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Abstract

This paper establishes existence of a stationary Markov perfect equilibrium in general stochastic games with noise — a component of the state that is nonatomically distributed and not directly affected by the previous period's state and actions. Noise may be simply a payoff-irrelevant public randomization device, delivering known results on existence of correlated equilibrium as a special case. More generally, noise can take the form of shocks that enter into players' stage payoffs and the transition probability on states. The existence result is applied to a model of industry dynamics and to a model of dynamic electoral competition.

1 Introduction

This paper proves existence of stationary Markov perfect equilibria in a class of stochastic games, a subset of dynamic games in which isomorphic subgames are indexed by a state variable that evolves according to a controlled Markov process. In each period, the current state is publicly observed and determines a stage game in which players simultaneously choose feasible actions, stage payoffs are realized, and a new state is drawn from a distribution depending on the current state and the players' actions. A natural starting point

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for strategic analysis in this setting is to consider equilibria that reflect the stationary structure of the environment, and so the issue of existence of stationary Markov perfect equilibria is of central interest. I establish existence of equilibrium for general stochastic games by adding noise — a component of the state state variable that is nonatomically distributed and not directly affected by the previous period's state and actions — in each period. I refer to games for which such a decomposition of states is possible as "noisy stochastic games." The presence of such noise is often innocuous from an applied point of view, where shocks to parameters of the game can increase modeling realism and are desirable for purposes of estimation. I give two examples to illustrate the application of the existence theorem: one is a dynamic model of firm entry, exit, and investment in an industry, where the noise component corresponds to demand or technology shocks, and another is a dynamic model of electoral competition with time-consistent policy choice, where noise is introduced via probabilistic voting, a standard assumption in the literature.

The literature on stochastic games has not yielded general existence results, even under the compactness and continuity conditions familiar from Debreu (1952), Fan (1952), and Glicksberg (1952) for static games. Indeed, the example in Section 2.1 of Harris et al. (1995) shows that compactness and continuity are not sufficient for existence of stationary Markov perfect equilibrium in stochastic games. That example is a relatively simple, two-period game in which the players' action sets are compact and payoffs are continuous in histories. Though not formulated explicitly as a stochastic game, the example can be formulated as one in which the state in the first period is an exogenous initial state, and the state in the second period is just the profile of actions taken in the first; then the mapping from action profiles in period 1 to the state in period 2 is just the identity mapping. The authors argue that there is no subgame perfect equilibrium in their example, and therefore there is no Markov perfect equilibrium when the game is viewed as a stochastic game. The approach taken by Harris et al. (1995) is to consider correlated equilibria in history-dependent strategies.

Another approach, followed in the literature on stochastic games, is to impose stronger continuity conditions on the transition probability. At issue is the fact that even if stage payoffs are continuous, discontinuities can conceivably be introduced by future behavior of the players, for players tomorrow may condition their responses to today's actions in a discontinuous way. The literature has accordingly assumed that next period's state is determined stochastically as a function of the current period's state and actions, and that the distribution of next period's state varies with current actions in a strongly continuous way, e.g., the probability of each measurable set of next period's states is continuous in this period's actions. This precludes deterministic transitions, as in the example of Harris et al. (1995), and it partially addresses the continuity problem in dynamic games. This stochastic element alone does not, however, deliver general existence results, for difficult technical problems arise when the set of states is uncountably infinite. Existence arguments for stochastic games typically involve selections from a correspondence (of induced equilibrium payoff vectors of stage games) defined on states, and when there is a continuum of states, these selections will live in an infinite-dimensional space, the set of selections need not be closed (even if the underlying correspondence has closed graph), and it need not vary upper hemicontinuously in its parameters (even if the graph of the correspondence does). To obtain both of these properties, the underlying correspondence must have convex values: when there is a continuum of states, the role of convexity is intertwined with upper hemicontinuity.

The papers in the extant literature closest to the current one are Nowak and Raghavan (1992), who prove the existence of correlated stationary Markov perfect equilibria, and Duffie et al. (1994), who additionally deduce ergodic properties of equilibria under stronger conditions. These papers essentially assume that the players observe the outcome of a public randomization device before choosing their actions in each period, convexifying payoffs in every state and delivering the convexity needed to obtain upper hemicontinuity needed for their arguments. The drawback is that the "sunspot" on which players coordinate is payoff-irrelevant and may be unnatural or unmotivated in applications. The innovation of the current paper is to replace sunspots with shocks that appear explicitly as a component of the state, along with a standard component, and that enter into the stage payoffs of the players and the transition probability on states. I refer to these shocks as the "noise" component of the state, because in contrast to the standard component, the distribution of noise next period is not directly affected by this period's state and actions. (The noise component can be correlated with the standard component, allowing indirect dependence on the state and actions this period.) The noise component could be simply an iid draw of a payoff-irrelevant, continuously distributed random variable, i.e., a public randomization device, thereby obtaining the existence results of Nowak and Raghavan (1992) and

Duffie et al. (1994) as special cases. More generally, the noise component can affect stage payoffs and the state transition, and it may play a natural role in many applications. Even so, it delivers the needed convexity for existence.

The usefulness of the noise component is that it permits the fixed point argument to be framed in the space of "interim" continuation values, which are defined over the standard component alone rather than the entire state, and these interim continuation values can be written as an integral over the noise component for each realization of the standard component. Given a realization of the standard component, integration over the non-atomic noise component provides convexity via Lyapunov's theorem, and the theorem of Artstein (1989) is used to address measurability issues across realizations. This convexifies the correspondence from interim continuation values to updated interim continuation values. At the level of technique, a contribution of the current paper is to explicitly formulate the correspondence of updated continuation values as a Bochner integral; then the requisite upper hemicontinuity of this correspondence is established with the help of a fundamental result of Artstein (1979) on Fatou's lemma in infinite dimensions.

Section 2 provides a literature review. Section 3 presents the noisy stochastic game model and the existence theorem. Section 4 gives two applications, one oriented toward industrial organization and the other toward political economy. Section 5 is devoted to an informal discussion of the proof approach. Section 6 contains the proof of the main theorem.

2 Literature Review

Existence of stationary Markov perfect equilibrium is a central issue in the literature on stochastic games beginning with Shapley (1953), who proved existence for finite, two-player, zero-sum games. Existence in general finite stochastic games follows from the straightforward application of Kakutani's fixed point theorem in finite dimensions (cf. Fink (1964), Rogers (1969), and Sobel (1971)), while Takahashi (1964) proves existence when the set of states is finite and action sets are compact. Haller and Lagunoff (2000) prove that the set of stationary Markov perfect equilibria in finite games is generically finite; Herings and Peeters (2004) develop an computational algorithm and show that the number of equilibria is generically odd; and Doraszelski and Escobar (2010) prove generic strong stability and purifiability of stationary equilibria. Parthasarathy (1973) extends the framework of Shapley to two-player, non-zero sum games with finite action sets and a countable set of states.

General results on existence, however, have been elusive and have relied on the imposition of relatively special structure or departures from the concept of stationary Markov perfect equilibrium, and all known results impose some form of strong continuity on transition probabilities. Letting s denote a state and a denote a profile of actions, a transition probability is a measurable mapping $(s, a) \to \mu_t(\cdot|s, a)$ from state-action pairs to probability measures on the set of states that can conceivably vary with the time period t. Next, in increasing strength, are some assumptions used in the literature.

(A1) μ_t is set-wise continuous in a,[1]

(A2) μ_t is norm-continuous in a,[2]

(A3) μ_t is norm-continuous in a and absolutely continuous with respect to some fixed probability measure ν_t,

(A4) μ_t is norm-continuous in a and absolutely continuous with respect to a fixed, non-atomic probability measure ν_t,

(A5) μ_t has a jointly measurable density $f(s'|s, a)$ with respect to Lebesgue measure that is continuous in a.

In the analysis of stationary stochastic games, as in the current paper, it is further assumed that the transition probability is fixed across time and the subscript t dropped. Note that even the weakest of the above assumptions, (A1), is inconsistent with deterministic transitions (precluding the example of Harris et al. (1995)) when action sets are uncountably infinite. Of course, (A1) and (A2) hold when the players' action sets are finite.

In finite-horizon stochastic games, Rieder (1979) establishes existence of Markov perfect equilibrium under (A1). Incorporating time in the state vari-

[1]For each state s, each measurable set Z of states, and each sequence $a_m \to a$, we have $\mu_t(Z|s, a^m) \to \mu_t(Z|s, a)$.

[2]For each state s and each sequence $a^m \to a$, we have $||\mu_t(Z|s, a^m) - \mu_t(Z|s, a)|| \to 0$. Here, given an arbitrary signed measure μ on a measurable space $(S, \mathcal{S})$, the *total variation norm* $||\mu||$ is the supremum of $\sum_k |\mu(S_k)|$ over the finite, measurable partitions $\{S_k\}$ of S.

able of a finite-horizon game, we may in fact view Rieder's equilibrium as stationary. Also assuming (A1), Federgruen (1978), Whitt (1980), and Escobar (2006) prove existence for countable state spaces and uncountable action sets. Himmelberg et al. (1976) prove existence of stationary p-equilibria for two-player games with an uncountable state space but assuming finite action sets and strong separability conditions on stage payoffs and the transition probability, and Parthasarathy (1982) gives additional conditions under which the result of the latter paper delivers a stationary Markov perfect equilibrium. Existence of equilibrium for multi-player games with uncountable state space is proved in Parthasarathy and Sinha (1989) under the assumptions of finite action sets and state-independent transitions. Under continuity assumptions on the transition probability akin to (A5), Amir (1996, 2002)), Curtat (1996), and Nowak (2007) prove existence of stationary Markov perfect equilibria in games possessing strategic complementarities with uncountable state and action spaces.[3] Assuming that the state transition is a convex combination of a fixed finite set of probability measures, Nowak (2003) gives sufficient conditions related to (A1) for existence of stationary Markov perfect equilibrium.

Otherwise more general results have been obtained by weakening stationarity or considering weaker notions of equilibrium. Most closely related to the current paper, Nowak and Raghavan (1992) prove existence of stationary Markov perfect equilibria with public randomization under (A3), and Duffie et al. (1994) add mutual absolute continuity of transition probabilities and show that the equilibrium induces an ergodic process. Mertens and Parthasarathy (1991) assume finite action sets and deduce existence of equilibria that are nearly Markovian, in the sense that the players' strategies in period t can depend not only on the current state but the previous state as well. Mertens and Parthasarathy (1987, 2003) allow for infinite action sets and deduce equilibria in which players use history-dependent strategies such that each player's mixture over actions is the same following any two histories ending in the same state and generating identical continuation values.

