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Valuation of Debt Indexed to Real Values I

The case of the Argentinean Growth Coupon: a Simple Model[1], [2]

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Abstract

This paper is the first of a series of works whose aim is trying to provide a framework for the understanding and valuation of debt indexed to real (generally non-tradable) variables. In particular, in the present paper we develop a methodology to analytically value the new GDP-linked Argentinean warrant.

[1]The views and opinions expressed in this publication are those of the authors and are not necessarily those of Universidad del CEMA.

[2]This work was first presented in a conference at Universidad Ort of Uruguay on April 7th, 2005. I would like to thank very useful comments made by the discussant Julio deBrun.

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1 Introduction

By January 2005, Argentina accumulated more than USD 100 billions of non-performing debt. On February 28$^{\text{th}}$ of this year, the biggest restructuring of a sovereign defaulted debt ended. Only USD 81.8 billions was recognized as eligible to be exchanged by new debt. The Argentinean offer included an original ingredient: for each new bond, creditors will receive a unit of GDP-linked warrants, best known as "GDP Kickers" (or just coupons for short). They are tied to the bonds for the first 180 days, but after that day they will trade independently. The consensus value as of March 2005 was about 2 cents. At those values the coupons already represented a market of USD 1.6 billions, and under some favorable circumstances they could imply payments by the Argentinean government of up to USD 40 billions.

The "coupons" will be emitted based on the USD 81.8 billions notional value (which is equal to the nominal residual value plus interests earned but unpaid until December 31$^{\text{th}}$, 2001), but only for the eligible debt actually exchanged.

The cash flow corresponding to year t (CF_t) depends on the behavior of the GDP in year t (P_t), but will be paid at the end of year $t+1$. More specifically, on December 15$^{\text{th}}$ of year $t+1$. The determination of the variables that dictate the actual amount to be paid will be made on November 1$^{\text{st}}$ of this same year ($t+1$). For example, the coupons start on year 2005, so if the cash flow corresponding to that year happens to be nonzero, the determination of the variables and the precise amount to be paid will be made on November 1$^{\text{st}}$ of 2006, and the first payment will be made on December 15$^{\text{th}}$ of 2006. We will refer to the year 2005 as $t = 1$ and to the year 2034 (last year of the life of the coupons) as $t = 30$ (the payment corresponding to the performance of year 2034 will be made effective on December of 2035, if it were nonzero).

Argentina will make a positive payment to the coupon owners in year $t{+}1$ (corresponding to the cash flow $\mathrm{CF_t}$) only if the following three conditions are met:

1. The average real GDP of year t, P_t, is greater than the base GDP of the same year (PB_t):

$$CF_t > 0 \Rightarrow P_t > PB_t, \quad \text{and if} \quad \mathrm{P_t \leq PB_t \Rightarrow CF_t = 0} \tag{1}$$

   (the cash flows are never negative).

2. The actual growth of the average real GDP in year t, $g_t = P_t/P_{t-1} - 1$, must be greater than the "base growth" (corresponding to the base GDP of that year and the previous one: $gb_t = PB_t/PB_{t-1} - 1$).

$$CF_t > 0 \Rightarrow g_t > gb_t, \quad \text{and if} \quad \mathrm{g_t \leq gb_t \Rightarrow CF_t = 0} \tag{2}$$

3. The sum of payments to the coupon owners should not be greater than a maximum value (MaxV) of 48 cents of the relevant currency.

$$CF_t > 0 \Rightarrow \sum_{i=1}^{t} CF_i < \mathrm{MaxV}. \tag{3}$$

The vector whose elements correspond to the base GDP, PB_t, from year 2005 to year 2034 and the vector of the corresponding "base growths" are given in Appendix A.

If these three conditions are met, the amount to be paid in year $t+1$ is given by the following formula:

$$CF_t = \frac{0.05 \cdot (P_t - PB_t) \cdot D_t \cdot u}{FX_t} \tag{4}$$

where D_t is the price index or GDP price index of year t (that transforms real pesos into current (year t) pesos) while u transforms the total amount to be paid into *per coupon* amount. If the GDP is measured in million pesos, u is given by Table 1 for the respective currencies. Finally, FX_t transforms the cash flow in pesos into the relevant currency. It

Table 1: u transforms the total payment to into per coupon payment.

Currency	u
USD	$1/81,800 = 0.0001222$
EUR	$1/81,800 * (1/0,7945) = 0.00015381$
ARP	$1/81,800 * (1/2,91750) = 0.00004189$

corresponds to the average exchange rate in the free market of pesos of the last 15 days of the reference year.

According to condition 3, the total amount per coupon to be paid during the life of the coupons will not be greater than MaxV. If such maximum limit is reached in a year before 2034, the coupons will be considered terminated this same year[4].

The Argentinean coupon constitutes an example of a debt instrument indexed to a real variable, and it has equity-like properties. The indexation of debt to real variables is not a new topic in the economic literature. For more than 20 years, authors have suggested tying debt payments to variables that are relatively endogenous to the fiscal sector, such as GDP (Shiller, [1], [2]; Obstfeld and Peri, [3]; Dreze, [4], [5]; and Varsavsky and Braun, [6]), exports (Bailey, [7]), and other variables (Lessard and Williamson, [8]). For instance, Barro [9] suggested the indexation of debt to public spending, highlighting the moral hazard of the government as well as the advantages of taking GDP as a reference. Other authors have suggested tying debt payments to exogenous variables, such as the price of exported commodities (see, for example, Krugman, [10]; Froot, Scharfstein, Stein [11]; Haldane, [12]; and Daniel, [13]). However, even though there is agreement on the advantages of this system, there has not been (with a few exceptions) a practical implementation of the idea

[4]If in a given year the total payment where greater than the amount necessary to reach the maximum limit, the Argentinean government will nevertheless honor that total payment.

until the Argentinean “coupon”.

There are few previous cases of sovereign debt tied to a real variable. One example is Bulgaria, which tied the payoff of its bonds to the level of its GDP[5],[6]. Similarly, Bosnia Herzegovina and Costa Rica indexed part of their debt to their GDP level. Mexico in 1986 used a contingency clause in a stand-by agreement with the IMF through which the authorities could demand greater financing from the World Bank and commercial banks if the GDP would fall below certain level. All these cases, however, where far smaller than the Argentinean case.

The literature in favor of indexing is large and well founded. Even from a simple, intuitive point of view, there are obvious advantages in indexing the sovereign obligations to the GDP and other variables: it constitutes a hedge against crises. It is likely, for example, than the Argentinean 2001 collapse would have been much smother, or even nonexistent, if its debt would have been indexed to the GDP. Moreover, as the Argentinean, the Russian, or many others sovereign defaults have taught us, the debt of emerging countries is in fact always contingent. Why, then, there have been so few and unimportant cases of explicitly contingent debt?

In an excellent survey on the subject, Borensztein and Mauro [14] (see also [15]) point out four reasons:

a. Greater volatility in the return of bonds that are already very volatile.

b. Lower incentives for the authorities to adopt pro-growth policies.

c. The possibility that the authorities declare lower growth than the real one.

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[5]One of the authors, SP, would like to thank Federico Sturzenegger for pointing this out.

[6]If the real Bulgarian GDP were to surpass a basis level (equivalent to 125% of its 1996 GDP) during the life of the bonds, the bondholders would receive 50bps of additional interests for each additional percentage point of growth after that year.

d. A complex architecture of difficult valuation.

Regarding point **a.**, it is true that the volatility of the bonds itself increases if their payment is tied to a real variable, but it is also true that there are many institutional investors that routinely invest in far more volatile instruments such as emerging markets equity, futures, options, etc. Some fixed income investors will not like very much these coupons, that is why Argentina decided to allow investors to untie such coupons from the bonds (as mentioned before, debt holders will be able to sell independently the GDP kickers 180 days after the bonds are received). The greater volatility will simply change the risk profile of the average investor in emerging debt.

Some authors have suggested that debt indexed to real variables reduces the incentives for the authorities to adopt pro-growth policies [10] (point **b.**). We believe that this does not apply to cases such as the Argentinean coupons, in which the real variable is GDP growth in the context of a democracy. National GDP growth is probably one of the economic variables that correlates most strongly with government electoral success, since greater growth typically implies less poverty, higher standards of living and social stability.

