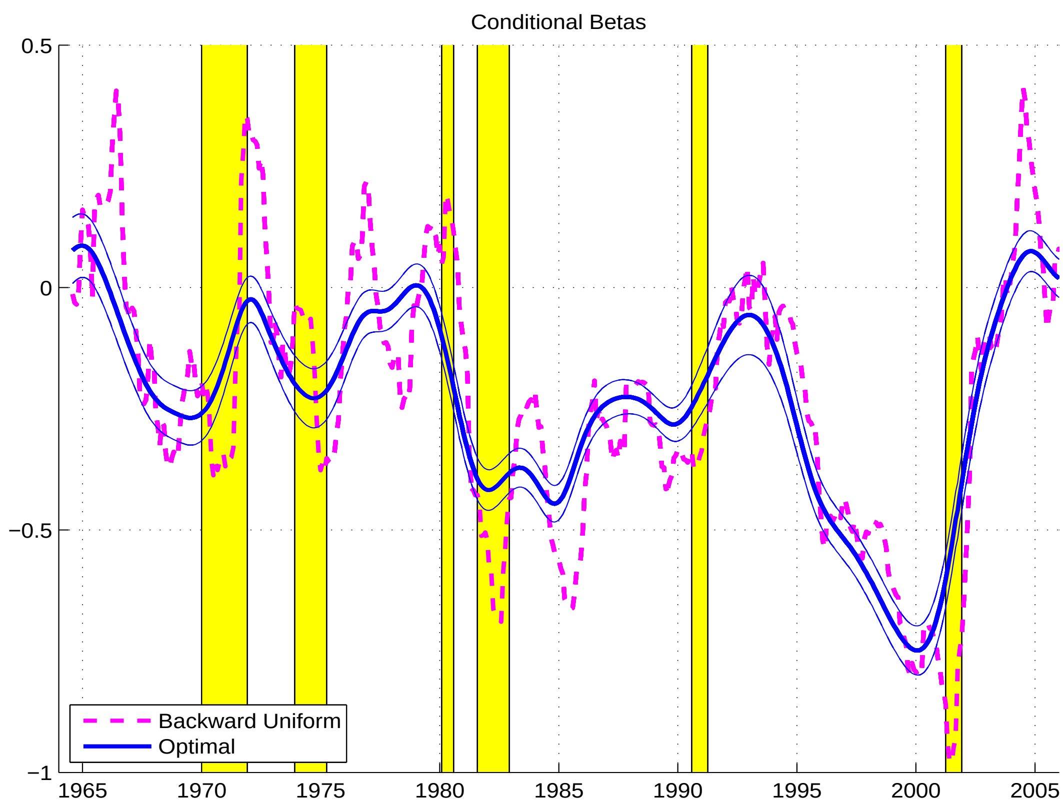

We plot long-run alphas implied by a conditional CAPM and OLS alphas for the book-to-market portfolios. We plot the long-run alphas using squares with 95% confidence intervals displayed by the solid error bars. The point estimates of the OLS alphas are plotted as circles with 95% confidence intervals in dashed lines. Portfolios 1-10 on the $x$-axis represent the growth to value decile portfolios. Portfolio 11 is the book-to-market strategy, which goes long portfolio 10 and short portfolio 1. The long-run conditional and OLS alphas are annualized by multiplying by 252.

