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STRUCTURAL RFV: RECOVERY FORM AND DEFAULTABLE DEBT ANALYSIS

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Structural RFV: Recovery Form and Defaultable Debt Analysis

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PRELIMINARY AND INCOMPLETE

Abstract

Receiving the same fractional recovery of par at default for bonds of the same issuer and seniority, regardless of remaining maturity, has been labelled in the academic literature as a Recovery of Face Value at Default (RFV). Such a recovery form results from language found in typical bond indentures and is supported by empirical evidence from defaulted bond values. We incorporate RFV into an exogenous boundary structural credit risk model and compare its effect to more typical recovery forms found in such models. We find that the chosen recovery form can significantly affect valuation and the sensitivities produced by these models, thus having important implications for empirical studies attempting to validate structural credit risk models. We show that some features of existing structural models are a result of the recovery form assumed in the model and do not necessarily hold under an RFV recovery form. Some of our results complement those found in the literature which examines the endogeneity of the default boundary. We find that some features that may have been solely attributed to modelling the boundary as an optimal decision by the firm can be obtained in an exogenous boundary framework with RFV. This has direct implications for studies which attempt to determine whether endogenous or exogenous models are better supported empirically. We extend our results to incorporate a multifactor default-free term structure model and examine the impact of the recovery form in estimating the cost of debt capital within a structural model framework.

JEL-Classification: G12, G13, G33.

Keywords: Recovery, Corporate Bonds, Credit Risk, Cost of Capital.

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1 Introduction

Receiving the same fractional recovery of par at default for bonds of the same issue and seniority, regardless of remaining maturity, has been labelled in the academic literature as a Recovery of Face Value at Default (RFV). Such a recovery form is a result, in theory, from the institutional framework that U.S. corporate bonds are subject to, such as the bond indenture and bankruptcy code. Most notably, the debt acceleration clause found in typical bond indentures leads to the principal amount of all outstanding debt to be due immediately. Recent empirical work by Guha (2002) shows that the RFV assumption is supported by data on defaulted bond values, providing strong evidence that it is the appropriate recovery form to describe corporate bonds upon default.

Such ex-post evidence suggests that it may be accurate to incorporate RFV into defaultable debt valuation models. This paper examines the effect of the RFV recovery form on ex-ante corporate bond valuation and hedging, and more generally analyzes the importance of the recovery form assumption within structural credit risk models. We find that in both constant and stochastic interest rate settings the RFV form can generate credit spread predictions which vary with respect to other recovery forms, especially for low credit quality bonds. In particular, the relative level of default-free interest rates to the coupon level is a primary determinant in comparing spreads across different recovery forms. For bonds with coupon rates higher than the appropriate default-free rate the RFV recovery form generates spreads higher than the recovery form most typically found in structural models. We find that the shape of the default-free yield curve also affects the relative comparison of predicted spreads. The interaction between recovery form and both the level and shape of the default-free term structure in determining spreads has important implications for empirical studies attempting to validate structural credit risk models. If we believe that RFV is the correct ex-post description of defaulted bond values we need to interpret data via a model assuming a recovery form which can generate such a pattern. Alternative specifications could lead to biased conclusions.

Furthermore, we find that certain features of previous structural models result from the specific recovery form assumed. Collin-Dufresne and Goldstein (2001) introduce a mean-reverting leverage ratio model, unlike typical structural models, and generate an upward-sloping term structure of credit spreads for speculative-grade bonds, consistent with empirical evidence found in Helwege and Turner (1999). However, we show that such a result comes from the recovery form assumed in their framework. Their model and parameter choices combined with an RFV assumption lead to a downward-sloping credit spread term structure for speculative-grade bonds. Moreover, we find that the RFV recovery form generates features that up to now have only been seen in models where default is modelled as an endogenous policy. Under certain parameter choices RFV corporate bond values can actually increase with decreases in firm value and increases in asset volatility because bondholders are better off if default becomes more likely. In addition, the RFV recovery form can cause both low credit quality bonds to have comparatively lower durations versus alternative recovery forms and bond prices to be concave functions of interest rates. These results have been seen in the Leland and Toft (1996) endogenous structural model. The RFV form is also able to match an interest rate sensitivity pattern consistent with empirical evidence found in Duffee (1998) and seen in the model of Acharya and Carpenter (2002). In their paper the pattern is interpreted to back the claim that endogenous models are better supported by data compared to typical exogenous default boundary models. We provide evidence that the recovery form assumed within the exogenous model is crucial in such an interpretation. Finally, we show that the recovery form is important in an relevant application of structural credit risk models: estimating the cost of debt capital for a firm.

1.1 Related Literature

Corporate bond models tend to be classified as either *intensity-based* models[1] which presume that default is a surprise event and the risk-neutral default probability as exogenously given, or *structural-based* models,[2] which provide a more fundamental framework for valu-

[1] A partial list of such models would include Litterman and Iben (1991), Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull (1997), Lando (1998), and Duffie and Singleton (1999).

[2] Duffie and Lando(2001) show the link between intensity and structural-based models by considering, very reasonably, that firm asset values are imperfectly observed.

ing the credit risk inherent in corporate debt. Starting from Black and Scholes (1973) and Merton (1974) this latter approach treats the securities of a firm as contingent claims on its asset value. Although the general empirical failure of this first set of models led to many important extensions[3], the basic intuition has remained the same.

Duffie (1998) and Lando (1998) were the initial papers to incorporate RFV into a defaultable debt model, both within an intensity-based framework[4]. An initial motivation for this study is the fact that several recovery forms are seen in the defaultable debt literature. In the intensity models of Jarrow and Turnbull (1995) and Jarrow, Lando, and Turnbull (1997) and in the structural model of Collin-Dufresne and Goldstein (2001) debtholders receive a fixed fraction of the face amount *at the maturity* of the defaultable bond. We label this recovery form the Recovery of Treasury-Face Value (RT-F) recovery form. In the intensity model of Duffie and Singleton (1999) debtholders receive a fraction of the market value of the defaultable bond just prior to default. Consistent with their paper we label this recovery form a Recovery of Market Value (RMV). Structural models with exogenous default boundaries such as Longstaff and Schwartz (1995), Cathcart and El-Jahal (1998), and Saa-Requejo and Santa-Clara (1999), assume the recovery amount is a fixed fraction of an equivalent coupon and maturity risk-free bond. We label this the Recovery of Treasury (RT) recovery form. Another recovery form is where the debtholder receives a fraction of the firm value at default as seen in the structural models of Merton (1974), Black and Cox (1976), Leland (1994), and Leland and Toft (1996).

In the existent literature the comparative effect of the different recovery forms on credit risk pricing has only been examined in the context of intensity-based models. Duffie and Singleton (1999) compare the pricing effects of RMV and RFV within an intensity-based model, while in a detailed empirical study Bakshi, Madan, and Zhang (2002) compare the RT-F, RMV, and RFV recovery forms within an intensity-based model using individual corporate bond price data. They find that their data better supports the RT-F assumption

[3]See Huang and Huang (2002) for a good overview of some of the main theoretical extensions to the basic structural model setup, along with references.

[4]Duffie (1998) cites Brennan and Schwartz (1980) as an early example that used such a recovery assumption in a (convertible) debt pricing model.

in terms of pricing accuracy, while they obtain more stable implied recovery rates using the RFV assumption. We use these papers as motivation for addressing this topic within the other major class of credit risk models[5].

From an economic perspective structural credit risk models are worth studying as they give us a setting in which we can link asset prices to corporate financial policies. There are also important practical reasons to focus on this particular class. First, these models can be used to impute estimated default probabilities from market equity prices, as seen in commercial applications e.g. Moody's/KMV. Second, in principle these models supply information on how one could hedge the default risk in corporate bonds using the securities of the firm, say its equity. While these two applications have been apparent since the original structural models were developed the growth in the credit derivatives market has increased the use of such models.[6] These facts imply that it is useful to understand the comparative pricing and hedging effects different recovery forms have on structural defaultable bond pricing, and in particular those models which can reliably be used in practical applications.

We consider the largest subset of the structural model literature where default is defined by the first time that the firm's asset value hits some default boundary. Black and Cox (1976), the first to introduce this idea, motivate the boundary in two different ways: 1) it can be *exogenous* due to safety covenants found in the bond indenture; and 2) it can be *endogenous* due to an optimal decision policy by the firm. We focus in this paper on the exogenous boundary specification. The endogenous approach, further developed by Leland (1994), Leland and Toft (1996), and Leland (1998), allows one to address the issue of an optimal capital structure in the presence of taxes and bankruptcy costs and is in general more economically sensible. One limitation of this approach, however, is that

[5]Delianedis and Lagnado (2002) and Finkelstein (1999) compare the effect of recovery form on credit derivative prices and find that it does matter, but neither does so within the context of a structural model

[6]The use of the EDF (default probability) measure by Moodys/KMV is well-cited. See their website: www.moodyskmv.com for more details. The cover article of the December 2002 Euromoney magazine titled "And Now for Capital Structure Arbitrage" describes the increased recent interest in using structural models for trading debt securities (or credit derivatives) versus their equity counterpart.

it places strong restrictions on the capital structure of the firm issuing the securities to be valued. This makes such models cumbersome to apply to individual bonds issued by firms with complex capital structures. Typical exogenous boundary models, starting from Longstaff and Schwartz (1995), assume that cross-default provisions lead to a simultaneous default for the different debt securities. This assumption allows such models to easily deal with complex capital structures. Another reason to favor exogenous models for practical implementation is the flexibility they allow in specifying bond recovery values. While in endogenous models the bondholder receives what is left of the firm value after any default costs; in exogenous models the expected recovery rate (i.e. the fraction of the form-varying recovery claim) is typically an input into the model which can, consistent with data, vary across industry (see e.g. Altman and Kishore (1996)) and seniority of the bond issue (see e.g. Franks and Torous (1994)).

We compare the valuation and hedging implications of the two recovery forms typically seen in this class of models, RT and RT-F,[7] with the RFV recovery form. While the RFV assumption, to the best of our knowledge, has not been explicitly labelled as such within the structural modelling framework, it is a special case of the recovery forms seen previously in Black and Cox (1976) and Leland and Toft (1996). In these models when the firm asset value falls to a potentially endogenous boundary the bondholder receives a fraction of the firm asset value *at* the default date. If we consider the boundary to be exogenously given and that the recovery claim to be the face value of the individual bond rather than the firm asset value we obtain what we label the *Structural RFV* model. The recovery form in this case is that of a type of barrier option. While the previous papers deserve full credit for applying the barrier option technology in modelling recovery, we explicitly recognize that such a recovery can be consistent with RFV, and more generally with recovery forms seen in the ex-post data and implied by certain institutional features.

The balance of the paper is organized as follows. Section 2 describes the basic structural model setup for valuing default risk as well as the different recovery forms which will be incorporated into the model. Section 3 compares and analyzes spreads and sensitivities

[7]The other major recovery form seen in intensity-based models is the RMV assumption. Since we do not consider jumps in our setting we avoid this particular form.

across the different recovery forms for coupon-paying bonds. Section 4 discusses how our results are related to the literature, with special emphasis on results found in the endogenous boundary literature. Section 5 extends the base case constant interest rate setting to one in which we can analyze the effect of both stochastic interest rates and yield curve shape in comparing the different recovery forms. Section 6 discusses the importance of recovery form in estimating the cost of debt capital for a firm using structural models. Section 7 offers some concluding remarks and planned extensions.

2 Structural Default Risk and Recovery Forms

This section introduces the default risk model and bond pricing setup we will use in our analysis. We formalize the different recovery forms which can be incorporated into this setup. First we show the recovery forms which have been traditionally used in the literature and then we introduce recovery of face value at default within a structural credit risk model, which we term the *Structural RFV* model. We consider the simplest of structural models which can reasonably be applied to coupon-paying bonds[8], a first-passage time model of a firm value process with constant interest rates and a constant default boundary. This simple model is considered the "base case" model in Huang and Huang (2002) against which more complicated models are judged.

There are several reasons why we are initially motivated to consider such a model. Firstly, Huang and Huang (2002) demonstrated in their paper that even when more realistic and economically sensible features are introduced to their base case model the main problems that structural models have in matching data, such as spread underestimation, remain the same. This leads us to believe that our results for studying comparative recovery forms under this simple model, in general, should hold under more complicated settings. Later, we perform some robustness checks by extending the base case in perhaps the most obvious way: by incorporating stochastic interest rates. Secondly, the setting

[8]For this reason we avoid the Merton(1974) model, the original and most widely cited structural model, which can be cumbersome to apply to coupon-paying bonds. We discuss this model further in the section on estimating the cost of debt capital.

allows us to fully concentrate on the comparative effects of the different recovery forms, a primary focus of this paper, rather than on issues such as sensible default boundary specifications or other important features which have been embedded into the structural modelling framework. Lastly, the simple setting leads to tractable closed-form solutions which greatly eases our analysis from a computational standpoint.

2.1 Structural Model for Default

The underlying stochastic variable for default risk is the firm asset value process whose dynamics under the risk-neutral measure are modelled as follows:

$$dV_t = (r - \delta) V_t dt + \sigma_V V_t dz_V^Q \tag{1}$$

where r is the constant default-free interest rate, δ is the assumed constant firm payout rate, and σ_V is the assumed constant volatility of firm's asset value, and z_V^Q is a standard Brownian motion under the risk-neutral probability measure. Default occurs at the first-passage time τ of V hitting the constant default boundary, K.

$$\tau \equiv \inf \left(u > 0, x_u \equiv \ln \left(\frac{V_u}{K} \right) = 0 \right) \tag{2}$$

Such a boundary is usually economically justified by the presence of positive net worth or safety covenants (Black and Cox (1976)). Defining x_t as $\ln \frac{V_t}{K}$,the log of the inverse leverage ratio[9] and employing Ito's lemma, we can write the risk-neutral cumulative default probability $Q_t\left(\tau < T\right)$ as the probability of the following process for x_t hitting zero between times t and T starting from an initial value x_0 assumed greater than zero:

$$dx_t = (r - \delta - \frac{\sigma_V^2}{2})dt + \sigma_V dz_V^Q \tag{3}$$

Using the first-passage time density of x we have the well-known result (see Musiela and Rutkowski (1997)) :

[9]This interpretation is correct if we assume the level of the default boundary to proxy for the value of total liabilities of the firm.

$$Q_t(\tau < T) = N\left(\frac{-x_0 - \mu_*(T-t)}{\sigma_V\sqrt{T-t}}\right) + e^{-\frac{2\mu_* x_0}{\sigma_v^2}} N\left(\frac{-x_0 + \mu_*(T-t)}{\sigma_V\sqrt{T-t}}\right) \tag{4}$$

where N is the standard normal cumulative distribution function and μ_* is the risk-neutral drift for x_t: $\mu_* \equiv r - \delta - \frac{\sigma_V^2}{2}$.