Far more important is the emergence of incentives for the authorities to sub-declare growth (point **c.**). However, if they were to juggle with the official data, the authorities would be playing with their reputation and this, in turn, would affect their access to capital markets. Additionally, the room for systematic data corruption is limited by the exhaustive and frequent monitoring of variables such as GDP growth by consultants, banks, and international organizations.

Another argument to downplay the role of moral hazard is the fact that there are already many financial instruments tied to another real variable: inflation. Inflation indexed debt exists in many countries, even emerging ones, and for it, the incentives to juggle with the data are identical to those present in the case of debt indexed to GDP growth. Yet this has not prevented the emergence of large markets for these instruments. We should note however that in the case of the Argentinean coupon there are some specific circumstances in which the incentives to alter GDP statistics are very high. These scenarios will be analyzed later in the present paper.

Finally, it is unfortunate that if we want to design debt tied to real variables that could serve as effective hedges to the issuer (and this is the most important reason to issue such debt), the resulting instrument will be intrinsically complex. Such complexity, and the consequent difficulty in valuating it, clearly do not contribute to the emergence of markets for it (point **d.**). In this respect, as we have seen, the Argentinean coupon is certainly not an example of simplicity. The difficulty and uncertainty when valuating new instruments such as the coupon typically translate into higher discount rates, which in turn makes them less attractive to the issuer. The potential lack of liquidity that could characterize complex products without predecessors could justify an even greater discount rate.

The magnitude of the GDP kickers emission in the Argentinean case could contribute to partially offset these problems. Moreover, Argentina grew in the last three years at a rate that doubles the minimum rate for the coupons to pay something. The payments adjust with inflation and it is inherently in local money in a country in which the consensus forecast is an appreciation of the real exchange rate of at least 25%. These features should promote the interest of investment banks in the coupon, and we presume that the international financial markets will take a close look at it, searching for better models of valuation and analysis.

The main purpose of this research program, of which the present work is just the first paper, is to provide a framework for the understanding and valuation of debt indexed to real (generally non-tradable) variables. Such type of debt could serve as effective hedges to prevent crises in emerging countries, crises such as the Argentinean one during $2001-2003$, in which the poverty level increased from 25% to 50% in just a few months.

We will use the Argentinean coupon as an example to develop the technology to value *any* potential debt indexed to real, non-tradable variables. We conjecture that any other future debt, issued by Argentina or other countries, will share with the coupons the imposition of contingency conditions (as we have seen, the coupons pay something if the long run performance of Argentinean economy is good, and if the performance during the specific year is also good). Such conditions make them useful hedges. That is why we make the effort to solve the problems associated with the imposition of these conditions in the valuation of the coupons starting always from general principles. These principles could apply equally well to other debt instruments indexed to real variables, and subject to different contingency conditions.

This work then has both an academic and a practical objective. Accepting from the existing literature the potential value and economic convenience of indexed debt, our academic objective is to contribute to eliminate factors **c.** and **d.** listed above. We believe these are the most important factors behind the present lack of markets for this type of debt. Our practical objective is to develop a methodology to value the Argentinean coupon, which we believe will constitute a sizable market. In this regard we would like to point out that in this paper we do not consider condition 3 on the coupon payments since we believe such condition is specific to Argentinean warrant and will not be shared by other debt indexed to real variables. Therefore only conditions 1 and 2 are analytically analyzed in this work.

The rest of the paper will be organized as follows. In the next section we provide some qualitative considerations that we believe are very important to value the warrants. In section 3 we propose a simple model for the GDP dynamics and value the coupon. In particular in subsection 3.1 we value the coupon applying the first condition only while in subsection 3.2 we value it imposing both the first and the second condition. The difference between these two valuations will give us insights into the relevance of the second condition and the inadequacy of a pure Black-Sholes approach for valuing the coupon. In section 4 we present the formulae of a more malleable model that allows much more flexibility to adjust parameters. In section 5 we study the dependence of the coupon valuation on several variables. We also present some econometric studies that will help us fix the parameters of the model according to well defined criteria. Finally, in section 6 we discuss our general findings and their implications, setting up the basis for the rest of our research program.

2 Qualitative Considerations

Currently practitioners value the coupon either with Monte Carlo simulations or simply by application of the Black-Scholes formula. It is not necessary to say much about the advantages of analytical valuations as opposed to Monte Carlo valuations. The insights one gains and the double checks one is able to do with analytical valuations are just not attainable with numerical simulations. Moreover, through an analytical approach we are going to be able to draw consequences for instruments with conditionalities beyond the specific one of the Argentinean GDP Warrant. Regarding the Black-Sholes formula, in section 3 we will analyze in detail its conceptual and practical limitations to value the coupon. For the time being, suffice it to say that it just give a wrong answers to the valuation of the coupons.

We will value the Argentinean coupon (with its first and second condition only) through a Present Value calculation, which for this type of sovereign debt, in our opinion requires three steps:

1. Calculation of the Expected Value of the cash flows.

2. An estimation of the rate at which these cash flows should be discounted.

3. A quantification of the consequences of the government's moral hazard.

We devote the present paper to the calculation of the expected value of the cash flows (step 1.). This requires in turn two steps:

a. The proposition of a stochastic process to model the GDP dynamics.

b. The proper calculation of the expected value of the cash flows for such stochastic process imposing the "boundary conditions" given by the contingency conditions 1, 2 and 3 for coupon payments (only the first two will be considered in this work).

In this paper we calculate the expected value of the cash flows for two stochastic processes as models of GDP dynamics. This step is probably the most challenging technically, given the nature of the conditions imposed over the coupon cash flows. In particular, the expected value of the cash flows calculation imposing the condition through which the coupon pays something only if the short run GDP growth exceeds a certain threshold (second condition), is, as far as we know, an original exercise. To our knowledge there is no other financial instrument with these characteristics.

As we said before, any other future debt indexed to real variables is likely to share with the coupons the imposition of contingency conditions of this type. That is what makes these instruments useful hedges. This is because, in an emerging country, the available money to pay creditors not only depends on the long term growth (first condition for the coupons to pay) but also in the short run performance, since the governments face strong pressure to spend unused money.

The stochastic processes analyzed in this work have the virtue of simplicity, which is very useful for developing the technology. In a different work, stochastic processes more closely based on the empirical evidence for the GDP growth of emerging economies will be analyzed. As it is shown in such work, although the level of technical difficulty increases considerably, the basic mathematical techniques to impose these conditions are conceptually the same.

Step 2., the estimation of the rate at which these expected values should be discounted, requires a separate paper presently in preparation. For the purpose of the valuation made in this paper, the discount rate is just a variable $r(t)$ to fix by hand. However two observations are relevant here to select a "reasonable" discount rate value:

a. The coupon cash flows are very insensitive to Argentina's default risk.

b. The coupon cash flows correlation with the returns of any reasonable global portfolio is basically zero.

By design the coupon pays a nonzero quantity only in scenarios in which Argentina grows at a compounded annualized rate of more than 3%. There is not a single case in history that we know of in which a country growing always at that rate suddenly defaults (point **a.**). This certainly does not mean that Argentina will not default in the next thirty years. But it does mean that if Argentina defaults on its sovereign bonds, it will do so in scenarios in which the coupons do not pay anything anyway[7]. So they are already accounted for in the calculation of the expected cash flows. The discount rate should not then adjust for this risk since it would be double counting.

The argument above does not mean that we should discount the coupon cash flows at a rate close to the risk free rate. This would not be correct simply because these cash flows

[7]This is not strictly correct, because in our calculations we are certainly including paths that, having fall by more than, say, 10%, grow again very fast and eventually pay again. We believe that those paths should not be considered because if an event of default happens, then we expect that the coupons will not pay anything after that even if the economy recovers strongly. In any case, the measure of such paths is very small in our stochastic processes. These issues will be analyzed in detail in a different paper.

are variable and people prefer a dollar to a $50-50$ bet of two dollars or zero. But a typical investor is interested in adding to her portfolio, instruments that increase the expected cash flows of the total portfolio and that reduce its volatility. A good measure of the contribution of the coupon to the volatility of the portfolio of such investor is given by the correlation (or β) of the coupon cash flows and the cash flows of the original portfolio (on this fact, among other things, are based Portfolio Theory and the CAPM, see for example [16]). The lower this correlation, the lower the volatility of the final portfolio. As pointed out in item **b.** above, it just happens to be the case that the correlation between the growth of the GDP of *any* emerging economy and the returns of *any* reasonable global portfolio such as for example the S&P 500 is basically zero[8].