Building on Rieder's (1979) result for finite-horizon games, Dutta and Sundaram (1998) prove existence of (possibly non-stationary) Markov perfect ϵ-equilibria under (A1). Assuming stage utilities and the state transition are

[3] Nowak (2007) gives conditions based on concavity of the stage game and a decomposition of the transition probability. Jovanovic and Rosenthal (1988), Bergin and Bernhardt (1992), and Horst (2005) restrict the way players' actions affect each others' payoffs.

continuous in the state variable, Whitt (1980) gives conditions related to (A2) for existence of a Markov perfect ϵ-equilibrium in stationary strategies, and Nowak (1985) drops continuity of the state transition in the state variable and increases (A2) to (A4) to obtain a stationary Markov perfect ϵ-equilibrium.

3 Existence Theorem

A *stochastic game* is a list $\Gamma = (N, (S, \mathcal{S}), (X_i, A_i, u_i, \delta_i)_{i \in N}, \mu)$, where N is a finite set of n players, denoted i or j; $(S, \mathcal{S})$ is a measurable space of states s; X_i is a compact metric space of actions a_i for player i, with $X = \prod_i X_i$ endowed with the product topology; $A_i \colon S \rightrightarrows X_i$ is a lower measurable correspondence from S into nonempty, compact feasible sets $A_i(s)$ of actions for player i;[4] $u_i \colon S \times X \to \Re$ is a bounded stage-payoff function such that $u_i(s, a)$ is measurable in s for each $a = (a_1, \ldots, a_n) \in X$ and continuous in a for each s; $\delta_i \in [0, 1)$ is player i's discount factor; and $\mu \colon S \times A \times \mathcal{S} \to [0, 1]$ is a transition probability on states representing the law of motion, i.e., $\mu(\cdot|s, a)$ is a probability measure on $(S, \mathcal{S})$ for all s and all a, and for all $Z \in \mathcal{S}$, $\mu(Z|s, a)$ is jointly measurable in (s, a). Here, $\mu(Z|s, a)$ is the probability that next period's state belongs to Z given state s and action vector a in the current period. Write $u(s, a) = (u_1(s, a), \ldots, u_n(s, a))$ for the vector of stage payoffs of the players. This is the standard definition of a general stochastic game.

The next step is to introduce a noise structure into the model. To this end, assume: (i) the set of states can be decomposed as $S = Q \times R$ and $\mathcal{S} = \mathcal{Q} \otimes \mathcal{R}$, where Q and R are complete, separable metric spaces and $\mathcal{Q}$ and $\mathcal{R}$ are the respective Borel sigma-algebras. Letting $\mu_q(\cdot|s, a)$ denote the marginal of $\mu(\cdot|s, a)$ on q, assume: (ii) there is a fixed probability measure κ on $(Q, \mathcal{Q})$ such that for all s and all a, $\mu_q(\cdot|s, a)$ is absolutely continuous with respect to κ. By the Radon-Nikodym theorem (see Theorem 13.20 of Aliprantis and Border (2006)), we can write $g(\cdot|s, a)$ for the density of $\mu_q(\cdot|s, a)$ with respect to κ. Moreover, assume: (iii) for all s, the mapping $a \to \mu_q(\cdot|s, a)$ is norm-continuous, i.e., for all s, all a, and each sequence $\{a^m\}$ of action profiles converging to a, the sequence $\{\mu_q(\cdot|s, a_m)\}$ converges to $\mu_q(\cdot|s, a)$ in total variation. Thus, the first component of the state satisfies

[4] A correspondence $\varphi \colon S \to X$ from a measurable space $(S, \mathcal{S})$ to a topological space X is *lower measurable* if for all open sets $G \subseteq X$, we have $\{s \in S \mid \varphi(s) \cap G \neq \emptyset\} \in \mathcal{S}$.

one of the typical continuity conditions assumed in the literature.

Furthermore, assume: (iv) conditional on next period's q', the distribution of r' next period is independent of the current state and actions. Specifically, assume there exists $\mu_r\colon Q\times\mathcal{R} \to [0,1]$ such that the mapping $q' \to \mu_r(\cdot|q')$ is a regular conditional probability for r following each state-action pair (see Rao (1993), p.46). In particular, for all s, all a, and all $Z \in \mathcal{S}$, we have $\mu(Z|s,a) = \int_{q'}\int_{r'} I_Z(q',r')\mu_r(dr'|q')\mu_q(dq'|s,a)$. Assume: (v) for κ-almost all q, $\mu_r(\cdot|q)$ is absolutely continuous with respect to a fixed, atomless probability measure λ on $(R,\mathcal{R})$; by the Radon-Nikodym theorem, this conditional probability has a density $h(\cdot|q)$, and moreover we can choose the density $h(r|q)$ to be jointly measurable in (r,q) (see Proposition 1.1 of Orey (1971)). For later use, define the product probability measure $\nu = \kappa \otimes \lambda$. A *noisy stochastic game* is a stochastic game satisfying conditions (i)–(v).

A stationary Markov strategy for i is a measurable mapping $\sigma_i\colon S \to \mathcal{P}(X_i)$ such that for all s, $\sigma_i(s)$ is a Borel probability measure on X_i that places probability one on $A_i(s)$.[5] Given a strategy profile $\sigma = (\sigma_1,\ldots,\sigma_n)$, let $\sigma(s)$ denote the product probability measure $\sigma_1(s)\otimes\cdots\otimes\sigma_n(s)$ on action vectors induced by the players' strategies, where $\sigma(\cdot|s)$ takes Borel measurable sets of action vectors. Continuation values $v(\cdot;\sigma)$, which are placed in the space $L^n_\infty(S,\mathcal{S},\nu)$ of ν-equivalence classes of essentially bounded, measurable functions from S to $\Re^n$, are uniquely defined by the following recursion:[6]

$$v_i(s;\sigma) \;=\; \int_a \left[(1-\delta_i)u_i(s,a) + \delta_i \int_{s'} v_i(s';\sigma)\mu(ds'|s,a)\right]\sigma(da|s).$$

A strategy vector σ is a *stationary Markov perfect equilibrium* if each player i's strategy maximizes i's discounted expected payoff in every state s, i.e.,

$$\begin{aligned} &v_i(s;\sigma) \\ &\quad = \sup_{a_i\in A_i(s)} \int_{a_{-i}} \left[(1-\delta_i)u_i(s,a) + \delta_i \int_{s} v_i(s';\sigma)\mu(ds'|s,a)\right]\sigma_{-i}(da_{-i}|s), \end{aligned}$$

where $\sigma_{-i}(s)$ is the product probability measure $\sigma_1(s)\otimes\cdots\otimes\sigma_{i-1}(s)\otimes\sigma_{i+1}(s)\otimes\cdots\otimes\sigma_n(s)$ over vectors a_{-i} of actions of players other than i. By the

[5] Given any metric space X, $\mathcal{P}(X)$ denotes the set of Borel probability measures endowed with the weak* topology.

[6] By the Riesz-Fischer theorem (see Theorem 13.5 of Aliprantis and Border (2006)), $L^n_\infty(S,\mathcal{S},\nu)$ is a Banach space with the essential supremum norm $||\cdot||_\infty$; then uniqueness follows from a standard contraction mapping argument.

one-shot deviation principle, every stationary Markov perfect equilibrium is, in particular, subgame perfect.

Theorem: Every noisy stochastic game possesses a stationary Markov perfect equilibrium.

In practice, conditions (i)–(v) will be widely satisfied in applications, where typically: Q and R would be compact subsets of finite-dimensional Euclidean space, the distribution over next period's r' would be given by a jointly measurable density $h(r'|q')$ (with respect to Lebesgue measure) that is conditioned only on q', and the marginal over q' would be given by a density $g(q'|s,a)$ that is measurable in s and continuous in (q',a).[7] A special case of interest is the situation in which r is identically and independently distributed across periods and u_i is constant in r, so the noise component of the state is payoff-irrelevant. Then r acts as a public randomization device, and every stationary Markov perfect equilibrium of the model can be viewed as a correlated equilibrium in the sense of Nowak and Raghavan (1992) or Duffie et al. (1994), and vice versa.[8] Thus, for any stochastic game satisfying the assumptions of the latter papers, we can extend that game by specifying a noise component r uniformly and independently distributed in each state (and stage payoffs constant in r) to obtain a stationary Markov perfect equilibrium of the extended game, which delivers a correlated equilibrium of the original game. But the formulation of this paper allows for noise that is payoff-relevant, capturing many economic and political models of interest.

4 Applications

4.1 Firm Exit, Entry, and Investment

This subsection provides a dynamic model of firm entry, exit, and investment in an industry. Each period begins with a set of firms active in the market, a vector of capital stocks for each firm, and a vector of demand or technology

[7] Compactness of Q and joint continuity of $g(q'|s,a)$ in (q',a) is more than sufficient for norm-continuity of $\mu_q(\cdot|s,a)$ in a.

[8] The setup here generalizes Duffie et al. (1994) in that Q is Polish but not assumed compact, while Nowak and Raghavan (1992) assume only that $\mathcal{S}$ is countably generated.

shocks. Each firm, active or inactive, must decide whether to enter or remain in the industry, and conditional on having entered in the previous period, a firm must choose a production plan. The firms' output and investment plans determine profits for the current period and a distribution over capital stocks next period, reflecting uncertain depreciation and returns to investment. The model is comparable to those of Hopenhayn (1992) or Bergin and Bernhardt (2008), where firms make entry and exit decisions over time and are subject to exogenous technology shocks. In contrast, those models assume a continuum of price-taking firms, while here there is a (possibly large) finite number of firms competing oligopolistically; and those models either fix capital or treat it as a variable input, while here firms make investment decisions and accumulate capital over time. In fact, firms' production plans and capital stocks can be multidimensional, and a firm's decisions can affect future production technology through current investment, so that technology evolves endogenously.