2.2 Defaultable Bond Pricing

Expressions for defaultable bond prices are straightforward in this simple setting. Our main objective is to compare the implications of the different recovery assumptions. To make this clear we partition the value of a defaultable bond with a face value of F maturing at time T paying semi-annual coupons[10] at an annual rate of c into two components: 1) valuation of payments in the states where no default occurs, and 2) valuation of payments in states where default occurs.

$$P_{0,c,T} = P_{0,c,T}^{ND} + P_{0,c,T}^{D} \tag{5}$$

The latter portion will vary with the recovery form we choose while the former part is independent of the recovery value. In other words, the first expression is equal to the value of the bond assuming a zero recovery value which can be written in closed-form in our setting:

$$P_{0,c,T}^{ND} = D(0,T)F\left[1 - Q_0(\tau < T)\right] + \frac{c}{2}\sum_{i=1}^{2T} D(0,T_{i/2})\left[1 - Q_0(\tau < T_{i/2})\right] \tag{6}$$

where $D(0,T_{i/2})$ is the value of the risk-free discount bond maturing at $T_{i/2}$, or in this setting: $\exp\left(-rT_{i/2}\right)$. The partitioned value for the default states will depend on the recovery form. We first present the two forms seen up to now in the literature: RT-F and RT. We focus on the RFV assumption in the following section. In all case an expected recovery rate of ω which is independent of firm asset value process is assumed.

[10] Clearly zero-coupon bond prices can be considered in this setting by setting the coupon rate to zero.

Recovery of Treasury - Face Value (RTF):

$$P_{0,c,T}^{D,RT-F} = E_0^Q \left[\mathbf{1}_{\{\tau<T\}} \exp(-rT)\omega F \right]$$
$$= \omega D(0,T) F Q_0(\tau < T) \tag{7}$$

Recovery of Treasury (RT):

$$P_{0,c,T}^{D,RT} = \omega \left[D(0,T) F Q_0(\tau < T) + \frac{c}{2} \sum_{i=1}^{2T} D(0, T_{i/2}) Q_0(\tau < T_{i/2}) \right] \tag{8}$$

As discussed previously and apparent from the equations here, at default, RT-F bondholders receive the face value of the bond discounted to the promised maturity at the risk-free rate, while RT bondholders receive the value of a bond with the same contractual features of the defaultable bond but priced discounting at the risk-free rate. In both cases recovery payments are assumed to occur at the promised coupon or bond redemption dates. In the case of a zero coupon bond it is clear from above that the two recovery forms are the same. With the model bond prices given as

$$P_{0,c,T}^{RT-F} = P_{0,c,T}^{ND} + P_{0,c,T}^{D,RT-F} \tag{9}$$
$$P_{0,c,T}^{RT} = P_{0,c,T}^{ND} + P_{0,c,T}^{D,RT}$$

we can compute the respective yield to maturity $Y_{0,c,T}$ implicitly and the associated yield spreads $s_{0,c,T}$ as

$$P_{0,c,T}^{j} = e^{-Y_{0,c,T}^{j} T} + \frac{c}{2} \sum_{i=1}^{2T} e^{-Y_{0,c,T}^{j} T_{i/2}} \tag{10}$$
$$s_{0,c,T}^{j} = Y_{0,c,T}^{j} - r$$

for $j = \{RT - F, RT, RFV\}$.

2.3 Structural RFV Model

We now consider the RFV case where the debtholder receives a fraction of the face value of the bond *at* the default date[11].

Recovery of Face Value at Default (RFV):

$$P_{0,c,T}^{D,RFV} = E_0^Q \left[\mathbf{1}_{\{\tau<T\}} \exp(-r\tau)\omega F \right] \quad (11)$$

Due to our assumptions regarding a constant expected recovery rate we can write this expression as ωF multiplied by the value of a unit payment received at default discounting to time zero. This latter term describes a particular type of barrier option referred to by some as the *down-and-in cash-at-hit option*[12] (see Haug (1998)). The closed-form solution is as follows:

$$E_0^Q \left[\mathbf{1}_{\{\tau<T\}} \exp(-r\tau) \right] = \left[e^{\frac{-x_0(\mu_*+\lambda)}{\sigma_v^2}} N\left(\frac{-x_0 + \lambda T}{\sigma_V \sqrt{T}} \right) + e^{\frac{-x_0(\mu_*-\lambda)}{\sigma_v^2}} N\left(\frac{-x_0 - \lambda T}{\sigma_V \sqrt{T}} \right) \right] \quad (12)$$

$$\text{where } \lambda \equiv \sqrt{\mu_*^2 + 2\sigma_V^2 r} \text{ and } \mu_* \equiv r - \delta - \frac{\sigma_V^2}{2}$$

and $N(\cdot)$ denotes the standard normal cumulative distribution function. A proof for the formula can be found in Nelken (1996). This type of expression has been applied before for corporate bond pricing before by Black and Cox (1976) and Leland and Toft (1996) where the bondholder receives a fraction of the firm value at default. However, neither source *explicitly* considers it as an application to model a RFV-type of recovery within a complex capital structure or state that such a recovery form may be consistent with data or certain institutional features which generate such a recovery form as discussed in Guha (2002). Later on we discuss how our results compare with these and other papers which consider

[11] In the more general form of RFV any accrued interest on the last coupon payment prior to default should be included in the recovery payment if stipulated as such in the bond indenture. We ignore this feature in our analysis.

[12] Also referred to as American binary/digital/bet options (Taleb (1997) and Zhang (1998)) or the rebate price for a down knock-out barrier option (Cox and Rubinstein (1985) and Sbuelz (1999)).

endogenous default boundaries. Next we analyze these different closed-form components that affect defaultable bond pricing in this simple setting, with a particular focus on the comparative differences between the $P^D_{0,c,T}$ terms.

3 Comparative Analysis: Coupon Bonds

In this section we compare spreads and sensitivities resulting from the three different recovery forms for the case of coupon-paying bonds. The vast majority of corporate bonds pay coupons and thus are the securities which are subject to empirical tests of model performance.

3.1 Comparative Credit Spreads

We first compare hypothetical credit spreads as given in (10) generated by the different recovery forms using the base case model we have assumed. It is useful to do this across different credit rating classes and various maturities. For choosing the parameters which do not vary across credit quality we refer to Huang and Huang (2002). These include the constant interest rate $r(8.00\%)$, the constant payout rate $\delta(6.00\%)$, the default boundary K, assumed to be 60% of the firm's total liabilities; and an expected recovery rate ω of 51.31%. For the parameters which would likely vary across credit quality we use mean estimates from Davydenko and Strebulaev (2002)[13]. These include the leverage ratio, as measured by the book value of total liabilities divided by the sum of the book value of total liabilities and the market value of equity, and the asset value volatility. Table I shows their estimates across the credit rating classes. While the credit spreads in absolute terms will depend highly on our choice of parameters it should be noted that we are concerned here with the *relative* difference in spreads across the different recovery forms. We consider a hypothetical semiannual coupon-paying bond under three different annual coupon rate

[13] In Huang and Huang(2002) the asset volatility parameter is calibrated to match the data, and thus vary for different maturities. While we could use their data on leverage ratios for the different credit rating classes, we decide to take these two parameters from a consistent source.

scenarios with the constant interest rate fixed at 8.00% : 1) 8.00% (*par bond*)[14], 2) 12.00% (*premium bond)*, and 3) 4.50% (*discount bond).*

Results are found in Table II for 6 different credit rating classes and three maturities (2, 10, and 30-year). Figure 1 plots the entire term structure of credit spreads for a Speculative-grade (B-rated) firm. In the top three credit rating classes (Aaa, Aa, A) the recovery form does not make much difference. This follows from default being an extremely unlikely event. This is especially true in the 2 and 10-year bonds. In the case of the A-rated 30-year maturity bond there is a 20 basis point difference between RT and RT-F for the par bond case and similar numbers in the premium and discount cases. One particular noticeable feature is how much higher the spreads in RT-F versus RT and RFV are as we increase maturity and as we move further down the credit rating class. For the B-rated 10-year maturity par bond the RT-F assumption results in a yield spread of 474 bps versus 319 and 324 bps for the RT and RFV forms respectively. This is not too surprising given that we have assumed fixed the expected recovery rate across all recovery forms[15]. As we increase the maturity the discounting effect leads to a smaller amount being recovered at default for RT-F and therefore significantly larger spreads.

Spreads in the RFV case, as expected, are always smaller than the RT-F case as the bondholder will receive the recovery amount at least as soon as in the RT-F case. Relative to the RT case, RFV spreads can be smaller or larger depending on whether the bonds are at a premium or discount. In the par bond case RFV and RT are virtually the same[16] with the largest difference between RFV and RT spreads being 5 bps in the 10-year maturity B-

[14]Of course the defaultable bonds would not be trading at par due to default risk, however we use the terminology to distinguish among the other cases.

[15]One potential criticism of the analysis that follows is that using the same expected recovery rate for the RT-F form as the others is inherently a bad assumption as it obviously implies a lower recovery value. However, we think it useful to compare the different recovery assumptions holding this parameter constant to concentrate on the form of the recovery. An alternative procedure would be to calibrate this parameter such that the RT-F bond price matched the RFV bond price. This will be considered in a future version.

[16]In the Huang-Huang(2002) paper the authors assume a coupon of 8.162% in their calibration exercises. This corresponds to a par coupon if yields are computed assuming semiannually compounding. Therefore, they effectively approximate the RFV assumption, although they price bonds in their base case using a RT assumption.

rated bond. This is made clear in Figure 1 which plots the term structure of credit spreads for a B-rated bond. For high coupon bonds we see that the RFV assumption can produce noticeably larger spreads versus the RT case especially as the credit rating worsens and the maturity increases. For the 10-year maturity B-rated premium bond RFV produces a spread of 387 bps versus 320 bps for the RT case. These results come directly from the fact that the assumed claim in default is higher than par in RT while the recovery claim in RFV is fixed at par. The opposite results are seen in the discount bond case. RFV spreads can become significantly lower than RT spreads. Again in the 10-year maturity B-rated bond case, the RT spread is 319 bps while the RFV spread is 250 bps. Figure 2 shows this point for the Ba-rated 10-year maturity bond. An interesting feature of the Structural RFV model is that for certain extreme parameters promised yield spreads can become negative. We discuss this point further when discussing the bond sensitivities to firm value.

The shape of the credit spread term structure generated by RT and RFV recovery forms is generally consistent to that found in previous structural credit risk models, upward sloping for investment-grade credits and downward sloping (starting from the 5-year maturity[17]) for speculative-grade credits. In the RFV case, increasing the coupon generates a less downward-sloping curve for lower grade credits, while lowering the coupon exacerbates the downward slope. The RT term structure shape is relatively unaffected by changes in the coupon. Given that corporate bonds tend to pay higher coupons than a risk-free bond issued at par (corresponding to our *premium bond* case) our results can be considered supportive of the empirical results found in Helwege and Turner (1999). They find that risky bonds can have upward-sloping credit yield curves. The RT-F assumption, interestingly, leads to upward-sloping credit spread curves in all cases with our parameter choices, even for speculative-grade credit ratings.

Within an RT-F recovery form setting Collin-Dufresne and Goldstein (2001) develop an exogenous default boundary model with mean-reverting leverage ratios. They claim

[17] Assuming simple geometric brownian motion for firm value implies that defaults are predictable. This leads to small default-related spreads at the short end of the maturity curve, thus the hump-shaped credit spread curve seen in such models.

that, unlike constant default boundary models, their mean-reversion feature results in an upward sloping term structure of credit spreads for low-grade companies, consistent with evidence found in Helwege and Turner (1999). Our results above contradict this claim as our base case RT-F model can generate an upward-sloping term structure of credit spreads. Further, we confirm that while mean-reversion does somewhat increase the slope of the credit spread term structure compared to constant default boundary models, the recovery form assumption is crucial in generating their results. In Figure 3 we consider their parameter choices and plot both our base case constant default boundary model and their model assuming RFV, RT, and RT-F recovery forms. We find that upward-sloping credit spread term structures are generated only in the RT-F version of their model [18]

While comparing spreads across recovery forms under such a simple environment may only be of moderate use, a few conclusions can be made. Firstly, RFV does not help much in addressing the problem of credit spread underestimation. This feature is most puzzling for high-grade credits where we find little difference between the spreads generated by the various recovery forms and indeed there are environments where RFV produces comparatively the lowest spreads. Secondly, RT-F produces significantly larger spreads than the other assumptions using the same expected recovery rate and a consistently positively sloped credit spread curve unlike RT and RFV. Thirdly, unless bonds have very high or low coupons relative to default-free yields the RT assumption provides a close approximation to RFV in producing spreads. Yet there seems to be little justification for using RT versus RFV especially since the latter embeds a theory generated by the bond indenture and bankruptcy code. From a computational standpoint both forms can be valued with closed-form equations in the constant interest rate case. In more realistic models where interest rates are driven by multiple factors and possibly correlated with the default process, advanced numerical methods would likely be needed in both recovery forms. Lastly, it does seem at a first glance from the coupon effects seen, that interest rate sensitivity will differ across all recovery forms, especially for low credit quality bonds. We explore these effects later.

[18]Under other reasonable parameter choices we can obtain downward-sloping term structure of credit spreads even in the RT-F case of their model.

3.2 Comparing Coupon Bond Sensitivities

3.2.1 Sensitivity to Firm Value

Coupon bond price sensitivities to firm value (i.e. deltas) tend to be an upward sloping function of maturity for high-grade bonds for all three different recovery forms. For low-grade bonds the RT and RFV deltas are humped and then slightly downward sloping similar to their credit spread curves versus maturity. In the par case, where the coupon is chosen to equal the risk-free rate, RT and RFV deltas are essentially identical whereas the RT-F deltas are consistently higher. The size of the coupon with respect to the constant risk-free rate affects the relative deltas among the recovery forms as it did for the spreads. When the coupon rate is higher than the risk-free rate the RFV deltas are higher than RT deltas while the opposite holds true when the coupon rate is lower, *ceteris paribus*.

The deltas for all three recovery forms increase in a convex manner as the credit quality decreases. Figure 4 shows this for a 10-year maturity bond assuming the coupon is equal to the risk-free interest rate. Differences in credit quality here is taken to mean different assumed leverage and asset volatility. In most cases the deltas for the coupon bonds increases with leverage, again in a convex fashion. However, the RFV delta can be a concave function of leverage and as we saw with spreads, can become negative. Figure 5 displays this for a 30-year maturity bond with a given set of parameters. The asset volatility (30.00%), the coupon rate (3.00% vs. the risk-free rate of 8.00%), and the expected recovery rate (60.00%) are all parameter choices within reason. The potentially negative delta of the Structural RFV model implies that the bondholder is better off if the firm defaults. Essentially, the present value benefit of the immediate recovery payment is greater than the increase in debt value due to the decrease in default risk if the firm value were to increase. This is also the reason why we saw that Structural RFV credit spreads could become negative. The fact that bondholders may actually prefer default was a point made initially made in Black and Cox (1976) but not explicitly generated in their model. With respect to the other variables, deltas increase linearly with coupon rates, decrease linearly with expected recovery rates, and decrease as a convex function of interest rates.

