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A Dynamic Auction for Differentiated Items under Price Rigidities

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Abstract: A number of heterogeneous items are to be sold to several bidders. Each bidder demands at most one item. The price of each item is not completely flexible and is restricted to some admissible interval. In such a market economy with price rigidities, a Walrasian equilibrium usually fails to exist. To facilitate the allocation of items to the bidders, we propose an ascending auction with rationing that yields a constrained Walrasian equilibrium outcome. The auctioneer starts with the lower bound price vector that specifies the lowest admissible price for each item, and each bidder responds with a set of items demanded at those prices. The auctioneer adjusts prices upwards for a minimal set of over-demanded items and chooses randomly a winning bidder for any item if the item is demanded by several bidders and its price has reached its highest admissible price. We prove that the auction finds a constrained Walrasian equilibrium outcome in a finite number of steps.

Keywords: Ascending auction, multi-item auction, constrained equilibrium, price rigidities, rationing.

JEL classification: D44.

1 Introduction

Economists have extensively studied market environments where there may be restrictions on the prices of commodities and services. There are many economic or political reasons for the existence of price rigidities. For instance, to prevent breakdown of stock markets, often ceilings and floors are imposed upon the price of each stock; price controls are used to

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reduce inflation or deflation; and minimum wages are employed to protect certain groups of the society; see e.g., Drèze (1975), Dehez and Drèze (1984), Cox (1980), van der Laan (1980), Kurz (1982), Azariadis and Stiglitz (1983), Weddepohl (1987), Herings et al. (1996) among many others. When there are fixed prices or price rigidities, rationing is needed to restrict how much agents are allowed to purchase or sell. As a result, rationing will help prices to facilitate the distribution of commodities among agents. However, efficiency cannot be fully attained in general.

While the literature has focused almost entirely on economic models with divisible goods, we will study the efficient allocation of heterogeneous indivisible goods such as houses or apartments among several agents. Each agent demands at most one item. The price of each item is restricted to an admissible interval and thus is assumed to be not perfectly flexible. The lowest admissible price of an item can be seen as a reservation price, while the highest admissible price of an item may be exogenously given by the government for the purpose of either preventing speculation or protecting low-income families so that they can afford to buy necessary goods, such as an apartment. In such a market economy, a Walrasian equilibrium usually fails to exist. To facilitate the allocation of items to the bidders (or agents), we propose an ascending auction with rationing that produces a constrained Walrasian equilibrium outcome. The auctioneer starts with the lower bound price vector that specifies the lowest admissible price for each item, and each bidder responds with a set of items demanded at those prices. The auctioneer adjusts prices upwards for a minimal set of over-demanded items and chooses randomly a winning bidder for an item if this item is demanded by several bidders and its price has reached its highest admissible price. We prove that the auction finds a constrained Walrasian equilibrium outcome in a finite number of steps. At such an equilibrium a bidder can be rationed on his demand for an item if the price of the item is on its upper bound, if the item is assigned to some other bidder, and if he will demand the item when the rationing is removed from him. Moreover, if an item is not assigned in equilibrium its price must be on its lower bound. The auction can be seen as a variant of the auction of Demange et al. (1985) for selling multiple items without price rigidities. Another related mechanism is due to Crawford and Knoer (1981). Maskin (2000) studied an auction for the allocation of one item when buyers may be significantly budget-constrained.

This paper proceeds as follows. Section 2 sets up the model. Section 3 presents the dynamic auction and establishes the main theorem.

2 The Model

An auctioneer wishes to sell a set of heterogeous items $N = \{0, 1, \cdots, n\}$, to a group of bidders $M = \{1, 2, \cdots, m\}$. The item 0 is a *dummy good* which can be assigned to more than one bidder. Without loss of generality, we assume that the auctioneer values every item at zero. Every bidder $i \in M$ attaches an integer monetary value to each item, namely, each bidder i has a utility function $V^i : N \to \mathbb{Z}_+$ with $V^i(0) = 0$. A *feasible allocation* π assigns every bidder $i \in M$ an item $\pi(i)$ such that no item in $N \setminus \{0\}$ is assigned to more than one bidder. Note that a feasible allocation may assign the dummy good to several bidders and a real item $j \neq 0$ may not be assigned to any bidder at all. An item $j \geq 1$ is *unassigned* at π if there is no bidder i such that $\pi(i) = j$. A feasible allocation π^* is *efficient* if $\sum_{i \in M} V^i(\pi^*(i)) \geq \sum_{i \in M} V^i(\pi(i))$ for every feasible allocation π.