The arguments of the two previous paragraphs point towards the notion that the rate at which we should discount the coupon cash flows should be close to the risk free rate. In any case, in section 5 we value the coupon for a whole range of discount rates, mostly including values far above the risk free rate.

The low correlation between the coupons and a global portfolio reflects a crucial argument for the convenience of debt indexed to real variables. As we mentioned before, this form of debt is a hedge on the part of Argentina against crises: it pays more if things go well and less if things go bad. From theory we know that risk management increases value only if the other party (the part that absorbs the risk that Argentina is hedging) has some advantage with respect to Argentina to absorb that risk. Why will the markets be better positioned than Argentina to absorb the risk of Argentinean crises? Because while

[8]In the arguments above we are disregarding the fact that the distribution of coupon cash flows is not Normal, even assuming that the distribution of GDP growth is so. Therefore the contribution of these cash flows to the volatility of the portfolio is not as simple as the β with respect to such portfolio. These issues will be analyzed in the mentioned work.

Argentina is "overexposed" to such risk, the market reduces volatility absorbing such risk (because of the low correlation of Argentinean growth with a global portfolio). This point deserves a much deeper analysis and as mentioned above a separate paper will be dedicated to these issues.

Finally, the third point (quantification of the moral hazard of the government) is obviously closely related to the previous one. It may happen that even though the Argentinean government honors its fixed obligations (so there is no default), it juggles with the statistics so as to pay less to the coupon owners. In yet another work this issue is being researched, analyzing in what scenarios it is likely that the government will be tempted to do this, how much it has to win in these scenarios and with what probability these scenarios will be realized. If we know all these quantities we could simply subtract its expected value to the expected cash flows.

In this line, we have identified one family of scenarios in which the incentives to juggle with the statistics are high. Suppose that the country grew for various years at a high rate, much higher than the base rate for the corresponding years, but that the year in consideration the growth is about the base growth for that year, say 3%. Then, an official growth of 2.99% implies that that year the payment is zero while if the official growth is 3.01%, the payment might be very large. In this scenario the incentives to distort the statistics are high. To quantify its effect in today coupon's price one needs to know the probability of these events and the losses if the statistics are distorted in such events. This kind of analysis is made in the separate work already mentioned.

This highlights a design defect of the coupons. Their payments are a continuous and relatively smooth function of the long run performance of the economy (condition 1), but a discontinuous function of the short run performance (condition 2). Moreover, as just pointed out, in some scenarios such discontinuity could be big, and this is what makes the coupons sensitive to moral hazard. In the work analyzing these issues different functions for the payoffs that avoid these problems will be proposed.

3 The Model

Consider the following geometric stochastic process as a model for a country GDP dynamics:

$$dP_t = \mu P_t dt + \sigma P_t dW_t \tag{5}$$

where μ, the drift, represents the geometric mean growth rate, and σ is the instantaneous standard deviation of such growth. dW_t represents the stochastic term.

Equation (5) as a model of a country GDP dynamics has some drawbacks, specially for an emerging country: one would not expect the mean growth to remain constant over a period of thirty years, and even less so for the volatility. To partially address this issue, in section 4 we present the solution of a time dependent version of the model where both μ and σ are (known a priori) functions of time. As we will see, the time dependent version is much more flexible and will allow to fix constants of the model according to diverse criteria. Moreover, as already mentioned, in a work still in preparation the pricing of the coupon is made with an underlying model of GDP dynamics inspired in econometric data. It is a mean reverting model of the type used among others by Vasicek [18] to model term structure dynamics.

In any case, (5), considered as a toy model, is an interestingly simple model to use in order to develop techniques to value the coupon. It was already proposed by Okseniuk [17] precisely for that purpose. Unfortunately the mathematical treatment in the mentioned work is just wrong.

In terms of the natural logarithm of the GDP we have, according to Ito's lema, the well known result

$$d\ln(P_t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t \tag{6}$$

in which the logarithmic growth is affected by the geometric volatility of the process. Instead of (5), from where we derive (6), we could start directly with

$$d\ln(P_t) = \eta dt + \sigma dW_t \tag{7}$$

Both models are of course equivalent since going from model (5) (or (6)) to model (7) simply amounts to the relabeling $\eta = \mu - \sigma^2/2$. We will generally use the notation in equation (7).

3.1 The First Condition

We need to obtain the probability distribution that such model implies for the logarithm of the GDP for each year over the next thirty years. Such probability depends, of course, of the boundary conditions. In our case these conditions reduce to impose that at time $t = 0$ we have the certainty that the GDP is P_0 (i.e., Argentinean GDP by the end of 2004). Mathematically, we demand that the probability distribution of $\ln(P_t/P_0)$, $p(\ln(P_t/P_0) = x)$, converges to Dirac's delta function $\delta(x)$ as $t \longrightarrow 0$.

The probability distribution compatible with (7) and the stated boundary condition is known as the **fundamental solution** of the stochastic process. For our case it is a Normal with mean ηt and variance $\sigma^2 t$. This is a well known result. However, for future reference we point out that one way to obtain such distribution (also known as the *transition probability*) is to derive for the proposed stochastic process the so-called **Kolmogorov's forward** or the **Fokker-Planck equation** (i.e., the partial differential equation for the probability distribution itself), and solve it with the mentioned boundary condition. For the process (7), such equation is

$$\frac{\partial p}{\partial t} + \eta \frac{\partial p}{\partial x} - \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2} = 0 \tag{8}$$

The reader can check that a Normal with mean ηt and variance $\sigma^2 t$, i.e., the function

$$p(\ln(P_t/P_0) = x_t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left[-\frac{(x_t - \eta t)^2}{2\sigma^2 t}\right] \tag{9}$$

satisfies this equations and converges to Dirac's delta function as $t \longrightarrow 0$.

Equation (9) is then the desired probability distribution of the logarithm of the GDP at time t, P_t, consistent with the stochastic process (7). In this section we want to calculate the expected value of the coupon cash flows imposing the condition 1 only. It amounts, according to equation (4), to the calculation of the expected value with the probability distribution (9), of the function

$$f_t(x_t) = \frac{0.05 \cdot D_t \cdot u}{FX_t} \cdot (P_0 e^{x_t} - PB_t)\, \theta\left(x_t - \ln\left(\frac{PB_t}{P_0}\right)\right) \tag{10}$$

where $P_0 e^{x_t} = P_t$, so that x_t is precisely the variable $\ln(P_t/P_0)$, of which we have in (9) the probability distribution. $\theta(x_t - \ln(PB_t/P_0))$ represents the step function, equal to zero if $x_t < \ln(PB_t/P_0)$ (or if $P_t < PB_t$) and to one otherwise (this is the first condition).

The expected payment at time $t+1$ (cash flow t) is then

$$\begin{aligned}
E_p(CF_t) &= \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} f_t(x_t) \exp\left[\frac{(x_t - \eta t)^2}{2\sigma^2 t}\right] dx_t && (11)\\
&= \frac{0.05 \cdot D_t \cdot u}{FX_t} \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{\ln(PB_t/P_0)}^{\infty} (P_0 e^{x_t} - PB_t) \exp\left[-\frac{(x_t - \eta t)^2}{2\sigma^2 t}\right] dx_t && (12)\\
&= \frac{0.05 \cdot D_t \cdot u}{FX_t} \left[P_0 e^{(\eta + \sigma^2/2)t} \cdot N(d_1) - PB_t \cdot N(d_2)\right] && (13)
\end{aligned}$$

where in the last line

$$\begin{aligned}
d_1 &= \frac{\ln(P_0/PB_t) + (\eta + \sigma^2)t}{\sigma\sqrt{t}} && (14)\\
d_2 &= d_1 - \sigma\sqrt{t} && (15)
\end{aligned}$$