3.2.2 Sensitivity to Volatility

In the case of investment-grade bonds the sensitivities of coupon bond prices to the volatility parameter (i.e. vegas) are, in general, negative decreasing functions versus maturity across all recovery forms. For low-grade bonds the vegas are much less sensitive to maturity. When the coupon rate is set equal to the interest rate the sensitivities produced by RT and RFV are essentially the same while the RT-F vegas show a more negative sensitivity. When the coupon rate is higher than the risk-free rate the RFV form has more negative vega than the RT form, while the opposite holds true when the coupon rate is lower than the risk-free rate. Generally speaking the vegas for all recovery forms become more negative as the credit quality decreases. However, vegas can increase in all recovery forms with both the assumed leverage ratio and asset volatility when the leverage and volatility levels are high. In the RT and RT-F forms the vegas are always negative, that is bond prices decrease with increases in asset volatility. In the RFV recovery form, however, the vegas can become positive under certain parameters. This is the same effect discussed above in which bondholders benefit from default occurring. Figure 6 shows this for a 30-year maturity bond with leverage and asset volatility parameters to match a hypothetical B-rated company. The coupon is chosen to be a discount bond (4.50% vs. the risk-free rate of 8.00%). For quite reasonable expected recovery rates the RFV recovery form can produce nonnegative vegas.

3.2.3 Sensitivity to Expected Recovery Rate

Figure 7 plots the sensitivity of coupon bond price to the expected recovery rate parameter across different maturities for a Baa-rated company using the three recovery forms. The coupon is chosen to equal the risk-free rate so again we see the equivalence between the RT and RFV forms. The sensitivity in these two forms are considerably higher than the RT-F form. These sensitivities are independent of the initial recovery rate assumed. Thus, if one were to assume the RT-F form of recovery in a model but with a substantially higher initial ω, the differences seen in these recovery sensitivities would still remain. As is apparent in the plot the RT-F recovery sensitivity can decrease with maturity due to a discounting effect while RT and RFV are strictly increasing.

The sensitivity to the expected recovery rate increases with a decline in credit rating for all three forms though both RT and RFV have greater sensitivity to increases to leverage and asset volatility than the RT-F form. When the coupon rate is higher than the risk-free rate the RT form has a greater sensitivity to the recovery rate versus the RFV form, while the opposite holds true when coupon rates are lower than the risk-free rate. In fact, the recovery sensitivities for the RFV and RT-F recovery forms as defined are independent of the coupon rate, while the RT sensitivity increases linearly with the coupon rate.

3.2.4 Sensitivity to Interest Rate

Intuition suggests that due to differences in timing of the cash flow payments for the different recovery forms we should expect different interest rate sensitivities. An understanding of these sensitivities is important for bondholders as they represent in most cases the major risk which can be hedged. We consider two measures of this interest rate risk: 1) the derivative: $\frac{\partial P^j_{0,c,T}}{\partial r}$, and 2) the negative elasticity with respect to the *model* bond price, or the model modified duration:

$$Dur^j_{0,c,T} = -\frac{1}{P^j_{0,c,T}} \frac{\partial P^j_{0,c,T}}{\partial r} \tag{13}$$

for $j = \{RT - F, RT, RFV\}$. The negative of the derivative is also often referred to as the *dollar duration.* For investment-grade credit the sensitivities are essentially the same across all recovery forms due to default being an unlikely event. However, as the credit quality decreases the sensitivity as implied by the three recovery forms diverge. This is shown in Figure 8 which plots the derivative against maturity for a hypothetical speculative-grade (B-rated) bond. The coupon rate is chosen to be equal to the risk-free rate (8.00%). The same plot in terms of the model modified duration is shown in Figure 9. The RT form has a significantly more negative sensitivity to interest rates and a considerably higher modified duration. For example, in the 30-year maturity case the RT modified duration is 8.69 while the RFV and RT-F modified durations are 5.32 and 4.94 respectively. The RT-F recovery form has the least negative sensitivity, but once transformed into a modified duration measure it is higher than the RFV form except for very long-dated maturities. The RT-F interest rate sensitivities (durations) can actually increase (decrease) with maturity. This is driven by the recovery payment term, $P^D_{0,c,T}$. In RT-F, the claim paid at default is a fraction of a zero-coupon bond maturing at the final redemption date. Such a security can have an increasing (less negative) sensitivity with respect to interest rates as maturity increases, unlike the claims paid at default under the RT and RFV recovery forms.

An important point here is that the coupon in these figures is equal to the risk-free rate. Previously when this was the case the RT and RFV recovery forms produced nearly identical numbers for both spreads and sensitivities. However, with regard to arguably the most important sensitivity from an investor's point of view, the interest rate hedge ratio, they can imply significantly different numbers. In general, as the credit quality decreases the sensitivity (modified duration) increases (decreases) for all recovery forms, with the RFV recovery form being particularly affected. That is, as volatility and leverage increases the modified duration of an RFV bond decreases substantially more than RT or RT-F.

We finally plot the model bond prices against the risk-free interest rate. This will allow us to examine the implied bond *convexity* exhibited by the different recovery forms. When the credit rating is high there is a decreasing convex relation between bond prices and interest rates, as in the case with default-free bonds. However, as the credit quality decreases this behavior is not necessary. Figure 10 plots the bond prices versus the risk-free interest rate for 20-year maturity bond using parameters for a B-rated issuer. We find that while the RT and RT-F bonds are still convex, the RFV bond is slightly concave with respect to interest rates. This is a result of the following. If we consider only the recovery-independent portion of the bond, the $P^{ND}_{0,c,T}$ term described above, low-grade bonds would be concave with respect to interest rates due to a effect driven by the probability of default. A decrease in r would produce such an increase in the default probability that $P^{ND}_{0,c,T}$ increases weakly with respect to r, or even decreases. Considering now the recovery portion $P^D_{0,c,T}$, of the different recovery forms, RFV is the least convex due to the insensitivity of its recovery claim F to interest rates. RT and RT-F embed enough convexity in their

$P^{D}_{0,c,T}$ terms to offset the concavity of the $P^{ND}_{0,c,T}$ term.

To summarize this section, we have provided convincing evidence that the recovery form, in general, matters for the fundamental valuation and hedging of corporate debt within exogenous boundary structural credit risk models. This conclusion is most applicable for the case of long-maturity low credit quality bonds. Model interest rate sensitivities, in particular, look to be an important feature which can be affected by the recovery form chosen. This has obvious practical implications in terms of relative valuation and risk management.

4 Relationship to Endogenous Bankruptcy Literature

It is interesting to note that some of the results seen in the previous section are closely related to those found in the endogenous boundary literature. Black and Cox (1976) were the first to motivate the fact that the default boundary could be an endogenous outcome of a firm's optimal decision policy. Essentially, the optimal default boundary is the level of the asset value where the firm can no longer issue new securities, such as equity, to service the debt. In both their endogenous model and exogenous case if default happens bondholders receive the firm value *at* the default date. The modelling of the timing of the recovery payment leads to a type of barrier option which is seen in the Structural RFV model above. Leland (1994) extends the Black and Cox (1976) paper to include bankruptcy costs and taxes which leads to considering the problem of the optimal debt contract a firm should issue ex-ante. Thus, he is able to link debt values and optimal capital structure decisions to the firm's asset value, asset volatility, taxes, bankruptcy costs, and interest rates. As these types of papers attempt to focus on important economic issues rather than practical bond valuation they necessarily work in stylized settings. For example, these two last papers focus on a hypothetical consol bond as it considerably simplifies the valuation equation. However, these early papers have been extended somewhat to consider more realistic situations. We address how our results complement two such papers: Leland and Toft (1996) ("LT") and Acharya and Carpenter (2002) ("AC").

LT extend Leland (1994) to consider finite maturity debt that is constantly rolled over such that the debt structure remains time-independent. This allows them to additionally consider the optimal maturity of the debt contract. We focus here on some of their pricing and hedging results. Their model generates term structure of credit spreads consistent with the much of the structural model literature: upward-sloping for low leverage firms and downward-sloping after an initial hump for high leverage firms. One interesting result they obtain is that bond prices increase when the asset volatility parameter increases if leverage is very high. They attribute this feature to the endogeneity of the default boundary. However, we have shown in Figure 6 that under reasonable scenarios a purely exogenous model can also produce a similar result. In the LT model, as in Black and Cox (1976) and Leland (1994), the default boundary decreases with increases in asset volatility. Thus, the endogeneity surely would increase the magnitude of this sensitivity to volatility result but we show evidence that, in general, it is not a necessary condition[19] for it to exist. Another feature of their model that LT highlight is the sensitivity of their model debt values to interest rates. They show that for very risky bonds the *effective duration* becomes considerably shorter than the duration for a corresponding risk-free bond. They also show that very risky bond prices can be concave to interest rates. We can generate both of these results in the Structural RFV model. We already showed evidence in Figure 10 that the RFV recovery form can lead to concavity to a degree much higher than other recovery forms. In Figure 11 we show a similar plot to Figure 5 in LT (pg. 1005) plotting the modified durations of A-rated and B-rated bonds against modified durations of the corresponding default risk-free bonds for both the RT and RFV recovery forms. We show, similar to the LT result, that when an RFV recovery form is assumed that the modified duration becomes insensitive to the maturity of the bond. While the setting of the LT paper is different than ours and has alternative aims, the consistent pricing and hedging results found in both lead us to conjecture that the similar timing of the recovery payment is playing a fundamental part.

AC make the same general point we have done which is that the bankruptcy rule affects

[19]LT does provide some comparative results assuming the boundary is exogenously fixed, but it is not entirely clear if the volatility result holds or does not hold in such a setting.

corporate bond sensitivity to interest rates and firm value. They work in a setting where a firm has a single bond outstanding with a finite-maturity and paying a fixed continuous coupon. This implies that their assumed endogenous default boundary is time-dependent. They assume interest rates to be a one-factor diffusion process and also allow the bond to be callable in addition to defaultable. In such a setting they are able to treat both the default option and the issuer call option as options on an underlying *host bond.* This gives them the ability to understand the non-negligible interaction between the two options. One result of their model is that corporate bond price sensitivities to interest rates are guaranteed to be negative, a feature found neither in typical exogenous default boundary models nor in the endogenous boundary models of Leland (1994) and Leland and Toft (1996). The authors then go on to claim that endogenous models explain the empirical evidence regarding sensitivities to interest rates better than typical exogenous boundary models. In particular, one piece of evidence they cite from Duffee (1998) is that for noncallable bonds the sensitivity to interest rates becomes monotonically less negative as the credit quality improves. Plotting duration versus firm value in their model they are able to generate such a result while doing the same in the Longstaff and Schwartz (1995) model leads to potentially U-shape plots. In Figure 12 we plot the model modified duration versus firm value (assumed as a multiple of total liabilities) for a generic 8.00% coupon 10-year bond using both the RT and RFV recovery forms. Using the RFV recovery form we can generate the upward-sloping duration curve that they show in their Figure 4 for an endogenous model, while the RT case corresponds to their Figure 6 showing the LS model. This provides evidence that their criticism of typical exogenous models is more of a criticism of the RT recovery form found in typical exogenous models rather than exogeneity per se. In short, it is not decidedly clear from their results that endogenous models explain the data better than exogenous models, however, they do provide indirect evidence that the RFV recovery form matches sensitivities seen in the data better than other recovery forms.

To summarize this section, some of our results from comparing different recovery forms within a simple exogenous boundary models complement those seen in the endogenous boundary literature. In particular, the RFV recovery form seems to generate some features which previously may have been solely attributed to modelling the default boundary as an optimal policy by the firm. While studies such as Leland (1994) have studied the theoretical differences between the exogenous and endogenous boundary models in very stylized settings, work is needed to properly isolate the quantitative effects of endogeneity in more realistic settings. Leland (2002) accomplishes this to some extent by focusing on default probabilities, but more work is needed. Empirically speaking, we believe it is still an open question whether endogeneity better explains the data.

5 Robustness Check: Incorporating Stochastic Interest Rates

5.1 Choosing a Term Structure Model

It is useful to see how the different recovery forms affect pricing within the setting of stochastic default-free interest rates. First, it provides a robustness check on our results in the base-case setting. Second, in any practical implementation of valuing corporate debt, the stochastic nature of interest rates should be taken into account. This requires us to choose a sensible term structure model. While most of literature on structural credit risk pricing with stochastic interest rates assume simple one-factor models for the risk-free rate (e.g.Kim, Ramaswamy, and Sundaresan (1993), Longstaff and Schwartz (1995), Cathcart and El-Jahal (1998), Collin-Dufresne and Goldstein (2001), Acharya and Carpenter (2002)) we believe a model with multifactor dynamics is more sensible. The empirical literature shows vast evidence (e.g. Nelson and Schaefer (1983) , Litterman and Scheinkman (1991)) across all major interest rate markets that two or three factors are necessary to adequately describe the empirical behavior of yield curve movements. In addition, to jointly examine how different initial risk-free yield curve shapes can affect credit risk pricing across the different recovery forms, we would like that a model can be calibrated to fit the initial yield curve. Since the initial yield curve would be an input to the model we can determine whether recovery forms matter more in a certain interest rate environment, say an upward-sloping yield curve, versus another, say a humped-shaped yield curve. Our last requirement for choosing a term structure model is that it allows us to consider non-zero correlation between the stochastic terms driving the risk-free rate and those driving the default risk of the corporate bonds, in this case the firm-value process. Such covariation would be consistent with empirical studies (e.g. Duffee (1998) and Collin-Dufresne, Goldstein, and Martin (2001)) showing a negative relation between interest rate levels and credit spreads.

5.2 Term Structure Model

We incorporate the above elements in a relatively simple two-factor *affine* term structure model as described by Brown and Schaefer (1994) and Duffie and Kan (1996) , and which is *admissible* (see Dai and Singleton (2000)). The short rate path $r(t)$ is assumed as the sum of two stochastic factors, $l(t)$ and $s(t)$, and a deterministic factor $g(t)$. The dynamic system of equations, under the risk-neutral measure, is

$$\begin{aligned} dl(t) &= -\kappa_l l(t)dt + \sigma_l dz_l^Q \\ ds(t) &= -\kappa_s s(t)dt + \sigma_s dz_s^Q \\ r(t) &= g(t) + l(t) + s(t) \end{aligned} \tag{14}$$

where $E(dz_l^Q dz_s^Q) = \rho_{ls}dt, E(dz_l^Q dz_V^Q) = \rho_{lV}dt, E(dz_s^Q dz_V^Q) = \rho_{sV}dt$ and the initial values of both l and s are equal to zero. We can refer to κ_l and κ_s as *reversion* parameters of the two factors; while σ_l and σ_s can be termed *volatility* parameters. By making the deterministic term, $g(t)$, time dependent we can calibrate the model to the initial yield curve. A full description of this particular specification as well its solution and calibration procedure can be found in the Appendix.

Implementing the model requires values for the parameters which are assumed constant. In two-factor gaussian models as the one described, typically one factor can be thought of as the interest rate *level* factor while the other factor can be interpreted as a *slope* factor. The level factor is likely the one with a considerably smaller reversion parameter value, indeed usually close to zero. Using the estimates of Brown and Schaefer (2000) from US Treasury STRIP data we set $\kappa_l = 0.0393$ and $\kappa_s = 0.2060$ thus we consider $l(t)$ as the level factor and $s(t)$ the slope factor. For the volatility parameters we set $\sigma_l = 0.010$ and $\sigma_s = 0.015$, values found in both Brown and Schaefer (1994) and He (2000). We set the term structure correlation parameter, ρ_{ls}, equal to -0.336 as estimated by Brown and Schaefer (2000). Using our own estimates from a subset of companies and relevant interest rate data we set $\rho_{lV} = -0.15$ and $\rho_{sV} = 0.00$. Ideally we would like these latter correlation parameters to vary by credit rating, but given the size of data which the estimates would have been taken from we consider only one value. Given these parameters and the initial yield curve the term structure model has been fully specified.