Figure 2: Conditional Betas of Growth and Value Portfolios



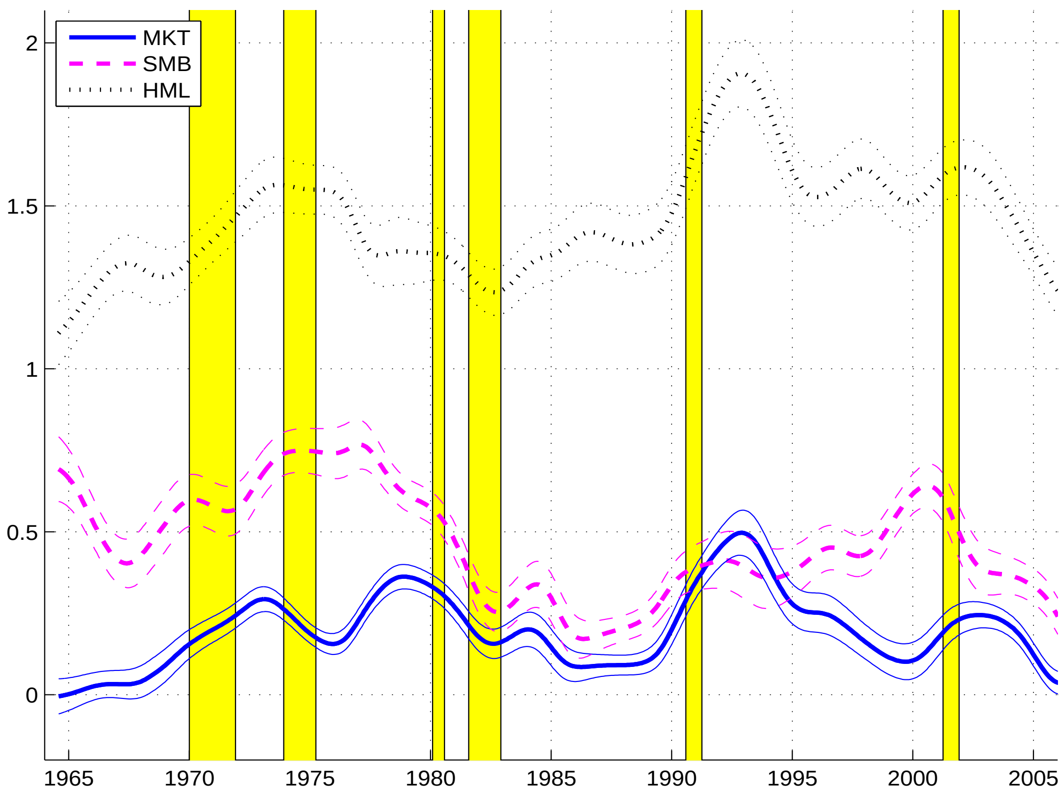

The figure shows monthly estimates of conditional conditional betas from a conditional CAPM of the first and tenth decile book-to-market portfolios (growth and value, respectively). We plot 95% confidence bands in dashed lines. The conditional alphas are annualized by multiplying by 252.