It is instructive to rewrite these equations in the notation of equation (6) (with $\eta = \mu - \sigma^2/2$), discounting at the risk free rate r, and omitting for the purpose of this exercise the fact that the payment is actually realized in year $t+1$ (i.e. assuming that it is realized in year t). With these assumptions, from (11), (14) and (15), the present value of the cash flow t *imposing condition* 1 *only* is:

$$PV(CF_t) = \frac{0.05 \cdot D_t \cdot u}{FX_t} \left[P_0 e^{(\mu - r)t} \cdot N(d_1) - PB_t e \cdot N(d_2) \right] \tag{16}$$

with

$$d_1 = \frac{\ln(P_0/PB_t) + (\mu + \sigma^2/2)t}{\sigma\sqrt{t}} \tag{17}$$

$$d_2 = d_1 - \sigma\sqrt{t} \tag{18}$$

Apart for the pre-factor $0.05 \cdot D_t \cdot u/FX_t$, which is specific of the coupon, equation (16) is formally very similar to the famous Black-Scholes formula for a call option with a strike price PB_t and an actual price of the "stock" equal to P_0. Actually, if we make $\mu = r$ it is identical to it. There is a reason for this. If we consider condition 1 only, each payment of the coupon is actually a call option on the real GDP. The fact that the real expected growth μ appears in these formulas instead of the risk free rate r is reflecting the fact that the real GDP is not a tradable. If it was a tradable (in a perfect market, etc.) we could have constructed a risk free portfolio out of shares of the GDP and the coupon. This procedure would have implied that the expected value should have to be computed with the risk neutral measure (where $\mu = r$) instead of computing it with the real probability as we did in (11).

As the GDP is not a tradable, however, not only we should take expected values with the real measure instead of the risk free measure, but there is no reason a priori to think that the discount rate r should be the risk free rate. As we mentioned in section 2, there may be other reasons to justify discounting the coupon at a rate close to the risk free rate, but that requires more investigation.

3.2 The Second Condition

As we have just seen, the imposition of the first condition is easy enough, since it just amounts to a replication (in the real measure instead of the risk free measure) of the Black-Scholes analysis for a non-tradable. The additional imposition of the second condition (that the actual growth of the average real GDP in year t must be greater than the "base growth" of that year) is, as far as we know, an original exercise, since there is no other financial instrument to our knowledge with these characteristics. As mentioned in the Introduction, this condition is essential to make the coupons useful hedges.

That is why we make the effort to solve the problems associated with the imposition of this condition starting always from general principles. These principles could apply equally well to other debt instruments indexed to real variables, and subject to different contingency conditions.

To impose both the first and second condition simultaneously in the calculation of the expected value of the cash flows, and moreover, to do so in a way that extends to other models as well, as it is the explicit purpose of this paper, it is convenient to remember some basic concepts of stochastic calculus.

The process (5) (or (7)) is a Markov process, i.e., one in which if we know the state of the system at a particular time t_0 (the present), additional information about the states of the system at earlier times $t < t_0$ (the past) has no effect on our knowledge of the probable development of the system at $T > t_0$ (the future). Of course one may question whether a Markov process is appropriate to model GDP growth, but we already discussed this issue. It is enough to add here that even a non-Markov process (in which for example the behavior of the GDP a few years back can have an effect over the growth today) can be transformed into a Markov one by appropriately formally extending the space over which such process is defined. This is precisely what will be done when analyzing a more realistic model in a future paper.

For general Markov processes it is valid the so-called Chapman-Kolmogorov equation for the fundamental solution (that in our case is (9)). Calling $X_t = \ln(P_t/P_0)$, such equation for our process takes the form

$$p(X_t = x_t|X_0 = 0) = \int_{-\infty}^{\infty} p(X_t = x_t|X_s = x_s)p(X_s = x_s|X_0 = 0)dx_s \tag{19}$$

where

$$p(X_t = x_t|X_s = x_s) = \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} \exp\left[-\frac{(x_t - x_s - \eta t)^2}{2\sigma^2(t-s)}\right] \tag{20}$$

is known as the *transition probability* for the process. The reader can explicitly check the validity of (19).

In imprecise language, equation (19) means that for the stochastic process under consideration, the probability of taking the value x_t at time t starting from zero at time $t = 0$ is equal to the probability of taking the value x_s at the intermediate time s starting from zero at time $t = 0$, times the probability of going from x_s to x_t in the interval of time between s and t, summed over all intermediate x_s values.

From the Chapman-Kolmogorov equation one can derive a fundamental theorem which marks the significance of the transition probabilities for Markov processes: all finite dimensional distributions can be obtained from them and from the initial distribution at time t_0. More precisely, we have the following theorem:

Theorem. If X_t is a Markov Process in the time interval $[t_0, T]$, if $p(X_t = x_t|X_s = x_s)$ is its transition probability, and if $f(x_0)$ is the distribution at t_0, then, for finite dimensional distributions such as

$$p(X_{t_1} \in B_1, \ldots, X_{t_{n-1}} \in B_{n-1}, X_{t_n} = x_n), \quad t_0 \le t_1 < \cdots < t_n \le T, \tag{21}$$

we have

$$\begin{aligned} p(X_{t_1} \in B_1, \ldots, X_{t_{n-1}} \in B_{n-1}, X_{t_n} = x_n) &= \int_{B_{n-1}} \cdots \int_{B_1} \int_{-\infty}^{\infty} dx_{n-1} \ldots dx_1 dx_0 \\ p(X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1}) &\cdot p(X_{t_{n-1}} = x_{n-1} | X_{t_{n-2}} = x_{n-2}) \\ \ldots p(X_{t_2} = x_2 | X_{t_1} = x_1) &\cdot p(X_{t_1} = x_1 | X_{t_0} = x_0) \cdot f(x_0) \end{aligned} \tag{22}$$

For a proof see for example [19].

The theorem is in fact more general than expressed above, but as it is in (22) is enough for our purposes. In words, it says that the probability distribution for the stochastic process of being at x_n at time t_n starting at x_0 at time t_0 with probability $f(x_0)$, having been in the set B_1 at time t_1, in the set B_2 at time t_2,..., and in the set B_{n-1} at time t_{n-1}, can be calculated form the transition probability. It is given by the product of the initial distribution $f(x_0)$ times the transition probability of going from x_0 to x_1 between times t_0 and t_1,..., times the transition probability of going from x_{n-1} to x_n between times t_{n-1} and t_n, integrated over all x_0, over x_1 in the set B_1,..., and over x_{n-1} in the set B_{n-1} (see figure 1).

Equipped with the Markov property of the stochastic process (7), the Chapman-Kolmogorov equation (19), and the theorem above, we can now devise a strategy to impose conditions 1 and 2 simultaneously. The basic strategy is to compute the mean value of the cash flows (10) (which incorporates condition 1) with a distribution that only includes paths in which the growth over the last year is at least bg_t (in order to incorporate condition 2). Considering GDP paths that end up at time t with a GDP P_t having grown during the last year at least bg_t, is equivalent to considering paths in the $x_t = \ln(P_t/P_0)$ space that end up at time t at x_t having been at time $t' = t-1$ at a "position" $x_{t'} = x_t - \ln(1+bg_t)$ or lower. To see this simply note that if the growth was at least bg_t, then $P_t/P_{t'} \geq 1+bg_t$. Therefore $\ln(P_t/P_{t'}) \geq \ln(1+bg_t)$. But $\ln(P_t/P_{t'}) = \ln(P_t/P_0) - \ln(P_{t'}/P_0)$,