A price vector $p \in \mathbb{R}^N_+$ indicates a price for every good. The price of each good $j \in N \setminus \{0\}$ is not completely flexible and is restricted to an interval $[\underline{p}_j, \bar{p}_j]$, where $\underline{p}_j$ and $\bar{p}_j$ are integers and $0 \leq \underline{p}_j < \bar{p}_j$. The set

$$P = \{p \in \mathbb{R}^N \mid p_0 = 0,\ \underline{p}_j \leq p_j \leq \bar{p}_j,\ j = 1, \cdots, n\ \}$$

denotes the set of admissible prices. The price of the dummy good is always fixed at zero.

When rationing does not take place, the demand set of bidder $i \in M$ at any price vector $p \in \mathbb{R}^N_+$ is given by

$$D^i(p) = \{j \mid V^i(j) - p_j \geq V^i(k) - p_k \text{ for every } k \in N\}.$$

A *Walrasian equilibrium* consists of a price vector $p \in \mathbb{R}^N_+$ and a feasible allocation π such that $\pi(i) \in D^i(p)$ for all $i \in M$ and $p_j = 0$ for any unassigned good j at π. It is well-known that a Walarasian equilibrium exists in the economy when there are no price rigidities.

In the case of price restrictions, a Walrasian equilibrium may not exist since the equilibrium price vector may not be admissible. In this case we may introduce a rationing scheme

$R^i \in \{0,1\}^N$ for each bidder $i \in M$ with $R^i_0 = 1$. For $i \in M$, the vector R^i dictates which goods bidder $i \in M$ can demand and which goods bidder i cannot demand, namely, $R^i_j = 1$ means that bidder i is allowed to demand good j, while $R^i_j = 0$ means that bidder i is not allowed to demand good j. Given the rationing scheme R^i and an item j with $R^i_j = 0$, the vector R^i_{-j} will denote that R^i_j is being ignored and bidder i is allowed to demand item j. A list $R = (R^1, R^2, \cdots, R^m)$ of all bidders' rationing schemes is called *a rationing system.*

At admissible price vector $p \in P$ and rationing system $R = (R^1, R^2, \cdots, R^m)$, the constrained demand set of bidder $i \in M$ is given by

$$D^i(p, R^i) = \{j \in N \mid R^i_j = 1 \text{ and } V^i(j) - p_j = \max\{V^i(h) - p_h \mid R^i_h = 1\}\}.$$

Now we can adapt the classical notion of Walrasian equilibrium to the current model under price rigidities.

Definition 2.1 *A tuple* (p^*, R^*, π^*) *constitutes* a constrained Walrasian equilibrium *if*

(i) π^* *is a feasible allocation,* p^* *is an admissible price vector, and* R^* *is a rationing system*;

(ii) $\pi^*(i) \in D^i(p^*, R^{i*})$ *for* $i \in M$;

(iii) $p^*_j = \underline{p}_j$ *if* $\pi^*(i) \neq j$ *for all* $i \in M$;

(iv) $p^*_j = \bar{p}_j$ *and* $\pi^*(h) = j$ *for some* $h \in M$ *if* $R^{i*}_j = 0$ *for some* $i \in M$;

(v) $j \in D^i(p^*, R^{i*}_{-j})$ *if* $R^{i*}_j = 0$.

Conditions (i) and (ii) need no explanation. Condition (iii) says that the price of an unassigned item must be equal to its lower bound price. Condition (iv) states that a bidder can be rationed on an item if the item is assigned to some other bidder and if its price is on its upper bound. Condition (v) says that a bidder can be rationed on an item only if without rationing on that item, the bidder will demand the item.

It is well-known (see e.g., Drèze (1975)) that an equilibrium allocation in a constrained Walrasian equilibrium in economies with divisible goods is not efficient when there are binding rationings on purchase or sale. It is, however, easy to show by example that this observation may not be true for economies with indivisible goods as studied here.

3 The dynamic auction

We now establish the existence of a constrained Walrasian equilibrium for the economy under price rigidities.

Theorem 3.1 *There exists at least one constrained Walrasian equilibrium in the model under price rigidities.*

We shall design a dynamic auction that can actually find in a finite number of steps a constrained Walrasian equilibrium in the economy. Roughly speaking, the auctioneer starts the auction at the lower bound prices of the items being sold. Then the bidders respond with their demand sets. The auctioneer accordingly eliminates sets of over-demanded items by increasing their prices or by a lottery to determine a rationing system so that no price of any item crosses its upper bound. The auction stops when there are no set over-demanded items left, which gives a constrained equilibrium as will be shown below. As a result, this yields a constructive proof of the above theorem.