5.3 Implementation and Results

We price defaultable bonds by jointly simulating the default variable process found in (3) [20] and the interest factor processes. The three different recovery forms (RT, RT-F, and RFV) are taken into account when determining cash flows path by path. Details of the simulation procedure and the appropriate variance reduction techniques used for reducing pricing bias are discussed in the Appendix. We examine 4 different initial yield curve shapes in terms of the instantaneous forward rate across different maturities: 1) flat curve; 2) upward-sloping curve; 3) humped-shape curve; and 4) downward-sloping curve. The curves are chosen such that the average forward rate over the first ten years is equal to 0.08, thus somewhat comparable to our base results found in the previous section which assumed a constant interest rate across maturities of 0.08. Figure 13 plots the 4 different term structure shapes. Details on how the yield curves are produced using the Nelson and Siegel (1987) methodology is found in the Appendix.

Table III shows the results in terms of basis point spreads over the default-free promised yield corresponding to the particular interest rate environment. The first noticeable result is that when we assume an initial flat curve the spreads are little unchanged from our base case result with constant interest rates, despite the fact that the firm value process is correlated with the interest rate process. This result is especially valid when we are

[20] Now with the short rate a stochastic variable rather than a deterministic constant.

concerned with the relative effects of the different recovery forms.

The results from the other interest rate environments assumed support our argument that the form of the recovery assumption can affect pricing significantly. This is true even if the coupon rate of the bond is not markedly high or low. Consistent with our base case result is that spreads produced by the RT-F assumption are consistently and in some cases dramatically higher than those produced by RT and RFV. Going forward we will concentrate on comparing the RT and RFV assumptions. As we saw in our base case analysis, as the maturity of the bond increases and the credit quality decreases the model spreads for the various recovery forms can differ significantly. Here, in addition, we can examine the effect of different yield curve shapes.

In the upward-sloping yield curve, the respective default-free promised yields for the three different maturities are: 4.67%(2-year), 8.00%(10-year), and 9.07% (30-year). For the short-dated bond the RFV spread is slightly higher than RT spread in the B-rated bond due to the coupon of the bond considered being at a significant premium. In the medium term bond, while the bond coupon rate is the same relative to the default-free promised yield the RT spread is higher than RFV, by up to 30 basis points for the B-rated bond. This is a result of the fact that since the payments upon default will be paid at their promised maturity in the RT case, the higher long-dated interest rates decreases the value of the RT bond relative to the RFV bond

In the humped-shape yield curve, the respective default-free promised yields for the three different maturities are: 9.46%(2-year), 8.25%(10-year), and 7.74% (30-year). The results in the case are similar to the flat curve case since the forward rates assumed flatten out to a constant 7.00% after the humped-shape rise in the early maturities. The positive difference between the RFV and RT assumptions in the 2 and 10-year case even though the hypothetical bond would be a discount coupon bond can be attributed to the higher interest rates assumed in the shorter maturities making any immediate recovery attained in an RFV setting worth relatively less. The difference between the two forms increases as the maturity increases to 30 years as the bond considered becomes a premium coupon bond.

In the downward-sloping yield curve, the respective default-free promised yields for the three different maturities are: 9.59%(2-year), 8.36%(10-year), and 6.07% (30-year). Contrasting with the upward-sloping case we find that for the medium term bonds RFV produces a slightly higher spread compared with the RT assumption despite the fact that the bond is at slight discount to the promised default-free yield. This is the opposite effect seen in the upward sloping case. Lower interest rates increase the recovery value under an RT assumption on a relative basis versus RFV. In the case of the 30-year maturity bond the significant premium of the coupon rate versus the default-free yield (8.00% versus 6.07%) exacerbates the difference between RFV and RT spreads.

5.4 Implication of Results

Several conclusions can be made from our analysis under a stochastic interest rate setting. First, our base case results, though assuming constant interest rates, are in general robust as long as the yield curve is not too extensively upward or downward-sloping. This implies that using the closed-form solutions presented earlier can be used with confidence in defaultable debt analysis in many interest rate environments. Second, while the coupon premium/discount effect is strong, the yield curve shape can also affect how the recovery form affects model credit spreads. *Ceteris paribus*, in an upward-sloping curve spreads under RT will be higher than RFV spreads, while the opposite holds in a downward-sloping yield curve. This has clear implication for practitioners using exogenous boundary structural credit risk models for relative valuation and risk management. In addition, these results are potentially important for researchers doing empirical work using such models.

Papers such as Eom, Helwege, and Huang (2002) empirically test such models using individual corporate bond price data and often study the prediction errors. Understanding the source of these errors is necessary if we would like to improve on current structural models. However, as these studies are typically done over a long period of time, the term structure setting could vary considerably from one period to another. If recovery forms have an impact on model spreads in a way which depend on both the level and shape of the default-free term structure of interest rates, it is possible that conclusions from such studies may be affected due to the recovery form one is assuming within the model. If the evidence were to convincingly show that one recovery form was consistently used by the market to value bonds, then the results of such studies could be biased.

6 Estimating the Cost of Debt Capital

The final piece of evidence we show in arguing the importance of the recovery form assumption is in its effect on estimating the cost of debt capital for a firm. Since the first structural models were developed it was understood that such models could be used for estimating the expected return premium on debt. A recent paper by Cooper and Davydenko (2002) (" CD") advocates the use of such models for this task as well as estimating the expected risk premium on equity by extracting them from corporate yield spreads. Extraction of the expected return premium or cost of debt capital comes from the following decomposition

$$\text{promised yield spread} = \text{expected default loss} + \text{tax effect} + \text{liquidity effect} + \text{expected return premium}$$

A primary motivation is the fact that a firm's cost of debt is used in calculating its overall cost of capital. As CD mention, this latter number is used in valuation, capital budgeting, goal-setting, performance measurement, regulation, and is perhaps the most important number in corporate finance. While for many firms the promised yield spread can provide a good approximation for this cost of debt, for firms with a significant probability of default this can significantly overestimate the expected return on debt. CD apply the Merton model on individual bond credit spread data taking into account non-default sources of premia, such as liquidity and taxes and are able to obtain sensible estimates.

6.1 Using the Merton Model

There are clear advantages for using a model as simple as the Merton model for such an application. In this model the equity of the firm is directly priced as well. Thus, one can more easily calibrate the model to more observable variables such as equity volatility and equity risk premia rather than unobservable variables such as asset volatility and asset risk premia. In fact, in the calibration method CD propose, only four observable inputs are needed to estimate the cost of debt capital, abstracting from measuring non-default sources of premia: 1) leverage of the firm; 2) bond yield spread; 3) equity volatility; and 4) equity risk premium. They are able to do this by calibrating the maturity of the debt to observable variables. This is due to the fact that the Merton model assumes a single class of zero-coupon debt which makes it quite stylized relative to capital structures observed in reality. However, a single calibrated maturity may be an unneeded constraint in typical capital budgeting applications since financial managers may face project-specific time horizons. Exogenous boundary structural models, as described here, help in overcoming such rigidities. In particular, we can incorporate the contractual details of a firm's individual bond such as the maturity and coupon rate when calculating the expected return premium. This may be particularly relevant if a firm's bonds exhibit a non-flat term structure of credit spreads. In cases where a term structure of expected return premia can be calculated, the cost of debt capital can be chosen to match the project-specific time horizon or be computed as a proper average of these premia.

6.2 Impact of Recovery Forms

We follow both CD and Huang and Huang (2002) ("HH") in calculating the expected return on debt over the holding period of the bond[21]. To do this we need to rewrite the dynamics of the underlying firm asset value under the objective measure:

$$dV_t = (r + \pi - \delta) V_t dt + \sigma_V V_t dz_V^{\mathbb{P}} \tag{15}$$

where π is the asset risk premium. As we are again primarily concerned about the impact of the recovery form we assume this to be constant[22]. The objective cumulative probability of default is

$$\mathbb{P}_t (\tau < T) = N \left(\frac{-x_0 - (\mu_* + \pi)(T - t)}{\sigma_V \sqrt{T - t}} \right) + e^{-\frac{2(\mu_* + \pi) x_0}{\sigma_v^2}} N \left(\frac{-x_0 + (\mu_* + \pi)(T - t)}{\sigma_V \sqrt{T - t}} \right) \tag{16}$$

[21] In contrast to an instantaneous expected return. HH compute the bond risk premia in order to relate equity risk premium to asset risk premium in their calibration exercise.

[22] HH extend their base case to model the asset risk premia as a stochastic variable.

We assume in our analysis that all the parameters are estimated from observable variables.

Using the objective probability of default and the structure of the different recovery forms we can compute the expected return premium of an individual bond given the market credit spread. The detailed equations are found in the Appendix, however, we outline the process here without considering tax or liquidity effects[23]. Given the market spread the bond price is calculated easily. We then implicitly solve for the annualized continuously compounded expected return for each recovery form which matches the market bond price. The spread of this expected return over a comparable default-free bond yield is the expected return premium or cost of debt capital as implied by the individual bond.

Figure 14 plots the expected return premia across the different credit ratings for a hypothetical 10-year maturity 8.00% coupon bond. The relevant information needed for each credit rating (market credit spread, leverage ratio, asset volatility) are all taken from Table I. We assume that the asset risk premium is constant across all credit rating classes at 4.50%. This is consistent with evidence found in CD. As we have before we consider the expected recovery rate to be 51.31% and the default boundary to be 60% of total liabilities. The figure also plots the corresponding promised yield spread for each credit rating. The results are consistent with CD which find that for low credit ratings the promised yield spread can differ substantially from the expected return premium. Only in the lowest two credit ratings, Ba and B, do we find that the expected return premium is noticeably different from the promised yield spread. This is especially the case for the B-rating class. In this case the promised yield spread is taken as 400 bps while the cost of debt capital for the three recovery forms are 167 bps (RT), 187 bps (RFV), and 60 bps (RTF). It follows that the recovery form can affect the cost of debt capital as estimated within these types of model in the relevant case of low-grade bonds.

We earlier hypothesized that a case where using an exogenous default boundary approach might have an advantage over the CD implementation of the Merton model is if a firm has a non-flat term structure of credit spreads. It is useful to know how important the maturity parameter is in affecting the results. Figure 15 and Table IV show the calcu-

[23]In practice, doing so would be important but such factors do not help in distinguishing between the different recovery forms. If one were to want to do this the methods proposed by CD could be implemented.

lated cost of debt capital versus maturity for a 8.00% coupon B-rated bond for the three different recovery forms assuming that the promised yield spread stays fixed at 400 bps and the risk-free interest rate is 8.00%. We see that the maturity of the bond considered does matter in calculating the cost of debt within these models. For example, in the RFV case if we take the spread from a 30-year bond the expected return premium calculated will be 277 bps while for a 5-year bond it will be 151 bps. An interesting feature seen in this figure is that, although the parameters are chosen such that the RT and RFV recovery forms would produce identical theoretical prices, they produce different model expected return premia (RFV is higher than RT) under the objective measure. This is a result of calibrating the cost of debt capital to market prices. For the case examined, the sensitivities of the expected payoffs to the implied expected return (y^*) is more negative for RT than RFV, a similar point to that seen in Figure 8. As a result, since expected payoffs have changed due to moving from the risk-neutral to the objective measure, the y^* that makes the discounted value of such payoffs equal to the market price will need to increase more in the RFV case compared to the RT case. In turn, for this particular example, the RFV expected return premia is higher than that implied by the RT recovery form.

As a point of reference we also show how results from the Merton model would change if we were to assume the same relevant parameters and changed the maturity. The reason that the Merton model generates lower return premia at early maturities is due to our assuming the default boundary in the base case exogenous boundary model to be 60% of the total liabilities rather than 100%. As the maturity increases the potential for a default occurring before maturity (which is restricted in the Merton model) increases the default portion of the boundary models making the expected return premium higher for the Merton model.

However, our objective here is not to compare and contrast exogenous boundary models with the Merton model. That is an empirical question which can only be answered with a suitable time series of data. We focused on showing that the recovery form can matter in estimating the cost of debt capital using a structural model. There are trade-offs one has to make in deciding whether to use the Merton model as suggested by CD or the slightly more complicated exogenous boundary model. The former leads to a more straightforward[24] implementation using observable variables while the latter can take into account the contractual features of the individual bond which serves as the input to the calculation.

7 Concluding Remarks

We set out in this paper to answer the question: "How important is the RFV assumption for the fundamental valuation and hedging of corporate debt securities?" This was done by analyzing the impact of different recovery forms on prices and sensitivities within exogenous boundary structural credit risk models. We found that, indeed, the recovery form can be extremely important for those bonds with a non-trivial probability of default. Different recovery forms have been seen in exogenous boundary models found in the literature and we have shown here that such an assumption can have implications for the predictions of the models. Previous literature may have attributed features to a particular model that were directly related to a chosen recovery form. We provided most results in a simple constant boundary setting holding interest rates constant. However, our extension to a multifactor stochastic default-free term structure provided a robustness check for our main results.

Our results on comparative prices and sensitivities answer our primary question. Further contributions are shown in this paper. We demonstrated that implications from assuming a RFV recovery form within an exogenous model can generate certain results which up to now have only been seen in endogenous models. Our results from the stochastic interest rate setting showed that empirical studies on exogenous boundary models need to understand the effect of the recovery form assumed before making any definite conclusions. On the practical side, our research should have direct relevance for the relative valuation and risk management of junk bonds as well as the estimation of the cost of debt

[24]It should be noted that within the exogenous boundary models that we can impute the asset volatility from equity volatility if we make an assumption about how equity is valued.

capital for low-grade firms.

There are some direct extensions to our work here. One issue we ignored for exposition is the inclusion of any accrued interest from the last due coupon payment prior to default. Many indentures allow debtholders this partial coupon as part of their claim. The effect of such an extension can easily be analyzed in our simulation framework. A major assumption we have made, as is typical in this literature, is that the expected recovery rate is constant and the same for all recovery forms. Using a constant value for this parameter allows us to easily use historical information on recovery rates, which indeed are typically measured as a recovery of face value at default. However, evidence shows that recovery rates decrease with default rates and in general are subject to economic conditions which may be firm-specific, industry-specific, as well as economy-wide. Although we have not attempted to construct such a robust specification of the recovery process, we argue that such a model would be affected by the choice of the recovery form. More specifically, the chosen form needs to be consistent with empirical evidence on defaulted bond values. We believe RFV is the most natural assumption to make given the evidence.

A criticism of the constant exogenous default boundary model we analyzed is that it implies that expected leverage ratios will decline exponentially over time. We made such an assumption for computational tractability and because we were mainly interested in comparing the alternative recovery forms. Figure 3 provides evidence that our analysis is robust to richer specifications of the default boundary process, such as that found in Collin-Dufresne and Goldstein (2001).