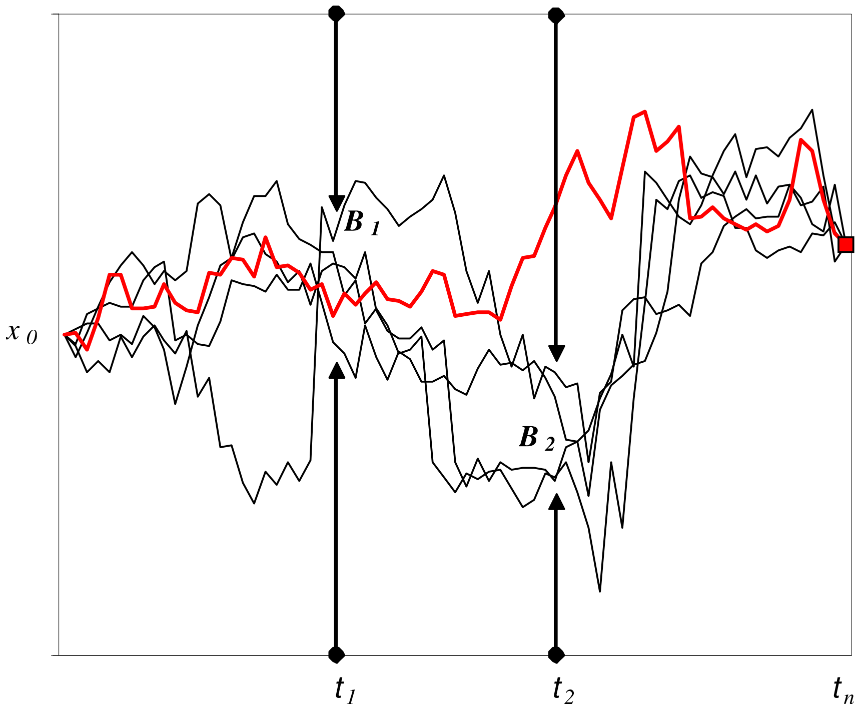

Figure 1: In equation (22) we are including the black (thin) paths that go through both windows, but not paths like the red (thick) one since it does not pass through window 2.

so $\ln(P_{t'}/P_0) \leq \ln(P_t/P_0) - \ln(1+bg_t)$. This is exactly what we wanted to prove. For simplicity of notation let us call cbg_t (*continuous* base growth) to $\ln(1+bg_t)$.

So our problem is to find the measure that only incorporates paths for which $x_{t'} \leq x_t - cbg_t$. But this condition is of the form of the windows considered in the general theorem encapsulated in equation (22). From that theorem then, the desired distribution is

$$p(X_t = x_t, X_{t'} \in B_{t'}, X_0 = 0) = \int_{-\infty}^{x_t - cbg_t} p(X_t = x_t | X_{t'} = x_{t'}) p(X_{t'} = x_{t'} | X_0 = 0) dx_{t'} \quad (23)$$

where the upper limit of integration ensures that $x_{t'}$ belongs to $B_{t'}$, defined as

$$B_{t'} = \{x_{t'} : x_{t'} \leq x_t - cbg_t\} \quad (24)$$

see figure 2.

The mean value of the cash flow t with both conditions 1 and 2 is formally,

$$\begin{aligned} E(CF_t) &= \int_{-\infty}^{\infty} f_t(x_t) p(X_t = x_t, X_{t'} \in B_{t'}, X_0 = 0) dx_t \quad (25) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{x_t - cbg_t} f_t(x_t) p(X_t = x_t | X_{t'} = x_{t'}) p(X_{t'} = x_{t'} | X_0 = 0) dx_{t'} dx_t \end{aligned}$$

To go from (25) to an explicit expression for the expected value it is convenient to go back and find an explicit expression for the probability distribution (23). Remembering that the transition probability is given by (20) and calculating the integral, one finds

$$\begin{aligned} p(X_t = x_t, X_{t'} \in B_{t'}, X_0 = 0) &= \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left[-\frac{(x_t - \eta t)^2}{2\sigma^2 t}\right] N[D(x_t)] \quad (26) \\ &= p(X_t = x_t | X_0 = 0) N[D(x_t)] \quad (27) \end{aligned}$$

where $N[D(x_t)]$ refers to the Normal Cumulative Distribution of $D(x_t)$, which in turn is given by

$$\begin{aligned} D(x_t) &= \frac{1}{\sqrt{\sigma^2 (t-t') t t'}} (x_t (t - t') - cbg_t t) \quad (28) \\ &= \sqrt{\frac{(t-t')t}{\sigma^2 t'}} (crag_t - cbag_t) \quad (29) \end{aligned}$$

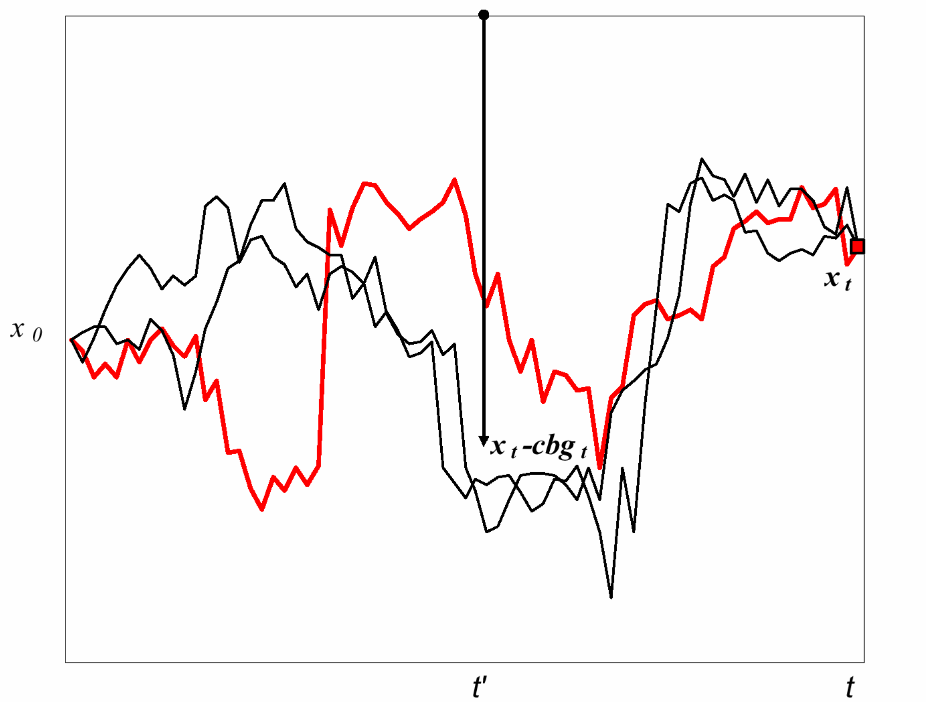

Figure 2: The imposition of condition 2 implies that only those paths that grow during the last year more than cbg_t are counted. For example, the black (thin) paths are counted while the red (thick) is not.

The variables appearing in line (28) have all been defined before: cbg_t is the continuous base growth for year t (generically, the base growth for the period $t - t'$), while x_t was defined by the relation $P_t = P_0 e^{x_t}$ (P_t is the GDP in year t). In line (29) we expressed these two quantities in annualized terms: $cbg_t = cbag_t \cdot (t - t')$ ($cbag$ stands for "continuous base annualized growth"), while $x_t = crag_t \cdot t$ ($crag$ stands for "continuous real annualized growth").

In (27), together with equation (29), we have a neat expression for the probability density of being at x_t at time t having started at $x_0 = 0$ at time $t_0 = 0$, and having grown at least cbg_t during the period between t' and t. It tells us that such density is equal to the probability density without condition 2 (i.e. the probability density of the GDP growth) times the Normal Cumulative Distribution of a variable proportional to the difference between continuous real annualized growth and the continuous base annualized growth during the period between t' and t. Such Normal Cumulative Distribution encapsulates the effect of the additional imposition of condition 2.

With the measure (26) (or (27)), together with equations (10) and (25) we can obtain an explicit expression for the expected value of the coupon cash flows subject to conditions 1 and 2. Such expression is:

$$E(CF_t) = \mathcal{F}_t \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{\ln(PB_t/P_0)}^{\infty} (P_0 e^{x_t} - PB_t) \exp\left[-\frac{(x_t - \eta t)^2}{2\sigma^2 t}\right] N[D(x_t)] dx_t \quad (30)$$

$$= \mathcal{F}_t [P_0 e^{(\eta + \sigma^2/2)t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-z^2/2} N(D_1(z)) dz \quad (31)$$

$$-PB_t \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-z^2/2} N(D_2(z)) dz] \quad (32)$$

where $\mathcal{F}_t = \frac{0.05 \cdot D_t \cdot u}{FX_t}$ and in line (30), $D(x_t)$ is given by equation (28). In lines (31) and (32), d_1 and d_2 are given by (14) and (15) respectively, while $D_1(z)$ and $D_2(z)$ are given by

$$D_1(z) = \sqrt{\frac{t - t'}{t'}} z + \sqrt{\frac{t(t - t')}{t'}} \left(\frac{\eta}{\sigma} + \sigma\right) - \sqrt{\frac{t}{(t - t')t'}} \frac{cbg_t}{\sigma} \quad (33)$$

$$D_2(z) = D_1(z) - \sqrt{\frac{t(t-t')}{t'}}\sigma \quad (34)$$

and $N(D_j(z))$ refers to the Normal Cumulative Distribution of the respective variables. Remember that by imposing condition 1 only we obtained as a result expected values a la Black-Scholes but in the real measure instead of the risk neutral measure (equation (13)). In lines (31) and (32) we can then appreciate the effect of imposing condition 2 in modifying the Black-Scholes result (in (13), $N(d_j) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_j} e^{-z^2/2} dz$). Such equations represent the main result of this paper.