A set of real items $S \subseteq N \setminus \{0\}$ is *over-demanded* at a price vector $p \in \mathbb{R}^N$, if the number of bidders who demand only items in S is strictly greater than the number of items in S, i.e., $|\{i \in M \mid D^i(p) \subseteq S\}| > |S|$. An over-demanded set S is said to be *minimal* if no strict subset of S is an over-demanded set. Now we are ready to describe the dynamic auction under price rigidities. Note that in the auction process, since the set of bidders and the set of items are shrinking, the demand set of each bidder and the over-demanded sets need to be adapted accordingly.

The Dynamic Auction under Price Rigidities

Step 1: The auctioneer announces the set of items $N = \{0, 1, \cdots, n\}$ for sale and the lower bound price vector $\underline{p}$. The bidders, denoted by $M = \{1, \cdots, m\}$, come to bid. Let $t := 0$ and $p^t := \underline{p}$. Go to Step 2.

Step 2: The auctioneer asks every remaining bidder i to report his demand set $D^i(p^t)$ on the remaining items and checks whether there is any over-demanded set of items at p^t. If there is no over-demanded set of items, the auction stops. Otherwise, there is at least one over-demanded set. The auctioneer first chooses a minimal over-demanded set S of items and next checks whether the price of any item in the set S has reached its upper bound. Let $\bar{S} := \{j \in S \mid p^t_j = \bar{p}_j\}$. If $\bar{S}$ is empty, the auctioneer increases the price of each item in S by one unit and keeps the prices of all other items unchanged. Let $t := t + 1$ and return to Step 2. If $\bar{S}$ is not empty, go to Step 3.

Step 3: The auctioneer picks an item at random from $\bar{S}$ and asks all bidders who demand the item to draw lots for the right to buy the item. Then the (unique) winning bidder gets the item by paying its current price and exits from the auction. Delete this bidder from M and delete his won item from N. If $M = \emptyset$ or $N = \emptyset$, the auction stops. Otherwise, let $t := t + 1$ and return to Step 2.

Before proving the convergence of the auction, we illustrate by example how the auction actually operates.

Example 1: Suppose that there are five bidders (1, 2, 3, 4, 5) and four items (0, 1, 2, 3, 4) in a market. The lower and upper bound price vectors are $\underline{p} = (0, 5, 4, 1, 5)$, and $\bar{p} = (0, 6, 6, 4, 7)$. Bidders' values are given in Table 1.

Table 1: Bidders' values on each item.

Items	0	1	2	3	4
Bidder 1	0	4	3	5	7
Bidder 2	0	7	6	8	3
Bidder 3	0	5	5	7	7
Bidder 4	0	9	4	3	2
Bidder 5	0	6	2	4	10

The auction starts at the price vector $p^0 = (0, 5, 4, 1, 5)$. Then bidders report their demand sets: $D^1(p^0) = \{3\}$, $D^2(p^0) = \{3\}$, $D^3(p^0) = \{3\}$, $D^4(p^0) = \{1\}$ and $D^5(p^0) = \{4\}$. The set $S = \{3\}$ is a minimal over-demanded set and the auctioneer adjusts p^0 to $p^1 = (0, 5, 4, 2, 5)$. The demand sets and price vectors and other relevant data generated by the auction are illustrated in Table 2. In Step 3, the price of item 3 has reached its upper bound 4. The auctioneer assigns randomly item 3, say, to bidder 2. So bidder 2 gets item 3 by paying 4 dollars and leaves the auction. Then we have $M = \{1, 3, 4, 5\}$ and $N = \{0, 1, 2, 4\}$. The auctioneer adjusts p^3 to $p^4 = (0, 5, 4, 5)$. In Step 6, there is no over-demanded set of items at p^6 and the auctioneer can assign item 2 to bidder 3, item 1 to bidder 4, and item 4 to bidder 5. In the end, bidder 1 gets no item and pays nothing; bidder 2 gets item 3 and pays 4; bidder 3 gets item 2 and pays 4; Bidder 4 gets item 1 and pays 5; Bidder 5 gets item 4 and pays 7. Letting $p^* = (0, 5, 4, 4, 7)$, $\pi^* = (0, 3, 2, 1, 4)$, $R^{1*} = (1, 1, 1, 0, 1)$, $R^{2*} = (1, 1, 1, 1, 1)$, $R^{3*} = (1, 1, 1, 0, 1)$, $R^{4*} = (1, 1, 1, 1, 1)$, and $R^{5*} = (1, 1, 1, 1, 1)$, (p^*, π^*, R^*) is a constrained equilibrium.

Table 2: The data generated by the auction in Example 2.

