CentER

Discussion Paper

No. 2007–90

NEW AXIOMATIZATIONS AND AN IMPLEMENTATION OF THE SHAPLEY VALUE

By Takumi Kongo, Yukihiko Funaki, Stef Tijs

November 2007

ISSN 0924-7815

Tilburg University

New Axiomatizations and an Implementation of the Shapley Value

Takumi Kongo[*], Yukihiko Funaki[†] and Stef Tijs[‡]

November 22, 2007

Abstract

Some new axiomatic characterizations and recursive formulas of the Shapley value are presented. In the results, dual games and the self-duality of the value implicitly play an important role. A set of non-cooperative games which implement the Shapley value on the class of all games is given.

JEL classification: C71, C72

Keywords: Shapley value, axiomatization, implementation,

1 Introduction

Axiomatic characterizations of the Shapley value (Shapley (1953)) have been studied by many researchers. For example, Shapley (1953), Myerson (1980), Young (1985), Hart and Mas-Colell (1989), Chun (1989), Hamiache (2001) and Brink (2001) characterized it on the class of all games. On subclasses of games, Dubey (1975) characterized it on the class of simple games and Neyman (1989) showed that the set of axioms used in the original Shapley's axiomatic characterizations also characterize the Shapley value on the additive class spanned by a single game.

Those various characterizations give us new viewpoints of the Shapley value. For instance, by consistency used in Hart and Mas-Colell (1989), the Shapley value is represented in a recursive manner (see Hart and Mas-Colell (1989) and Maschler and Owen (1989)). In addition, by the recursive representation, Pérez-Castrillo and Wettstein (2001) construct a non-cooperative game whose subgame perfect equilibrium payoffs coincide with the Shapley value. The non-cooperative game is called the bidding mechanism.

[*]Graduate School of Economics, Waseda University, 1-6-1 Nishi-Waseda, Shinjuku-Ku, Tokyo, 169-8-50 Japan. E-mail: kongo_takumi@toki.waseda.jp

[†]School of Political Science and Economics, Waseda University, Japan.

[‡]CentER for Economic Research and Department of Econometrics and Operations Research, Tilburg University, the Netherlands.

In this paper, we give some new axiomatic characterizations of the Shapley value. Given a cooperative game and a coalition, we define two modified games: the marginal game and the extended marginal game. The two games differ only in the player set. For the two games, we give some new axioms related to the balanced contributions property. The marginal games are closely related to the dual games; hence, in our results, the dual games and the self-duality of the value implicitly play an important role. By our axiomatic characterizations, some new recursive formulas for the Shapley value are given. One of the recursive formulas can be seen as a dual game representation of the above mentioned recursive formula. Moreover, we give a set of non-cooperative games which implement the Shapley value as subgame perfect equilibrium payoffs on the class of all games. That game is a modification of the bidding mechanism and the difference is that when someone rejects the offer made by a proposer, the remaining players play the marginal game instead of just a subgame.

The paper is organized as follows. Notations and definitions are presented in Section 2. The axiomatic characterizations of the Shapley value are given in Section 3. The recursive formulas of the Shapley value are provided in Section 4. The implementation of the Shapley value is presented in Section 5. Further discussions are included in Section 6.

2 Preliminaries

A pair (N, v) is a *cooperative game* or a *TU game* where $N \subseteq \mathbb{N}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is a *characteristic function*. Let $|N| = n$ where $|\cdot|$ represents the cardinality of a set. A subset S of N is called a *coalition*. For any $S \subseteq N$, $v(S)$ represents the worth of the coalition. For simplicity, each singleton is represented as i instead of $\{i\}$ when there exists no fear of confusion. For any $S \subseteq N$, the subgame of (N, v) on S is a pair $(S, v|_S)$ where $v|_S(T) = v(T)$ for any $T \subseteq S$.

Let $\mathcal{G}$ be a set of all cooperative games. A *value* is a mapping from $\mathcal{G}$ into $|N|$-dimensional vector $(x_i)_{i\in N}$ that satisfies $\sum_{i\in N} x_i = v(N)$. One of the well-known values on the class of cooperative games is the Shapley value introduced by Shapley (1953). Let π be a permutation on N and Π be a set of all permutations on N. Given $(N, v) \in \mathcal{G}$, the *Shapley value* $\phi(N, v)$ is defined as follows: For each $i \in N$,

$$\phi_i(N, v) = \frac{1}{n!} \sum_{\pi\in\Pi} \Big(v(\{j|\pi(j) \leq \pi(i)\}) - v(\{j|\pi(j) < \pi(i)\})\Big).$$

The Shapley value satisfies the following *balanced contributions property* (see Myerson (1980)): For any $i, j \in N$ with $i \neq j$,

$$\phi_i(N, v) - \phi_i(N\backslash j, v) = \phi_j(N, v) - \phi_j(N\backslash i, v).$$

The *dual game* (N, v^*) of (N, v) is the game that assigns to each coalition $S \subseteq N$ the worth that is lost by the grand coalition N if S leaves N, that is, for each $S \subseteq N$,

$$v^*(S) = v(N) - v(N\backslash S).$$

A value φ is *self-dual* if $\varphi(N,v) = \varphi(N,v^*)$ for each (N,v). It is well-known that the Shapley value is self-dual.