Three comments are in order relative to the measure (27) and the expected value (30):

1. Expression (27) shows that the imposition of condition 2 is quantitatively strong. From expression (29) we see that for trajectories where the real annualized growth is equal to the base annualized growth, condition 2 modifies the results of condition 1 alone by a factor of 1/2 ($N[0] = 1/2$).

2. We see from (27) and (28) that in the limit where $cbg_t \longrightarrow -\infty$, condition 2 is irrelevant ($N[\infty] \longrightarrow 1$) and only condition 1 remains binding. This is as it should be, since in that limit condition 2 effectively demands that during the period $t - t'$ the growth should be greater than $-\infty$, which will happen with probability one.

3. During the first year (where $t = 1$ and $t' = 0$), the first condition says that to have a positive cash flow it is necessary that $P_1 > PB_1$, or equivalently $x_1 = \ln(P_1/P_0) > \ln(PB_1/P_0)$. On the other hand, the second condition says that to have a positive cash flow it is necessary that $g_1 > bg_1$, or equivalently $P_1/P_0 > PB_1/PB_0$, which in turn implies $x_1 = \ln(P_1/P_0) > \ln(PB_1/PB_0) = cbg_1$. In another words, if $P_0 > PB_0$, condition 2 is binding and condition 1 is irrelevant, while if $P_0 < PB_0$ only condition 1 is binding and condition 2 is irrelevant. Is our calculation consistent with this fact?[9]

[9] In reality $PB_0 = 275276.01$ million pesos while $P_0 = 279141.3$ million pesos, so that condition

From expression (30) for the expected value of the cash flows and (28) for $D(x_t)$, we see that as $t' \longrightarrow 0^+$ (from the right, since the process is defined for $t \geq 0$), $D(x_t) \longrightarrow \infty$ for paths ending in an x_t greater than the base growth for the first year cbg_1, and to $D(x_t) \longrightarrow -\infty$ for paths ending in an x_t smaller than cbg_1. So $N[D(x_t)] \longrightarrow 1$ for $x_1 > cbg_1$ and $N[D(x_t)] \longrightarrow 0$ for $x_1 < cbg_1$. Therefore the integrand in (30) is zero for values of $x_1 < \ln(PB_1/PB_0)$ (condition 2). However, due to the lower limit of integration in (30) (which constitutes condition 1), condition 2 is irrelevant if $P_0 < PB_0$. Conversely, if $P_0 > PB_0$ the lower limit of integration (condition 1) becomes irrelevant because the integrand is still zero for values of x_1 greater than $\ln(PB_1/P_0)$ but smaller than $\ln(PB_1/PB_0)$. This is exactly what we wanted to prove.

Finally, since the discounted cash flow t is simply $E(CF_t) \cdot \exp(-r_{t+1}(t+1))$, the present value of the coupon is

$$\begin{aligned} PV(\text{coupon}) &= \sum_{t=1}^{30} \mathcal{F}_t e^{-r_{t+1}(t+1)} \frac{1}{\sqrt{2\pi\sigma^2 t}} \\ &\quad \cdot \int_{\ln(PB_t/P_0)}^{\infty} (P_0 e^{x_t} - PB_t) \exp\left[-\frac{(x_t - \eta t)^2}{2\sigma^2 t}\right] N[d(x_t)] dx_t \\ &= \sum_{t=1}^{30} \mathcal{F}_t e^{-r_{t+1}(t+1)} [P_0 e^{(\eta+\sigma^2/2)t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-z^2/2} N(D_1(z)) dz \\ &\quad - PB_t \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-z^2/2} N(D_2(z)) dz] \end{aligned} \tag{35}$$

4 Time Dependent Model

Consider a model similar to (7) but where η and σ are (known a priori) functions of time:

$$d\ln(P_t) = \eta(t)dt + \sigma(t)dW_t \tag{36}$$

2 is binding and condition 1 is irrelevant. In any case the formula has to work both ways.

(alternatively we could have started in the notation of equation (5). As in section 3, the relationship between notations is given by $\eta(t) = \mu(t) - \sigma^2(t)/2$).

The Kolmogorov's forward equation is, as in (8),

$$\frac{\partial p}{\partial t} + \eta(t)\frac{\partial p}{\partial x} - \frac{1}{2}\sigma^2(t)\frac{\partial^2 p}{\partial x^2} = 0 \tag{37}$$

and the fundamental solution is

$$p(X_t = x_t | X_s = x_{t_s}) = \frac{1}{\sqrt{2\pi \int_{t_s}^{t} \sigma^2(u)du}} \exp\left[-\frac{(x_t - x_s - \int_{t_s}^{t} \eta(u)du)^2}{2\int_{t_s}^{t} \sigma^2(u)du}\right] \tag{38}$$

where $X_t = \ln(P_t/P_0)$. As $t \longrightarrow t_s^+$ (38) converges to Dirac's delta function $\delta(x_t - x_s)$.

The Chapman-Kolmogorov equation is of course valid for our time dependent process, so the distribution (23) with $B(t')$ given in equation (24) imposes condition 2 in the time dependent model as well (but with $p(X_t = x_t|X_s = x_{t_s})$ given in (38) instead of (20)).

An explicit expression for the distribution of being at x_t at time t having started at $x_0 = 0$ at time $t_0 = 0$, and having been at $x_{t'}$ at time $t' = t - 1$ with $x_{t'}$ belonging to the set $B_{t'}$ (given in equation (24)), i.e., satisfying condition 2, is given by

$$p(X_t = x_t, X_{t'} \in B_{t'}, X_0 = 0) = \frac{1}{\sqrt{2\pi \int_0^t \sigma^2(u)du}} \exp\left[-\frac{(x_t - \int_0^t \eta(u)du)^2}{2\int_0^t \sigma^2(u)du}\right] N[D(x_t)] \tag{39}$$

instead of (26). In turn, $D(x_t)$ is given by

$$\begin{aligned} D(x_t) \quad &= \quad \left[\frac{\int_0^t \sigma^2(u)du}{\int_{t'}^t \sigma^2(u)du \cdot \int_0^{t'} \sigma^2(u)du}\right]^{1/2} \\ &\cdot \left[\frac{\int_{t'}^t \sigma^2(u)du}{\int_0^t \sigma^2(u)du} x_t - cbg_t + \int_{t'}^{t} \eta(u)du \frac{\int_0^{t'} \sigma^2(u)du}{\int_0^t \sigma^2(u)du} - \int_0^{t'} \eta(u)du \frac{\int_{t'}^t \sigma^2(u)du}{\int_0^t \sigma^2(u)du}\right] \end{aligned} \tag{40}$$

instead of (28). Naturally, both (39) and (40) reduce respectively to (26) and (28) in the case in which both $\sigma(t)$ and $\eta(t)$ are constants.

