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By Gerwald van Gulick, Peter Borm, Anja De Waegenaere, Ruud Hendrickx

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TILBURG UNIVERSITY

Deposit Games with Reinvestment

Gerwald van Gulick[1,2], Peter Borm[1], Anja De Waegenaere[1], Ruud Hendrickx[1,3]

Abstract

In a deposit game coalitions are formed by players combining their capital. The proceeds of their investments then have to be divided among those players. The current model extends earlier work on capital deposits by allowing reinvestment of returns. Two specific subclasses of deposit games are introduced. It is seen that each term dependent deposit game possesses a core element. Capital dependent deposit games are also shown to have a core element and even a population monotonic allocation scheme if the revenue function exhibits increasing returns to scale. Furthermore, it is shown that all superadditive games are deposit games if one allows for debt.

JEL classification: C71

Keywords: cooperative game theory, deposit games, core elements, population monotonic allocation schemes, superadditive games

1 Introduction

A deposit problem is a decision problem where an individual has a certain amount of capital at his disposal to deposit at a bank. This individual aims to maximise the return on his investment. When we allow for cooperation between individuals by means of joint investment, they can increase their joint return. However, this gives rise to the additional question of how to allocate the proceeds among the individuals. In this paper, we analyse cooperation in deposit situations and in particular, we explore whether an allocation exists such that all players will want to cooperate and form the grand coalition.

Deposit games have previously been studied by Borm, De Waegenaere, Rafels, Suijs, Tijs, and Timmer (2001), Tan (2000) and De Waegenaere, Suijs, and Tijs (2005). Borm et al. (2001) define a deposit as a fixed amount of capital at a bank for a certain amount of time, and use a revenue function that describes the revenue of a deposit in a fixed end period, after which no more deposits can be made. In particular, they do not allow for reinvestment of intermediate revenue. Tan (2000) extends this approach and adds the possibility of borrowing.

The approach of De Waegenaere et al. (2005) is more general. They allow reinvestment and a broader range of investment products than deposits. Among other things, the money invested in a single investment product is allowed to change every period, rather than being a fixed amount of capital. The drawback of this approach is that because of its generality, it is harder to draw firm conclusions.

Lemaire (1983) and Izquierdo and Rafels (1996) analyse related issues. Lemaire (1983) gives an overview of the use of game theory in financial issues. Izquierdo and Rafels (1996) analyse games that are related to capital dependent deposit games.

[1]CentER and Department of Econometrics and Operations Research, Tilburg University.

[2]Corresponding author: P.O. Box 90153, 5000 LE Tilburg, The Netherlands, E-mail: G.vanGulick@uvt.nl.

[3]This author acknowledges financial support from the Netherlands Organisation for Scientific Research (NWO).

In this paper we take the approach that reinvestment should be possible, while still allowing for a particular form of the revenue function. We start by modelling the decisions of individuals during a discrete and finite number of periods of time. Any deposit will lead to a non-negative revenue. This non-negativity condition is justified because, by the nature of deposits, it is always possible to privately save any amount of money at no cost. There will be no default risk involved in any deposit.

We allow coalitions of individuals to deposit money jointly by cooperating. Clearly, the revenue of a coalition will never be lower than the sum of all individual revenues. We are interested in finding a core element, *i.e.* we want an allocation for the grand coalition such that no subcoalition has any incentive to split off.

Particular forms of the revenue function are explored by introducing two subclasses: term dependent deposit games and capital dependent deposit games. These classes are also used by Borm et al. (2001). Term dependent deposit games are deposit games where the return only depends on how long capital is deposited, and not on how much is deposited. We show that term dependent deposit games are totally balanced. Capital dependent deposit games are the natural counterpart of term dependent deposit games. Here, deposits lead to a different rate of return only if a different amount of capital is deposited; longer terms do not influence the return. If the rate of return in capital dependent deposit games is increasing in the amount of capital deposited, a core element exists. In fact, we can explicitly construct a population monotonic allocation scheme *a la* Sprumont (1990).

In a further extension of the model, we allow individuals to also have debt, and we show that all non-negative superadditive games are deposit games of this type.

The remainder of this paper is structured as follows. In Section 2 we introduce deposit problems and deposit situations with reinvestment and define corresponding deposit games. In Section 3 we compare the current model to the one analysed by Borm et al. (2001) and show that it indeed is an extension. In the next two sections we analyse two specific subclasses of deposit games. Term dependent deposit games are analysed in Section 4 and in Section 5 capital dependent deposit games are considered. We show how superadditive games can be rewritten as deposit games with reinvestment and debt in Section 6.

2 Deposit Problems and Deposit Games with Reinvestment

Similar to Borm et al. (2001), a deposit is defined as a positive, fixed amount of capital c that is at a bank during a prearranged and consecutive number of periods $t_1, t_1+1, \ldots, t_2$, where $1 \leq t_1 \leq t_2 \leq \tau$. Here τ is the final period in which a deposit can be made. The scope of a deposit problem is thus a discrete and finite timespan $\{1, \ldots, \tau\}$. The time interval over which money is deposited is called the term of the deposit

$$T = \{t_1, t_1+1, \ldots, t_2\}.$$

The set of all possible terms of a deposit is given by

$$\mathcal{T} = \big\{ T \subset \{1, \ldots, \tau\} \,|\, \exists t_1, t_2 \in \{1, \ldots, \tau\} : T = \{t_1, t_1+1, \ldots, t_2\} \,\big\}. \tag{1}$$

A deposit with capital c and term $T = \{t_1, t_1+1, \ldots, t_2\}$ can be represented as a vector over the periods $\{1, 2, \ldots, \tau+1\}$, where at the start of period t_1 an amount c is deposited and at the start of period t_2+1 the amount c is returned. Note that since capital is returned at t_2+1, we extend our model to include the period $\tau+1$.

To illustrate this, we look at a deposit where $\tau = 3$. A possible deposit can be written as $(0, 3, 0, -3)$, which means that three units of capital ($c = 3$) are deposited during periods $t_1 = 2$ and $t_2 = 3$, and this is returned at the beginning of period 4, which is t_2+1.

The set of all possible deposits is denoted

$$\Delta = \left\{ \delta \in \mathbb{R}^{\tau+1} \,|\exists c > 0, T \in \mathcal{T} : \delta = c \cdot h(T) \right\}, \tag{2}$$

where we define the function $h : \mathcal{T} \to \mathbb{R}^{\tau+1}$ as a deposit of 1 unit of capital over term $T = \{t_1, t_1+1, \ldots, t_2\}$, so for all $t \in \{1, 2, \ldots, \tau+1\}$ and all $T \in \mathcal{T}$ we have

$$h_t(T) = \begin{cases} 1, & \text{if } t = t_1, \\ -1, & \text{if } t = t_2 + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

We assume there is a revenue function $P : \Delta \to \mathbb{R}_+^{\tau+1}$ that assigns to each deposit $\delta \in \Delta$ a non-negative revenue in each period of time in $\{1, \ldots, \tau+1\}$. No explicit formula for the revenue function is assumed at this time. Borm et al. (2001) allow revenue to be obtained only in period $\tau+1$. This corresponds to $P_t(\delta) = 0$ for all $t < \tau+1$ and all deposits $\delta \in \Delta$. If we consider again $\tau = 3$ and the deposit $\delta = (0, 3, 0, -3)$ and assume that we get a 4% interest on it, the revenue is $P(\delta) = (0, 0, 0.12, 0.12)$.

To avoid complications, we assume that money has to be deposited first, and only in a later period a revenue can be attained[1].

Assumption 2.1 *For all $\delta = c \cdot h(\{t_1, t_1+1, \ldots, t_2\}) \in \Delta$ and all $t \leq t_1$ we have $P_t(\delta) = 0$.*

The capital each individual has available to deposit can come from two sources. The endowments of the individual constitute the first source of capital, which consists for each time period of the income of the individual reduced by the consumption of the individual. The endowments are exogenous and express the net income an individual has available to use for deposits. The second source is payback of capital that was previously deposited plus revenue from deposits.

The endowments of an individual are given by $m \in \mathbb{R}^\tau$. Although the endowment can be negative at some point in time, we assume that for an individual at each time period the cumulative endowment is non-negative. This boils down to the following assumption.

Assumption 2.2 *For all $t \in \{1, \ldots, \tau\}$, we have $\sum_{s=1}^{t} m_s \geq 0$.*

A *deposit problem* with final period τ, set of deposits Δ, revenue function P and endowments m is denoted by (τ, Δ, P, m).