Step	Prices	N	M	S	$D^1(p)$	$D^2(p)$	$D^3(p)$	$D^4(p)$	$D^5(p)$
0	$(0, 5, 4, 1, 5)$	$\{0, 1, 2, 3, 4\}$	$\{1, 2, 3, 4, 5\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{1\}$	$\{4\}$
1	$(0, 5, 4, 2, 5)$	$\{0, 1, 2, 3, 4\}$	$\{1, 2, 3, 4, 5\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{1\}$	$\{4\}$
2	$(0, 5, 4, 3, 5)$	$\{0, 1, 2, 3, 4\}$	$\{1, 2, 3, 4, 5\}$	$\{3\}$	$\{3, 4\}$	$\{3\}$	$\{3\}$	$\{1\}$	$\{4\}$
3	$(0, 5, 4, 4, 5)$	$\{0, 1, 2, 3, 4\}$	$\{1, 2, 3, 4, 5\}$	$\emptyset$	$\{4\}$	$\{3\}$	$\{3\}$	$\{1\}$	$\{4\}$
4	$(0, 5, 4, 5)$	$\{0, 1, 2, 4\}$	$\{1, 3, 4, 5\}$	$\{4\}$	$\{4\}$	$\{3\}$	$\{4\}$	$\{1\}$	$\{4\}$
5	$(0, 5, 4, 6)$	$\{0, 1, 2, 4\}$	$\{1, 3, 4, 5\}$	$\{4\}$	$\{4\}$	$\{3\}$	$\{2, 4\}$	$\{1\}$	$\{4\}$
6	$(0, 5, 4, 7)$	$\{0, 1, 2, 4\}$	$\{1, 3, 4, 5\}$	$\emptyset$	$\{0, 4\}$	$\{3\}$	$\{2\}$	$\{1\}$	$\{4\}$

We can establish the following finite convergence theorem for the auction.

Theorem 3.2 *The dynamic auction finds a constrained equilibrium in a finite number of steps.*

Proof: It is clear that the auction will stop at some step t. Let p^* and π^* be the price vector and the allocation at step t generated by the auction. We will show that there is a rationing system R^* such that (p^*, π^*, R^*) constitutes a constrained Walrasian equilibrium. To achieve this, we construct an equivalent version of the dynamic auction that yields the same allocation π^* and price vector p^* and in addition generates a rationing system R^*. At price vector $p \in \mathbb{R}^N$ and rationing system R a set of real items $S \subseteq N \setminus \{0\}$ is said to be *over-demanded* if the number of bidders who given the rationing demand only items in S is strictly greater than the number of items in S, i.e., $|\{i \in M \mid D^i(p, R^i) \subseteq S\}| > |S|$. An over-demanded set S is said to be *minimal* if no strict subset of S is an over-demanded set. We can now describe the equivalent version of the dynamic auction under price rigidities.

The Equivalent Dynamic Auction under Price Rigidities

Step 1: The auctioneer announces the set of items $N = \{0, 1, \cdots, n\}$ for sale and the lower bound price vector $\underline{p}$. The bidders, denoted by $M = \{1, \cdots, m\}$, come to bid. The auctioneer sets rationing system R with $R^i_j = 1$ for every $i \in M$ and $j \in N$ and price $p := \underline{p}$. The set W of winning bidders and$O$ of sold items are both empty and go to Step 2.

Step 2: The auctioneer asks every bidder i to report his demand set $D^i(p, R)$. The auctioneer sets $R^i_j = 0$ for any bidder $i \notin W$ and any $j \in O \cap D^i(p, R)$, and asks such bidder i to resubmit his demand set at p and this adjusted rationing system R. Then the auctioneer checks whether there is any over-demanded set of goods at p and R. If there is no over-demanded set of items, the auction stops. Otherwise, there is at least one over-demanded set. The auctioneer chooses a minimal over-demanded set S of items and checks whether the price of any item in the set S has reached its upper bound. Let $\bar{S} := \{j \in S \mid p_j = \bar{p}_j\}$. If $\bar{S}$ is empty, the auctioneer increases in p the price of each item in S by one unit and keeps the prices of all other items and the rationing system R unchanged. Return to Step 2. If $\bar{S}$ is not empty, go to Step 3.

Step 3: The auctioneer picks an item $j \in \bar{S}$ at random and asks all bidders who demand the item to draw lots for the right to buy the item. The seller sets $R^i_j := 0$ for every losing bidder i who demanded item j. The winning bidder and item j are added to the sets W and O, respectively, and go to Step 2.

Clearly, both auctions produce the same price vector p^* and the same allocation π^*. The difference is that the second auction also produces a rationing system R^*. Following Roth and Sotomayer (1990, Section 8.3), it is not difficult to verify that (p^*, π^*, R^*) is indeed a constrained Walrasian equilibrium.

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