Let $S \subseteq N$. The *S-marginal game* $(N\backslash S, v^S)$ of (N,v) is the game that assigns to each coalition $T \subseteq N\backslash S$ the worth of $T \cup S$ minus the worth of S, that is, for each $T \subseteq N\backslash S$,

$$v^S(T) = v(S \cup T) - v(S).$$

In the S-marginal game, any subset of $N\backslash S$ can win the cooperation of S by paying the value $v(S)$ to S.

Between the dual games and the marginal games, the following holds.

Proposition 1. *For any $S \subseteq N$, it holds $v^S = (v^*|_{N\backslash S})^*$.*

Proof. For any $S \subseteq N$ and any $T \subseteq N\backslash S$,

$$\begin{aligned}(v^*|_{N\backslash S})^*(T) &= v^*(N\backslash S) - v^*(N\backslash S\backslash T)\\ &= v(N) - v(S) - v(N) + v(S \cup T)\\ &= v^S(T). \quad \square\end{aligned}$$

A *dummy coalition* S of (N,v) is a coalition that satisfies the following condition: for any $T \subseteq N\backslash S$, $v(S \cup T) = v(S) + v(T)$. In particular, if S is a singleton, we call it a *dummy player*.

Let $S \subseteq N$. The *extended S-marginal game* $(N, \bar{v}^S)$ of (N,v) is the game that assigns to each coalition $T \subseteq N$ the worth of $T \cup S$ minus the worth of $T\backslash S$, that is, for each $T \subseteq N$,

$$\bar{v}^S(T) = v(S \cup T) - v(S\backslash T).$$

By definition, given $(N,v) \in \mathcal{G}$ and $S \subseteq N$, for any $T \subseteq N\backslash S$,

$$\begin{aligned}\bar{v}^S(T \cup S) &= v(S \cup T) - v(\emptyset)\\ &= v(S \cup T) - v(S) + v(S) - v(\emptyset)\\ &= v(S \cup T) - v(S\backslash T) + v(S \cup S) - v(S\backslash S)\\ &= \bar{v}^S(T) + \bar{v}^S(S).\end{aligned}$$

Thus, S is a dummy coalition of $(N, \bar{v}^S)$. Moreover,

$$\bar{v}^S|_{N\backslash S}(T) = v(S \cup T) - v(S) = v^S(T),$$

that is, the difference between the extended S-marginal game and the S-marginal game is whether or not a dummy coalition S is included in the player set. For the Shapley value of the two games, the following holds.

Proposition 2. *Given $(N,v) \in \mathcal{G}$, for any $S \subseteq N$ and any $i \in N$,*

$$\phi_i(N, \bar{v}^S) = \begin{cases} \phi_i(S, v) & \textit{if } i \in S\\ \phi_i(N\backslash S, v^S) & \textit{if } i \notin S. \end{cases}$$

Proof. Given $(N, v) \in \mathcal{G}$ and $S \subseteq N$, let

$$w^S(T) = v(S) - v(S \backslash T)$$

and

$$u^S(T) = v(S \cup T) - v(S)$$

for any $T \subseteq N$. Then, $w^S + u^S = \bar{v}^S$. By additivity of the Shapley value,

$$\phi(N, \bar{v}^S) = \phi(N, w^S) + \phi(N, u^S).$$

By definition, any $i \notin S$ is a null player in (N, w^S) and any player $i \in S$ is a null player in (N, u^S), respectively.[1] The null player property of the Shapley value implies that

$$\phi_i(N, \bar{v}^S) = \begin{cases} \phi_i(N, w^S) & \text{if } i \in S \\ \phi_i(N, u^S) & \text{if } i \notin S. \end{cases}$$

By the fact that any null player in (N, v) is also a null player in any subgame of (N, v), the null player property and the balanced contributions property, we obtain that for any player $i \in N$ and any null player $j \neq i$,

$$\phi_i(N, v) - \phi_i(N \backslash j, v) = \phi_j(N, v) - \phi_j(N \backslash i, v) = 0 - 0 = 0,$$

which implies $\phi_i(N, v) = \phi_i(N \backslash j, v)$ for any $i \neq j$ when j is a null player, that is, in the Shapley value, any null player has no effect on the other players.