Formally, let N be a finite set of n firms (or potential firms) in an industry, and suppose that in each period, firms must decide whether to enter or remain in the market and, conditional on having previously entered the market, must make output and investment decisions. At the beginning of any period, let $z \in \{0,1\}^n$ summarize the firms active in the industry, with $z_i = 1$ indicating that i is active and $z_i = 0$ indicating i is inactive; let $k_i \in \Re^\ell_+$ denote the capital stock of firm i and $k = (k_1, \ldots, k_n)$ the vector of stocks; and let $r = (r_1, \ldots, r_m)$ be a vector of shocks to demand or production technology belonging to a compact subset $R \subseteq \Re^m$ with positive Lebesgue measure. For tractability, the level of capital stock of each firm is bounded above by $\overline{k}$ in each coordinate. The state of the industry is then summarized by the state variable (z, k, r), where (z, k) is influenced by the actions of the firms, and r is distributed iid across periods. A decision for firm i is a pair (e_i, p_i), where $e_i \in \{0, 1\}$ is firm i's entry/exit decision, with $e_i = 1$ indicating i is will be in the market next period and $e_i = 0$ that it will not, and $p_i \in \Re^d$ is a multidimensional production plan for firm i. In state (z, k, r), the set of feasible production plans for firm i is a nonempty, compact subset $\phi_i(z, k, r) \subseteq \Re^d$, where $\phi_i \colon \{0,1\}^n \times [0, \overline{k}]^{n\ell} \times R \rightrightarrows \Re^d$ is lower measurable with compact range $\tilde{X}_i \subseteq \Re^d$. When firm i is inactive, i.e., $z_i = 0$, assume that $\phi_i(z, k, r) = \{0\}$ to indicate that the firm makes no output or investment decision. Then the decisions available to firm i in state (z, k, r) are $(e_i, p_i) \in \{0,1\} \times \phi_i(z, k, r)$, and the vector of firm decisions is denoted (e, p).

Given current state (z, k, r) and actions (e, p), next period's state, denoted (z', k', r'), is determined as follows. Entry and exit decisions determine each firm's status next period, so $z' = e$, while k' is a random variable assumed to be absolutely continuous with respect to a probability measure $\tilde{\kappa}$, defined as the product measure $\tilde{\kappa} = \tilde{\kappa}_1 \times \cdots \times \tilde{\kappa}_n$, where $\tilde{\kappa}_i$ is the equally weighted average of the uniform distribution on $[0, \overline{k}]^\ell$ and the unit mass on zero. Denote the density of k' with respect to $\tilde{\kappa}$ by $\tilde{g}(k'|(z,k),(e,p))$, which depends on the firms' current capital stocks and decisions. Assume that the density $\tilde{g}(k'|(z,k),(e,p))$ is measurable in (z,k) and continuous in (k', e, p). The distribution of next period's capital stock levels reflects the assumption that returns on investment to capital stock are subject to uncertainty, but because $\tilde{\kappa}_i$ places positive probability on zero, the model allows for the possibility that the capital stock of an inactive firm is fixed at zero. Assume $r' \in R$ is iid and has density h with respect to Lebesgue measure.

A firm i remains active in a period if $z_i = e_i = 1$. Let $\pi_i((z,k,r),(e_i,p))$ be the profit of a firm i that remains active given its own entry/exit decision e_i and production plans p in state (z,k,r), and assume $\pi_i((z,k,r),(e_i,p))$ is bounded, measurable in (z,k,r), and continuous in (e_i,p). The payoff of an active firm that decides to leave the market, i.e., $z_i = 1 = e_i + 1$, is $\iota_i((z,k,r),(e_i,p))$, which may reflect the scrap value of a firm leaving the market. Assume $\iota_i((z,k,r),(e_i,p))$ is bounded, measurable in (z,k,r), and continuous in (e_i,p). The payoff of an inactive firm that decides to enter the market, i.e., $z_i = 0 = e_i - 1$, is $\alpha_i(z,k,r)$, a bounded, measurable function that may reflect the setup cost of entry into the market. The payoff to an inactive firm that remains inactive, i.e., $z_i = e_i = 0$, is zero. Payoffs are discounted over time by the factor δ_i for each firm.

Though formulated generally, the expected structure can be imposed on these payoffs. In particular, it may be that capital k_i is one-dimensional, that $r = (r_1, \ldots, r_n, r_{n+1}, r_{n+2}) \in \Re^{n+2}$ consists of firm-specific production shocks $(r_1, \ldots, r_n)$, an aggregate output demand shock r_{n+1}, and an aggregate labor supply shock r_{n+2}, and that firms compete in a single output market. A production plan is then a pair $p_i = (\dot{k}_i, \ell_i)$ consisting of levels of capital investment and labor input. Firm i's output is then $y_i = F_i(k_i + \dot{k}_i, \ell_i, r_i)$, total labor demand is $L = \sum_i \ell_i$, and total output is $Y = \sum_i y_i$. The inverse demand for output is given by $P(Y, r_{n+1})$, and inverse supply of labor is $W(L, r_{n+2})$. Then for firms currently active in the market and remaining in the market next period, i.e., $z_i = e_i = 1$, we have

$$\begin{aligned}
\pi_i((z,k,r),(1,p)) &= P(Y, r_{n+1})y_i - W(L, r_{n+2})\ell_i \\
Y &= \sum_j F_j(k_j + \dot{k}_j, \ell_j, r_j) \\
L &= \sum_j \ell_j,
\end{aligned}$$

with suitable continuity assumptions on production, inverse demand, and supply functions, and with bounds on investment and labor inputs to ensure compactness. The industry then transitions to (z', k', r'), where z' is determined by entry/exit decisions, k' is drawn from $g(k'|(z,k),(e,p))$ reflecting depreciation on capital and returns to investment, and new shocks r' are drawn independently from h.

At issue is the existence of a stationary Markov perfect equilibrium in this model, which is addressed in the next proposition.

Proposition 1: In the dynamic model of firm exit, entry, and investment, there exists a stationary Markov perfect equilibrium.

To apply the main theorem of the paper, the model must be recast as a noisy stochastic game. The set of players is N, the set of states is $S = Q \times R$, where $Q = \{0,1\}^n \times [0, \overline{k}]^{n\ell}$ and R is as above, and the set of conceivable actions for firm i is $X_i = \{0,1\} \times \tilde{X}_i$. The correspondence of feasible actions for firm i is defined by $A_i(s) = \{0,1\} \times \phi(z,k,r)$, which is lower measurable with nonempty, compact values contained in X_i, and an action for firm i is $a_i = (e_i, p_i)$. The stage payoff of firm i is then

$$u_i(s,a) = \begin{cases} \pi_i(s,(e_i,p)) & \text{if } z_i = e_i = 1 \\ \iota_i(s,(e_i,p)) & \text{if } z_i = 1, e_i = 0, \\ \alpha_i(s) & \text{if } z_i = 0, e_i = 1, \\ 0 & \text{if } z_i = e_i = 0, \end{cases}$$

which is bounded, measurable in s, and continuous in a. Discount factors are as in the original model, and the law of motion μ is as described above. Letting $\hat{\kappa}$ be the uniform distribution on $\{0,1\}^n$ and defining $\kappa = \hat{\kappa} \times \tilde{\kappa}$, the marginal $\mu_q(\cdot|s,a)$ is absolutely continuous with respect to κ and norm-continuous in a; and letting λ be the uniform distribution on R, the conditional $\mu_r(\cdot|q)$ is absolutely continuous with respect to λ. Thus, conditions (i)–(v) are satisfied, and the main existence theorem yields Proposition 1.

4.2 Electoral Competition and Time-consistent Policy

This subsection provides a dynamic model of elections in which politicians cannot make binding commitments, and a representative voter chooses among candidates to fill a finite number of offices in each period.[9] At the beginning of each period, a state of the economy e is given, and the voter chooses from a finite number of politician types for each office; then the winning candidates play a political game that determines a policy p, and the voter makes a simultaneous consumption decision c; and these choices then stochastically determine a new state leading into the next election. To apply the main theorem of the paper, I impose the further structure that the voter's preferences contain an idiosyncratic component that is realized at the beginning of each period prior to the election but after play of the political game in the previous period, so voting is essentially probabilistic; consistent with that assumption, the politicians' preferences in the political game are also subject to type-specific shocks that are unobserved by the voter at the time of the election. The elected politicians then choose policy given their expectations of the voter's choices and the evolution of the economy, and the voters' choices in turn take expectations of political outcomes and future economic states as given.

In the special case that there is just one elected office (so the winning politician unilaterally determines policy) and just one politician type (so elections are trivial), the equilibrium analysis entails that the policy maker use time-consistent policies, i.e., she choose optimally given her expectations of the voter's choices, while the voter chooses under rational expectations of political outcomes (see Kydland and Prescott (1977)). In macroeconomic applications, it may be that the voter's decision c consists of choosing a level of employment, and the policy decision p is the rate of inflation, as in Barro and Gordon (1983). In contrast to the standard framework, however, the model here allows for the level of inflation in one period to affect real variables in

[9] The assumption of a representative voter is for tractability only. In general, it is important that voters eliminate weakly dominated strategies, a condition that is not implied by stationary Markov perfect equilibrium. This could be finessed in the stochastic game model by having voters vote sequentially, but it is simpler to assume a representative voter.

the next. Assuming two politician types (interpreted as parties), elections become non-trivial, and the parties are in competition with each other. Thus, a party's optimization problem must also take as given the expectations of the voters' future choices between the two parties and the opposing party's future policy choices when elected. More generally, there could be multiple offices, and the political game could be viewed as a reduced form for a legislative process; alternatively, the political game could reflect interaction among multiple branches of government, providing an infinite-horizon version of Alesina and Rosenthal's (1996) model of divided government with a general institutional environment and endogenous policy making.

An interesting incentive that can arise in equilibrium is that elected politicians may seek to influence future economic states to their advantage. In the two-party model, in particular, it may be that one party seeks to "tie the hands" of the other, or to engender economic states in which it is perceived favorably by the voter. The model has antecedents in Alesina (1988), which considers repeated elections with probabilistic voting, but there is no economic state variable in the setting of that paper; there, voting behavior is black-boxed (and does not depend on expectations of the voters about future policy choices); and Alesina considers equilibria in trigger strategies, rather than Markovian strategies. Alesina (1987) studies stationary equilibria in a model of macroeconomic policy making, where the party in power chooses a level of monetary expansion and rational wage setters anticipate monetary policy, but there parties are myopic and voting behavior is exogenous. Dixit et al. (2000) analyze a model in which farsighted parties compete in elections to divide a surplus, and in which a state variable evolves according to an exogenous Markov process. In contrast, the present model endogenizes voting behavior, and it allows the possibility that current policy decisions influence future states. As well, voting behavior is exogenous in their model, and those authors focus on efficient equilibria in history-dependent strategies.