Figure 3: Conditional Betas of the Book-to-Market Strategy



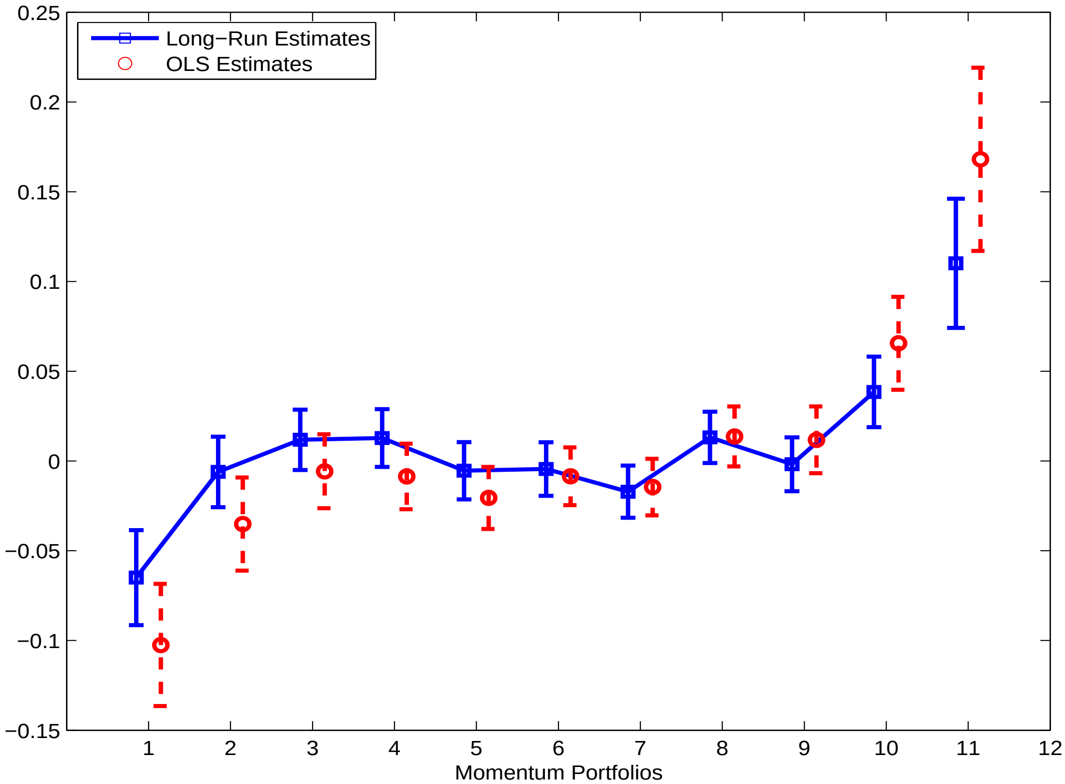

The figure shows monthly estimates of conditional conditional betas of the book-to-market strategy. We plot the optimal estimates in bold solid lines along with 95% confidence bands in regular solid lines. We also overlay the backward one-year uniform estimates in dashed lines. NBER recession periods are shaded in horizontal bars.

Figure 4: Conditional Fama-French (1993) Loadings of the Book-to-Market Strategy



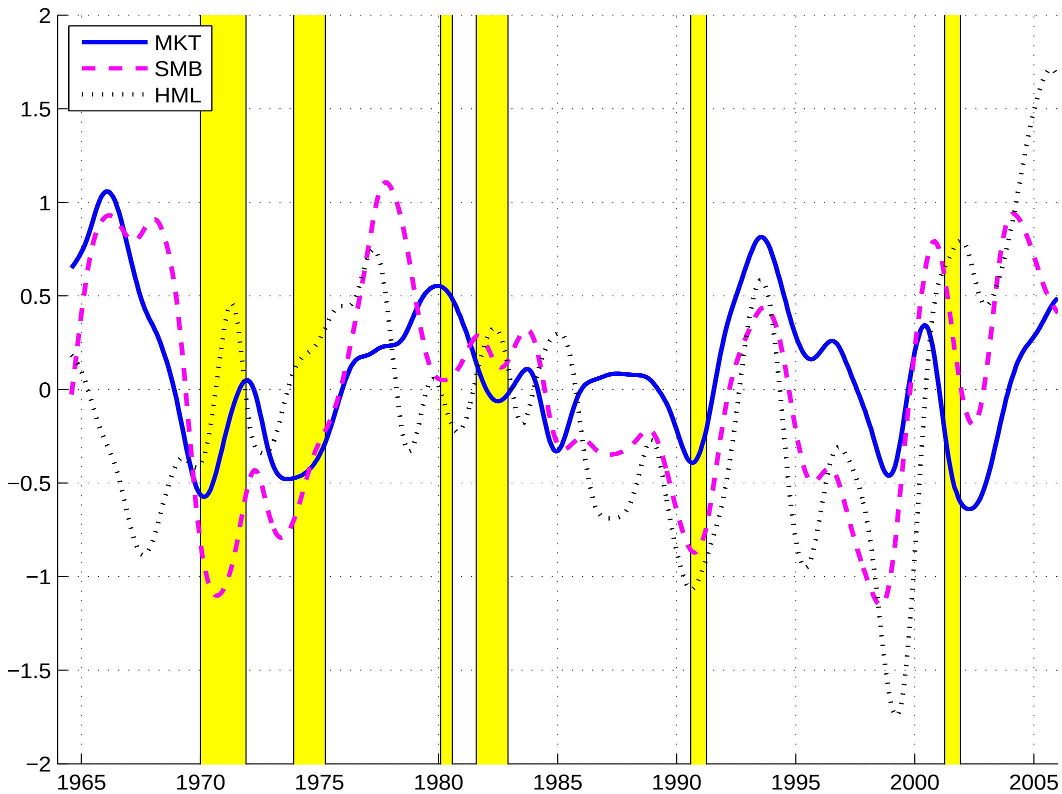

The figure shows monthly estimates of conditional Fama-French (1993) factor loadings of the book-to-market strategy, which goes long the 10th book-to-market decile portfolio and short the 1st book-to-market decile portfolio. We plot the optimal estimates in bold lines along with 95% confidence bands in regular lines. NBER recession periods are shaded in horizontal bars.