By definition,

$$(w^S|_S)^*(T) = w^S(S) - w^S(S \backslash T) = v(S) - v(S \backslash S) - v(S) + v(S \backslash (S \backslash T)) = v(T)$$

for any $T \subseteq S$ and $u^S(T) = v^S(T)$ for any $T \subseteq N \backslash S$. Thus, for any $i \in S$, since any $j \in N \backslash S$ is a null player in (N, w^S),

$$\phi_i(N, w^S) = \phi_i(S, w^S|_S) = \phi_i(S, (w^S|_S)^*) = \phi_i(S, v),$$

and for any $i \in N \backslash S$, since any $j \in S$ is a null player in (N, u^S),

$$\phi_i(N, u^S) = \phi_i(N \backslash S, u^S|_{N \backslash S}) = \phi_i(N \backslash S, v^S). \quad \square$$

3 Axiomatizations

By using the marginal games, the Shapley value is axiomatized in the following way. Let φ be a value for cooperative games.

Balanced M-contributions property: For each $(N, v) \in \mathcal{G}$ and any $i, j \in N$ with $i \neq j$,

$$\varphi_i(N, v) - \varphi_i(N \backslash j, v^j) = \varphi_j(N, v) - \varphi_j(N \backslash i, v^i).$$

[1] A *null player* in the game (N, v) is a player $i \in N$ that satisfies $v(S) = v(S \backslash i)$ for any $S \subseteq N$.

Theorem 1. *The Shapley value is the unique value which satisfies the balanced M-contributions property.*

Proof. First, we show that the Shapley value satisfies the balanced M-contributions property. In the case of $|N| = 1$ this is obvious. Let $|N| \geq 2$. For any $i, j \in N$ with $i \neq j$, by the fact that the Shapley value is self-dual, satisfies the balanced contributions property and by Proposition 1,

$$\begin{aligned}
\phi_i(N, v) - \phi_j(N, v) &= \phi_i(N, v^*) - \phi_j(N, v^*) \\
&= \phi_i(N \backslash j, v^*) - \phi_j(N \backslash i, v^*) \\
&= \phi_i(N \backslash j, (v^*|_{N \backslash j})^*) - \phi_j(N \backslash i, (v^*|_{N \backslash i})^*) \\
&= \phi_i(N \backslash j, v^j) - \phi_j(N \backslash i, v^i).
\end{aligned}$$

For the uniqueness, we use the induction with respect to the number of players. Let φ be a value on the class of cooperative games. In the case of $|N| = 1$, $\varphi_i(N, v) = v(i) = \phi_i(N, v)$ for $i \in N$. If $|N| = 2$, by the balanced M-contributions property,

$$\begin{aligned}
\varphi_i(N, v) - \varphi_j(N, v) &= \varphi_i(N \backslash j, v^j) - \varphi_j(N \backslash i, v^i) \\
&= v(N) - v(j) - v(N) + v(i) \\
&= v(i) - v(j),
\end{aligned}$$

which implies

$$\varphi_i(N, v) = \frac{v(N) + v(i) - v(j)}{2} \text{ and } \varphi_j(N, v) = \frac{v(N) - v(i) + v(j)}{2}.$$

This is exactly the Shapley value of (N, v).

Let $n \geq 2$ and suppose $\varphi = \phi$ in case of there are less than n players. Consider the case of n players. Fix $i \in N$; by the balanced M-contributions property and the supposition above, for any $j \in N \backslash i$,

$$\begin{aligned}
\varphi_i(N, v) - \varphi_j(N, v) &= \varphi_i(N \backslash j, v^j) - \varphi_j(N \backslash i, v^i) \\
&= \phi_i(N \backslash j, v^j) - \phi_j(N \backslash i, v^i) \\
&= \phi_i(N, v) - \phi_j(N, v).
\end{aligned}$$

Summing up the above equalities over $j \in N \backslash i$, we obtain

$$(n-1)\varphi_i(N, v) - \sum_{j \in N \backslash i} \varphi_j(N, v) = (n-1)\phi_i(N, v) - \sum_{j \in N \backslash i} \phi_j(N, v),$$

or

$$n\varphi_i(N, v) - v(N) = n\phi_i(N, v) - v(N).$$

Since $n \geq 2$, $\varphi_i(N, v) = \phi_i(N, v)$. For any $j \in N$, $\varphi_j(N, v) = \phi_j(N, v)$ in the same manner. Hence, $\varphi = \phi$ in the case of n players. $\square$

By using the extended marginal games instead of the marginal games, the following is obtained.