Formally, let $L = \{1, \ldots, \ell\}$ be a set of political offices and $M = \{1, \ldots, m\}$ a set of politician types, and let $N = \{0\} \cup (L \times M)$ consist of a representative voter, denoted 0, and ℓm agents corresponding to one type per office; that is, agent $(j, k) \in L \times M$ is a type $k \in M$ politician who may fill office $j \in L$.[10] In each period, the voter casts a ballot $b \in M^L$, where $b(j) \in M$ is

[10]Because politicians are policy-motivated and because of the focus on stationary equilibria, there is no loss of generality in identifying politicians of the same type holding the

the politician type elected to office $j \in L$, and elected politicians then decide policy. Accordingly, each period is divided into two phases: voting and policy making. At the beginning of the voting phase, an economic state e belonging to a subset $E \subseteq \Re^d$ is given, where E is compact and has positive Lebesgue measure; also given is a preference shock r for the voter belonging to the set $R = [0,1]^{L \times M}$; for now, $r(j,k)$ is to be interpreted as an additive shock to the voter's payoff when a type k politician is elected to office j. The voter then casts a ballot $b \in M^L(e)$, where $M^L(e) \subseteq M^L$ is a nonempty subset of ballots that varies lower measurably with e. A new economic state e' is realized according to the jointly measurable density $\tilde{g}(e'|e,b)$, and preference shocks $r' \in R$ are drawn from the jointly measurable density $h(r|e')$; now $r'(j,k)$ serves as a preference shock for the politician (j,k).

The game then enters the policy-making phase, where the elected politicians simultaneously make policy choices that determine the political outcome for the period, and simultaneously the voter makes a consumption decision. More precisely, politician type k elected to office j chooses a policy $p_{j,k} \in P_{j,k}(e) \subseteq P_{j,k}$, where $P_{j,k} \subseteq \Re^d$ is a compact policy space, and $P_{j,k}(e)$ is a nonempty, compact feasible set that is lower measurable in e; and the voter chooses $c \in C(e) \subseteq C$, where $C \subseteq \Re^d$ is a compact consumption set, and $C(e)$ is a nonempty, compact feasible set that is lower measurable in e. The resulting vector of policies chosen by elected politicians is $p = (p_{j,b(j)})_{j \in L}$. Finally a new economic state e'' is realized from the density $\tilde{g}(e''|b,e',p,c)$, which is measurable in (b,e') and continuous in (e'',p,c); new shocks r'' for the voter are drawn according to the jointly measurable density $h(r'|e'')$; and the game moves to the next period, where the process is repeated.

In the election phase, the voter receives payoff $\sum_{j \in L} r(j,b(j))$, while the politicians receive a zero payoff. In the policy-making phase, the voter receives a payoff $\tilde{u}_0(e',p,c)$ from policy p and consumption c in state e', and the payoff to politician (j,k) is $\tilde{u}_{j,k}(e',r'(j,k),p,c)$, where all stage utility functions are jointly measurable and continuous in (p,c). Payoffs are discounted after each period (at the end of the policy making phase) by $\tilde{\delta}_0$ and $\tilde{\delta}_{j,k}$, respectively, for the voter and politicians. Thus, the voter's payoff in a period is

$$\tilde{u}_0(e',p,c) + \sum_{j \in L} r(j,b(j)),$$

same political office after different histories.

so the voter's preference shocks act to perturb her payoffs from any ballot; as is common in the literature on probabilistic voting, these shocks can reflect the voter's perceptions of attributes unrelated to policy (such as the politicians' charisma). In the simple formulation above, the economic state e realized in the voting phase does not directly affect payoffs; it is simply a technical device that represents the voter's information about the economic state that will obtain in the subsequent policy making phase.

Proposition 2: In the dynamic model of electoral competition and time-consistent policy, there exists a stationary Markov perfect equilibrium.

To apply the main theorem, the model must be reformulated as a noisy stochastic game. The set of players remains $N = \{0\} \cup (L \times M)$, and the set of states is $S = Q \times R$, where $Q = (\{0\} \cup M^L) \times E$ and R is as above. Here, the first component of $q = (q_1, q_2)$ indicates whether a period corresponds to the voting phase ($q_1 = 0$) or the policy phase ($q_1 \in M^L$); and in the latter case, it also indicates the type of politician elected to each office. The sets of conceivable actions are $X_0 = M^L \cup C$ for the voter and $X_i = P_{j,k}$ for politician $i = (j, k)$. Disregarding inactive players, an action profile a in state $s = ((q_1, q_2), r)$ is either a ballot $a = b \in M^L$ when $q_1 = 0$ or a vector $a = (p, c) \in P \times C$ when $q_1 \in M^L$. The correspondence of feasible actions is defined as follows: when $q = (0, e)$, the voter's feasible actions are $A_0(s) = M^L(e)$; and when $q = (b, e)$, the feasible actions are $A_0(s) = C(e)$ and $A_i(s) = P_i(e)$ for the voter and politician $i = (j, b(j))$, respectively. Given $(s, a) = ((q, r), a)$, stage payoffs are defined for $i = 0$ as

$$u_0(s,a) \quad = \quad \begin{cases} \sum_{j \in L} r_0(j, b(j)) & \text{if } q = (0,e) \\ \frac{1}{\delta_0} \tilde{u}_0(e,p,c) & \text{if } q = (b,e) \text{ and } a = (p,c), \end{cases}$$

and for $i \in L \times M$ as

$$u_i(s,a) \quad = \quad \begin{cases} 0 & \text{if } q = (0,e) \\ \frac{1}{\delta_i} \tilde{u}_i(e, r(i), p, c) & \text{if } q = (b,e) \text{ and } a = (p,c), \end{cases}$$

and discount factors are the square roots, $\delta_i = \sqrt{\tilde{\delta}_i}$, of the original discount factors. A "period" in the stochastic game corresponds to a "phase" in the original model, so in contrast to the original, discounting now must occur between the voting and policy-making phases; this is undone by using square roots of the original discount factors and by inflating stage payoffs in the policy making stage accordingly. The law of motion μ is as above. Letting let κ be uniform on Q, the marginal $\mu_q(\cdot|s,a)$ is absolutely continuous with respect to κ and norm-continuous in a; and letting λ be uniform on R, the conditional $\mu_r(\cdot|q)$ is absolutely continuous with respect to λ. Thus, conditions (i)–(v) are satisfied, and the main existence theorem implies Proposition 2.

5 Discussion of Proof

To describe the method of analysis, I begin with the existence proof of Nowak and Raghavan (1992) for stationary Markov perfect equilibrium with public randomization. The argument takes place in a compact, convex space V of continuation value functions $v \colon S \to \Re^n$ endowed with the weak* topology. Given v, consider the induced game $\Gamma_v(s)$ with actions $A_i(s)$ and payoffs

$$(1-\delta_i)u_i(s,a) + \delta_i \int_{s'} v_i(s')\mu(ds'|s,a)$$

for each player i. Assuming the transition $\mu(\cdot|s,a)$ is norm-continuous in a, these payoffs are continuous in actions, and the theorem of Debreu-Fan-Glicksberg implies that there is at least one mixed strategy equilibrium of the induced game. Let $P_v(s)$ be the set of mixed strategy equilibrium payoff vectors of the induced game, and let $P_v^*(s)$ be the convex hull of that set. To update continuation values, let E_v consist of all selections $\hat{v}$ from the correspondence $s \to P_v^*(s)$, yielding a nonempty-valued correspondence $v \to E_v$ depicted in Figure 1. Because we select from the convex hull of induced equilibrium payoffs, E_v is clearly convex. Closed graph of $v \to E_v$ follows from both continuity assumptions imposed on the model and convex values of the correspondence; if $v \to E_v$ were not convex-valued, then closed graph would not follow. By the Debreu-Fan-Glicksberg theorem, the correspondence $v \to E_v$ has a fixed point $v^* \in E(v^*)$, and equilibrium strategies can be backed out from v^*, with care to ensure measurability.

The role of public randomization in convexifying equilibrium payoffs in induced games is critical in the foregoing. To eschew correlation, I assume a nonatomically distributed noise component of the state. The argument now takes place in a compact, convex set V of "interim" continuation values, which

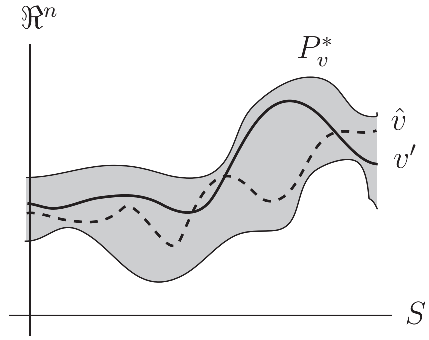

Figure 1: Correlation approach

are conditioned only on the realization of the standard component q, rather than the full state. To convey this notion more precisely, define the interim continuation $v\colon Q \to \Re^n$ generated by strategy profile σ by the recursion

$$v_i(q;\sigma) \\ = \int_r \left[\int_a \left[(1-\delta_i)u_i(s,a) + \delta_i \int_{q'} v_i(q';\sigma)\mu_q(dq'|s,a) \right] \sigma(da|s) \right] h(r|q)\lambda(dr),$$

where $s = (q,r)$. Given interim continuation value function v, define the induced game $\Gamma_v(s)$ with actions $A_i(s)$ and payoffs

$$(1-\delta_i)u_i(s,a) + \delta_i \int_{q'} v_i(q')\mu_q(dq'|s,a),$$

which are continuous in actions by norm-continuity of $\mu_q(\cdot|s,a)$. Let $P_v(s)$ be the (possibly non-convex) set of mixed strategy equilibrium payoffs in $\Gamma_v(s)$.