From (39) we can compute the expected cash flows.

$$E(CF_t) \quad = \quad \mathcal{F}_t \frac{1}{\sqrt{2\pi \int_0^t \sigma^2(u)du}} \int_{\ln(PB_t/P_0)}^{\infty} (P_0 e^{x_t} - PB_t) \exp\left[-\frac{(x_t - \int_0^t \eta(u)du)^2}{2\int_0^t \sigma^2(u)du}\right] N[D(x_t)]dx_t$$

$$= \mathcal{F}_t[P_0 e^{\int_0^t (\eta(u)+\sigma^2(u)/2)du} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-z^2/2} N(D_1(z))dz \quad (41)$$

$$-PB_t \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-z^2/2} N(D_2(z))dz \quad (42)$$

where $\mathcal{F}_t = \frac{0.05 \cdot D_t \cdot u}{FX_t}$ and in the first line $D(x_t)$ is given by (40). (41) and (42) are formally very similar to the corresponding equations of the time independent model ((31) and (32)), however d_1, d_2, D_1 and D_2 now mean

$$d_1 = \frac{\ln(P_0/PB_t) + \int_0^t (\eta(u) + \sigma^2(u))\, du}{\left(\int_0^t \sigma^2(u)du\right)^{1/2}} \quad (43)$$

$$d_2 = d_1 - \left(\int_0^t \sigma^2(u)du\right)^{1/2} \quad (44)$$

and

$$D_1(z) = \sqrt{\frac{\sigma^2_{t't}}{\sigma^2_{0t'}}}\, z + \sqrt{\frac{\sigma^2_{t't}}{\sigma^2_{0t'}}} \left(\frac{\eta_{0t}}{\sqrt{\sigma^2_{0t}}} + \sqrt{\sigma^2_{0t}}\right) + \sqrt{\frac{\sigma^2_{0t}}{\sigma^2_{0t'}\sigma^2_{t't}}} \left(-cbg_t + \eta_{t't}\frac{\sigma^2_{0t'}}{\sigma^2_{0t}} - \eta_{0t'}\frac{\sigma^2_{t't}}{\sigma^2_{0t}}\right) \quad (45)$$

$$D_2(z) = D_1(z) - \sqrt{\frac{\sigma^2_{0t}\sigma^2_{t't}}{\sigma^2_{0t'}}} \quad (46)$$

where the symbols σ^2_{ab} and η_{cd} mean

$$\sigma^2_{ab} \equiv \int_a^b \sigma^2(u)du \quad (47)$$

$$\eta_{cd} \equiv \int_c^d \eta(u)du \quad (48)$$

From the expression of the expected value of the cash flows in equations (41) and (42) it is a trivial matter to give an expression for the present value of the coupon:

$$PV(\text{coupon}) = \sum_{t=1}^{30} e^{r_{t+1}(t+1)} E(CF_t) \quad (49)$$

5 Sensitivity Analysis

In this section we use the model presented in the previous one to analyze how the GDP-warrant depends on the various parameters, and we value the instrument for different alternative scenarios. The evaluations are made for the GDP-warrant in pesos, and assume a base real GDP for 2004 of $275,276.01$ million pesos (the base GPD for years 2005 to 2034 are in Appendix A). The real Argentinean GDP for 2004 was $279,141.3$ million pesos, note that this number is greater than the base GDP for that year. The GDP price index for 2004 is 1.604.

Let us first see how the GDP warrant depends on the assumed growth level for the future years: in figure 3 we see the value of the GDP Warrant for different assumptions about the real GDP growth during $2006-2034$. During 2005 a growth of 7.5% with a volatility of 1%, and a 10% inflation is assumed. The real discount rate is assumed at 6% and the volatility for years $2006-2034$ is assumed at 3%. As can be appreciated, the value of the GDP is extremely sensitive to the assumed growth rate for years 2006 to 2034.

In figure 4 we display the value of the real GDP against volatility.

The dependence of the value of the GDP Warrant as a function of the real discount rate is explored in figure 5.

Finally, in table 2 the value of the GDP Warrant is calculated for values of the growth of the real GDP (μ) and the volatility of such growth (σ) corresponding to sets of countries that may be representative of the behavior of the Argentinean economy during the next 30 years. The sets are: "ELGC" stands for Extremely Low Growth Countries, and they correspond to the sets of countries that during the las 50 years have gown less. Unfortunately Argentina belonged to this group. Will Argentina's GDP growth during the next 30 years continue to be representative of this group? "LATAM" refers to the group of Latin American countries, to which Argentina naturally belongs. "WORLD" refers to the group of all the countries

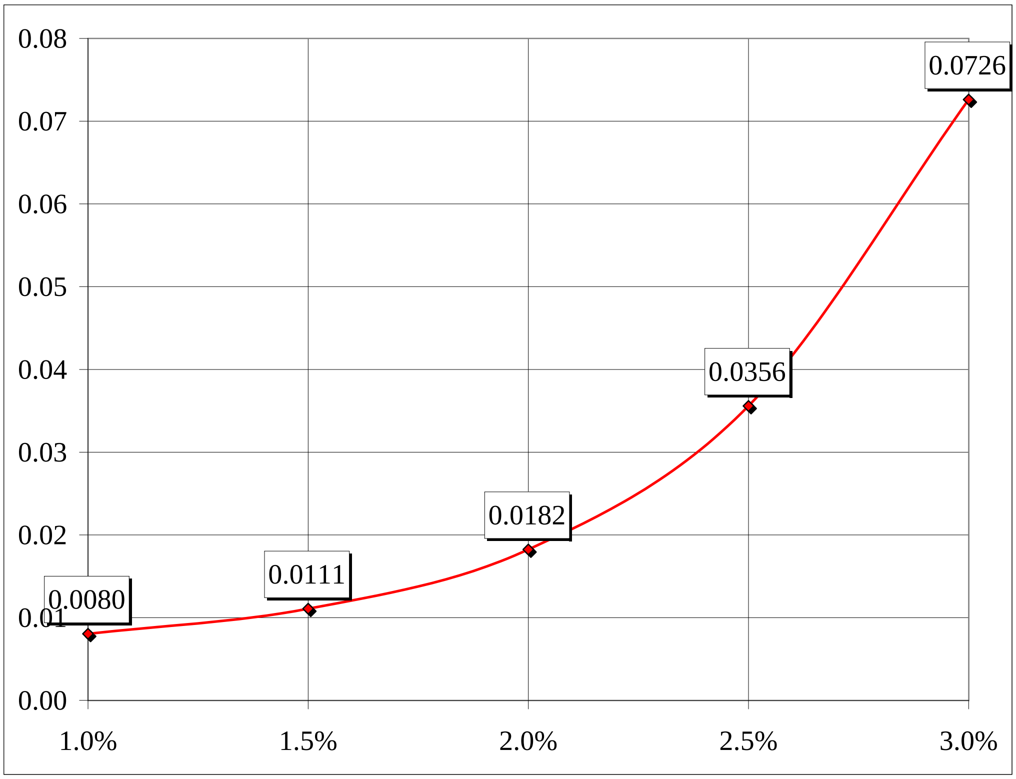

Figure 3: Value of the DGP Warrant for different assumptions about the real GDP growth during $2006-2034$. During 2005 a growth of 7.5% with a volatility of 1% and a 10% inflation is assumed. The real discount rate is assumed at 6% and the volatility for years $2006-2034$ is assumed at 3%

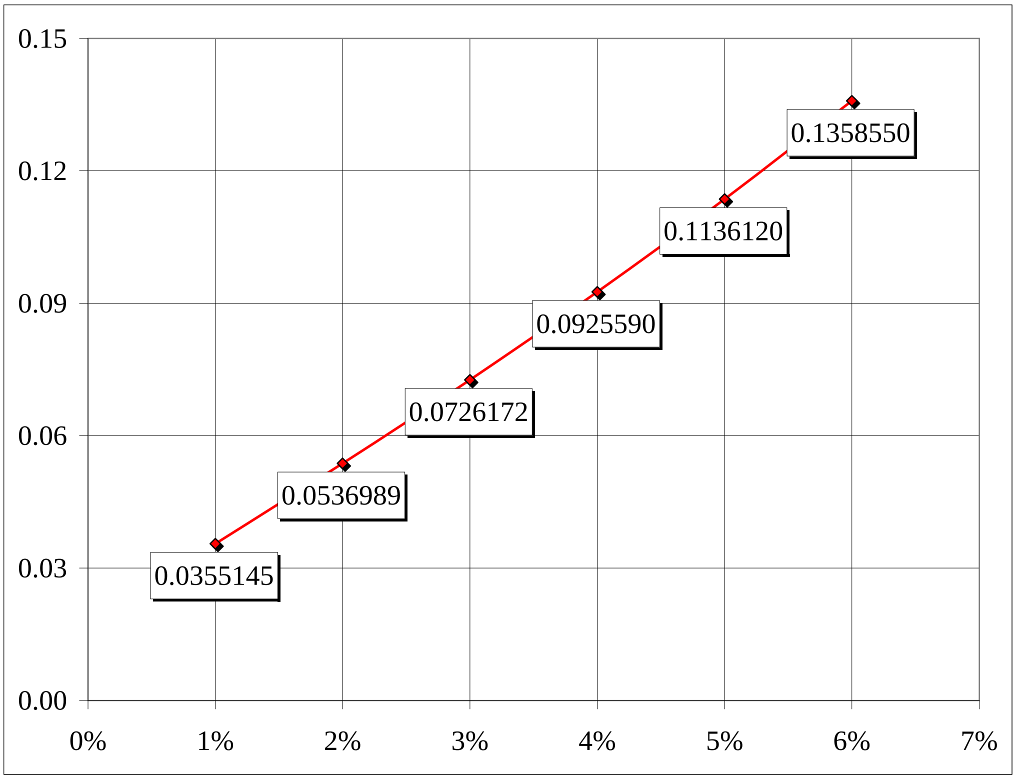

Figure 4: Value of the DGP Warrant for different assumptions about the volatility of the real GDP growth during $2006-2034$. During 2005 a growth of 7.5% with a volatility of 1% and a 10% inflation is assumed. The real discount rate is assumed at 6% and the real growth for years $2006-2034$ is assumed at 3%.