Balanced EM-contributions property: For each $(N, v) \in \mathcal{G}$ and any $i, j \in N$ with $i \neq j$,

$$\varphi_i(N, v) - \varphi_i(N, \bar{v}^j) = \varphi_j(N, v) - \varphi_j(N, \bar{v}^i).$$

Dummy invariance: For each $(N, v) \in \mathcal{G}$, any dummy player $j \in N$ of (N, v) and any $i \in N \backslash j$,

$$\varphi_i(N, v) = \varphi_i(N \backslash j, v).$$

Theorem 2. *The Shapley value is the unique value which satisfies the balanced EM-contributions property and dummy invariance.*

Proof. First, we show that the Shapley value satisfies the balanced EM-contributions property and dummy invariance, respectively. In the case that $|N| = 1$ this is obvious. If $|N| \geq 2$, for any $i, j \in N$ with $i \neq j$, by Proposition 2 and Theorem 1,

$$\begin{aligned}\phi_i(N, v) - \phi_i(N, \bar{v}^j) &= \phi_i(N, v) - \phi_i(N \backslash j, v^j) \\ &= \phi_j(N, v) - \phi_j(N \backslash i, v^i) \\ &= \phi_j(N, v) - \phi_j(N, \bar{v}^i).\end{aligned}$$

By the fact that a dummy player j of (N, v) is also a dummy player of $(N \backslash i, v)$, for any $i \neq j$, and since the Shapley value satisfies the dummy player property and the balanced contributions property, we have

$$\phi_i(N, v) - \phi_i(N \backslash j, v) = \phi_j(N, v) - \phi_j(N \backslash i, v) = v(j) - v(j) = 0,$$

which implies dummy invariance.

For the uniqueness, let φ be a value which satisfies the two axioms. In the case when $|N| = 1$, $\varphi_i(N, v) = v(i) = \phi_i(N, v)$ for $i \in N$. If $|N| \geq 2$, for any $i, j \in N$ with $i \neq j$, the dummy invariance together with the balanced EM-contributions property imply,

$$\begin{aligned}\varphi_i(N, v) - \varphi_i(N \backslash j, v^j) &= \varphi_i(N, v) - \varphi_i(N, \bar{v}^j) \\ &= \varphi_j(N, v) - \varphi_j(N, \bar{v}^i) \\ &= \varphi_j(N, v) - \varphi_j(N \backslash i, v^i).\end{aligned}$$

Thus, the value which satisfies dummy invariance and the balanced EM-contributions property must satisfies the balanced M-contributions property. By Theorem 1, the uniqueness is shown. □

For the independence of the two axioms, we consider the values χ^1 and χ^2 defined by for each $i \in N$,

$$\chi_i^1(N, v) = \begin{cases} v(N) & \text{if } i = \min_{j \in N} j \\ 0 & \text{if } i \neq \min_{j \in N} j, \end{cases}$$

and

$$\chi_i^2(N, v) = \begin{cases} \frac{v(N)}{|N \backslash D|} & \text{if } i \text{ is not a dummy player of } (N, v) \\ 0 & \text{if } i \text{ is a dummy player of } (N, v), \end{cases}$$

where D is a set of all dummy players in (N, v). It is easy to show that χ^1 satisfies the balanced EM-contributions property but not dummy invariance and that χ^2 satisfies dummy invariance but not the balanced EM-contributions property.

The following is obtained as a corollary of Theorem 2.

Corollary 1. *The Shapley value is the unique value which satisfies dummy invariance, additivity and the following property: For each $(N, v) \in \mathcal{G}$ and any $i, j \in N$ with $i \neq j$,*

$$\varphi_i(N, v - \bar{v}^j) = \varphi_j(N, v - \bar{v}^i),$$

where for any $S \subseteq N$, $(v - \bar{v}^k)(S) = v(S) - \bar{v}^k(S)$ for $k = i, j$.

4 Recursive formulas

It is well-known that the Shapley value is characterized in the following recursive manner. (See Maschler and Owen (1989) and Hart and Mas-Colell (1989)):

$$\phi_i(N, v) = \frac{1}{n}(v(N) - v(N\backslash i)) + \frac{1}{n} \sum_{j \in N\backslash i} \phi_i(N\backslash j, v). \tag{1}$$

By using the (extended) marginal games, we give other recursive representations of the Shapley value as follows.