To update interim continuation values, for each $s = (q,r)$, choose an element of $P_v(s)$. Intuitively, we then integrate across the noise component r to get a new interim continuation value $\hat{v}$. Repeating this for all possible selections of induced equilibrium payoffs, we have a correspondence $v \to E_v$ that maps any v to a set E_v of updated interim continuation values. The crux of the proof is to formalize this idea and establish the usual properties needed to deduce a fixed point. Technically, to define the correspondence, we take a selection $\phi(r)\colon Q \to \Re^n$ for each r of density-weighted equilibrium payoffs, i.e., for κ-almost all q, $\phi(r)(q) \in h(r|q)P_v(q,r)$, as depicted in Figure 2. The mapping $\phi\colon R \to V$ takes values in the function space V, and assuming it

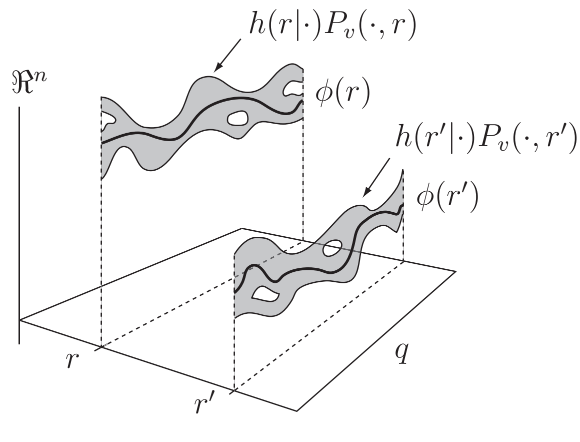

Figure 2: Noise approach

is measurable, the Bochner integral $\hat{v} = \int_r \phi(r)\lambda(dr)$ provides a new interim continuation value. We repeat this procedure for each measurable function ϕ taking selections of density-weighted equilibrium payoffs. More formally, letting $\Phi_v(r)$ be the set of density-weighted equilibrium payoff selections at r, the set of updated continuation values is $E_v = \int_r \Phi_v(r)\lambda(dr)$, the Bochner integral of the correspondence Φ_v.

The key to the existence proof is establishing that $v \to E_v$ has convex values and closed graph. Both properties rely on the observation that E_v can be equivalently defined by integrating over selections from $P_v^*(s)$. That is, letting $\Phi_v^*(r)$ be the set of density-weighted mixtures of equilibrium payoff selections as a function of q, we have $E_v = \int \Phi_v^*(r)\lambda(dr)$. The argument for the claim proceeds by arbitrarily choosing $\hat{v} \in \int_r \Phi_v^*(r)\lambda(dr)$ and considering each q separately. For κ-almost all q, we have $\hat{v}(q) \in \int_r h(r|q)P_v^*(q,r)\lambda(dr)$. Given such q, the correspondence $h(\cdot|q)P_v(q,\cdot)\colon R \to \Re^n$ may have nonconvex values, as depicted in Figure 3, but it maps to finite-dimensional Euclidean space. Thus, since λ is nonatomic, a version of Lyapunov's theorem yields the equality $\int_r h(r|q)P_v(q,r)\lambda(dr) = \int_r h(r|q)P_v^*(q,r)\lambda(dr)$, and in particular, there is a mapping $\psi(q)\colon R \to \Re^n$ that is measurable on R, integrates to $\hat{v}(q)$, and is a λ-almost everywhere selection from $h(\cdot|q)P_v(q,\cdot)$. The selections $\psi(q)$ are chosen independently for each q, and so the mapping $\psi\colon Q \to V$ so-defined need not be measurable, but the theorem of Artstein (1989) allows us to "sew up" these selections in a measurable way, giving us $\hat{v} \in \int_r \Phi_v(r)\lambda(dr)$, as required. Finally, using the fact that for each s, the

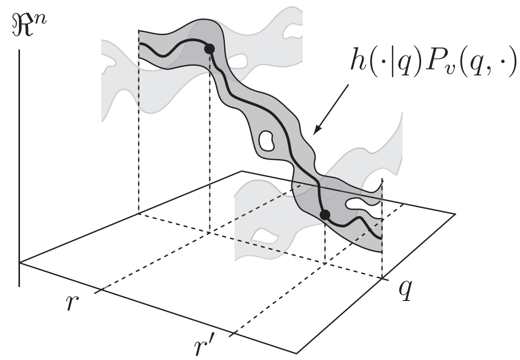

Figure 3: Applying Lyapunov

correspondence $v \to P_v^*(s)$ has closed graph and convex values, a result of Artstein (1979) is used to show closed graph of the correspondence $v \to E_v$. Therefore, $v \to E_v$ possesses a fixed point $v^* \in E_{v^*}$, and the final step of the proof is to back out equilibrium strategies corresponding to this value.

6 Proof of the Theorem

The proof of existence consists of a fixed point argument in a subset V of continuation value functions in the space $L_\infty^n \equiv L_\infty^n(Q, \mathcal{Q}, \kappa)$ of κ-equivalence classes of essentially bounded, measurable functions from Q to $\Re^n$, and it makes use of the space $L_1^n \equiv L_1^n(Q, \mathcal{Q}, \kappa)$ of κ-equivalence classes of integrable functions from Q to $\Re^n$. These spaces are equipped with the usual norms,

$$\begin{aligned} ||f||_\infty &= \inf\{c \in \Re \mid \kappa(\{q \mid ||f(q)|| \leq c\}) = 1\} \\ ||g||_1 &= \int_q ||g(q)||\kappa(dq), \end{aligned}$$

respectively. By the Riesz representation theorem (see Theorem 13.28 of Aliprantis and Border (2006)), L_∞^n consists of the continuous linear functionals on L_1^n, and it is endowed with the weak* topology $\sigma(L_\infty^n, L_1^n)$, i.e., a net $\{f_\alpha\}$ converges to f in L_∞^n if and only if for all $g \in L_1^n$, we have $\int f_\alpha(q)g(g)\kappa(dq) \to \int f(q)g(q)\kappa(dq)$.[11] By Lemma 3 (p.419) of Dunford and

[11]The choice of L_∞^n and the weak* topology for the space of continuation value functions is important in the continuity argument of Lemma 1, below.

Schwartz (1957), L_∞^n is then a locally convex, Hausdorff topological vector space. Let V consist of the functions $v \in L_\infty^n$ such that $||v(q)|| \leq C$ for κ-almost all q, where C is a fixed constant such that $||u(s,a)|| \leq C$ for all s and a. Obviously, V is nonempty and convex, and it follows from Alaoglu's theorem (see Theorem 6.21 of Aliprantis and Border (2006)) that V is compact. Since Q is a separable metric space, the sigma-algebra $\mathcal{Q}$ is countably generated, and it follows that L_1^n is separable (see Theorem 8.3.27 of Corbae et al. (2009)). Then Aliprantis and Border's (2006) Theorem 6.30 implies that V is metrizable in the weak* topology. Henceforth, universal quantifiers over continuation value functions will (unless otherwise specified) to range over V, and I work with sequences of continuation value functions rather than nets.

For each v, let $\Gamma_v(s)$ be the stage game induced by v at s, where each player i's action space is $A_i(s)$ and i's payoff from $a \in \prod_j A_j(s)$ is

$$U_i(s,a;v) \;\; = \;\; (1-\delta_i)u_i(s,a) + \delta_i \int_{q'} v_i(q')\mu_q(dq'|s,a).$$

Let $U(s,a;v) = (U_1(s,a;v),\ldots,U_n(s,a;v))$ be the vector of payoffs. A mixed strategy for player i in $\Gamma_v(s)$ is a probability measure $\alpha_i \in \mathcal{P}(A_i(s))$; but we may equivalently view α_i as an element of $\mathcal{P}(X_i)$ satisfying $\alpha_i(A_i(s)) = 1$, and mixed strategies for all players then determine a product probability measure $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_n \in \bigotimes_i \mathcal{P}(X_i)$. The spaces $\bigotimes_i \mathcal{P}(X_i)$ and $\bigotimes_i \mathcal{P}(A_i(s))$ are endowed with the relative weak* topology inherited from $\mathcal{P}(\prod_i X_i)$, so by Billingsley's (1968) Theorem 3.2, convergence of a sequence $\{\alpha^m\}$ to α is equivalent to convergence of the marginals $\{\alpha_i^m\}$ to α_i for all i. Define the extension $U(\cdot;v)\colon S \times \bigotimes_i \mathcal{P}(X_i) \to \Re^n$ to mixed strategies in the induced game by $U(s,\alpha;v) = \int_a U(s,a;v)\alpha(da)$.

Lemma 1: For each v and for all α, $U(s,\alpha;v)$ is measurable in s; and for all s, $U(s,\alpha;v)$ is jointly continuous in (α,v).

Proof: First, fix v and a, and note that $\mu_q(\cdot|\cdot,a)$ is a transition probability, i.e., for all s, $\mu_q(\cdot|s,a)$ is a probability measure, and for each $Z \in \mathcal{Q}$, $\mu_q(Z|s,a)$ is measurable in s. Since v is essentially bounded, Theorem 19.7 of Aliprantis and Border (2006) implies that $\int_{q'} v_i(q')\mu_q(dq'|s,a)$ is measurable in s, which delivers measurability of $U(s,a;v)$ in s. Now fix s, and consider a sequence $\{(a^m,v^m)\}$ converging to (a,v) in $X \times V$. Then for all i, we have

$$\left| \int_{q'} v_i^m(q')\mu_q(dq'|s,a^m) - \int_{q'} v_i(q')\mu_q(dq'|s,a) \right|$$

$$
\begin{aligned}
&\leq \left|\int_{q'} v_i^m(q')\mu_q(dq'|s,a^m) - \int_{q'} v_i^m(q')\mu_q(dq'|s,a)\right| \\
&\quad + \left|\int_{q'} v_i^m(q')\mu_q(dq'|s,a) - \int_{q'} v_i(q')\mu_q(dq'|s,a)\right| \\
&\leq C||\mu_q(\cdot|s,a^m) - \mu_q(\cdot|s,a)|| + \left|\int_{q'} (v_i^m(q') - v_i(q'))g(q'|s,a)\nu(dq')\right| \\
&\to 0,
\end{aligned}
$$

where the first inequality uses the triangle inequality, the second follows since $\{v_i^m\}$ is essentially bounded by C, and the limit follows both from norm-continuity of $\mu_q(\cdot|s,a)$ and from weak* convergence of $\{v^m\}$ to v and the fact that the transition density $g(\cdot|s,a)$ of $\mu_q(\cdot|s,a)$ lies in L_1^n. Thus, $U(s,a;v)$ is continuous in (a,v), and Aliprantis and Border's (2006) Lemma 4.51 implies that U is jointly measurable. Given any α, it follows that $U(s,\alpha;v) = \int_a U(s,a;v)\alpha(da)$ is measurable in s. Moreover, for every sequence $\{(\alpha^m, v^m)\}$ converging to (α, v), we have for all i,

$$
\begin{aligned}
|U_i(s,\alpha^m;v^m) - U_i(s,\alpha;v)| &\leq \left|\int_a (U_i(s,a;v^m) - U_i(s,a;v))\alpha^m(da)\right| \\
&\quad + \left|\int_a U_i(s,a;v)\alpha^m(da) - \int_a U_i(s,a;v)\alpha(da)\right| \to 0,
\end{aligned}
$$

where the inequality follows from the triangle inequality, and the limit follows from both Lebesgue's dominated convergence theorem (using the fact that $U(s,\cdot;v^m) \to U(s,\cdot;v)$ pointwise; see Theorem 11.21 of Aliprantis and Border (2006)) and weak* convergence of $\{\alpha^m\}$ to α, as required.