Because an individual has limited endowments, there are only certain combinations of deposits that he can invest in. A *portfolio* of deposits is denoted by a function $f : \Delta \to \mathbb{N} \cup \{0\}$, which expresses how many units of each deposit are used. A portfolio of deposits an individual is able to finance with his endowments is called feasible.[2] Note that we take into account the possibility of reinvestment of previously deposited capital and returns.[3] The set of all feasible portfolios for an endowment vector $m \in \mathbb{R}^\tau$ is [4]

$$\mathcal{F}(m) = \left\{ f : \Delta \to \mathbb{N} \cup \{0\} \,\middle|\, \forall t \in \{1, \ldots, \tau\} : \sum_{\delta \in \Delta} f(\delta)\delta_t = m_t + \sum_{\delta \in \Delta} f(\delta) P_t(\delta) \right\}.$$

[1] We also assume that arbitrage is excluded. Conditions to exclude arbitrage are beyond the scope of this paper.

[2] The endowment of an individual is capital that is available after consumption. It is therefore reasonable to assume that an individual invests *all* available capital in deposits in every period in $\{1, \ldots, \tau\}$, since the return on any deposit is non-negative and any amount of money can be deposited.

[3] Also, it is possible to get a feasible collection of deposits using the same deposit more than once. It is for instance possible to open two accounts at the bank and deposit the same amount in each account.

[4] Even though Δ is non-denumerable, either side of the equality in the feasibility condition is bounded. At $t = 1$ there is no revenue by Assumption 2.1, so the right hand side of the equality in the feasibility condition is a finite number. This implies that a feasible portfolio $f : \Delta \to \mathbb{N} \cup \{0\}$ is non-zero only for a denumerable number of deposits

The left hand side of the equality[5] in the feasibility condition represents the net investment in deposits in period t. It contains not only the capital required for investment in new deposits at time t, but also the capital returns in period t, according to the definition of deposits in (2) and (3). Consequently this part of the expression can be negative if less capital is deposited in new deposits than is paid back from previous deposits.

The right hand side of the equality consists of two parts. The first part are the endowments, which can be negative. The second part is the sum of the revenue over all deposits at time t, which is non-negative. Thus the equality states that the net change in investment in deposits is equal to the endowments plus revenue.

We now define the optimisation problem for an individual in the same way as Borm et al. (2001). The natural objective is to maximise the total revenue. However, the fact that at any period there might be some revenue presents us with some difficulties. Because of the non-negativity of the revenue function it is possible to carry over all intermediate revenue to period $\tau + 1$. Also, from our definition of a feasible portfolio we see that all available capital is deposited each period. Hence the natural objective is to maximise the total revenue at time $\tau + 1$.

The total capital of an individual at time $\tau + 1$ consists of two parts; the revenue of deposits at time $\tau + 1$ and the payback of deposits at $\tau + 1$. Thus the total capital at time $\tau + 1$, Π, as a function of a feasible portfolio $f \in \mathcal{F}(m)$ equals

$$\Pi(f) = \sum_{\delta \in \Delta} f(\delta)[P(\delta) - \delta]_{\tau+1}.$$

Note that the total capital at time $\tau + 1$ also contains the intermediate revenues.

Observe that for two portfolios $f \in \mathcal{F}(m_1)$ and $g \in \mathcal{F}(m_2)$ we have $f + g \in \mathcal{F}(m_1 + m_2)$ and

$$\Pi(f) + \Pi(g) = \sum_{\delta \in \Delta} (f(\delta) + g(\delta))[P(\delta) - \delta]_{\tau+1} = \Pi(f + g). \tag{4}$$

With $m \in \mathbb{R}^\tau$ define the maximal revenue in period $\tau + 1$ by means of a feasible portfolio as

$$\pi(m) = \sup \left\{ \Pi(f) - \sum_{t=1}^{\tau} m_t \,\middle|\, f \in \mathcal{F}(m) \right\}.$$

We assume that for all $m \in \mathbb{R}^{\tau+1}$ this supremum exists.[6] Note that $\pi(m) \geq 0$.

Within a cooperative framework, where a group of agents $N = \{1, \ldots, n\}$ combines efforts, it is possible for individuals to form coalitions and deposit money jointly. If for instance interest rates, defined as the revenue divided by the amount of capital deposited, are higher when a larger sum of money is invested, or money is deposited over a longer period, it might be attractive for individuals to cooperate. In this way, coalitions can possibly make more money than the individuals can when they act alone. In this way we consider a *deposit situation* $(N, \tau, \Delta, P, \{m(i)\}_{i \in N})$, where agent $i \in N$ has endowment vector $m(i) \in \mathbb{R}^\tau$ available to deposit.

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with period 1 included in their term. Because of the exclusion of arbitrage, it is impossible to get an unbounded revenue with a finite amount of capital. Therefore, also in the next period, $t = 2$, the revenue is bounded, and again only a denumerable amount of deposits including period 2 can be non-zero. This argument holds for all periods, so f can only be non-zero for a denumerable amount of deposits. Thus, the sum is well defined.

[5] In our definition of $\mathcal{F}(m)$ it is in fact more natural to consider the restriction as an inequality, $\sum_{\delta \in \Delta} f(\delta)\delta_t \leq m_t + \sum_{\delta \in \Delta} f(\delta)P_t(\delta)$ for all t. However, it will become apparent later that without loss of generality we can assume the expression to be an equality, because the non-negativity of the revenue function makes it possible to carry over any amount of capital to the next period without losses.

[6] The only concern here is the value of the supremum being infinity. This can only happen if there is the possibility of arbitrage, which is assumed to be excluded.

The available capital $m(S)$ to a coalition $S \subset N$, $S \neq \emptyset$ is simply the sum of all capital available to each member of the coalition. So for a coalition $S \subset N$, $S \neq \emptyset$

$$m(S) = \sum_{i \in S} m(i).$$

Obviously the maximal joint revenue of a coalition $S \subset N$ in period $\tau + 1$ is given by

$$v(S) = \sup\left\{ \Pi(f) - \sum_{t=1}^{\tau} m_t(S) \,\middle|\, f \in \mathcal{F}(m(S)) \right\}. \tag{5}$$

A *deposit game with reinvestment*[7] corresponding to a deposit situation $(N, \tau, \Delta, P, \{m(i)\}_{i \in N})$ is the transferable utility game (N, v) with characteristic function v as given by (5) for all $S \subset N$, $S \neq \emptyset$. By assumption $v(\emptyset) = 0$.

The next theorem states that all deposit games are superadditive, *i.e.* $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subset N$ such that $S \cap T = \emptyset$. The intuition behind this is that all players in a coalition are at least able to invest in the same deposits as they would optimally do as smaller coalitions.

Theorem 2.3 Let $(N, \tau, \Delta, P, \{m(i)\}_{i \in N})$ be a deposit situation with corresponding deposit game (N, v). Then (N, v) is superadditive.

Proof: Let $S, T \subset N$ be such that $S \cap T = \emptyset$. Then

$$\begin{aligned}
v(S) + v(T) &= \sup\left\{ \Pi(f^S) - \sum_{t=1}^{\tau} m_t(S) \,\middle|\, f^S \in \mathcal{F}(m(S)) \right\} \\
&\quad + \sup\left\{ \Pi(f^T) - \sum_{t=1}^{\tau} m_t(T) \,\middle|\, f^T \in \mathcal{F}(m(T)) \right\} \\
&= \sup\left\{ \Pi(f^S) + \Pi(f^T) - \sum_{t=1}^{\tau} m_t(S) - \sum_{t=1}^{\tau} m_t(T) \,\middle|\, f^S \in \mathcal{F}(m(S)), f^T \in \mathcal{F}(m(T)) \right\} \\
&= \sup\left\{ \Pi(f^S + f^T) - \sum_{t=1}^{\tau} m_t(S \cup T) \,\middle|\, f^S \in \mathcal{F}(m(S)), f^T \in \mathcal{F}(m(T)) \right\} \\
&\leq \sup\left\{ \Pi(f) - \sum_{t=1}^{\tau} m_t(S \cup T) \,\middle|\, f \in \mathcal{F}(m(S \cup T)) \right\} \\
&= v(S \cup T),
\end{aligned}$$

where the inequality follows from the fact that $\mathcal{F}(m(S)) \cup \mathcal{F}(m(T)) \subset \mathcal{F}(m(S) + m(T))$. $\square$

The natural question that arises is how to divide the maximal joint revenue over the individual players. To tackle this question we analyse core allocations of the corresponding game. The core of a cooperative game (N, v) is defined by

$$C(v) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i \in N} x_i = v(N), \forall S \subset N : \sum_{i \in S} x_i \geq v(S) \right\},$$

[7] We add the qualification "with reinvestment" to make the distinction with deposit games as defined by Borm et al. (2001). Throughout the remainder, where no confusion arises this qualification is dropped.

and a core allocation is a vector $x \in C(v)$. A core allocation is stable against coalitional deviations, because every coalition is allocated at least what it can obtain by depositing on its own.