Proposition 3. *For each $(N, v) \in \mathcal{G}$ and any $i \in N$,*

$$\phi_i(N, v) = \frac{1}{n}v(i) + \frac{1}{n} \sum_{j \in N\backslash i} \phi_i(N\backslash j, v^j).$$

Proof. By (1), the self-duality of the Shapley value and Proposition 1,

$$\begin{aligned}
\phi_i(N, v) &= \phi_i(N, v^*) \\
&= \frac{1}{n}(v^*(N) - v^*(N\backslash i)) + \frac{1}{n} \sum_{j \in N\backslash i} \phi_i(N\backslash j, v^*) \\
&= \frac{1}{n}v(i) + \frac{1}{n} \sum_{j \in N\backslash i} \phi_i(N\backslash j, (v^*|_{N\backslash j})^*) \\
&= \frac{1}{n}v(i) + \frac{1}{n} \sum_{j \in N\backslash i} \phi_i(N\backslash j, v^j). \quad \square
\end{aligned}$$

We notice that Proposition 3 can be seen as the particular case of the following Proposition 4 when $r = 1$.

Proposition 4. *Let $r = |R|$ be fixed, where $R \subseteq N$. Then for each $(N, v) \in \mathcal{G}$ and each $i \in N$,*

$$\phi_i(N, v) = \frac{1}{\binom{n}{r}} \sum_{R \subseteq N, |R|=r, R \ni i} \phi_i(R, v) + \frac{1}{\binom{n}{r}} \sum_{R \subseteq N, |R|=r, R \not\ni i} \phi_i(N\backslash R, v^R).$$

Proof. Let $r = |R|$ be fixed, where $R \subseteq N$. Let $\Pi^R = \{\pi \in \Pi | \pi(j) \leq r \text{ for each } j \in R\}$. Then for any $R \neq R'$ which satisfy $|R| = |R'| = r$, we have $\Pi^R \cap \Pi^{R'} = \emptyset$ and $\bigcup_{R \subseteq N, |R|=r} \Pi^R = \Pi$. Thus,

$$\begin{aligned}
\phi_i(N, v) &= \frac{1}{n!} \sum_{R \subseteq N, |R|=r} \sum_{\pi \in \Pi^R} \Big(v(\{j | \pi(j) \leq \pi(i)\}) - v(\{j | \pi(j) < \pi(i)\}) \Big) \\
&= \frac{1}{n!} \sum_{R \subseteq N, |R|=r, R \ni i} \sum_{\pi \in \Pi^R} \Big(v|_R(\{j | \pi(j) \leq \pi(i)\}) - v|_R(\{j | \pi(j) < \pi(i)\}) \Big) \\
&\quad + \frac{1}{n!} \sum_{R \subseteq N, |R|=r, R \not\ni i} \sum_{\pi \in \Pi^R} \Big(v^R(\{j | r < \pi(j) \leq \pi(i)\}) - v^R(\{j | r < \pi(j) < \pi(i)\}) \Big) \\
&= \frac{1}{n!} \sum_{R \subseteq N, |R|=r, R \ni i} r!(n-r)! \phi_i(R, v) + \frac{1}{n!} \sum_{R \subseteq N, |R|=r, R \not\ni i} r!(n-r)! \phi_i(N \backslash R, v^R) \\
&= \frac{1}{\binom{n}{r}} \sum_{R \subseteq N, |R|=r, R \ni i} \phi_i(R, v) + \frac{1}{\binom{n}{r}} \sum_{R \subseteq N, |R|=r, R \not\ni i} \phi_i(N \backslash R, v^R). \quad \Box
\end{aligned}$$

The following is obtained as a corollary of Propositions 2 and 4.

Corollary 2. *Let $r = |R|$ be fixed, where $R \subseteq N$. For each $(N, v) \in \mathcal{G}$ and each $i \in N$,*

$$\phi_i(N, v) = \frac{1}{\binom{n}{r}} \sum_{R \subseteq N, |R|=r} \phi_i(N, \bar{v}^R).$$

5 Implementation

In this section, given a cooperative game, we consider a non-cooperative game which implements the Shapley value of the cooperative game as equilibrium payoffs. In that non-cooperative game, the marginal games and the recursive formula we mentioned in the previous section play important roles.

Given a cooperative game $(N, v) \in \mathcal{G}$, the non-cooperative game $\Gamma(N, v)$ is defined in the following recursive manner.

In case $|N| = 1$, player $i \in N$ obtains $v(i)$ and the game is over.

Assume that the non-cooperative game is known when there are less than n players. We define the case where there are n players.

t=1 Each player $i \in N$ makes bids $b^i_j \in \mathbb{R}$ for every $j \neq i$.

For each $i \in N$, the net bid B^i is the sum of the bids he made minus the sum of the bids the others made to him, that is, $B^i = \sum_{j \neq i} b^i_j - \sum_{j \neq i} b^j_i$. Let $\alpha = argmax_i B^i$, where in the case of multiple maximizers, one of them is randomly chosen. The chosen player α pays b^α_j to every player $j \neq \alpha$.

t=2 Player α makes an offer $x^\alpha_j \in \mathbb{R}$ to every player $j \in N \backslash \alpha$.

t=3 Players in $N \backslash \alpha$ respond to the offer in a sequential manner, say $(j_1, \ldots, j_{n-1})$. An order of the players makes no matter. Response is either "accept it" or "reject it".