The payoff function $U(s,\cdot;v)$ restricted to $\prod_i A_i(s)$ is continuous, and with compactness of each $A_i(s)$, the Debreu-Fan-Glicksberg theorem implies that the set of mixed strategy Nash equilibria of $\Gamma_v(s)$, denoted $N_v(s)$, is a nonempty, compact subset of $\bigotimes_i \mathcal{P}(X_i)$. Let $P_v(s)$ denote the payoffs generated by equilibria in $N_v(s)$, i.e., $P_v(s) = U(s, N_v(s); v) = \{U(s,\alpha;v) \mid \alpha \in N_v(s)\}$. Note the immediate implication of Lemma 1 that the correspondences $v \to N_v(s)$ and $v \to P_v(s)$ have weak* closed graph. The next lemma is essentially Lemmas 5 and 6 of Nowak and Raghavan (1992).

Lemma 2: For each v, the correspondences $s \to N_v(s)$ and $s \to P_v(s)$ are lower measurable.

Proof: Fix v. Note that the correspondence $s \to A_i(s)$ is measurable by Theorem 18.10 of Aliprantis and Border (2006), so Theorem 3 of Himmelberg and van Vleck (1975) implies that the correspondence $s \to \mathcal{P}(A_i(s))$ is lower measurable for all i. Part 2 of Aliprantis and Border's (2006) Lemma 18.4 then implies that $\Delta\colon S \rightrightarrows \prod_i \mathcal{P}(X_i)$ defined by $\Delta(s) = \prod_i \mathcal{P}(A_i(s))$ is lower measurable. Letting $\pi\colon \prod_i \mathcal{P}(X_i) \to \bigotimes_i \mathcal{P}(X_i)$ be the homeomorphism defined by $\pi(\alpha_1, \ldots, \alpha_n) = \alpha_1 \otimes \cdots \otimes \alpha_n$, it follows that $s \to \bigotimes_i \mathcal{P}(A_i(s)) = \pi(\Delta(s))$ is lower measurable. Now define $\xi\colon S \times \bigotimes_i \mathcal{P}(X_i) \to \Re$ by

$$\xi(s, \alpha) \;=\; \sum_i [U_i(s, \alpha; v) - \max_{a_i \in A_i(s)} U_i(s, a_i, \alpha_{-i};)].$$

By Lemma 1 and the theorem of the maximum (see Theorem 17.31 of Aliprantis and Border (2006)), $\xi(s, \alpha)$ is continuous in α; and by Theorem 18.19 of Aliprantis and Border (2006), it is measurable in s. Thus, ξ is Caratheodory. Defining $\Xi\colon S \rightrightarrows \bigotimes_i \mathcal{P}(X_i)$ by

$$\Xi(s) \;=\; \left\{ \alpha \in \bigotimes_i \mathcal{P}(X_i) \mid \xi(s, \alpha) = 0 \right\},$$

Corollary 18.8 of Aliprantis and Border (2006) implies that Ξ is lower measurable. Then, by part 3 of Aliprantis and Border's (2006) Lemma 18.4, $s \to N_v(s) = \Xi(s) \cap \mathcal{P}(A_i(s))$ is lower measurable. That $s \to P_v(s)$ is lower measurable then follows from the fact that $P_v(s) = U(s, N_v(s); v)$ is the image of $N_v(s)$ under a continuous function, as required.

For all r, let $\Phi_v(r)$ be the set of κ-integrable, density-weighted, equilibrium payoff selections as a function of q: specifically, $f \in L_1^n$ belongs to $\Phi_v(r)$ if and only if for κ-almost all q, $f(q) \in h(r|q)P_v(q, r)$, i.e., there is an equilibrium payoff vector $y \in P_v(q, r)$ of the stage game induced by v at (q, r) such that $f(q) = h(r|q)y$.[12] The remainder of the proof considers properties of the integral of the correspondence $r \to \Phi_v(r)$, suitably defined. A function $\phi\colon R \to L_1^n$ is *Bochner integrable* if two conditions hold: it can be approximated by a sequence $\{\varphi^m\}$ of simple functions, in the sense that $||\phi(r) - \varphi^m(r)||_1 \to 0$ for λ-almost all r; and it is λ-integrable, in the sense that $\int_r ||\phi(r)||_1 \lambda(dr) < \infty$ (see Definition 11.42 and Theorem 11.44

[12] The choice of L_1^n for the range of Φ_v is important for the existence of a λ-integrable selection in the proof of Lemma 4, because $h(r|\cdot)$ need not be essentially bounded for λ-almost all r.

of Aliprantis and Border (2006)). In this case, the Bochner integral of ϕ is $\lim_{m\to\infty}\int_r \varphi^m(r)\lambda(dr)$, and the Bochner integral of a general correspondence $\Phi\colon R \rightrightarrows L_1^n$ is the set of integrals of λ-almost everywhere selections, i.e.,

$$\int_r \Phi(r)\lambda(dr) \;=\; \left\{\int_r \phi(r)\lambda(dr) \mid \begin{array}{c}\phi \text{ is a Bochner integrable,}\\ \lambda\text{-a.e. selection of } \Phi\end{array}\right\}.$$

For all v, define $E_v = \int_r \Phi_v(r)\lambda(dr)$ as the Bochner integral of the correspondence $r \to \Phi_v(r)$ with respect to λ, parameterized by v. The next lemma establishes that the correspondence $v \to E_v$ maps to subsets of V.

Lemma 3: For each v, $E_v \subseteq V$.

Proof: Given v, consider any $f \in E_v$. Then there exists a Bochner integrable selection $\phi\colon R \to L_1^n$ such that $\phi(r) \in \Phi_v(r)$ for λ-almost all r and such that $f = \int_r \phi(r)\lambda(dr)$. Moreover, for all r with $\phi(r) \in \Phi_v(r)$ and for κ-almost all q, $\phi(r)(q) \in h(r|q)P_v(q,r)$. By part 1 of Aliprantis and Border's (2006) Theorem 11.47, there is a $\nu = \kappa \otimes \lambda$-integrable $F\colon Q \times R \to \Re^n$ such that

(a) for λ-almost all r, we have for κ-almost all q, $\phi(r)(q) = F(q,r)$,

(b) for κ-almost all q, $F(q,\cdot)$ is λ-integrable and

$$\left(\int_r \phi(r)\lambda(dr)\right)(q) \;=\; \int_r F(q,r)\lambda(dr).$$

An implication of (a), with the fact that $\phi(r) \in \Phi_v(r)$ for λ-almost all r, is that for κ-almost all q and for λ-almost all r, $F(q,r) \in h(r|q)P_v(q,r)$. With (b), this implies that there is a set $Q^0 \in \mathfrak{Q}$ with $\kappa(Q^0) = 0$ such that for all $q \in Q \setminus Q^0$, we have (i) for λ-almost all r, $F(q,r) \in h(r|q)P_v(q,r)$, and (ii) $(\int_r \phi(r)\lambda(dr))(q) = \int_r F(q,r)\lambda(dr)$. Then for all $q \in Q \setminus Q^0$, we have

$$\begin{aligned} ||f(q)|| \;&=\; \left|\left|\left(\int_r \phi(r)\lambda(dr)\right)(q)\right|\right| \;=\; \left|\left|\int_r F(q,r)\lambda(dr)\right|\right| \\ &\leq\; \int_r ||F(q,r)||\lambda(dr) \;\leq\; \int_r Ch(r|q)\lambda(dr) \;\leq\; C, \end{aligned}$$

where the first equality follows from $f = \int_r \phi(r)\lambda(dr)$, the second equality from (ii), the first inequality from Jensen's inequality (see Theorem 11.24 of Aliprantis and Border (2006)), the second inequality from (i), and the last inequality from the fact that $h(\cdot|q)$ is a density. This implies $f \in V$ and therefore $E_v \subseteq V$, as required.

To see that the correspondence $v \to E_v$ has nonempty values, note the implication of Lemma 2, via the Kuratowski-Ryll-Nardzewski selection theorem (see Theorem 18.13 of Aliprantis and Border (2006)), that for each v, the correspondence $s \to P_v(s)$ admits an measurable selection. With some care to ensure integrability, this implies the desired result.

Lemma 4: For each v, $E_v \neq \emptyset$.