The following example shows that within the general setting of deposit games as developed above, where we have not put any restrictions on the revenue function P other than the exclusion of arbitrage opportunities and Assumption 2.1, core elements need not exist. In the next sections we impose some natural conditions on the revenue function P that ensure stability.

Example 2.1 Consider a three player deposit situation $(N, \tau, \Delta, P, \{m(i)\}_{i \in N})$ with two periods, so $\tau = 2$, $N = \{1, 2, 3\}$ and the set of deposits Δ is given by (2). Furthermore, assume that these three players are identical, in the sense that they each have the same endowments. Assume that the endowments of player $i \in N$ are given by $m(i) = (300, -50)$. Note that $m(i)$ satisfies Assumption 2.2.

The setting we consider consists of only two deposits with non-zero revenue. The first one is a two-year bond of 500 yielding an interest rate of 6% per period. The second one is a two-year bond of 250 yielding 1% interest. So, in our model the revenue function is given by

$$P(\delta) = \begin{cases} (0, 2.5, 2.5), & \text{if } \delta = (250, 0, -250), \\ (0, 30, 30), & \text{if } \delta = (500, 0, -500), \\ (0, 0, 0), & \text{otherwise.} \end{cases}$$

This revenue function satisfies Assumption 2.1.

We now calculate the value of each coalition. Obviously, for every two-player coalition $\{i, j\}$ it is optimal to buy one two-year bond of 500. The remaining 100 in the first period can not be used to get any revenue, but can be carried over to the next period (without interest). In the second period, the coalition still has 500 deposited, has 100 carried over, receives endowments of -100 and receives a revenue of 30. This intermediate revenue is carried over. The optimal feasible portfolio f for any two-player coalition is thus given by[8]

$$f(\delta) = \begin{cases} 1, & \text{if } \delta \in \{(500, 0, -500), (100, -100, 0), (0, 30, -30)\}, \\ 0, & \text{otherwise.} \end{cases}$$

This gives all two-player coalitions a value of $v(\{i, j\}) = (30 - (-500)) + (0 - (-30)) - (600 - 100) = 60$.

Observe that any single player $i \in N$ has insufficient capital to deposit 500. Clearly depositing in the two-year bond of 250 is optimal, together with a deposit that carries over the intermediate revenue from $t = 2$ to $t = 3$. This results in a value $v(\{i\}) = 5$. The grand coalition can buy one two-year bond of 500 and one of 250 in $t = 1$ and carries over the remaining capital of 150. The intermediate revenue at period $t = 2$ is also carried over to period $t = 3$. This leads to $v(N) = 65$.

If we want to construct a core element $x \in \mathbb{R}^3$ for this deposit game, it needs to satisfy several inequalities. For the two-player coalitions, these are: $x_1 + x_2 \geq 60$, $x_1 + x_3 \geq 60$ and $x_2 + x_3 \geq 60$. If we add these three inequalities together, we get $2(x_1 + x_2 + x_3) \geq 180$, or $x_1 + x_2 + x_3 \geq 90$. However, efficiency tells us that a core allocation x should satisfy $x_1 + x_2 + x_3 = 65 < 90$. So the core is empty for this particular deposit game. $\triangleleft$

[8] To clarify further, had the endowments been $(300, -60)$, the deposit $(500, 0, -500)$ would still have been feasible. In period $t = 2$ the endowments of the coalition is only 480, however with the intermediate return of 30 from the mentioned deposit itself, 510 would be available. With endowments $(300, -70)$ however, the deposit would not have been feasible, since at $t = 2$ only 490 can be invested in deposits.

3 Deposit Games with and without Reinvestment

This section compares deposit games with reinvestment to those without reinvestment. To do so, we first recall the notation used originally by Borm et al. (2001).

The scope of a deposit problem without reinvestment is a finite and discrete timespan $\{1, \ldots, \rho\}$. The set of all possible terms of a deposit $\mathcal{T}$ is denoted by (1) with $\tau = \rho$.

Borm et al. (2001) denote deposits by $d = c \cdot e(T)$, where

$$e_t(T) = \begin{cases} 1, & \text{if } t_1 \leq t \leq t_2, \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

for all $T = \{t_1, t_1 + 1, \ldots, t_2\} \in \mathcal{T}$, $t \in \{1, \ldots, \tau\}$. The set of all deposits is given by

$$D = \left\{ d \in \mathbb{R}^{\rho}_{+} \,|\, \exists c \geq 0, T \in \mathcal{T} : d = c \cdot e(T) \right\}.$$

The revenue function is denoted by $R : D \to \mathbb{R}_+$ with $R(0) = 0$. $R(d)$ is the revenue of a deposit d, which is received in period $\rho + 1$. Let N denote the set of all players. The endowments of player $i \in N$ are denoted by a vector $\omega(i) \in \mathbb{R}^{\rho}$. $\omega_t(i)$ denotes the total amount of capital available to player $i \in N$ to deposit in period t. For $S \subset N$, $S \neq \emptyset$ we define $\omega(S) = \sum_{i \in S} \omega(i)$. A *deposit situation without reinvestment* is denoted by $(N, \rho, D, R, \{\omega(i)\}_{i \in N})$.

Finally the value of a coalition $S \subset N$ in the corresponding deposit game without reinvestment is defined by

$$w(S) = \sup \left\{ \sum_{k=1}^{\ell} R(d^k) \,\middle|\, \exists \ell \in \mathbb{N}, \exists d^1, \ldots, d^{\ell} \in D : \forall t \in \{1, 2, \ldots, \tau\} : \sum_{k=1}^{\ell} d^k_t \leq \omega_t(S) \right\}.$$

Theorem 3.1 Every deposit game without reinvestment is a deposit game with reinvestment.

Proof: Consider a deposit game without reinvestment (N, w) corresponding to a deposit situation $(N, \rho, D, R, \{\omega(i)\}_{i \in N})$. We now construct a deposit situation with reinvestment $(N, \tau, \Delta, P, \{m(i)\}_{i \in N})$ for which the corresponding deposit game (N, v) is equal to (N, w).

Let $\tau = \rho$, let the set of possible terms $\mathcal{T}$ be defined by (1) and let the set of all deposits Δ be defined by (2). Note that the interpretation of deposits is the same in both models, so an amount c is deposited over term T in both cases, so the deposit defined as $d = c \cdot e(T) \in D$ naturally corresponds to $\delta = c \cdot h(T) \in \Delta$. In particular, from (3) and (6) we find for all $T \in \mathcal{T}$

$$h_t(T) = \begin{cases} e_1(T), & \text{if } t = 1, \\ e_t(T) - e_{t-1}(T), & \text{if } t \in \{2, \ldots, \tau\}. \end{cases}$$

Moreover, define the revenue function $P : \Delta \to \mathbb{R}^{\tau+1}_{+}$ by

$$P_t(c \cdot h(T)) = \begin{cases} R(c \cdot e(T)), & \text{if } t = \tau + 1, \\ 0, & \text{otherwise,} \end{cases}$$

for all $c > 0$ and all $T \in \mathcal{T}$. For all $i \in N$, the endowments $m(i) \in \mathbb{R}^{\tau}$ are recursively defined by

$$m_t(i) = \begin{cases} \omega_1(i), & \text{if } t = 1, \\ \omega_t(i) - \omega_{t-1}(i), & \text{if } t \in \{2, \ldots, \tau\}. \end{cases}$$

Since there are no intermediate revenues, for a feasible portfolio $f \in \mathcal{F}(m)$ with $m \in \mathbb{R}^\tau$ we have

$$\Pi(f) = \sum_{\delta\in\Delta} f(\delta)[P(\delta) - \delta]_{\tau+1} = \sum_{\delta\in\Delta} f(\delta)P_{\tau+1}(\delta) + \sum_{t=1}^{\tau} m_t.$$