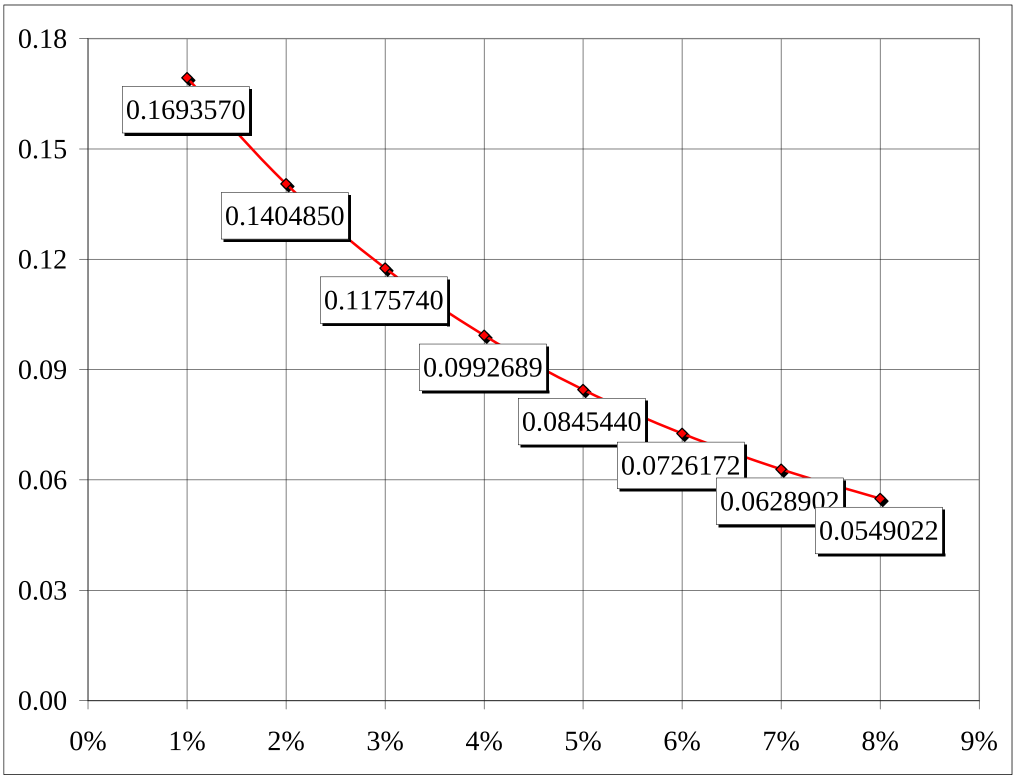

Figure 5: Value of the DGP Warrant for different assumptions about the discount rate. During 2005 a growth of 7.5% with a volatility of 1% and a 10% inflation is assumed. The real growth for years 2006 – 2034 is assumed at 3%.

Table 2: Prices of the coupon for various assumptions about the expected growth and its volatility for years $2005-2034$. For 2005 a growth of 7.5% with a volatility of 1% and a 10% inflation is assumed. The real discount rate is assumed at 6%. The different labels correspond to groups of countries which one can naturally associate with Argentina.

	μ	σ	Value
ELGC	1.66%	6.11	0.047
LATAM	2.79%	4.91%	0.092
WORLD	3.47%	4.70%	0.164
EM	3.44%	5.166	0.170

of the world except some extremely small countries. Finally, "EM" refers to the group of emerging countries, Argentina naturally belongs to this group too. As can be appreciated in this table, different assumptions about the likely behavior of Argentina's GDP implies great differences in the valuation of the GDP Warrant.

6 Conclusion

As we mentioned in the Introduction, the main purpose of this research program, of which the present paper is just the first work, is to provide a framework for the understanding and valuation of debt indexed to real (generally non-tradable) variables, which can display equity-like characteristics. This type of debt is of interest because it may serve as an effective hedge to prevent crises in emerging countries, crises such as the Argentinean one during $2001-2003$, in which poverty increased from 25% to 50% of the population in just a few months.

The Argentinean GDP Warrant constitutes the first important example of debt indexed to GDP growth and we used it here to develop the technology to value *any* potential debt instrument indexed to real, non-tradable variables. As we have mentioned, the main advantage of this type of debt is its hedging properties against poor long and short-term economic performance. For this reason it seems likely that future debt indexed to GDP growth, issued by Argentina or by other country, will share with the Argentinean coupon the imposition of contingency conditions similar to conditions 1 and 2.

Many authors have suggested that difficulties in valuation are one of the reasons why debt instruments indexed to real variables have not encountered many adherents. While one can always use Monte Carlo simulations to value anything, such numerical methods fail to provide key insights about the consequences of the conditionalities imposed by these instruments.

In this sense, the purpose of this research program can also be stated as to contribute to a detailed theory of value of debt indexed to real variables, or equity-like debt. Such theory has at least three ingredients: 1. The calculation of expected values of cash flows with the type of contingency conditions likely to appear in these instruments. 2. The calculation of the discount rate for the expected cash flows. 3. A detailed analysis of the effects of possible distortions of key input statistics.

In this paper we intended to contribute to step 1, by providing a general analytic method for imposing contingency conditions likely to appear in any debt with hedging purposes. The methodology developed here works as long as the underlying stochastic process is Markovian. Yet, as we repeatedly mentioned during the paper, this condition is not very restrictive. We also gave some hindsight as to how to address steps 2 and 3, and specifically analyzed the consequences for the Argentinean GDP Warrant. However, much more work is necessary to complete steps 2 and 3, and we are currently working in this direction.

Finally, we obtained values for the Argentinean instrument that were well in excess to the values given at the time by the "when and if" market when this work started to circulate. Not surprisingly the market has already made great corrections in the direction predicted by this paper.

7 Appendix A

The vector whose elements correspond to the base GDP PB_t from year 2005 to year 2034 is given by (in units of million real pesos):

$$\begin{aligned} PB = & (\$287,012.52; \$297,211.54; \$307,369.47; \$317,520.47; \$327,968.83; \\ & \$338,675.94; \$349,720.39; \$361,124.97; \$372,753.73; \$384,033.32; \\ & \$395,554.32; \$407,420.95; \$419,643.58; \$432,232.88; \$445,199.87; \\ & \$458,555.87; \$472,312.54; \$486,481.92; \$501,076.38; \$516,108.67; \\ & \$531,591.93; \$547,539.69; \$563,965.88; \$580,884.85; \$598,311.40; \\ & \$616,260.74; \$634,748.56; \$653,791.02; \$673,404.75; \\ & \$693,606.89) \end{aligned} \tag{50}$$

corresponding to a vector of "base growths" given by

$$bg = (4.26\%; 3.55\%; 3.42\%; 3.30\%; 3.29\%; 3.26\%; 3.26\%; 3.26\%; 3.22\%; 3.03\%; 3.00\%...) \tag{51}$$

where the dots mean that the other components correspond all of them to 3.00%. The component t of the vector gb corresponds to $gb_t = PB_t/PB_{t-1} - 1$, and the first component is computed with a base 2004 GDP of \$275,276.01 $(287,012.52/275,276.01 - 1 = 0.0426)$.

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