Proof: It suffices to deduce a Bochner integrable mapping $\phi\colon R \to L_1^n$ such that $\phi(r) \in \Phi_v(r)$ for λ-almost all r. By Lemma 2 and the Kuratowski-Ryll-Nardzewski selection theorem, there is a measurable mapping $y\colon S \to \Re$ satisfying $y(s) \in P_v(s)$ for $\nu = \kappa \otimes \lambda$-almost all s. In particular, $||y(q,r)|| \leq C$ for $\kappa \otimes \lambda$-almost all (q,r). Obviously, $\int_q \int_r h(r|q)\lambda(dr)\kappa(dq) = 1$, and Tonelli's theorem (see Theorem 11.28 of Aliprantis and Border (2006)) implies that $h(r|q)$ is $\kappa \otimes \lambda$-integrable and $\int_r \int_q h(r|q)\kappa(dq)\lambda(dr) = 1$. Thus, $\int_q h(r|q)\kappa(dq) < \infty$ for λ-almost all r. Let $R^0 \in \mathcal{R}$ satisfy $\lambda(R^0) = 0$ and for all $r \in R \setminus R^0$, $\int_q h(r|q)\kappa(dq) < \infty$, and define $\phi\colon R \to L_1^n$ so that $\phi(r)(q) = h(r|q)y(q,r)$ for all $r \in R \setminus R^0$ and all q. Then $\phi(r) \in \Phi_v(r)$ for λ-almost all r, and $h(r|q)y(q,r)$ is $\kappa \otimes \lambda$-integrable, so part 2 of Aliprantis and Border's (2006) Theorem 11.47 implies ϕ is Bochner integrable, as required.

To establish convex values and closed graph of $v \to E_v$, it will be useful to define the following auxiliary correspondences. For each v, let $P_v^*(s)$ denote the convex hull of $P_v(s)$, and let $\Phi_v^*(r)$ be the set of κ-integrable, density-weighted, convex combinations of equilibrium payoff selections as a function of q: specifically, $f \in L_1^n$ belongs to $\Phi_v^*(r)$ if and only if for κ-almost all q, $f(q) \in h(r|q)P_v^*(q,r)$, i.e., there is a convex combination $y \in P_v^*(q,r)$ of equilibrium payoff vectors in the induced game such that $f(q) = h(r|q)y$. In contrast to $\Phi_v(r)$, the set $\Phi_v^*(r)$ must be convex, and it follows immediately that the Bochner integral $\int_r \Phi_v^*(r)\lambda(dr)$ is convex as well.

The usefulness of the latter observation lies in the fact, shown next, that E_v can be written as the integral of Φ_v^*, i.e., $\int_r \Phi_v(r)\lambda(dr) = \int_r \Phi_v^*(r)\lambda(dr)$. One direction of this inclusion is obvious. For the less trivial $\supseteq$ inclusion, we would like to apply a version of Lyapunov's theorem for correspondences (e.g., Theorem 4, p.64, of Hildenbrand (1974)) with respect to a nonatomic measure: for correspondences mapping to $\Re^n$, Lyapunov's theorem implies that the integral of a correspondence with respect to a nonatomic measure is equal to the integral of the convex hull of the correspondence. But Lyapunov's theorem does not hold in infinite-dimensional settings, so this direct avenue is not open. Instead, the approach I use relies on the fact that the correspondence $r \to \Phi_v(r)$ has a product structure, in that given r, $P_v(q, r)$ is defined independently for each q; the selection of equilibrium payoffs in the induced game $\Gamma(q, r)$ does not restrict (beyond considerations of measurability) the selection at $\Gamma(q', r)$. This permits the application of Lyapunov's theorem separately for each q. Thus, the proof "goes down" from the Bochner integral $\int_r \Phi^*_v(r)\lambda(dr)$ to integrals of the correspondence $(q, r) \to P^*_v(q, r)$ defined on (q, r) pairs. I then apply Lyapunov's convexity theorem for correspondences mapping to subsets of $\Re^n$, integrating across r one q at a time. Finally the proof "goes up" to the Bochner integral. A technical issue is that in the latter step, we have one integral $\int_r P_v(q, r)h(r|q)\lambda(dr)$ for each q, and thus one selection from $r \to h(r|q)P_v(q, r)$ for each q. To return to the Bochner integral, we have to "sew up" these selections in a measurable way, a task facilitated by a theorem of Artstein (1989).[13]

Lemma 5: For each v, $E_v = \int_r \Phi^*_v(r)\lambda(dr)$; in particular, E_v is convex.

Proof: Clearly, $E_v \subseteq \int_r \Phi^*_v(r)\lambda(dr)$. Now consider any $\hat{v} \in \int_r \Phi^*_v(r)\lambda(dr)$, so there exists a λ-integrable mapping $\phi\colon R \to L^n_1$ such that $\phi(r) \in \Phi^*_v(r)$ for λ-almost all r and $\hat{v} = \int_r \phi(r)\lambda(dr)$. By Aliprantis and Border's (2006) Theorem 11.47, part 1, there is a $\kappa \otimes \lambda$-integrable function $F\colon Q \times R \to \Re^n$ satisfying (a) and (b) in the proof of Lemma 3. An implication of (a), with the fact that $\phi(r) \in \Phi^*_v(r)$ for λ-almost all r, is that for κ-almost all q and for λ-almost all r, $F(q, r) \in h(r|q)P^*_v(q, r)$. With (b), this implies that there is a set $Q^0 \in \mathcal{Q}$ with $\kappa(Q^0) = 0$ such that for all $q \in Q \setminus Q^0$, we have (i) for λ-almost all r, $F(q, r) \in h(r|q)P^*_v(q, r)$, and (ii) $(\int_r \phi(r)\lambda(dr))(q) = \int_r F(q, r)\lambda(dr)$. Then for all $q \in Q \setminus Q^0$, we have

$$\hat{v}(q) = \int_r F(q, r)\lambda(dr) \in \int_r P^*_v(q, r)h(r|q)\lambda(dr) = \int_r P_v(q, r)h(r|q)\lambda(dr),$$

where the first equality follows from $\hat{v} = \int_r \phi(r)\lambda(dr)$ and (ii), the inclusion from (i), and the last equality from Hildenbrand's (1974) Theorem 4 (p.64). It follows that for κ-almost all q, $\hat{v}(q) \in \int_r P_v(q, r)h(r|q)\lambda(dr)$. To apply the

[13] Alternatively, this lemma can be deduced from part 2 of the more general theorem of Mertens (2003). The proof of Lemma 5 here is straightforward relative to Mertens' proof.

theorem of Artstein (1989), note that Q and R are complete, separable metric spaces, and that the correspondence $(q,r) \to P_v(q,r)$ is lower measurable, by Lemma 2, and has nonempty, compact values. Thus, by Aliprantis and Border's (2006) Theorem 18.10, the correspondence $(q,r) \to P_v(q,r)$ is in fact measurable. Note also that for κ-almost all q, $|| \int_r h(r|q)P_v(q,r)\lambda(dr)|| \leq C$, so the correspondence $r \to h(r|q)P_v(q,r)$ is λ-integrably bounded. We identify our R with Artsein's S, our Q with his T, our λ with his β_t (constant in t), and our $P_v(q,r)$ with his $F(t,s)$. Then Artstein's theorem yields a measurable mapping $G\colon Q \times R \to \Re^n$ such that for κ-almost all q, $\hat{v}(q) = \int_r G(q,r)\lambda(dr)$ and for λ-almost all r, $G(q,r) \in h(r|q)P_v(q,r)$. Of course, G is $\kappa \otimes \lambda$-integrable, so by part 2 of Aliprantis and Border's (2006) Theorem 11.47, the mapping $\psi\colon R \to L_1^n$ defined so that $\psi(r)(q) = G(q,r)$ for $\kappa \otimes \lambda$-almost all (q,r) is Bochner integrable, and for κ-almost all q,

$$\hat{v}(q) \;=\; \int_r G(q,r)\lambda(dr) \;=\; \left(\int_r \psi(r)\lambda(dr)\right)(q),$$

so $\hat{v} = \int_r \psi(r)\lambda(dr)$. Furthermore, $\psi(r) \in \Phi_v(r)$ for λ-almost all r, and we conclude that $\hat{v} \in \int_r \Phi_v(r)\lambda(dr) = E_v$, as required.

The next lemma shows that the set of selections of $(q,r) \to h(r|q)P_v^*(q,r)$ has closed graph with respect to v. Specifically, the correspondence analyzed is $\Psi^*\colon V \rightrightarrows L_1^n(S,\mathcal{S},\nu)$ defined by

$$\Psi^*(v) \;=\; \left\{F \in L_1^n(S,\mathcal{S},\nu) \mid \text{ for } \nu\text{-a.e. } s=(q,r),\, F(s) \in h(r|q)P_v^*(s) \;\right\},$$

with the weak* topology $\sigma(L_\infty^n, L_1^n)$ on its domain and the weak topology $\sigma(L_1^n(S,\mathcal{S},\nu), L_\infty^n(S,\mathcal{S},\nu))$ on its range. The result is related to Lemma 7 of Nowak and Raghavan (1992), but here we must frame the result in $L_1^n(S,\mathcal{S},\nu)$ instead of $L_\infty^n(S,\mathcal{S},\nu)$, and the proof uses Proposition C of Artstein (1979). A technical issue is that the closed unit ball in $L_1^n(S,\mathcal{S},\nu)$ is not metrizable (unless ν has finite support), so the lemma is stated only for sequentially closed graph.

Lemma 6: The correspondence $v \to \Psi^*(v)$ has sequentially closed graph.

Proof: Note that the correspondence $v \to P_v(s)$ has closed graph by Lemma 1, and then part 2 of Aliprantis and Border's (2006) Theorem 17.35 implies that $v \to P_v^*(s)$ has closed graph as well. Consider a sequence $\{(v^m,F^m)\}$ converging to (v,F) in $V \times L_1^n(S,\mathcal{S},\nu)$ such that for all m,

$F^m \in \Psi^*(v^m)$, i.e., F^m is a ν-almost everywhere selection from $P^*_{v^m}$. By Tonelli's theorem, $h(r|q)$ is ν-integrable as a function of $s = (q,r)$, and note that $||F^m(s)|| \leq h(r|q)C$ for all m, implying the sequence $\{F^m\}$ is uniformly integrable. Then Proposition C of Artstein (1979) implies that for ν-almost every s, $F(s)$ is a convex combination of accumulation points of $\{F^m(s)\}$. Let $Z^0 \in \mathcal{S}$ be a set of full measure for which this property holds. For each m, let $Z^m \in \mathcal{S}$ be a set of full measure on which F^m selects from $P^*_{v^m}$, i.e., $Z^m \subseteq \{s \mid F^m(s) \in P^*_{v^m}(s)\}$, and define $Z = \bigcap_{m=0}^{\infty} Z^m$, which also has full measure. Considering any $s \in Z$, we may write $F(s) \in \mathrm{co}\{y^1, \ldots, y^k\}$, where for each $j = 1, \ldots, k$, there is a subsequence $\{F^{m_\ell}\}$ such that $y^j = \lim_{\ell \to \infty} F^{m_\ell}(s)$. And since $F^{m_\ell}(s) \in P^*_{v^{m_\ell}}(s)$ for all ℓ, this implies $y^j \in P^*_v(s)$, which implies that $F(s) \in P^*_v(s)$. Thus, F is a ν-almost everywhere selection from $s \to P^*_v(s)$, i.e., $F \in \Psi^*(v)$, as required.