Let $S \subset N$. Then

$$\begin{aligned}
v(S) &= \sup\left\{ \Pi(f) - \sum_{t=1}^{\tau} m_t(S) \;\middle|\; f \in \mathcal{F}(m(S)) \right\} \\
&= \sup\left\{ \sum_{\delta\in\Delta} f(\delta)P_{\tau+1}(\delta) \;\middle|\; f : \Delta \to \mathbb{N}\cup\{0\}, \forall t \in \{1,\dots,\tau\} : \right. \\
&\qquad\qquad \left. \sum_{\delta\in\Delta} f(\delta)\delta_t = m_t(S) + \sum_{\delta\in\Delta} f(\delta)P_t(\delta) \right\} \\
&= \sup\left\{ \sum_{\delta\in\Delta} f(\delta)P_{\tau+1}(\delta) \;\middle|\; f : \Delta \to \mathbb{N}\cup\{0\}, \forall t \in \{1,\dots,\tau\} : \sum_{\delta\in\Delta} f(\delta)\delta_t = m_t(S) \right\} \\
&=^* \sup\left\{ \sum_{k=1}^{\ell} P_{\tau+1}(\delta^k) \;\middle|\; \exists \ell \in \mathbb{N}\cup\{0\}, \exists \delta^1,\dots,\delta^\ell \in \Delta : \forall t \in \{1,\dots,\tau\} : \sum_{k=1}^{\ell} \delta^k_t = m_t(S) \right\} \\
&= \sup\left\{ \sum_{k=1}^{\ell} P_{\tau+1}(c^k \cdot h(T^k)) \;\middle|\; \exists \ell \in \mathbb{N}\cup\{0\}, \exists c^1,\dots,c^\ell > 0, \exists T^1,\dots,T^\ell \in \mathcal{T} : \right. \\
&\qquad\qquad \left. \forall t \in \{1,\dots,\tau\} : \sum_{k=1}^{\ell} c^k \cdot h_t(T^k) = m_t(S) \right\} \\
&= \sup\left\{ \sum_{k=1}^{\ell} P_{\tau+1}(c^k \cdot h(T^k)) \;\middle|\; \exists \ell \in \mathbb{N}\cup\{0\}, \exists c^1,\dots,c^\ell > 0, \exists T^1,\dots,T^\ell \in \mathcal{T} : \right. \\
&\qquad\qquad \left. \forall t \in \{1,\dots,\tau\} : \sum_{r=1}^{t}\sum_{k=1}^{\ell} c^k \cdot h_r(T^k) = \sum_{r=1}^{t} m_r(S) \right\} \\
&= \sup\left\{ \sum_{k=1}^{\ell} P_{\tau+1}(c^k \cdot h(T^k)) \;\middle|\; \exists \ell \in \mathbb{N}\cup\{0\}, \exists c^1,\dots,c^\ell > 0, \exists T^1,\dots,T^\ell \in \mathcal{T} : \right. \\
&\qquad\qquad \left. \forall t \in \{1,\dots,\tau\} : \sum_{k=1}^{\ell} c^k \sum_{r=1}^{t} h_r(T^k) = \sum_{r=1}^{t} m_r(S) \right\} \\
&= \sup\left\{ \sum_{k=1}^{\ell} P_{\tau+1}(c^k \cdot h(T^k)) \;\middle|\; \exists \ell \in \mathbb{N}\cup\{0\}, \exists c^1,\dots,c^\ell > 0, \exists T^1,\dots,T^\ell \in \mathcal{T} : \right. \\
&\qquad\qquad \left. \forall t \in \{1,\dots,\tau\} : \sum_{k=1}^{\ell} c^k \cdot e_t(T^k) = \omega_t(S) \right\} \\
&= \sup\left\{ \sum_{k=1}^{\ell} R(c^k \cdot e(T^k)) \;\middle|\; \exists \ell \in \mathbb{N}, \exists c^1,\dots,c^\ell \geq 0, \exists T^1,\dots,T^\ell \in \mathcal{T} : \right. \\
&\qquad\qquad \left. \forall t \in \{1,\dots,\tau\} : \sum_{k=1}^{\ell} c^k \cdot e_t(T^k) = \omega_t(S) \right\}
\end{aligned}$$

$$= \sup\left\{ \sum_{k=1}^{\ell} R(d^k) \,\middle|\, \exists \ell \in \mathbb{N}, \exists d^1, \ldots, d^\ell \in D : \forall t \in \{1, \ldots, \tau\} : \sum_{k=1}^{\ell} d_t^k = \omega_t(S) \right\}$$
$$= \sup\left\{ \sum_{k=1}^{\ell} R(d^k) \,\middle|\, \exists \ell \in \mathbb{N}, \exists d^1, \ldots, d^\ell \in D : \forall t \in \{1, \ldots, \tau\} : \sum_{k=1}^{\ell} d_t^k \leq \omega_t(S) \right\}$$
$$= w(S),$$

where at (*) we use that because the feasible portfolio is non-zero for a denumerable number of deposits only, the supremum over the revenue of these denumerable many deposits is equal to the supremum of the sum of the revenue of any finite number of deposits. $\square$

4 Term Dependent Deposit Games

From Example 2.1 it is clear that in a deposit situation $(N, \tau, \Delta, P, \{m(i)\}_{i \in N})$ we need a further assumption on the revenue function P for the corresponding deposit game to have a non-empty core. One such assumption leads to term dependent deposit games as first introduced by Borm et al. (2001). In the underlying term dependent deposit situation, the rate of return on a deposit only depends on the term of the deposit and not on the amount of money deposited.

Definition A revenue function $P : \Delta \rightarrow \mathbb{R}_+^{\tau+1}$ is called *term dependent* if it holds for all $t \in \{1, \ldots, \tau + 1\}$, for all deposits $\delta \in \Delta$ and for all $\alpha > 0$, that $P_t(\alpha\delta) = \alpha P_t(\delta)$. If the underlying revenue function is term dependent, also the corresponding deposit situation and deposit game are called term dependent.

Note that in a term dependent deposit situation, for all terms $T \in \mathcal{T}$, all $\ell \in \mathbb{N}$ and all $c^1, \ldots, c^\ell > 0$ we have

$$\sum_{k=1}^{\ell} P\Big(c^k \cdot h(T)\Big) = P\Big(h(T) \sum_{k=1}^{\ell} c^k\Big).$$

So without loss of generality we can assume that all deposits with the same term $t \in \mathcal{T}$ can be combined into an aggregate portfolio deposit.

The following lemma shows that if we have a feasible portfolio for some endowments, then if we scale these endowments the portfolio that consists of the scaled deposits is feasible for the new problem, and furthermore its revenue is also scaled in the same manner.

Lemma 4.1 *Let $m \in \mathbb{R}^\tau$, $f \in \mathcal{F}(m)$ and $\lambda > 0$ and define $g(\lambda \cdot \delta) = f(\delta)$ for all $\delta \in \Delta$. Then $g \in \mathcal{F}(\lambda \cdot m)$ and $\Pi(g) = \lambda \cdot \Pi(f)$.*

Proof: We first show that $g \in \mathcal{F}(\lambda m)$. For all $t \in \{1, \ldots, \tau\}$ we have

$$\begin{aligned}
\sum_{\delta \in \Delta} g(\delta)\delta_t &= \sum_{\lambda\delta \in \Delta} g(\lambda\delta)\lambda\delta_t \\
&= \lambda \sum_{\lambda\delta \in \Delta} f(\delta)\delta_t \\
&= \lambda \sum_{\delta \in \Delta} f(\delta)\delta_t
\end{aligned}$$

$$
\begin{aligned}
&= \lambda \left(m_t + \sum_{\delta \in \Delta} f(\delta) P_t(\delta) \right) \\
&= \lambda m_t + \lambda \sum_{\frac{1}{\lambda}\delta \in \Delta} f\left(\frac{1}{\lambda}\delta\right) P_t\left(\frac{1}{\lambda}\delta)\right) \\
&= \lambda m_t + \sum_{\frac{1}{\lambda}\delta \in \Delta} g(\delta) P_t(\delta)) \\
&= \lambda m_t + \sum_{\delta \in \Delta} g(\delta) P_t(\delta)).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\Pi(g) &= \sum_{\delta \in \Delta} g(\delta)[P(\delta) - \delta]_{\tau+1} \\
&= \sum_{\lambda\delta \in \Delta} g(\lambda\delta)[P(\lambda\delta) - \lambda\delta]_{\tau+1} \\
&= \lambda \sum_{\lambda\delta \in \Delta} f(\delta)[P(\delta) - \delta]_{\tau+1} \\
&= \lambda \sum_{\delta \in \Delta} f(\delta)[P(\delta) - \delta]_{\tau+1} \\
&= \lambda \Pi(f),
\end{aligned}
$$

which proves the second statement. □

We now show that for each term dependent deposit game there is a core allocation. For this we use the notion of balancedness. A cooperative game (N, v) is called *balanced* if $\sum_{S \subset N} \lambda(S) v(S) \leq v(N)$ for all functions $\lambda : 2^N \to \mathbb{R}_+$ satisfying $\sum_{S \subset N : i \in S} \lambda(S) = 1$. A game is called *totally balanced* if every subgame $(S, v|_S)$ is balanced, where for all $S \subset N$, $S \neq \emptyset$ the subgame with respect to S is defined by $v|_S(T) = v(T)$ for all $T \subset S$. Bondareva (1963) and Shapley (1967) derived the following result about the core and balancedness.