In case player j_h accepts the offer, the next player j_{h+1} responds to it. If every j_h accepts the offer, the players come to an agreement. If there is some rejection, an agreement is not reached.

When an agreement is reached, proposer α pays the proposed payoff x_j for any $j \in N \backslash \alpha$ in return for obtaining the value of their total cooperation, $v(N)$. Thus, the payoff for responder j is

$$b_j^\alpha + x_j^\alpha$$

and the payoff for proposer α is

$$v(N) - \sum_{j \neq \alpha} b_j^\alpha - \sum_{j \neq \alpha} x_j^\alpha.$$

Then the game is over.

On the other hand, when an agreement is not reached, the proposer is weakly split off by the other players. He leaves the game with obtaining $v(\alpha)$. Then, the remaining players $N \backslash \alpha$ continue a non-cooperative game $\Gamma(N \backslash \alpha, v^\alpha)$.

This non-cooperative game is inspired by the *bidding mechanism* presented in Pérez-Castrillo and Wettstein (2001). The difference between the above mentioned non-cooperative game and the biding mechanism is what will happen when someone rejects an offer. In our games, in the case of rejection, the proposer α is weakly split off by the other players and the remaining players play the α-marginal game $(N \backslash \alpha, v^\alpha)$, while in the bidding mechanism, the proposer just leaves the game and the remaining players play just the subgame $(N \backslash \alpha, v)$.

We obtain the following theorem.

Theorem 3. $\Gamma(N, v)$ *produces the Shapley value payoff for* (N, v) *in any subgame perfect equilibrium.*

Proof. The proof proceeds by induction with respect to the number of players. If $|N| = 1$, the Shapley value is equal to the value of his stand-alone coalition; hence, the theorem holds. Assume that the theorem holds in case there are less than n players and consider the case when there are n players.

First, we show that there exists an SPE whose payoff coincides with the Shapley value of the game (N, v). Consider the following strategy for each player.

t=1 Each player $i \in N$ announces $b_j^i = \phi_j(N, v) - \phi_j(N \backslash i, v^i)$ for every $j \neq i$.

t=2 A proposer α offers $x_j^\alpha = \phi_j(N \backslash \alpha, v^\alpha)$ for every $j \in N \backslash \alpha$.

t=3 A responder j accepts the offer if $x_j^\alpha \geqq \phi_j(N \backslash \alpha, v^\alpha)$ and rejects it otherwise.

If all players take the above strategies, an agreement is formed at t=3 and the game is over. It is clear that the above strategy profile yields the Shapley value for any player who is not the proposer α since $b_j^\alpha + x_j^\alpha = \phi_j(N,v)$ for any $j \neq \alpha$. The proposer obtains $v(N) - \sum_{j\neq\alpha} b_j^\alpha - \sum_{j\neq\alpha} x_j^\alpha = v(N) - \sum_{j\neq\alpha}\phi_j(N,v) = \phi_i(N,v)$. Note that all players obtain their Shapley value whether or not the player is a proposer. In other words, given the strategies, an outcome is the same regardless of who is chosen as a proposer.

To check whether the above strategies constitute an SPE, first, we show that the strategies at t=3 are best responses for each of the players. Let j_{n-1} be the last player who has to decide whether accept or reject the offer. If no other players reject an offer, player j_{n-1}'s best response is accept the offer if $x^\alpha_{j_{n-1}} \geqq \phi_{j_{n-1}}(N\backslash\alpha, v^\alpha)$ and reject it otherwise.[2] Knowing that the above mentioned reaction of the last player, the second last player j_{n-2}'s best response is accept the offer if $x^\alpha_{j_{n-2}} \geqq \phi_{j_{n-2}}(N\backslash\alpha, v^\alpha)$ and reject it otherwise. Using the same argument to go backward, we can show that the strategies mentioned above constitute an SPE of the subgame starting from t=3.

Next, we prove that the strategies at t=2 are best responses for each of them. By this strategy, the proposer obtains $v(N) - \sum_{j\neq\alpha}\phi_j(N\backslash\alpha, v^\alpha) = v(N) - v^\alpha(N\backslash\alpha) = v(\alpha)$ in the subgame starting from t=2. If he offers some player j the value $\bar{x}_j^\alpha$ less than $\phi_j(N\backslash\alpha, v^\alpha)$, the offer is rejected by the player and the proposer obtains $v(\alpha)$ which is not strictly better off. If he offers some player j the value $\hat{x}_j^\alpha$ larger than $\phi_j(N\backslash\alpha, v^\alpha)$ without lowering the offer to the other players, the offer is accepted but the share of the proposer is strictly worse off. Thus, the above mentioned strategies constitute a SPE of the subgame starting from t=2.