Other planned extensions involve empirical data. Although there may be evidence that RFV is the most appropriate recovery form to describe corporate debt securities at default, we cannot necessarily say that this is *the* recovery form incorporated into bond prices prior to default. Both in RT and RT-F the expected recovery rate can potentially be made a function of time to produce RFV-consistent recovery values at default. Indeed, Bakshi, Madan, and Zhang (2002) found in an intensity-model framework that a robust implementation of RT-F fit their sample of bond prices the best. Performing a similar study in a structural credit risk market would be useful. A way which we can compare the recovery forms that the previous study does not consider is by determining which recovery form best matches empirical sensitivities. As an example, studies have consistently shown that credit spreads and interest rates are negatively related. An often ignored important feature of structural credit risk models is that they provide an economic theory on how interest rates should affect credit spreads. We have shown here that the recovery form plays an important part in this theory but what remains to be seen is if the data supports the sensitivity to interest rates of one form versus another. Acharya and Carpenter (2002) have provided some indirect evidence concerning this which supports RFV, but more work is needed directly applying such models on individual corporate bond prices in a dynamic term structure setting. In future research we plan to conduct such an analysis.

While an empirical-based comparison and validation of the different recovery forms is useful, another approach could be taken. As seen in Table III the different recovery forms can generate much different term structure of credit spreads for low-grade firms under realistic term structure environments. With an appropriate dataset we could use the model spreads predicted by each recovery form as a relative valuation tool considering bonds of the same issuer and seniority. Using a feasible trading strategy and comparing the excess returns from implementing models using each recovery form would provide an additional test.

Table I: Parameter Values for Various Credit Grades

The table shows the leverage ratio and asset volatility parameters across credit ratings used for computing model spreads for representative companies of that particular credit rating. Assumed "market" credit spreads for the different credit ratings are also given. The numbers are taken from Davydenko and Strebulaev(2002). Parameters used in the model which are invariant across credit rating classes are taken from Huang and Huang (2002).

Credit Rating	**Leverage Ratio**	**Asset Vol**	**Credit Spread**
Aaa	.12	.22	48
Aa	.15	.24	55
A	.29	.24	81
Baa	.36	.25	120
Ba	.45	.28	223
B	.64	.37	400

Table II: Recovery Form and Credit Spreads: Constant Interest Rates

Calculated yield spreads are shown in basis points across different credit rating classes for 2, 10, and 30-year maturities using a first passage defaultable debt model. Parameter assumptions are described in Table I and Section 3. The hypothetical bond pays a semi-annual coupon at an annual rate of either 8.00% (Par Bond), 12.00% (Premium Bond), or 4.50% (Discount Bond). The recovery assumptions considered are: 1) Recovery of Treasury (RT), 2) Recovery of Treasury - Face Value (RT-F), and 3) Recovery of Face Value (RFV).

Panel A: 2-Year Maturity

	Par Bond			**Premium Bond**			**Discount Bond**		
	RT	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*
Aaa	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Aa	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
A	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Baa	0.05	0.05	0.05	0.04	0.04	0.04	0.04	0.04	0.04
Ba	2.97	3.13	3.07	2.91	3.12	3.05	3.05	3.13	3.08
B	224.73	240.15	229.62	221.29	243.38	233.33	228.01	237.06	226.08

Panel B: 10-Year Maturity

	Par Bond			**Premium Bond**			**Discount Bond**		
	RT	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*
Aaa	0.07	0.08	0.07	0.06	0.07	0.07	0.08	0.08	0.07
Aa	0.78	0.88	0.79	0.70	0.83	0.76	0.87	0.94	0.83
A	10.58	12.57	10.76	9.77	12.31	10.78	11.55	12.88	10.74
Baa	27.67	33.91	28.13	25.94	33.86	28.97	29.79	33.95	27.12
Ba	83.55	107.94	84.90	80.08	111.04	91.46	87.82	104.14	76.88
B	319.45	473.63	324.31	320.14	517.02	386.64	318.59	420.74	250.63

Panel C: 30-Year Maturity

	Par Bond			**Premium Bond**			**Discount Bond**		
	RT	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*
Aaa	2.67	4.00	2.70	2.39	3.84	2.89	3.18	4.31	2.35
Aa	8.14	12.86	8.22	7.49	12.61	9.25	9.33	13.32	6.36
A	27.13	46.67	27.33	26.01	47.15	33.12	29.18	45.79	16.94
Baa	46.20	83.15	46.50	45.06	84.98	58.28	48.30	79.77	25.58
Ba	92.70	180.48	93.20	92.27	186.94	122.54	93.49	168.45	42.73
B	249.71	617.81	250.76	255.22	652.11	364.25	239.21	551.45	80.97

Table III: Recovery and Credit Spreads: Stochastic Interest Rates

Calculated yield spreads are shown in basis points across different credit rating classes for 2, 10, and 30-year maturities using the first passage firm value model described in Section 2 combined with the default-free multifactor term structure model described in Section 5. The parameters for the firm value model are described in Table I while the parameters for the term structure model are described in Section 5. We consider a hypothetical bond paying a semi-annual coupon at an annual rate of 8.00%. We consider four different initial interest rate environments as described in the Appendix: 1) Flat Curve, 2)Upward Sloping Curve, 3)Humped Shape Curve, and 4)Downward Sloping Curve. The respective default-free promised yields for the 2,10, and 30-year maturity bonds under the different environments are: 1)8.00%(2-year), 8.00%(10-year), 8.00%(30-year); 2)4.67%, 8.00%, 9.07%; 3)9.46%, 8.25%, 7.74%; and 4)9.59%, 8.36%, 6.07%. The recovery assumptions considered are: 1) Recovery of Treasury (RT), 2)Recovery of Treasury - Face Value (RT-F), and 3) Recovery of Face Value (RFV).

Panel A: 2-Year Maturity

	Flat Curve			**Upward Sloping**			**Humped Shape**			**Downward Sloping**		
	RT	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*
Aaa	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.03	0.03	0.02	0.02	0.02
Aa	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.02	0.02	0.00	0.00	0.00
A	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.02	0.02	0.02	0.02	0.02
Baa	0.00	0.00	0.00	0.17	0.17	0.17	0.07	0.08	0.07	0.02	0.02	0.02
Ba	2.53	2.65	2.59	4.57	4.79	4.71	2.07	2.15	2.11	2.18	2.27	2.22
B	226.97	242.49	231.70	279.29	298.51	289.08	204.74	219.08	205.77	204.90	218.96	207.65

Panel A: 10-Year Maturity

	Flat Curve			**Upward Sloping**			**Humped Shape**			**Downward Sloping**		
	RT	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*
Aaa	0.25	0.26	0.25	0.21	0.22	0.21	0.16	0.16	0.16	0.03	0.03	0.03
Aa	0.93	1.03	0.94	0.84	0.95	0.81	0.64	0.73	0.67	0.41	0.48	0.44
A	9.98	11.79	10.18	10.27	12.53	9.78	10.33	12.12	10.70	9.99	11.69	10.43
Baa	26.75	32.87	27.24	27.01	33.97	25.45	26.82	32.45	27.94	24.52	29.53	25.61
Ba	83.22	107.32	84.22	82.65	110.17	76.27	76.28	96.86	79.27	77.11	98.21	80.25
B	315.85	467.62	318.48	328.43	504.93	299.58	307.38	448.29	317.68	306.73	447.28	315.66

Panel A: 30-Year Maturity

	Flat Curve			**Upward Sloping**			**Humped Shape**			**Downward Sloping**		
	RT	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*	*RT*	*RT-F*	*RFV*
Aaa	3.46	5.23	3.71	1.28	1.77	1.17	4.91	7.33	5.59	17.67	23.91	23.29
Aa	9.85	15.24	10.27	3.25	5.36	2.69	13.02	19.89	14.39	31.76	45.35	43.31
A	29.63	50.70	30.68	14.64	26.67	10.77	35.51	59.61	39.76	67.72	104.12	95.29
Baa	47.74	85.47	48.88	29.14	54.59	20.54	55.31	97.64	61.88	89.52	145.64	128.47
Ba	93.70	182.24	94.97	71.71	143.45	48.35	100.73	192.27	112.52	134.75	239.87	197.62
B	246.70	608.51	246.03	240.81	610.73	158.81	250.35	605.49	278.83	266.79	604.24	400.51

Table IV: Cost of Debt Capital versus Maturity: B-rated Bond

The table the expected return premium (cost of debt capital) in basis points as described in Section 6 for a hypothetical B-rated 8.00% coupon bond for different maturities. The promised yield spread is assumed equal to 400 bps across all maturities. Three different recovery forms within the base case exogenous default boundary model are considered: Recovery of Treasury (RT), Recovery of Face Value at Default (RFV), and Recovery of Treasury-Face Value (RT-F). Also presented are calculations using the Merton (1974) model. The promised yield spread, leverage ratio and asset volatility for the B credit rating are taken from Table I. The asset risk premium is assumed constant at 4.50%. The remaining parameter are the constant interest rate (8.00%), payout rate (6.00%), default boundary (60% of total liabilities), and expected recovery rate (51.31%).

Maturity (yrs)	**RT**	**RFV**	**RT-F**	**Merton**
2	235	234	224	115
5	147	151	96	141
10	167	187	60	220
20	201	251	22	306
30	211	277	-8	344

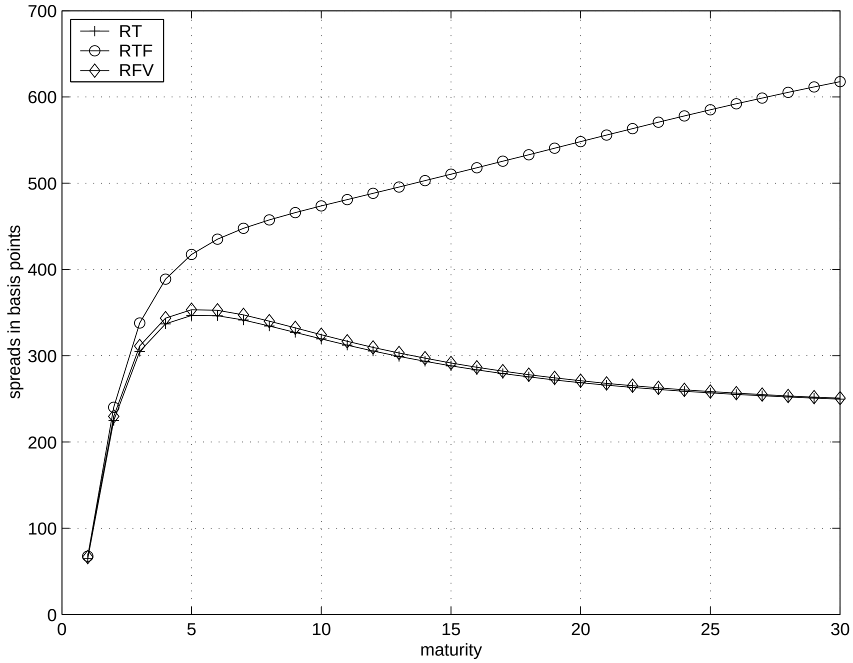

Figure 1: Term Structure of Credit Spreads: B Rating

The plot shows the term structure of credit spreads of a hypothetical company with an speculative-grade B-credit rating for the Recovery of Treasury (RT), Recovery of Treasury-Face Value (RT-F), and Recovery of Face Value (RFV) recovery assumptions. The coupon (8.00%) is chosen to equal the risk-free interest rate. The leverage ratio (64%) and asset volatility (37%) parameters are taken from Table I. Other parameters, include the constant interest rate (8.00%), constant payout ratio (6.00%), default boundary (60% of total liabilities) and expected recovery rate (51.31%).

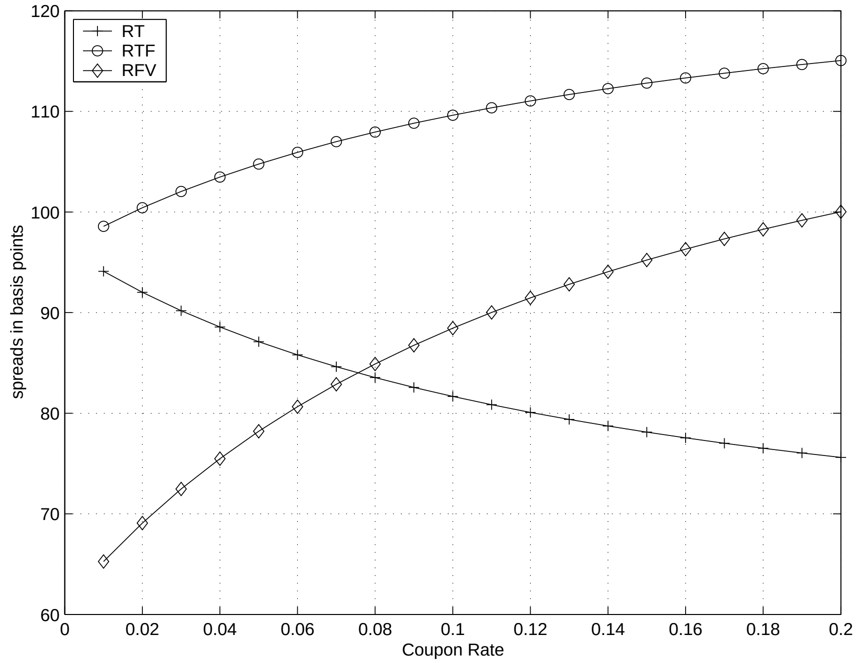

Figure 2: Credit Spreads vs. Coupon Rates: 10-Year Ba Rated Bond

The plot shows credit spreads of a hypothetical 10-year bond issued by a Ba-rated firm with different coupon rates assuming a constant risk-free interest rate of 8.00%. The Recovery of Treasury (RT), Recovery of Treasury-Face Value (RT-F) and Recovery of Face Value (RFV) recovery assumptions are shown. The leverage ratio (45%) and asset volatility (28%) parameters are taken from Table I. Other parameters, include the constant payout ratio (6.00%), default boundary (60% of total liabilities) and expected recovery rate (51.31%).