Theorem 4.2 (Bondareva (1963) and Shapley (1967)) Let (N, v) be a cooperative game. Then $C(v) \neq \emptyset$ if and only if (N, v) is balanced.

Theorem 4.3 Every term dependent deposit game with reinvestment is totally balanced.

Proof: Let (N, v) be a deposit game corresponding to a term dependent deposit situation $(N, \tau, \Delta, P, \{m(i)\}_{i \in N})$. First we show that (N, v) is balanced. Take $\lambda : 2^N \to \mathbb{R}_+$ such that $\sum_{S \subset N : i \in S} \lambda(S) = 1$ for all $i \in N$. Then for every $t \in \{1, \ldots, \tau\}$

$$
\begin{aligned}
\sum_{S \subset N} \lambda(S) m_t(S) &= \sum_{S \subset N} \lambda(S) \sum_{i \in S} m_t(i) \\
&= \sum_{i \in N} \sum_{S \subset N : i \in S} \lambda(S) m_t(i) \\
&= \sum_{i \in N} m_t(i) \sum_{S \subset N : i \in S} \lambda(S) \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i \in N} m_t(i) \\
&= m_t(N).
\end{aligned}
$$

Consequently, using (4), (7) and Lemma 4.1,

$$
\begin{aligned}
\sum_{S \subset N} \lambda(S) v(S) &= \sum_{S \subset N} \lambda(S) \sup \left\{ \Pi(f^S) - \sum_{t=1}^{\tau} m_t(S) \,\middle|\, f^S \in \mathcal{F}(m(S)) \right\} \\
&= \sup \left\{ \sum_{S \subset N} \lambda(S) \Pi(f^S) - \sum_{S \subset N} \lambda(S) \sum_{t=1}^{\tau} m_t(S) \,\middle|\, \forall S \subset N : f^S \in \mathcal{F}(m(S)) \right\} \\
&= \sup \left\{ \sum_{S \subset N} \lambda(S) \Pi(f^S) - \sum_{t=1}^{\tau} m_t(N) \,\middle|\, \forall S \subset N : f^S \in \mathcal{F}(m(S)) \right\} \\
&= \sup \left\{ \sum_{S \subset N} \Pi(f^S) - \sum_{t=1}^{\tau} m_t(N) \,\middle|\, \forall S \subset N : f^S \in \mathcal{F}(\lambda(S) m(S)) \right\} \\
&= \sup \left\{ \Pi(\sum_{S \subset N} f^S) - \sum_{t=1}^{\tau} m_t(N) \,\middle|\, \forall S \subset N : f^S \in \mathcal{F}(\lambda(S) m(S)) \right\} \\
&\leq \sup \left\{ \Pi(f) - \sum_{t=1}^{\tau} m_t(N) \,\middle|\, f \in \mathcal{F}(\sum_{S \subset N} \lambda(S) m(S)) \right\} \\
&= \sup \left\{ \Pi(f) - \sum_{t=1}^{\tau} m_t(N) \,\middle|\, f \in \mathcal{F}(m(N)) \right\} \\
&= v(N).
\end{aligned}
$$

Clearly every subgame of a term dependent deposit game is again a term dependent deposit game corresponding to a term dependent deposit situation. Hence, a term dependent deposit game is totally balanced. □

In fact deposit games with reinvestment span the whole class of non-negative totally balanced games.

Theorem 4.4 A non-negative cooperative game is totally balanced if and only if it is a term dependent deposit game with reinvestment.

Proof: Theorem 4.3 already implies one part of this theorem. Moreover Borm et al. (2001) show that each non-negative totally balanced cooperative game is a term dependent deposit game without reinvestment, and hence using Theorem 3.1 also a deposit game with reinvestment. Moreover, term dependency of the revenue function of a deposit situation without reinvestment translates directly into term dependency of the revenue function of the associated deposit situation with reinvestment as constructed in the proof of Theorem 3.1. This concludes the proof. □

Theorem 4.4 implies that the classes of term dependent deposit games without reinvestment, term dependent deposit games with reinvestment and non-negative totally balanced games coincide.

5 Capital Dependent Deposit Games

Another subclass of deposit games is the class of capital dependent deposit games. In this subclass, revenues are not assumed to depend on the term of the deposit but just on the amount of capital invested[9]. Depositing over multiple periods does not yield any extra revenue compared to investing only one period ahead several times. Without reinvestment this class was first considered by Borm et al. (2001).

In an optimum, depositing for longer than one period is never necessary, so capital dependency can be modelled by assuming that the revenue of deposits with a term longer than one period is zero.

Definition A revenue function $P : \Delta \to \mathbb{R}_+^{\tau+1}$ is called *capital dependent* if for all $T \in \mathcal{T}$ with $|T| > 1$ we have $P_t(c \cdot h(T)) = 0$ for all $t \in \{1, \ldots, \tau + 1\}$ and $c > 0$. If the underlying revenue function is capital dependent, also the corresponding deposit situation and deposit game are called capital dependent.

Following the construction used in the proof of Theorem 3.1 we readily see that if a deposit situation without reinvestment is capital dependent, then so is the constructed deposit situation with reinvestment.

To refine this subclass even further, we make one additional assumption. If more capital is deposited in a certain period, more revenue per unit of capital deposited is obtained. This boils down to the following definition.

Definition A revenue function $P : \Delta \to \mathbb{R}_+^{\tau+1}$ has *increasing returns to scale* if the function $\frac{P_t(c \cdot h(T))}{c}$ is non-decreasing in c for $c > 0$ and for all $t \in \{1, \ldots, \tau + 1\}$ and $T \in \mathcal{T}$. If the underlying revenue function has increasing returns to scale, also the corresponding deposit situation and deposit game are said to have increasing returns to scale.

Throughout the remainder of this section, let (N, v) be a capital dependent deposit game with increasing returns to scale corresponding to a deposit situation $(N, \tau, \Delta, P, \{m(i)\}_{i \in N})$. Moreover, to avoid unnecessary complications, we replace Assumption 2.2 by a stronger assumption. For all $t \in \{1, \ldots, \tau\}$ and all $i \in N$ we assume $\sum_{s=1}^{t} m_s(i) > 0$. This ensures that each coalition indeed makes a deposit each period.

For each coalition $S \subset N$ it is obviously optimal to make at most one deposit each period with a term of one period, and to make it as large as possible. Denote the optimal deposits for coalition S in period $t \in \{1, \ldots, \tau\}$ by

$$\delta^{t,S} = c^{t,S} \cdot h(\{t\}), \tag{8}$$

where $c^{t,S}$ is defined by

$$c^{t,S} = \sum_{s=1}^{t} \left(m_s(S) + \sum_{j=1}^{s-1} P_s(\delta^{j,S}) \right). \tag{9}$$

Because we assumed $\sum_{s=1}^{t} m_s(i) > 0$, for all $t \in \{1, \ldots, \tau\}$ and all $S \subset N$ we have $c^{t,S} > 0$. The

[9] The restriction that the revenue of deposits is independent of their term may not seem straightforward if we only consider deposits at a bank. However, if we expand our view to a broader class of investment products, such as physical capital, such a restriction is useful.

corresponding optimal feasible portfolio for coalition $S \subset N$ is given by

$$f^S(\delta) = \begin{cases} 1, & \text{if } \delta = \delta^{t,S},\ t \in \{1,\dots,\tau\}, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Theorem 5.4 shows that each deposit game (N,v) has a *population monotonic allocation scheme* (pmas). We first recall the definition of a pmas. According to Sprumont (1990) an allocation scheme $\{x^S\}_{S\subset N, S\neq\emptyset}$ with $x^S \in \mathbb{R}^S$ for all $S \subset N, S \neq \emptyset$ is called a pmas for a cooperative game (N,w) if it satisfies efficiency and monotonicity. Here $\{x^S\}_{S\subset N, S\neq\emptyset}$ satisfies efficiency if

$$\sum_{i\in S} x_i^S = w(S),$$

for all $S \subset N$, $S \neq \emptyset$, and monotonicity if for each $U \supset S$ and $i \in S$ it holds that

$$x_i^U \geq x_i^S.$$

Note that if $\{x^S\}_{S\subset N, S\neq\emptyset}$ is a pmas, then for all $S \subset N, S \neq \emptyset$ x^S belongs to the core of the game restricted to S and hence v is totally balanced.