Then, we show that the strategies t=1 are best responses for each of them. Given the strategies, for any $i \in N$,

$$B^i = \sum_{j\neq i} b_j^i - \sum_{j\neq i} b_i^j = \sum_{j\neq i}(\phi_j(N,v) - \phi_j(N\backslash i, v^i)) - \sum_{j\neq i}(\phi_i(N,v) - \phi_i(N\backslash j, v^j)) = 0,$$

since ϕ satisfies the balanced M-contributions property. Hence, all players are chosen to be a proposer with probability $\frac{1}{n}$. As seen before, the outcome is the same regardless of who is chosen as a proposer. Given the above mentioned strategies, consider the case that player i changes his strategy to $\bar{b}_j^i = b_j^i + a_j$ for each of $j \neq i$. If $\sum_{j\neq i} a_j < 0$, i is not chosen as a proposer; hence, his final payoff is unchanged. If $\sum_{j\neq i} a_j = 0$, i may be chosen to be a proposer. In the case that he is not chosen as a proposer, his final payoff is unchanged. In the case that he is chosen as a proposer, his final payoff is

$$v(N) - \sum_{j\neq i}\bar{b}_j^i - \sum_{j\neq i}\phi_j(N\backslash i, v^i) = v(N) - \sum_{j\neq i} b_j^i - \sum_{j\neq i}\phi_j(N\backslash i, v^i) = \phi_i(N,v),$$

which means his final payoff is unchanged. If $\sum_{j\neq i} a_j > 0$, i must be chosen to be a proposer. However, by the previous result, he obtains

$$v(N) - \sum_{j\neq i}\bar{b}_j^i - \sum_{j\neq i}\phi_j(N\backslash i, v^i) < v(N) - \sum_{j\neq i} b_j^i - \sum_{j\neq i}\phi_j(N\backslash i, v^i) = \phi_i(N,v).$$

[2]Note that it is not a unique best response.

Thus, his share is strictly worse off. Therefore, the above mentioned strategies constitute a SPE.

Next, we prove that any SPE implements the Shapley value payoff as an equilibrium outcome by the following series of claims.

Claim 1: In any subgame starting from t=2, a proposer α obtains $v(\alpha)$ and each of the other players obtains his Shapley value of the game $(N\backslash\alpha, v^{\alpha})$ in any SPE.

Let α be a proposer. There are two types of SPEs: (a) SPEs in which someone rejects the offer at t=3 and (b) SPEs in which players reach an agreement at t=3.

In case (a), by the definition of the non-cooperative game $\Gamma(N, v)$ and the induction hypothesis, α obtains $v(\alpha)$ and each of the other players obtains his Shapley value of the game $(N\backslash\alpha, v^{\alpha})$.

By the induction hypothesis, each player $j \neq \alpha$ surely obtains $\phi_j(N\backslash\alpha, v^{\alpha})$ by rejecting the offer. Hence, in case (b), each player $j \neq \alpha$ obtains not less than $\phi_j(N\backslash\alpha, v^{\alpha})$. Thus, the proposer α obtains at most $v(\alpha)$ since $v(N) - \sum_{j\neq\alpha} \phi_j(N\backslash\alpha, v^{\alpha}) = v(N) - v^{\alpha}(N\backslash\alpha) = v(\alpha)$. But, the proposer α surely obtains $v(\alpha)$ when the offer is rejected. Hence, he must obtain $v(\alpha)$ in case (b). Therefore, the claim also holds in this case.

Claim 2: In any SPE, $B^i = \sum_{j\neq i} b^i_j - \sum_{j\neq i} b^j_i = 0$ for any $i \in N$.

Claim 3: In any SPE, each player's payoff is the same regardless of who is chosen as a proposer.

The above two claims are the same as Claim (c) and (d) of Pérez-Castrillo and Wettstein (2001), and are shown in the same manner, respectively.

Claim 4: In any SPE, the final payoff coincides with the Shapley value.