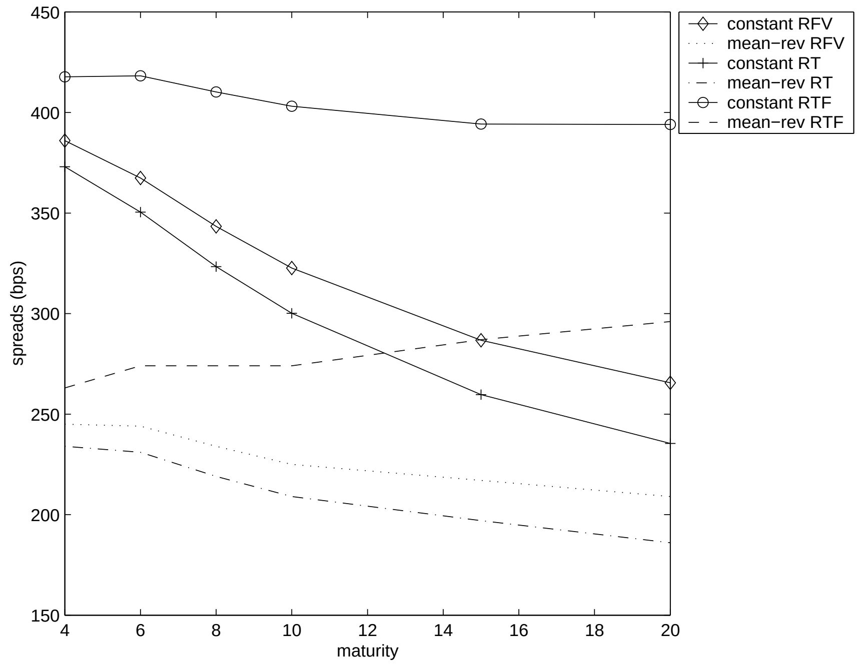

Figure 3: Constant Default Boundary vs. Mean-Reversion Model: Speculative-Grade Bonds

The plot shows the term structure of credit spreads for speculative-grade bonds in both the base-case constant default boundary model and the mean-reversion model of Collin-Dufresne and Goldstein (2001)("CDG") for three different recovery forms (RFV, RT, RT-F). In their paper CDG assume an RT-F recovery form only. The parameters used are taken from Figure 3 in their paper. Parameters common to both models include the constant risk-free interest rate (6.00%), coupon rate (7.50%), payout rate (3.00%), asset volatility (20%), expected recovery rate (44%), and the initial leverage ratio (65%). For the constant boundary model the default boundary is set to equal 100% of total liabilities. For the mean-reversion model the long-term leverage ratio is set equal to 40% and the mean-reversion parameter λ is set to 0.18.

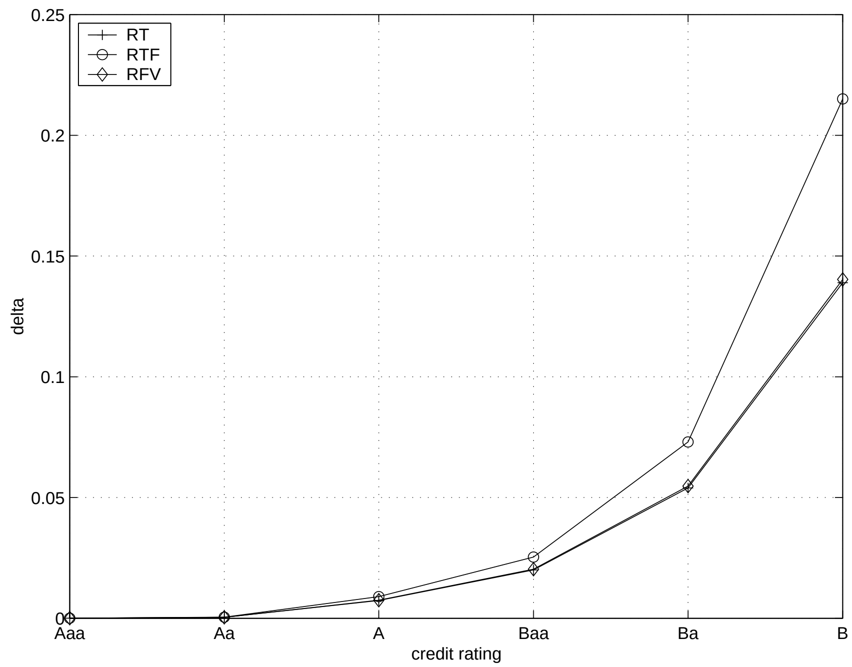

Figure 4: Coupon Bond Deltas vs. Credit Rating: 10-year Maturity

The plot shows the sensitivities to firm asset value (deltas) across different credit ratings for a 10-year maturity bond with a coupon equal to the risk-free interest rate (8.00%). The Recovery of Treasury (RT), Recovery of Treasury-Face Value (RT-F), and Recovery of Face Value (RFV) recovery assumptions are considered. The assumed leverage ratio and asset volatility parameters which vary by credit rating are given in Table I. The remaining parameters assume are the constant payout ratio (6.00%), default boundary (60% of total liabilities) and expected recovery rate (51.31%).

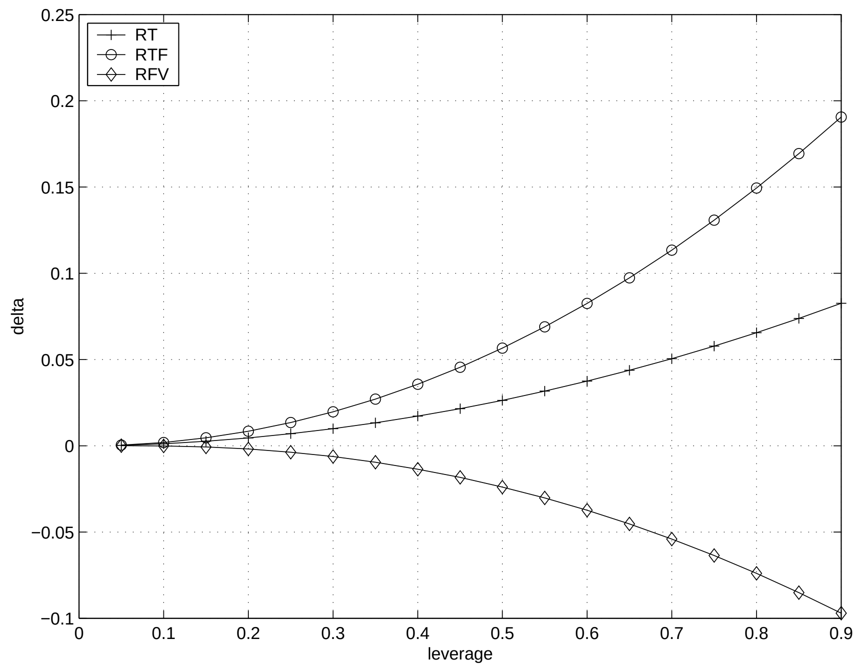

Figure 5: Coupon Bond Deltas vs. Leverage: 30-year Maturity

The plot shows the sensitivities to firm asset value (deltas) against the leverage ratio for a 30-year maturity bond. The coupon rate assumed for the bond is 3.00%. The Recovery of Treasury (RT), Recovery of Treasury-Face Value (RT-F), and Recovery of Face Value (RFV) recovery assumptions are considered. The parameters assumed are the constant risk-free interest rate (8.00%), the asset volatility (30.00%), the constant payout ratio (6.00%), default boundary (60% of total liabilities) and expected recovery rate (60.00%).

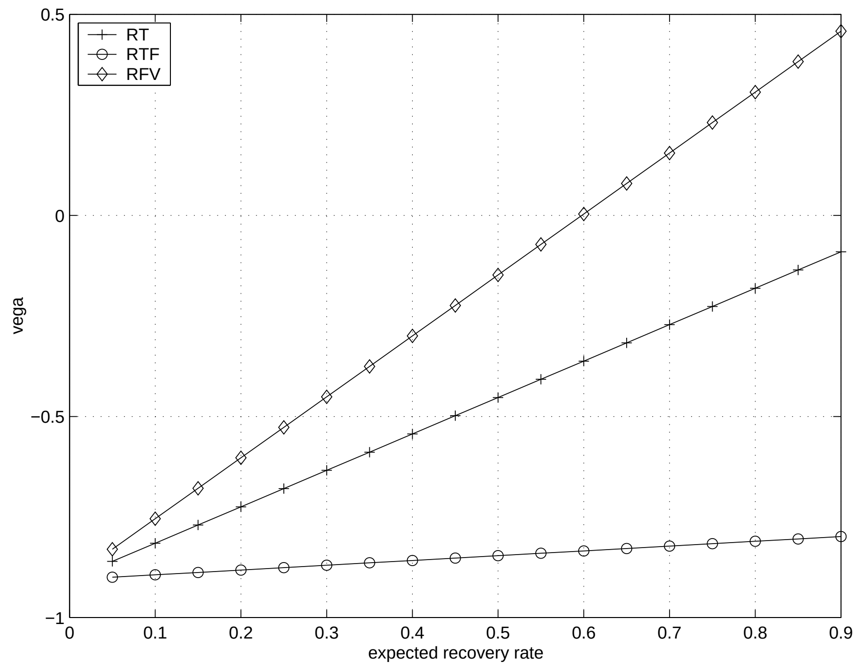

Figure 6: Coupon Bond Vegas vs. Expected Recovery Rate: 30-year Maturity B-Rated Bond

The plot shows the sensitivities to firm asset volatility(vegas) against the expected recovery rate for a 30-year maturity bond. The coupon rate assumed for the bond is 4.50%. The Recovery of Treasury (RT), Recovery of Treasury-Face Value (RT-F), and Recovery of Face Value (RFV) recovery assumptions are considered. The leverage (65%) and asset volatility (37%) parameters are chosen to match a B-rated company. The other parameters assumed are the constant risk-free interest rate (8.00%), the constant payout ratio (6.00%), and default boundary (60% of total liabilities).

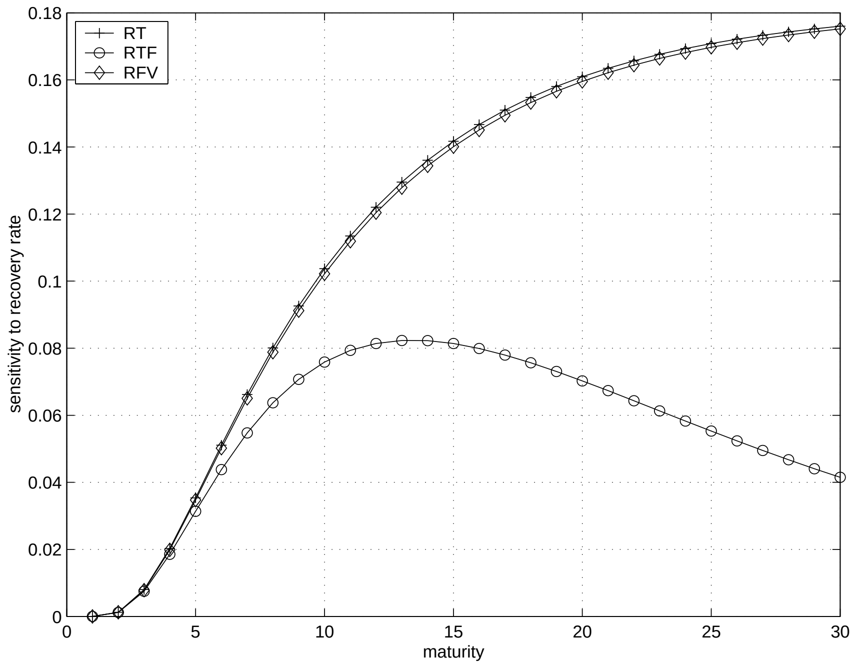

Figure 7: Term Structure of Recovery Sensitivity: Baa Rating

The plot shows the term structure of sensitivities to the expected recovery rate parameter of a hypothetical company with a Baa credit rating for the Recovery of Treasury (RT), Recovery of Treasury-Face Value (RT-F), and Recovery of Face Value (RFV) recovery assumptions. The coupon (8.00%) is chosen to equal the risk-free interest rate. The leverage ratio (36%) and asset volatility (25%) parameters are taken from Table I. Other parameters, including the constant interest rate (8.00%), constant payout ratio (6.00%), default boundary (60% of total liabilities) and expected recovery rate (51.31%) are taken from Huang and Huang(2002).

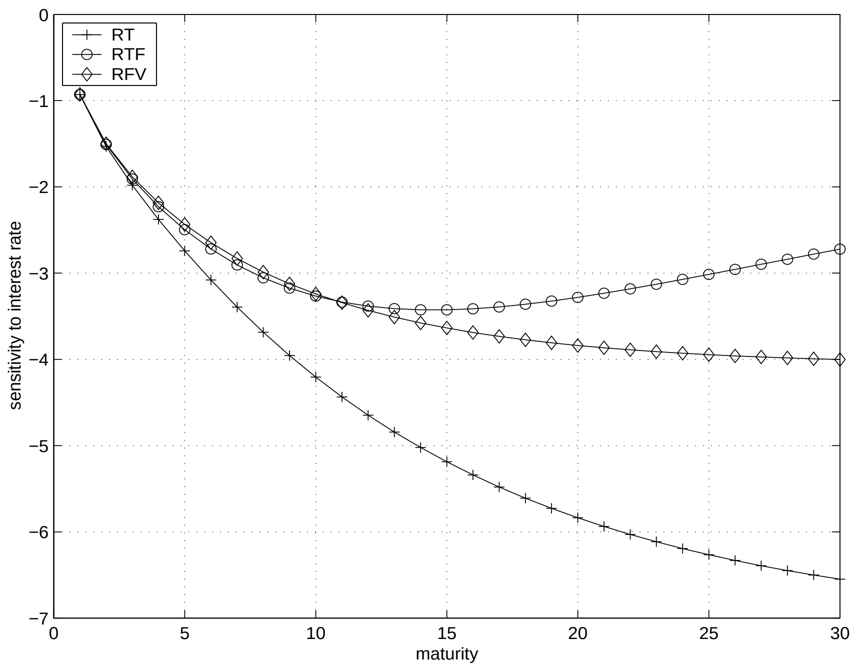

Figure 8: Term Structure of Interest Rate Sensitivity: B Rating

The plot shows the term structure of sensitivities of coupon bond prices to the interest rate for a hypothetical company with a speculative-grade B credit rating using the Recovery of Treasury (RT), Recovery of Treasury-Face Value (RT-F), and Recovery of Face Value (RFV) recovery assumptions. The negative of the derivative values shown on the y-axis can be interpreted as the dollar duration of the relevant bond. The coupon (8.00%) is chosen to equal the risk-free interest rate. The leverage ratio (64%) and asset volatility (37%) parameters are taken from Table I. Other parameters, include the constant interest rate (8.00%), constant payout ratio (6.00%), default boundary (60% of total liabilities) and expected recovery rate (51.31%).