The proof of Theorem 5.4 uses three lemmas. The first lemma provides an explicit formula for the coalitional values in terms of the optimal deposits for coalitions as stated in (8) and (9). The second lemma states that larger coalitions have higher returns. The third lemma constructs weights for each player in a coalition, which are used to determine how to split the revenue of a coalition over its members.

Lemma 5.1 *For all $S \subset N$, it holds that*[10]

$$v(S) = \sum_{t=1}^{\tau+1}\sum_{j=1}^{t-1} P_t(\delta^{j,S}).$$

Proof: Let $S \subset N$. With f^S as in (10) we have

$$\begin{aligned} v(S) &= \Pi(f^S) - \sum_{t=1}^{\tau} m_t(S) \\ &= \left[\left(\sum_{t=1}^{\tau} P(\delta^{t,S})\right) - \delta^{\tau,S}\right]_{\tau+1} - \sum_{t=1}^{\tau} m_t(S) \\ &= \left(\sum_{t=1}^{\tau} P_{\tau+1}(\delta^{t,S})\right) + \sum_{t=1}^{\tau}\left[m_t(S) + \sum_{j=1}^{t-1} P_t(\delta^{j,S})\right] - \sum_{t=1}^{\tau} m_t(S) \\ &= \sum_{t=1}^{\tau}\left(\sum_{j=1}^{t-1} P_t(\delta^{j,S}) + P_{\tau+1}(\delta^{t,S})\right) \\ &= \sum_{t=1}^{\tau+1}\sum_{j=1}^{t-1} P_t(\delta^{j,S}). \end{aligned}$$

□

[10]This lemma does not require increasing returns to scale and holds for all capital dependent deposit games.

Example 5.1 Consider a deposit situation $(N, \tau, \Delta, P, \{m(i)\}_{i\in N})$ with $N = \{1, 2, 3\}$, $\tau = 3$, Δ as in (2), where the revenue function P satisfies increasing returns to scale and moreover for all $c > 0$ and for all $t \in \{1, \ldots, \tau + 1\}$

$$P_t(c \cdot h(\{t-1\})) = \begin{cases} 0.05c, & \text{if } c < 1500, \\ 0.10c, & \text{if } c \geq 1500, \end{cases}$$

and $P_t(c \cdot h(T)) = 0$ for all $|T| > 1$, and finally with $m(1) = (800, 200, -450)$, $m(2) = (400, -300, 0)$ and $m(3) = (200, 600, -200)$. Observe that the revenue function is capital dependent. Note that the revenue is only non-zero for deposits made in the previous period, although a capital dependent revenue function in general does not require this.

Before we calculate the value of the grand coalition, we determine the optimal deposits from (8) and (9). These deposits are given by (8) where

$$\begin{aligned} c^{1,N} &= m_1(N) = 800 + 400 + 200 = 1400, \\ c^{2,N} &= c^{1,N} + m_2(N) + P_2(\delta^{1,N}) = 1400 + (200 - 300 + 600) + 70 = 1970, \\ c^{3,N} &= c^{2,N} + m_3(N) + P_3(\delta^{1,N}) + P_3(\delta^{2,N}) = 1970 - 650 + 0 + 197 = 1517. \end{aligned}$$

It immediately follows from Lemma 5.1 that

$$\begin{aligned} v(N) &= P_2(\delta^{1,N}) + P_3(\delta^{1,N}) + P_3(\delta^{2,N}) + P_4(\delta^{1,N}) + P_4(\delta^{2,N}) + P_4(\delta^{3,N}) \\ &= 70 + 0 + 197 + 0 + 0 + 151.70 = 418.70, \end{aligned}$$

which is the value of the grand coalition. ◁

The next lemma shows that the optimal amount deposited, $c^{t,S}$, and the rate of return are both monotonic in S.

Lemma 5.2 *For all $t \in \{1, \ldots, \tau\}$ and $S \subset U \subset N$:*

$$c^{t,S} \leq c^{t,U},$$

and also for all $j, t \in \{1, \ldots, \tau\}$

$$\frac{P_j(\delta^{t,S})}{c^{t,S}} \leq \frac{P_j(\delta^{t,U})}{c^{t,U}}.$$

Proof: The first statement follows immediately from the fact that $\mathcal{F}(m(S)) \subset \mathcal{F}(m(U))$. Moreover, because of increasing returns to scale we find for all $j \in \{1, \ldots, \tau + 1\}$

$$\frac{P_j(\delta^{t,S})}{c^{t,S}} = \frac{P_j(c^{t,S} \cdot h(\{t\}))}{c^{t,S}} \leq \frac{P_j(c^{t,U} \cdot h(\{t\}))}{c^{t,U}} = \frac{P_j(\delta^{t,U})}{c^{t,U}}.$$

□

We now construct a specific weight scheme $\{\theta^S\}_{S\subset N, S\neq\emptyset}$ with $\theta^S \in (\mathbb{R}^S_+)^\tau$. These weights represent for each period in time and for each player within each coalition, how large his contribution is relative to the total capital of the coalition at that time. For all $S \subset N$, $i \in S$ and $t \in \{1, \ldots, \tau\}$, define recursively

$$\theta^S_t(i) = \begin{cases} \frac{m_1(i)}{m_1(S)}, & \text{if } t = 1, \\ \frac{\sum_{s=1}^t m_s(i) + \sum_{s=1}^t \sum_{j=1}^{s-1} \theta^S_j(i) P_s(\delta^{j,S})}{c^{t,S}}, & \text{if } t \in \{2, \ldots, \tau\}. \end{cases} \tag{11}$$

Lemma 5.3 *For all $t \in \{1, \dots, \tau\}$ and $S \subset N$,*

$$\sum_{i \in S} \theta_t^S(i) = 1. \tag{12}$$

Moreover for all $S \subset U \subset N$ with $i \in S$,

$$c^{t,U} \cdot \theta_t^U(i) \geq c^{t,S} \cdot \theta_t^S(i). \tag{13}$$

Proof: We show (12) by induction on the period. Let $S \subset N$. For $t = 1$ we see

$$\sum_{i \in S} \theta_1^S(i) = \frac{\sum_{i \in S} m_1(i)}{c^{1,S}} = 1.$$

Let $t \in \{2, \dots, \tau\}$ and assume for all $j \in \{1, \dots, t-1\}$ that $\sum_{i \in S} \theta_j^S(i) = 1$, then

$$\begin{aligned}
\sum_{i \in S} \theta_t^S(i) &= \sum_{i \in S} \frac{\sum_{s=1}^t m_s(i) + \sum_{s=1}^t \sum_{j=1}^{s-1} \theta_j^S(i) P_s(\delta^{j,S})}{c^{t,S}} \\
&= \frac{\sum_{s=1}^t \sum_{i \in S} m_s(i) + \sum_{s=1}^t \sum_{j=1}^{s-1} \sum_{i \in S} \theta_j^S(i) P_s(\delta^{j,S})}{c^{t,S}} \\
&= \frac{\sum_{s=1}^t m_s(S) + \sum_{s=1}^t \sum_{j=1}^{s-1} P_s(\delta^{j,S})}{c^{t,S}} \\
&= \frac{c^{t,S}}{c^{t,S}} \\
&= 1.
\end{aligned}$$

For (13) we also use induction. Let $S \subset U \subset N$ and $i \in S$. For $t = 1$ we see

$$c^{1,U} \cdot \theta_1^U(i) = m_1(i) = c^{1,S} \cdot \theta_1^S(i).$$

Let $t \in \{2, \dots, \tau\}$ and assume that for all $j \leq t-1$ it holds that $c^{j,U} \theta_j^U(i) \geq c^{j,S} \theta_j^S(i)$. Then, also using (11) and Lemma 5.2, we find

$$\begin{aligned}
c^{t,U} \cdot \theta_t^U(i) &= \sum_{s=1}^t m_s(i) + \sum_{s=1}^t \sum_{j=1}^{s-1} \theta_j^U(i) P_s(\delta^{j,U}) \\
&= \sum_{s=1}^t m_s(i) + \sum_{s=1}^t \sum_{j=1}^{s-1} c^{j,U} \cdot \theta_j^U(i) \frac{P_s(\delta^{j,U})}{c^{j,U}} \\
&\geq \sum_{s=1}^t m_s(i) + \sum_{s=1}^t \sum_{j=1}^{s-1} c^{j,U} \cdot \theta_j^U(i) \frac{P_s(\delta^{j,S})}{c^{j,S}} \\
&\geq \sum_{s=1}^t m_s(i) + \sum_{s=1}^t \sum_{j=1}^{s-1} c^{j,S} \cdot \theta_j^S(i) \frac{P_s(\delta^{j,S})}{c^{j,S}} \\
&= \sum_{s=1}^t m_s(i) + \sum_{s=1}^t \sum_{j=1}^{s-1} \theta_j^S(i) P_s(\delta^{j,S}) \\
&= c^{t,S} \cdot \theta_t^S(i).
\end{aligned}$$

$\square$

Theorem 5.4 Every capital dependent deposit game with reinvestment that has increasing returns to scale has a population monotonic allocation scheme.