Figure 5: Long-Run Fama-French (1993) Alphas versus OLS Alphas for the Momentum Portfolios



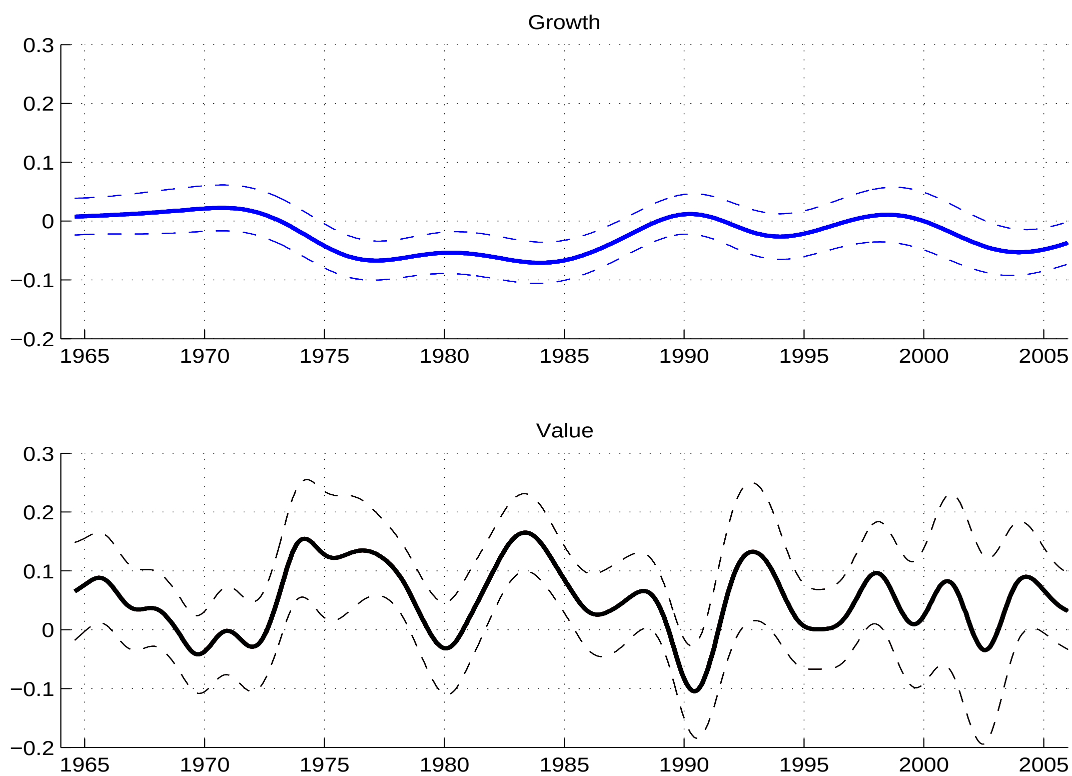

We plot long-run alphas from a conditional Fama and French (1993) model and OLS Fama-French alphas for the momentum portfolios. We plot the long-run alphas using squares with 95% confidence intervals displayed in the error bars. The point estimates of the OLS alphas are plotted as circles with 95% confidence intervals in dashed lines. Portfolios 1-10 on the $x$-axis represent the loser to winner decile portfolios. Portfolio 11 is the momentum strategy, which goes long portfolio 10 and short portfolio 1. The long-run conditional and OLS alphas are annualized by multiplying by 252.

Figure 6: Conditional Fama-French (1993) Loadings of the Momentum Strategy




The figure shows monthly estimates of conditional Fama-French (1993) factor loadings of the momentum strategy, which goes long the 10th past return decile portfolio and short the 1st past return decile portfolio. NBER recession periods are shaded in horizontal bars.

Figure A-1: Conditional Alphas of Growth and Value Portfolios




The figure shows monthly estimates of conditional conditional alphas from a conditional CAPM of the first and tenth decile book-to-market portfolios (growth and value, respectively). We plot 95% confidence bands in dashed lines. The conditional alphas are annualized by multiplying by 252.