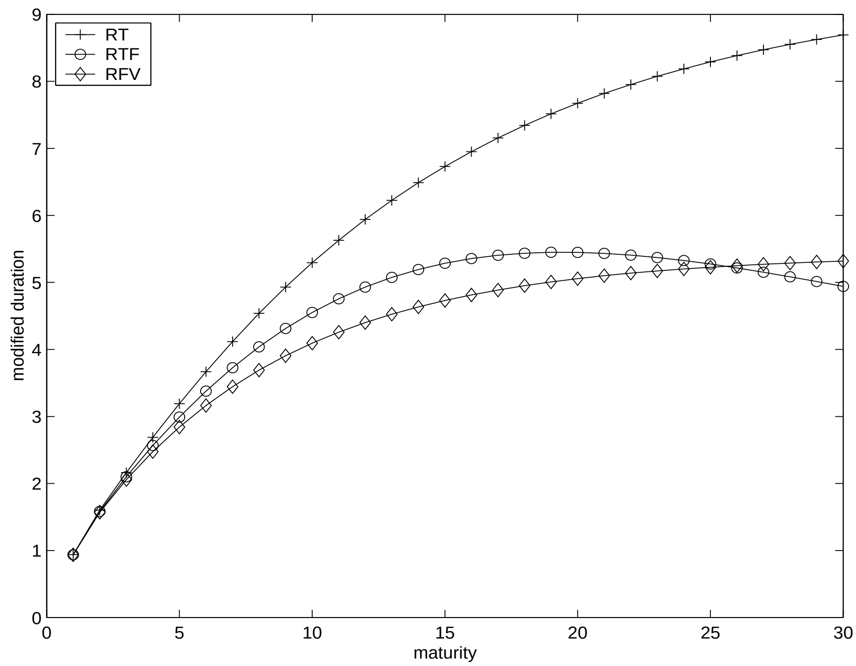

Figure 9: Term Structure of Modified Durations: B Rating

The plot shows the term structure of modified durations for a hypothetical company with a speculative-grade B credit rating using the Recovery of Treasury (RT), Recovery of Treasury-Face Value (RT-F), and Recovery of Face Value (RFV) recovery assumptions. The modified duration is defined as the negative of the derivative divided by the relevant model price. The coupon (8.00%) is chosen to equal the risk-free interest rate. The leverage ratio (64%) and asset volatility (37%) parameters are taken from Table I. Other parameters, include the constant interest rate (8.00%), constant payout ratio (6.00%), default boundary (60% of total liabilities) and expected recovery rate (51.31%).

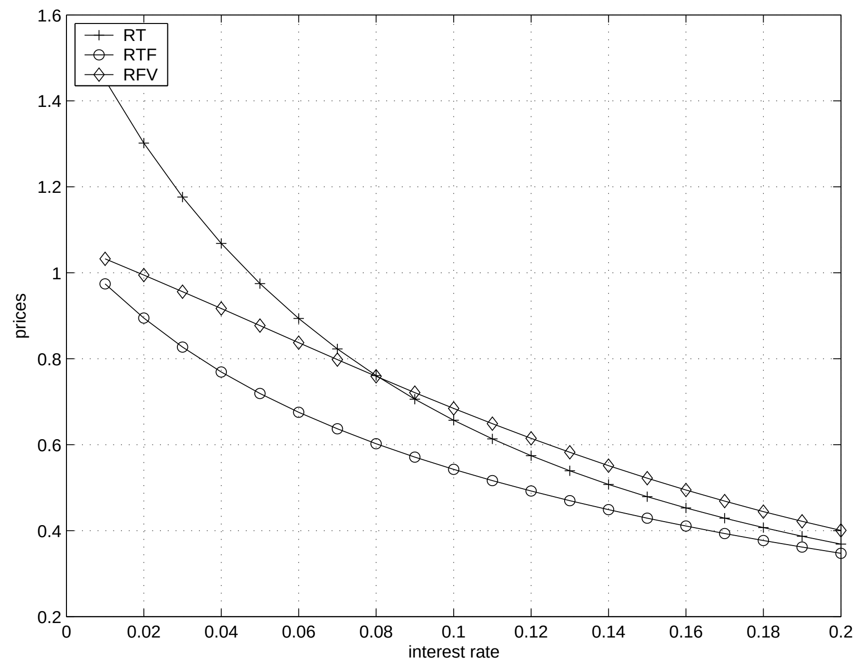

Figure 10: Coupon Bond Prices vs. Interest Rate: 20-year Maturity B-Rated Bond

The plot shows the model bond prices against the risk-free interest rate for a 20-year maturity bond. The leverage ratio (64%) and asset volatility (37%) are chosen to match that of a B-rated company. The coupon rate assumed for the bond is equal to the risk-free interest rate (8.00%). The Recovery of Treasury (RT), Recovery of Treasury-Face Value (RT-F), and Recovery of Face Value (RFV) recovery assumptions are considered. The remaining parameters assumed are the constant payout ratio (6.00%), default boundary (60% of total liabilities) and expected recovery rate (51.31%).

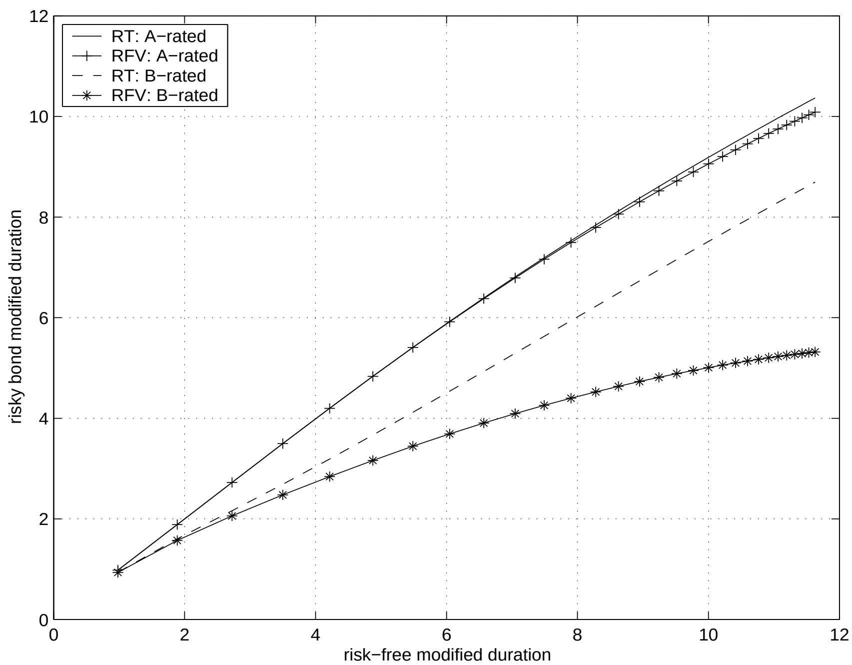

Figure 11: Risky Bond Modified Duration vs. Risk-Free Modified Duration

The plot shows the model modified duration as a function of the risk-free modified duration for A-rated and B-rated 8.00% coupon bonds using the RT and RFV recovery forms. The leverage ratio and asset volatility parameters for the different credit ratings come from Table I. The remaining parameters are: interest rate (8.00%), payout rate (6.00%), default boundary (60% of total liabilities) and expected recovery rate (51.31%). This plot is similar to Figure 5 found in Leland and Toft (1996) which considers an endogenous default boundary model in that the modified duration for the very risky bond flattens out considerably in the RFV case.

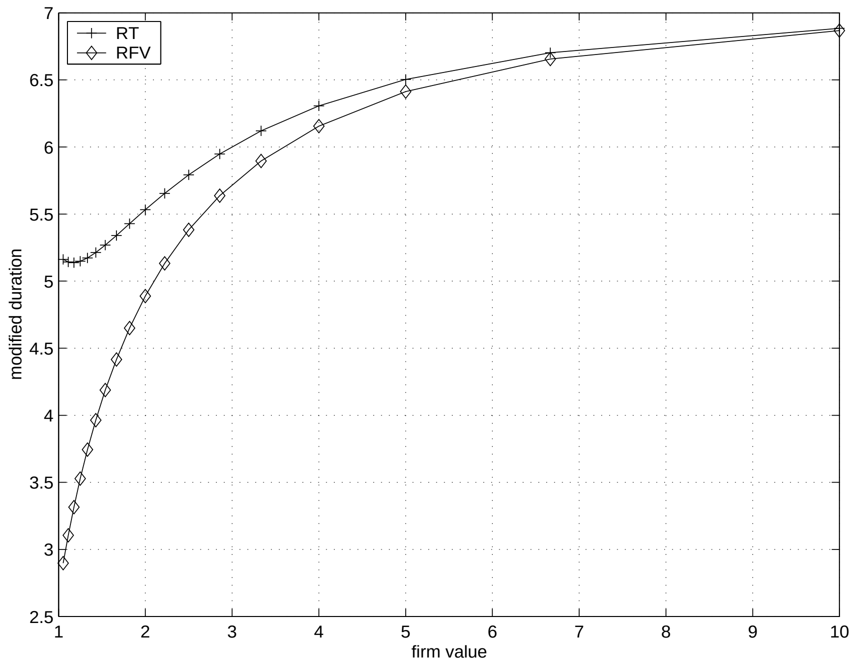

Figure 12: RT and RFV Modified Duration vs. Firm Value

The plot shows the model modified duration of a 8.00% 10-year maturity coupon bond as a function of firm value where firm value is defined as a multiple of total liabilities. We consider two recovery forms: RT and RFV. The parameters assumed are asset volatility (35.00%), interest rate (8.00%), payout rate (6.00%), default boundary (60% of total liabilities) and expected recovery rate (51.31%). This plot is complementary to Figure 4-6 of Acharya and Carpenter (2002) for a pure defaultable bond which compares durations implied by endogenous and exogenous default boundary models. RFV generates a similar shape to their endogenous model.

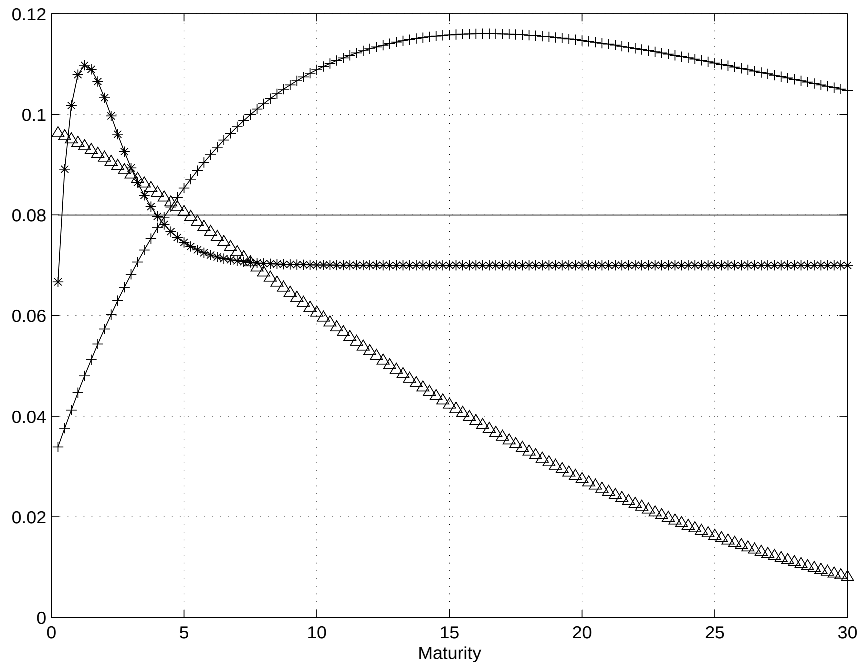

Figure 13: Hypothetical Yield Curve Shapes used in Bond Pricing

The plot shows the 4 different yield curve shapes we assume in comparing defaultable bond pricing in a stochastic interest rate environment. The yield curves are in terms of the instantaneous forward rates and are generated by Nelson-Sielgel(1987) methodology using the parameters given in the Appendix.

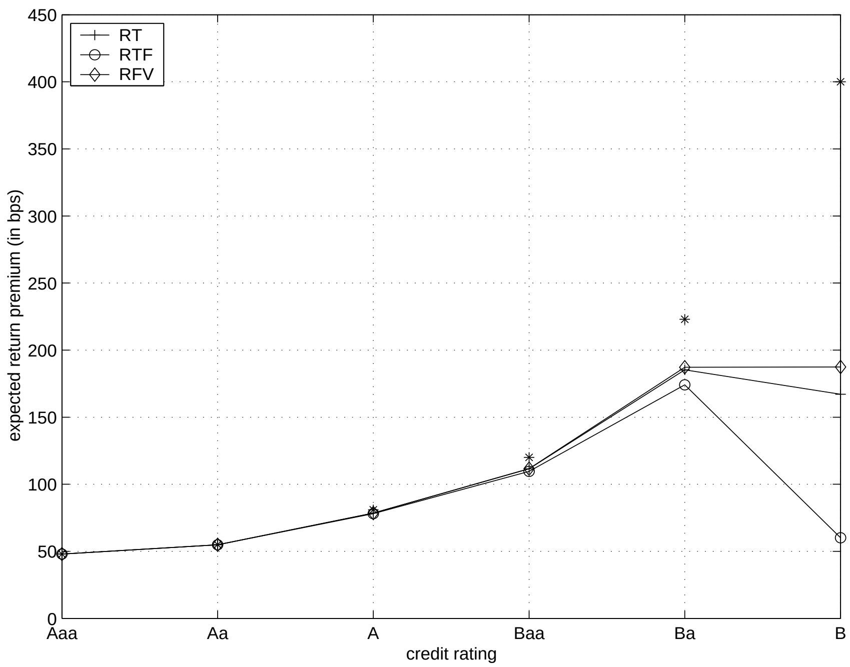

Figure 14: Cost of Debt Capital vs. Credit Rating

The plot shows the expected return premium (cost of debt capital) in basis points for a hypothetical 10-year maturity 8.00% coupon bond by credit rating using three different recovery forms within the base case model: Recovery of Treasury (RT), Recovery of Treasury-Face Value (RT-F), and Recovery of Face Value (RFV). The stars represent the promised yield spreads for each credit rating. The promised yield spread, leverage ratio, and asset volatility for each credit rating are taken from Table I. The asset risk premium is assumed constant across all credit ratings at 4.50%. The remaining parameter are the constant interest rate (8.00%), payout rate (6.00%), default boundary (60% of total liabilities), and expected recovery rate (51.31%).

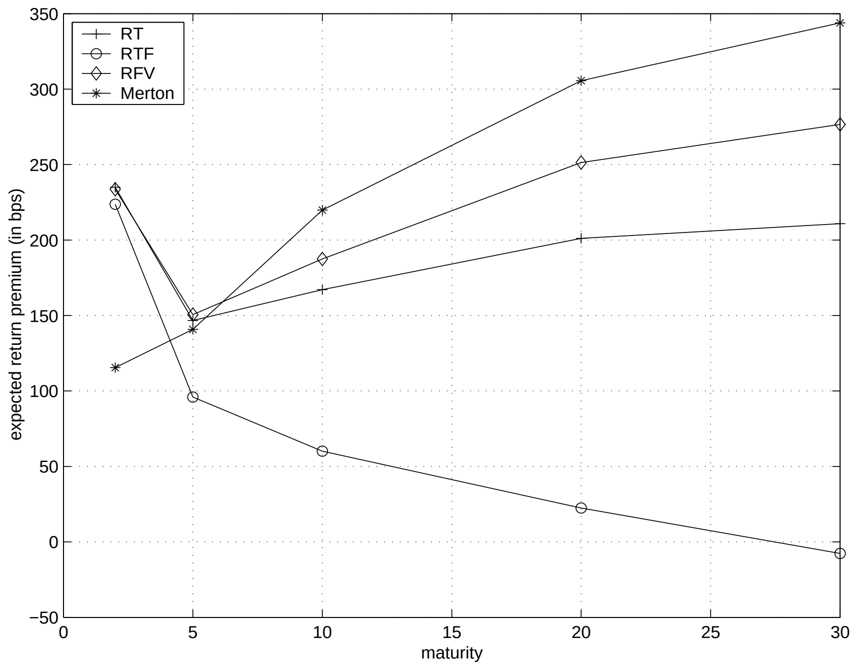

Figure 15: Cost of Debt Capital vs. Maturity: B-rated Bond

The plot shows the expected return premium (cost of debt capital) in basis points for a hypothetical B-rated 8.00% coupon bond versus maturity using three different recovery forms within the base case model: Recovery of Treasury (RT), Recovery of Treasury-Face Value (RT-F), and Recovery of Face Value (RFV), as well as using the Merton model. The promised yield spread, leverage ratio, and asset volatility for the B credit rating are taken from Table I. The asset risk premium is assumed constant at 4.50%. The remaining parameter are the constant interest rate (8.00%), payout rate (6.00%), default boundary (60% of total liabilities), and expected recovery rate (51.31%).