Proof: Let $\{x^S\}_{S\subset N, S\neq\emptyset}$ be defined by

$$x_i^S = \sum_{t=1}^{\tau+1}\sum_{j=1}^{t-1} \theta_j^S(i) P_t(\delta^{j,S}),$$

for all $S \subset N$, $i \in S$. We show that $\{x^S\}_{S\subset N, S\neq\emptyset}$ is a pmas. Let $S \subset N$. Recall that to obtain $v(S)$ we use Lemma 5.1, which states that the value of coalition S is the sum of all revenues using optimal deposits as in (8) and (9). The allocation scheme $\{x^S\}_{S\subset N, S\neq\emptyset}$ distributes the revenue in each period over the players according to the contributions of the players to the total capital of the coalition. Formally, by Lemma 5.3 we have

$$\begin{aligned}
\sum_{i\in S} x_i^S &= \sum_{i\in S}\left(\sum_{t=1}^{\tau+1}\sum_{j=1}^{t-1} \theta_j^S(i) P_t(\delta^{j,S})\right) \\
&= \sum_{t=1}^{\tau+1}\sum_{j=1}^{t-1} P_t(\delta^{j,S}) \left(\sum_{i\in S} \theta_j^S(i)\right) \\
&= \sum_{t=1}^{\tau+1}\sum_{j=1}^{t-1} P_t(\delta^{j,S}) \\
&= v(S).
\end{aligned}$$

Let $S \subset U \subset N$ and take $i \in S$. Then using Lemma 5.2 and Lemma 5.3

$$\begin{aligned}
x_i^U &= \sum_{t=1}^{\tau+1}\sum_{j=1}^{t-1} \theta_j^U(i) P_t(\delta^{j,U}) \\
&= \sum_{t=1}^{\tau+1}\sum_{j=1}^{t-1} c^{j,U} \cdot \theta_j^U(i) \frac{P_t(\delta^{j,U})}{c^{j,U}} \\
&\geq \sum_{t=1}^{\tau+1}\sum_{j=1}^{t-1} c^{j,U} \cdot \theta_j^U(i) \frac{P_t(\delta^{j,S})}{c^{j,S}} \\
&\geq \sum_{t=1}^{\tau+1}\sum_{j=1}^{t-1} c^{j,S} \cdot \theta_j^S(i) \frac{P_t(\delta^{j,S})}{c^{j,S}} \\
&= \sum_{t=1}^{\tau+1}\sum_{j=1}^{t-1} \theta_j^S(i) P_t(\delta^{j,S}) \\
&= x_i^S.
\end{aligned}$$

Hence, $\{x^S\}_{S\subset N, S\neq\emptyset}$ is a pmas. □

Sprumont (1990) shows that not all non-negative totally balanced games have a population monotonic allocation scheme. So the class of capital dependent deposit games with reinvestment and increasing returns to scale forms a proper subclass of deposit games with reinvestment, and these games are obviously totally balanced.

Example 5.2 Recall the deposit situation from Example 5.1 where we found the optimal deposits as well as the value for the grand coalition of the corresponding deposit game. To calculate the pmas provided in the proof of Theorem 5.4, we first calculate the weights θ^N of the weight scheme $\{\theta^S\}_{S\subset N, S\neq\emptyset}$ using (11).

The weights for the players in the coalition N for each period are given by

$$\theta_1^N(1) = \frac{m_1(1)}{m_1(N)} = \frac{800}{1400} = \frac{4}{7}, \qquad \theta_1^N(2) = \frac{2}{7}, \qquad \theta_1^N(3) = \frac{1}{7},$$
$$\theta_2^N(1) = \frac{800 + 200 + \frac{4}{7}\cdot 70}{1970} = \frac{104}{197}, \qquad \theta_2^N(2) = \frac{12}{197}, \qquad \theta_2^N(3) = \frac{81}{197},$$
$$\theta_3^N(1) = \frac{800 + 200 - 450 + \frac{4}{7}\cdot 70 + \frac{104}{197}\cdot 197}{1517} = \frac{694}{1517}, \quad \theta_3^N(2) = \frac{132}{1517}, \quad \theta_3^N(3) = \frac{691}{1517}.$$

With these weights we find that

$$x^N(1) = \frac{4}{7}\cdot 70 + \frac{104}{197}\cdot 197 + \frac{694}{1517}\cdot 151.70 = 213.40,$$
$$x^N(2) = 45.20,$$
$$x^N(3) = 160.10,$$

which is a core allocation of (N, v). It can be extended to the pmas as described in the proof of Theorem 5.4.

Coalition	Player 1	Player 2	Player 3
$\{1\}$	124.10	-	-
$\{2\}$	-	32.30	-
$\{3\}$	-	-	83.03
$\{1,2\}$	124.10	32.30	-
$\{1,3\}$	178.70	-	125.55
$\{2,3\}$	-	32.30	83.03
N	213.40	45.20	160.10

Table 1: A pmas for the game in Example 5.2.

It is readily observed that Table 1 provides indeed this population monotonic allocation scheme. $\triangleleft$

6 Reinvestment and Debt; a characterisation of all non-negative superadditive games

This section introduces deposit situations with reinvestment that also allow for debt and introduces the associated games. The main purpose is not to generalise the deposit games with reinvestment introduced before, but to characterise all non-negative superadditive games. Our framework is related to the borrowing model introduced by Tan (2000). We define loans as a negative, fixed amount of capital c that is borrowed from a bank during a prearranged and consecutive number of periods $t_1, t_1+1, \ldots, t_2$, where $1 \leq t_1 \leq t_2 \leq \tau$. We define τ as the final period in which a deposit can be made or any amount of money can be borrowed, so the finite and discrete timespan is $\{1, \ldots, \tau\}$. We define the set of all possible terms of a deposit or a loan $\mathcal{T}$ by (1). The set of all deposits is

$$\Delta^+ = \{\delta \in \mathbb{R}^{\tau+1} | \exists c > 0, T \in \mathcal{T} : \delta = c \cdot h(T)\}, \tag{14}$$

and the set of all loans is

$$\Delta^- = \{\delta \in \mathbb{R}^{\tau+1} | \exists c < 0, T \in \mathcal{T} : \delta = c \cdot h(T)\}, \tag{15}$$

where $h(T)$ is the function defined in (3). We define an endowment $m \in \mathbb{R}^\tau$ as before, and we assume that for all $t \in \{1, \ldots, \tau\}$, we have $\sum_{s=1}^t m_s \geq 0$. We consider a revenue function $P : \Delta^+ \cup \Delta^- \to \mathbb{R}^{\tau+1}$. We assume that for deposits revenue is non-negative, while loans are costly and thus the corresponding revenue is never positive. Formally assume for all $t \in \{1, \ldots, \tau + 1\}$

$$\begin{cases} P_t(\delta) \geq 0, & \text{if } \delta \in \Delta^+ \\ P_t(\delta) \leq 0, & \text{if } \delta \in \Delta^-. \end{cases}$$

A general assumption we make is that arbitrage is excluded. We also assume that for all $\delta = c \cdot h(\{t_1, t_1 + 1, \ldots, t_2\}) \in \Delta^+ \cup \Delta^-$ and all $t \leq t_1$ we have $P_t(\delta) = 0$.