Closed graph of the correspondence $v \to E_v$ can now be established by an infinite-dimensional Fatou's lemma argument. One possible approach would be to apply Yannelis's (1990) Theorem 3.2, with the proviso is that Yannelis works with a complete measure space (T, τ, μ), whereas our $(S, \mathcal{S}, \nu)$ is not assumed complete; but a close reading of his proof reveals that completeness is not used. Instead, I provide a direct proof based on Lemma 6.

Lemma 7: The correspondence $v \to E_v$ has closed graph.

Proof: Consider a sequence $\{(v^m, \hat{v}^m)\}$ converging to $(v, \hat{v})$ in $V \times V$ such that for all m, $\hat{v}^m \in E_{v^m} = \int_r \Phi_{v^m}(r)\lambda(dr)$. For each m, there is a Bochner integrable, λ-almost everywhere selection ϕ^m of $r \to \Phi_{v^m}(r)$ such that $\hat{v}^m = \int_r \phi^m(r)\lambda(dr)$. By part 1 of Aliprantis and Border's (2006) Theorem 11.47, there is a $\kappa \otimes \lambda$-integrable function $F^m \colon Q \times R \to \Re$ such that (a) for λ-almost all r, we have for κ-almost all q, $\phi^m(r)(q) = F^m(q,r)$, and (b) for κ-almost all q, $F^m(q, \cdot)$ is λ-integrable and $\hat{v}^m(q) = \int F^m(q,r)\lambda(dr)$. From (a), it follows that $F^m \in \Psi^*(v^m)$ for all m, and as in the proof of Lemma 6, $\{F^m\}$ is uniformly integrable. Furthermore, $||F^m||_1 \leq C$ for all m, so the sequence is norm-bounded. Then Theorem 15 (p.76) of Diestel and Uhl (1977) implies that the sequence is relatively compact. By the Eberlein-Smulian theorem (see Theorem 6.34 of Aliprantis and Border (2006)), it has a convergent subsequence $\{F^m\}$ (still indexed by m) with limit $F \in L^n_1(S, \mathcal{S}, \nu)$. By Lemma 6, we have $F \in \Psi^*(v)$, and moreover $||F(\cdot, r)||_1 < \infty$ for λ-almost all r, so we can define the mapping $\phi \colon R \to L^n_1$ so that $\phi(r)(q) = F(q,r)$ for κ-almost all q. Note that for λ-almost all r, we have for κ-almost all q,

$\phi(r)(q) \in P_v^*(q,r)$, i.e., $\phi(r) \in \Phi_v^*(r)$. By part 2 of Aliprantis and Border's (2006) Theorem 11.47, it follows that (c) ϕ is Bochner integrable, and that (d) $(\int_r \phi(r)\lambda(dr))(q) = \int_r F(q,r)\lambda(dr)$ for κ-almost all q. In particular, ϕ is a Bochner integrable, λ-almost everywhere selection from $r \rightarrow \Phi_v^*(r)$, so it suffices to prove that $\hat{v} = \int_r \phi(r)\lambda(dr)$. For this, since $\hat{v}^m \rightarrow \hat{v}$ weak* (and therefore weakly), it suffices to show that $\hat{v}^m \rightarrow \int_r \phi(r)\lambda(dr)$ weakly. To this end, consider any $f \in L_\infty^n$. Note that

$$
\begin{aligned}
\int_q \hat{v}^m(q)f(q)\kappa(dq) &= \int_q \left(\int_r F^m(q,r)\lambda(dr)\right) f(q)\kappa(dq) \\
&= \int_{(q,r)} F^m(q,r)f(q)(\kappa\otimes\lambda)(d(q,r)) \\
&\rightarrow \int_{(q,r)} F(q,r)f(q)(\kappa\otimes\lambda)(d(q,r)) \\
&= \int_q \left(\int_r F(q,r)\lambda(dr)\right) f(q)\kappa(dq) \\
&= \int_q \left(\int_r \phi(r)\lambda(dr)\right)(q) f(q)\kappa(dq),
\end{aligned}
$$

where the first equality follows from (b), the second equality from Fubini's theorem, the limit from $F^m \rightarrow F$ weakly (viewing f as defined on $Q \times R$ and constant in r), the third equality from Fubini's theorem, and the last from (d). Thus, $\hat{v} = \int_r \phi(r)\lambda(dr) \in \int_r \Phi_v^*(r)\lambda(dr) = \int_r \Phi_v(r)\lambda(dr) = E_v$, as required.

Collecting Lemmas 3, 4, 5, and 7, the correspondence $v \rightarrow E_v$ maps the nonempty, convex, weak* compact set V (a subset of a locally convex Hausdorff tvs) to nonempty, convex subsets of V, and it has closed graph in the weak* topology. By the Debreu-Fan-Glicksberg theorem, there exists a fixed point $v \in E_v$. The final step of the proof is to construct a stationary Markov perfect equilibrium. Let $\phi\colon R \rightarrow L_1^n$ be an λ-integrable mapping such that $\phi(r) \in \Phi_v(r)$ for λ-almost all r and such that $v = \int_r \phi(r)\lambda(dr)$. Using part 1 of Aliprantis and Border's (2006) Theorem 11.47 again, let $F\colon Q \times R \rightarrow \Re^n$ be a $\kappa\otimes\lambda$-integrable function satisfying (a) and (b) in the proof of Lemma 3. From (a), there is a set $S^0 \in \mathcal{S}$ with $\nu(S^0) = 0$ such that for all $s = (q,r) \in S \setminus S^0$, we have $F(s) \in h(r|q)P_v(q,r)$. Using the fact that $s \rightarrow P_v(s)$ is lower measurable, and therefore admits a measurable selection, we can modify F so that $F(s) \in h(r|q)P_v(q,r)$ for all $s \in S^0$ as well. From (b), it follows that for κ-almost all q, $v(q) = \int_r F(q,r)\lambda(dr)$.

By Lemmas 1 and 2, $U(\cdot; v)\colon S \times \bigotimes_i \mathcal{P}(X_i) \to \Re^n$ is a Caratheodory function and $s \to N_v(s)$ is lower measurable. Moreover, for all $s = (q, r)$, there exists $\alpha \in N_v(s)$ such that $F(s) = h(r|q)U(q, r, \alpha; v)$. Then Filippov's implicit function theorem (see Aliprantis and Border's (2006) Theorem 18.17) yields a measurable function $\xi\colon S \to \bigotimes_i \mathcal{P}(X_i)$ such that for all $s = (q, r)$, we have $\xi(s) \in N_v(s)$ and $F(s) = h(r|q)U(s, \xi(s); v)$. Define the strategy $\sigma_i\colon S \to \mathcal{P}(X_i)$ for each player i so that for all s, $\sigma_i(s)$ is the marginal of $\xi(s)$ on X_i, and write instead $\sigma(s)$ for $\xi(s)$, the product of the players' mixed strategies. By Aliprantis and Border's (2006) Theorem 19.7, measurability of ξ implies that $\xi_i(s)(Y_i) \equiv \xi(s)(Y_i \times X_{-i})$ is measurable in s for each Borel $Y_i \subseteq X_i$, which implies that the mapping $\sigma_i\colon S \to \mathcal{P}(X_i)$ is indeed measurable, so these strategies are well-defined.

Finally, I argue that σ is an equilibrium. For κ-almost all q and all i,

$$\begin{aligned} v_i(q) &= \int_r F_i(s)\lambda(dr) = \int_r U_i(s, \sigma(s); v)h(r|q)\lambda(dr) \\ &= \int_r \left[\int_a \left[(1-\delta_i)u_i(s,a) + \delta_i \int_{q'} v_i(q')\mu_q(dq'|s,a)\right]\sigma(da|s)\right]\mu_r(dr|q), \quad (1) \end{aligned}$$

where $s = (q, r)$. Define $w \in L^n_\infty(S, \mathcal{S}, \nu)$ so that for ν-almost all q and all i,

$$w_i(s) = \int_a \left[(1-\delta_i)u_i(s,a) + \delta_i \int_{q'} v_i(q')\mu_q(dq'|s,a)\right]\sigma(da|s). \quad (2)$$

Recall that the mapping $q \to \mu_r(\cdot|q)$ is a regular conditional probability for r following any state-action pair. By a generalized version of Fubini's theorem (see Proposition 2, p.47, of Rao (1993)), equations (1) and (2) then imply that for all s, all a, and all i,

$$\int_{q'} v_i(q')\mu_q(dq'|s,a) = \int_{s'} w_i(s')\mu(ds'|s,a), \quad (3)$$

and then substituting back into (2), we obtain for ν-almost all s and all i,

$$w_i(s) = \int_a \left[(1-\delta_i)u_i(s,a) + \delta_i \int_{s'} w_i(s')\mu(ds'|s,a)\right]\sigma(da|s).$$

Thus, w satisfies the recursion that uniquely defines $v(\cdot;\sigma)$, so $w = v(\cdot;\sigma)$. Using (2), the fact that $\sigma_i(s)$ is a best response to $\sigma_{-i}(s)$ in $\Gamma_v(s)$ implies

$$w_i(s) = \sup_{a_i \in A_i(s)} \int_{a_{-i}} \left[(1-\delta_i)u_i(s,a) + \delta_i \int_{s'} v_i(q')\mu_q(dq'|s,a)\right]\sigma_{-i}(da_{-i}|s).$$

Finally, using $w = v(\cdot;\sigma)$ and (3), the above equality yields

$$\begin{aligned} & v_i(s;\sigma) \\ &= \sup_{a_i \in A_i(s)} \int_{a_{-i}} \left[(1-\delta_i) u_i(s,a) + \delta_i \int_{s'} v_i(s';\sigma) \mu(ds'|s,a) \right] \sigma_{-i}(da_{-i}|s), \end{aligned}$$

for all s and all i. Therefore, σ is a stationary Markov perfect equilibrium.

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