Let u^i_j be j's equilibrium payoff when the proposer is i at t=1. By Claim 1,

$$u^i_i = \sum_{k\neq i} b^i_k + v(i)$$

and for each $j \neq i$,

$$u^j_i = b^j_i + \phi_i(N\backslash j, v^j).$$

Thus,

$$\sum_{k\in N} u^k_i = \sum_{k\neq i} b^i_k + v(i) + \sum_{k\neq i} b^k_i + \sum_{k\neq i} \phi_i(N\backslash k, v^k).$$

By Claim 2, the above equality is equivalent to

$$\sum_{k\in N} u^k_i = v(i) + \sum_{k\neq i} \phi_i(N\backslash k, v^k).$$

By Claim 3, $\sum_{k\in N} u^k_i = nu^j_i$ for each $j \in N$. Therefore, for each $j \in N$,

$$u^j_i = \frac{1}{n} v(i) + \frac{1}{n} \sum_{k\neq i} \phi_i(N\backslash k, v^k).$$

By Proposition 3, the right-hand side of the above equality coincides with $\phi_i(N, v)$. □

6 Concluding remarks

As we mentioned before, the difference between our non-cooperative game and the bidding mechanism is what will happen when someone rejects an offer. That difference yields the difference of the implementability of the class of games. Our non-cooperative game implements the Shapley value as an unique equilibrium payoff on the class of all games, while the bidding mechanism works on a class of zero-monotonic games. The point is that in our non-cooperative game, players always divide the value $v(N)$, that is, if an offer made by proposer α is rejected at t=3, the proposer obtains $v(\alpha)$ and the remaining players divide the value $v^{\alpha}(N\backslash\alpha) = v(N) - v(\alpha)$ among them. Thus, the sum of the payoffs over all players is $v(N)$. On the other hand, in the bidding mechanism, the amount of the value that players obtain changes if someone rejects an offer, for example, if an offer made by proposer α is rejected at t=3, the proposer obtains $v(\alpha)$ and the remaining players divide the value $v(N\backslash\alpha)$ among them. Generally, $v(\alpha) + v(N\backslash\alpha) \neq v(N)$.

In situation in which players discuss how to allocate the outcome generated by cooperation *before* they decide to cooperate or not, the setting in the bidding mechanism is appropriate. On the other hand, in the situation in which players discuss how to allocate the outcome that has *already* been generated by cooperation, the setting in our non-cooperative game is appropriate.

When we consider the game in which cooperation among players generates a positive effect, the position of the proposer differs in our non-cooperative game and the bidding mechanism. In our non-cooperative game, once chosen as a proposer, he obtains the value of his stand alone coalition in any SPE since then. Whereas, in the bidding mechanism, once chosen as a proposer, he obtains the value of the grand coalition minus the value of coalition including all players except him in any SPE since then. In our non-cooperative game, being chosen as a proposer is undesirable and the interpretation of the bidding in SPEs is *compensation* of being chosen as a proposer, while in the bidding mechanism, being chosen as a proposer is desirable and the interpretation of the bidding in SPEs is *expenditure* of being chosen as a proposer.

Our result can be generalized to the Owen value of cooperative games with coalition structures (Owen (1977)). Vidal-Puga and Bergantiños (2003) generalized the bidding mechanism to implement the Owen value as an equilibrium payoff. However, their result works only on the class of strictly superadditive games. Appropriate generalization of our non-cooperative game can implement the Owen value on the class of all games.

Acknowledgment: The authors thank Rodica Branzei for her helpful comments.

References

Brink, van den, R., 2001. An axiomatization of the Shapley value using a fairness property. International Journal of Game Theory 30, 309–319.

Chun Y., 1989. A new axiomatization of the Shapley value. Games and Economic Behavior 1, 119–130.

Dubey P., 1975. On the uniqueness of the Shapley value. International Journal of Game Theory 4, 131–139.

Hamiache G., 2001. Associated consistency and Shapley value. International Journal of Game Theory 30, 279–289.

Hart S., Mas-Colell A., 1989. Potential, value and consistency. Econometrica 57, 589–614.

Maschler M., Owen G., 1989. The consistent Shapley value for hyperplane games. International Journal of Game Theory 18, 389–407.

Myerson R.B., 1980. Conference structures and fair allocation rules. International Journal of Game Theory 9, 169–182.

Neyman A., 1989. Uniqueness of the Shapley value. Games and Economic Behavior 1, 116–118.

Owen G., 1977. Value of games with a priori unions. In: R. Henn, O. Moeschlin (Eds.), Mathematical Economics and Game Theory. Springer-Verlag.

Pérez-Castrillo D., Wettstein D., 2001. Bidding for the surplus. Journal of Economic Theory 100, 274–294.

Shapley L.S., 1953. A value for n-person games. In: A.E. Roth (Ed.), The Shapley Value. Cambridge University Press, pages 41–48.

Vidal-Puga J., Bergantiños G., 2003. An implementation of the Owen value. Games and Economic Behavior 44, 412–427.

Young H., 1985. Monotonic solutions of cooperative games. International Journal of Game Theory 14 (2), 65–72.