The set of feasible portfolios is in a natural way extended to

$$\mathcal{D}(m) = \left\{ f : \Delta^+ \cup \Delta^- \to \mathbb{N} \cup \{0\} \,\middle|\, \forall t \in \{1, \ldots, \tau\} : \sum_{\delta \in \Delta^+ \cup \Delta^-} f(\delta)\delta_t = m_t + \sum_{\delta \in \Delta^+ \cup \Delta^-} f(\delta) P_t(\delta) \right\}.$$

The total capital $\Pi(f)$ at $\tau + 1$ for a feasible portfolio $f \in \mathcal{D}(m)$ is described by

$$\Pi(f) = \sum_{\delta \in \Delta^+ \cup \Delta^-} f(\delta)[P(\delta) - \delta]_{\tau+1}.$$

We define a *deposit situation with debt* $(N, \tau, \Delta^+, \Delta^-, P, \{m(i)\}_{i \in N})$ for a group of players N, where each player $i \in N$ has $m(i) \in \mathbb{R}^\tau$ available to deposit. The value of a coalition $S \subset N$ in the corresponding *deposit game with reinvestment and debt* (N, v) is given by

$$v(S) = \sup \left\{ \Pi(f) - \sum_{t=1}^{\tau} m_t(S) \,\middle|\, f \in \mathcal{D}(m(S)) \right\},$$

where $m(S) = \sum_{i \in S} m(i)$. We assume this supremum exists.[11]

The inclusion of debt allows for a full characterisation of all non-negative superadditive games.

Theorem 6.1 Each deposit game with reinvestment and debt is superadditive. Moreover, every non-negative superadditive game is a deposit game with reinvestment and debt.

Proof: The proof of the first statement is identical to that of Theorem 2.3 if we replace the set of feasible portfolios $\mathcal{F}$ by the set of feasible portfolios $\mathcal{D}$ that allow for debt.

For the proof of the second statement, let (N, w) be a non-negative superadditive game. We construct a deposit situation with reinvestment and debt $(N, \tau, \Delta^+, \Delta^-, P, \{m(i)\}_{i \in N})$, such that the corresponding game equals (N, w). First define $\tau = 2^{|N|}$, and define Δ^+ as in (14) and Δ^- as in (15).

We associate with every period $t \in \{1, \ldots, \tau\}$ a coalition $S_t \subset N$, where $S_\tau = \emptyset$, $S_{t_1} \neq S_{t_2}$ for all $t_1 \neq t_2$, and $|S_{t_1}| \leq |S_{t_2}|$ for all $t_1 < t_2 < \tau$. The period associated with coalition S is denoted by $t_S \in \{1, \ldots, \tau\}$. Define the endowment $m(i)$ of player i recursively by

$$m_1(i) = \begin{cases} w(N), & \text{if } t_S = 1 \text{ for some } S \ni i, \\ 0, & \text{otherwise,} \end{cases}$$

[11] As before, the exclusion of arbitrage opportunities and the fact that endowments are bounded guarantees the existence of this supremum.

and for $t \in \{2, \dots, \tau\}$

$$m_t(i) = \begin{cases} w(N), & \text{if } t_S = t \text{ for some } S \ni i \text{ and } \sum_{r=1}^{t-1} m_r(i) = 0, \\ -w(N), & \text{if } t_S = t \text{ for some } S \notin i \text{ and } \sum_{r=1}^{t-1} m_r(i) = w(N), \\ 0, & \text{otherwise.} \end{cases}$$

So, during periods associated with coalitions to which player i belongs, i has $w(N)$ available to deposit. Take ε such that

$$0 < \varepsilon < \frac{w(N)}{2^\tau}.$$

The necessity of this particular upper bound for ε becomes apparent later on. The revenues are non-trivial for the following types of deposits:

$$\begin{cases} P_{\tau+1}\left(w(N)h(\{t_{\{i\}}\})\right) = w(\{i\}), & \text{for all } i \in N, \\ P_{t_S}\left([w(N) + 2^{t_S}\varepsilon]h(\{t_{\{i\}}\})\right) = 2^{t_S}\varepsilon, & \text{for all } i \in N \text{and for all } S \ni i, \\ P_{t_S+1}\left([-2^{t_S}\varepsilon]h(\{t_{\{i\}}\})\right) = -2^{t_S}\varepsilon, & \text{for all } i \in N \text{and for all } S \ni i, \\ P_{\tau+1}\left(|S|(w(N) + 2^{t_S}\varepsilon)h(\{t_S\})\right) = w(S), & \text{for all } S \subset N. \end{cases}$$

For all other $\delta \in \Delta^+$ we set $P_t(\delta) = 0$ for all $t \in \{1, \dots, \tau+1\}$ and for all other $\delta \in \Delta^-$ we set $P_t(\delta) = M$ for all $t \in \{1, \dots, \tau+1\}$, where M is negative and sufficiently small.

We show that it is optimal for each coalition S to make deposits and loans such that its total revenue at $\tau+1$ is $w(S)$. First we show that the costs of loans cancel out the intermediate revenues. Next we show that each coalition S can obtain at least $w(S)$. Finally we show that $w(S)$ is indeed the highest possible revenue for coalition S.

Let $S \subset N$. Since the objective is to maximise total revenue at time $\tau+1$, we make a distinction between intermediate revenue and revenue at $\tau+1$. The only deposits that have positive intermediate revenue are deposits of the form $[w(N) + 2^{t_U}\varepsilon]h(\{t_{\{i\}}\})$, for all $U \subset N$ and all $j \in N$. Since in period $t_{\{i\}}$ only player i has capital $w(N)$ to deposit, at least $2^{t_U}\varepsilon$ needs to be borrowed. However, the corresponding revenue is $2^{t_U}\varepsilon$ and this cancels out with the costs of the loan one period later which is $-2^{t_U}\varepsilon$. So no intermediate revenue can be carried over to period $\tau+1$.

The revenue at period $\tau+1$ can be obtained by two sorts of deposits. First, we consider the deposits of the form $w(N)h(\{t_{\{i\}}\})$ for all $i \in N$. If $i \in S$ this deposit can be made. Otherwise, $w(N)$ needs to be borrowed[12], costing $w(N)$ at some point. By non-negativity and superadditivity $w(\{i\}) \leq w(N)$, so it is never optimal to borrow capital to make this deposit. A total revenue of $\sum_{i \in S} w(\{i\})$ can be obtained by S using the deposits of this form.

For $|S| > 1$ to make the deposit $|S|(w(N) + 2^{t_S}\varepsilon)h(\{t_S\})$, each player $i \in S$ has to make a deposit $[w(N) + 2^{t_S}\varepsilon]h(\{t_{\{i\}}\})$ and receive $2^{t_S}\varepsilon$ revenue in period t_S. Therefore, the deposit can be made by borrowing $|S|2^{t_S}\varepsilon$, with revenue $-|S|2^{t_S}\varepsilon$ that can be paid back in period t_S+1. So the intermediate revenue and costs of the loans cancel out, and the coalition has revenue $w(S)$ at time $\tau+1$. No additional deposits can be made without borrowing at least $w(N)$, in particular $w(N)h(\{t_{\{i\}}\})$ for all $i \in S$ is no longer possible without borrowing, and this is never profitable. So, each player $i \in S$ can deposit in only one deposit with non-zero revenue at $\tau+1$.

[12]Because $2^{t_{\{i\}}}\varepsilon < w(N)$ it is never possible to carry player i's $w(N)$ through to a period $t_{\{j\}} > t_{\{i\}}$ for some $j \neq i$ without borrowing an additional $w(N)$.

The deposits of the form $|U|(w(N)+2^{t_U}\varepsilon)h(\{t_U\})$ for all $U \supsetneq S$ can only be made by borrowing $w(N) \cdot (|U| - |S|) > w(N)$. Because $w(U) - w(N) \leq 0$, this is not profitable.

For each coalition U such that $S \cap U \neq \emptyset$ and $S \not\subset U$, $U \not\subset S$, a similar argument holds. Because each player $i \in S \cup U$ can deposit in only one deposit with non-zero revenue at $\tau + 1$, making both the deposits at t_S and t_U is impossible without borrowing at least $w(N)$, which is not profitable.

It is clear that any coalition S can also obtain a total revenue of $\sum_{k=1}^{t} v(S_k)$ for any partition $\{S_1, S_2, \ldots, S_k\}$ of S. However, because of superadditivity the revenue is lower in this case, so the optimal total revenue for the coalition S is $w(S)$. So the deposit game corresponding to this deposit situation is (N, w). □

We see that the construction of Theorem 6.1 depends on loans. The presence of loans ensures that at the relevant periods in time, money is available to the players, although it has to be repaid (possibly with interest) later on. So it might be beneficial for some coalition to use loans. The key in this construction is that it is sometimes not beneficial to use loans, since it is costly, and only available for a period of time.

If we do not allow debt, it is unclear whether we still have a characterisation of all non-negative superadditive games. For three players, the classes of deposit games without reinvestment, with reinvestment and with reinvestment and debt are equal to the class of non-negative superadditive games. Whether this is true for more than three players remains an open question.

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