APPENDIX

A Default-Free Term Structure Model Specification

Under the standard assumption of no-arbitrage we can write the value of a default-free riskless zero coupon bond at time t with maturity date T as

$$P(t,T) = E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right)\right] \tag{A1}$$

where the expectation is taken under the risk-neutral probability measure Q, and $r(t)$ is the process for instantaneous riskless nominal short rate. We consider a term structure model in the multivariate affine class as described by Brown and Schaefer (1994) , Duffie and Kan (1996) and Dai and Singleton (2000). We introduce as the two factors: $l(t)$ and $s(t)$. The new dynamic system of equations, again under Q, is

$$\begin{aligned} dl(t) &= -\kappa_l l(t)dt + \sigma_l dz_l^Q \\ ds(t) &= -\kappa_s s(t)dt + \sigma_s dz_s^Q \\ r(t) &= g(t) + l(t) + s(t) \end{aligned} \tag{A2}$$

where $E(dz_l^Q dz_s^Q) = \rho_{ls}dt$ and the initial values of both l and s are equal to zero. We can refer to κ_l and κ_s as *reversion* parameters of the two factors; while σ_l and σ_s can be termed *volatility* parameters.

Solution and Calibration of Model We substitute our affine representation of the short rate into the basic bond valuation equation (A1)

$$\begin{aligned} P(t,T) &= E_t^Q\left[\exp\left(-\int_t^T (g(u) + l(u) + s(u))du\right)\right] = \\ &\exp\left(-\int_t^T g(u)du\right) \cdot E_t^Q\left[\exp\left(-\int_t^T (l(u) + s(u))du\right)\right] \end{aligned} \tag{A3}$$

We see from the right hand side of the equation that the bond value can be viewed as the product of a deterministic factor and a "bond price" with zero mean factors. The expectation on the right hand side has a well-known solution form as seen in Duffie and Kan (1996) so we can write the bond value as

$$P(t,T) = \exp\left(-\int_t^T g(u)du\right) \cdot \exp\left[A(T-t) - B_l(T-t)l(t) - B_s(T-t)s(t)\right] \tag{A4}$$

where after solving the PDE and denoting $\tau = T - t$, we find

$$\begin{aligned} B_l(\tau) = \frac{1-\exp(-\kappa_l\tau)}{\kappa_l} &\quad B_y(\tau) = \frac{1-\exp(-\kappa_s\tau)}{\kappa_s} \\ A(\tau) = \int_0^\tau A'(u)du \text{ where } A'(\tau) &= \frac{1}{2}\sigma_l^2 B_l^2(\tau) + \frac{1}{2}\sigma_s^2 B_s^2(\tau) + \rho_{ls}\sigma_l\sigma_s B_l(\tau)B_s(\tau) \end{aligned} \tag{A5}$$

We would like to calibrate the model to be able to fit the initial yield curve. First we write the equation for the short rate path which is the solution to the term structure model (A2)

$$r(t) = g(t) + \sigma_l e^{-\kappa_l t}\int_0^t e^{\kappa_l u}dz_l(u) + \sigma_s e^{-\kappa_s t}\int_0^t e^{\kappa_s u}dz_s(u) \tag{A6}$$

It is evident that the time dependent intercept $g(t)$ will embed any information regarding the initial yield curve. From (A4) we can express the initial yield curve as the set of zero coupon bond prices at time 0

$$P(0,T) = \exp\left[-\int_0^T g(u)du + A(T)\right] \tag{A7}$$

since $l_0 = 0$ and $s_0 = 0$ by construction. Taking logs and differentiating with respect to T on both sides of the equation and then substituting from (A5)we arrive at

$$g(T) = -\frac{d\log P(0,T)}{dT} + \frac{dA(T)}{dT} = \tag{A8}$$

$$f(0,T) + \frac{1}{2}\sigma_l^2 B_l^2(T) + \frac{1}{2}\sigma_s^2 B_s^2(T) + \rho_{ls}\sigma_l\sigma_s B_l(T)B_s(T) \tag{A9}$$

where $f(0,T)$ is the initial instantaneous forward rate curve. We can substitute this term into $[A6]$ to arrive at our new expression for the short rate path

$$\begin{aligned} r(t) \quad = \quad & f(0,t) + \frac{1}{2}\sigma_l^2 B_l^2(t) + \frac{1}{2}\sigma_s^2 B_s^2(t) + \rho_{ls}\sigma_l\sigma_s B_l(t)B_s(t) \\ & + \sigma_l e^{-\kappa_l t}\int_0^t e^{\kappa_l u} dz_l(u) + \sigma_s e^{-\kappa_s t}\int_0^t e^{\kappa_s u} dz_s(u) \end{aligned} \tag{A10}$$

From this expression it is clear how the short rate path used to value all interest rate sensitive claims can be calibrated to initial market data – via the forward rate curve. All that is needed for implementation are the time-invariant parameters: κ_l, $\kappa_s, \sigma_l, \sigma_s$, and ρ_{ls}.

B Simulation Pricing Procedure

This section describes the Monte Carlo simulation procedure used for pricing the defaultable bonds under a stochastic interest rate environment with the term structure model specified above in (14). The dynamic system of equations with the correlated processes are:

$$\begin{aligned} dl(t) &= -\kappa_l l(t)dt + \sigma_l dz_l^Q \\ ds(t) &= -\kappa_s s(t)dt + \sigma_s dz_s^Q \\ dx_t &= (r(t) - \delta - \frac{\sigma_V^2}{2})dt + \sigma_V dz_V^Q \\ r(t) &= g(t) + l(t) + s(t) \end{aligned} \tag{B1}$$

where $E(dz_l^Q dz_s^Q) = \rho_{ls}dt, E(dz_l^Q dz_V^Q) = \rho_{lV}dt, E(dz_s^Q dz_V^Q) = \rho_{sV}dt$. The initial values of $l(t)$ and $s(t)$ are equal to zero, while the initial value of $x(t)$ is x_0. Default occurs the first time $x(t)$ hits zero. It is useful to do a Cholesky factorization of the correlation matrix $\mathbf{\Sigma}$ of the random processes $z_i, i = \{V, l, s\}$ so we can write the system in terms of independent Wiener processes $w_i, i = \{V, l, s\}$. That is we find the lower triangular matrix $\mathbf{M}$ such that

$$\mathbf{z} = \mathbf{Mw}$$

and

$$\mathbf{MM'} = \mathbf{\Sigma}$$

After we have rewritten the dynamics in terms of the independent processes $w_i, i = \{V, l, s\}$ we discretize our continuous time equations at equal time steps Δt. For $x(t)$, we do a simple Euler discretization. For $l(t)$ and $s(t)$ we discretize their continuous closed-form solutions[25].

Once we have discretized our dynamic processes and chosen an appropriate time step we produce N random paths as follows. First we generate independent standard normal variates $\epsilon_i, i = \{l, s, V\}$ each

[25] The increments of the correlated processes would be replaced by independent Wiener processes multiplied by the Cholesky factorization matrix.

vectors of length $\frac{N}{2}$ at each time step for the three independent stochastic terms. We then use the antithetic variable technique which means that for the remaining $\frac{N}{2}$ paths we use -ϵ_i. Thus, for each time step we generate $\left[\begin{array}{c} \epsilon_i \\ -\epsilon_i \end{array}\right], i = \{l, s, V\}$. With this methodology we can generate N paths of x and the short rate r. Bond valuation is attained as follows. First, the promised cash flows would be generated for each path. Second, if default has occurred for a specific path the promised cash flows would be altered in a way specific to the recovery form assumed. Once the default-adjusted cash flow paths have been generated we discount each cash flow path at the *relevant* riskfree short rate path. That is the rate path which produced the default path associated with the default-adjusted cash flow path. The mean of these N values can be considered the *crude* Monte Carlo estimated bond price.

A problem with using the crude Monte Carlo price as the estimate is that in almost all practical cases there will be an overpricing bias. This is due to the fact that default, i.e. when $x(t)$ reaches the zero barrier, is in general a highly improbable event, and once we sample our continuous time processes at discrete time intervals we do not get as many "hits" in our simulation as we would expect under our risk-neutral pricing measure. This is due to the fact that for any nonzero time interval discretization induces a bias since many time points at which low values of the variable could occur are ignored. Narrowing the time step shrinks the bias toward zero, but time steps small enough to produce accurate answers may be computationally quite burdensome. To correct this bias we implement the technique developed by Beaglehole, Dybvig, and Zhou (1997). Their technique draws on the theory of the Brownian bridge. After we have generated the paths for $x(t)$ in our crude simulation we treat each discrete time interval as two ends of a Brownian bridge. We then draw the minimum (as we are interested in $x(t)$ reaching zero) of the path process on the interval using the known theoretical distribution of a Brownian bridge on an interval.

C Generating Hypothetical Yield Curves

We generate 4 yield curve shapes: 1) flat curve; 2) upward-sloping curve; 3) humped-shape curve; and 4) downward sloping curve. We produce curves in terms of the instantaneous forward rate curve. Generating the flat curve is straightforward. For the remaining types we used the Nelson and Siegel(1987) methodology which fits term structures using the following form for the initial instantaneous forward rate curve:

$$f(0, m) = \beta_0 + \beta_1 \exp\left(\frac{-m}{\tau}\right) + \beta_2\left(\frac{m}{\tau}\exp\left(\frac{-m}{\tau}\right)\right) \tag{C1}$$

where m is a particular maturity point on the yield curve while $\beta_0, \beta_1, \beta_2$,and τ are parameters. We choose parameters such that the sought for shapes are generated and that the average instantaneous forward rate over the first ten years is equal to 0.08. The curves are shown in Figure 13. The parameter values chosen are as follows:

Flat: $\beta_0 = 0.08, \beta_1 = 0, \beta_2 = 0$

Upward-Sloping: $\beta_0 = 0.08, \beta_1 = -0.05, \beta_2 = 0.14, \tau = 12.05$

Humped-Shape: $\beta_0 = 0.07, \beta_1 = -0.04, \beta_2 = 0.143, \tau = 1$

Downward-Sloping: $\beta_0 = -.01, \beta_1 = 0.1068, \beta_2 = 0.085, \tau = 10.00$

D Calculating Cost of Debt Capital

In this section we briefly discuss how we calculate the expected return premium or cost of debt capital for the different recovery forms. The input is the market credit spread, s, assumed given in continuously compounded percentage terms. The assumed constant interest rate is r. From this we calculate the price of the risky bond P_0 with a semiannual coupon paying an annual rate c and with a face value of F and a redemption maturity at time T

$$P_0 = F \exp\left(-(r+s)T\right) + \frac{c}{2}\sum_{i=1}^{2T} \exp\left(-(r+s)T_{i/2}\right) \tag{D1}$$

Given the market bond price of the risky bond we can solve implicitly for the expected return premia for the different recovery firms. First, let us consider a bond with the same contractual features of the risky bond but is default free. Its price is given by D_0. The risk-free yield for such a bond, y^*_{rf}, is defined as the solution to the equation

$$-D_0 + F\exp\left(-y^*_{rf}T\right) + \frac{c}{2}\sum_{i=1}^{2T}\exp\left(-y^*_{rf}T_{i/2}\right) = 0 \tag{D2}$$

D.1 Recovery of Treasury (RT)

The expected return y^*_{RT} under the RT case is defined as the solution to the equation

$$\begin{gathered}-P_0 + F\exp\left(-y^*_{RT}T\right)\left[1-\mathbb{P}_0\left(\tau<T\right)\right] + \\ \frac{c}{2}\sum_{i=1}^{2T}\exp\left(-y^*_{RT}T_{i/2}\right)\left[1-\mathbb{P}_0\left(\tau<T_{i/2}\right)\right] + \\ \omega\left[F\exp\left(-y^*_{RT}T\right)\mathbb{P}_0\left(\tau<T\right) + \frac{c}{2}\sum_{i=1}^{2T}\exp\left(-y^*_{RT}T_{i/2}\right)\mathbb{P}_0\left(\tau<T_{i/2}\right)\right] = 0,\end{gathered} \tag{D3}$$

where

$$\mathbb{P}_0\left(\tau<T\right) = N\left(\frac{-x_0-\left(\mu_*+\pi\right)T}{\sigma_v\sqrt{T}}\right) + e^{-\frac{2(\mu_*+\pi)x_0}{\sigma_v^2}}N\left(\frac{-x_0+\left(\mu_*+\pi\right)T}{\sigma_v\sqrt{T}}\right). \tag{D4}$$

which is the real cumulative probability of default. Given the expected return the RT expected return premium or cost of debt capital is

$$RP_{RT} = y^*_{RT} - y^*_{rf} \tag{D5}$$

D.2 Recovery of Treasury-Face Value (RT-F)

The expected return y^*_{RT-F} under the RT-F case is defined as the solution to the equation

$$\begin{gathered}-P_0 + F\exp\left(-y^*_{RT-F}T\right)\left[1-\mathbb{P}_0\left(\tau<T\right)\right] + \\ \frac{c}{2}\sum_{i=1}^{2T}\exp\left(-y^*_{RT-F}T_{i/2}\right)\left[1-\mathbb{P}_0\left(\tau<T_{i/2}\right)\right] + \\ \omega F\exp\left(-y^*_{RT-F}T\right)\mathbb{P}_0\left(\tau<T\right) = 0.\end{gathered} \tag{D6}$$

As above the RT-F expected return premium follows

$$RP_{RT-F} = y^*_{RT-F} - y^*_{rf} \tag{D7}$$

D.3 Recovery of Face Value (RFV)

The expected return y^*_{RFV} under the RFV case is defined as the solution to the equation

$$\begin{gathered}-P_0 + F\exp\left(-y^*_{RFV}T\right)\left[1-\mathbb{P}_0\left(\tau<T\right)\right] + \\ \frac{c}{2}\sum_{i=1}^{2T}\exp\left(-y^*_{RFV}T_{i/2}\right)\left[1-\mathbb{P}_0\left(\tau<T_{i/2}\right)\right] + \\ \omega F E_0^{\mathbb{P}}\left[\exp\left(-y^*_{RFV}\tau\right)\mathbf{1}_{\{\tau<T\}}\right]\end{gathered} \tag{D8}$$

where

$$\begin{gathered}E_0^{\mathbb{P}}\left[\exp\left(-y^*_{RFV}\tau\right)\mathbf{1}_{\{\tau<T\}}\right] = \left[e^{\frac{-x_0(\mu_*+\pi+\phi)}{\sigma_v^2}}N\left(\frac{-x_0+\phi T}{\sigma_V\sqrt{T}}\right) + e^{\frac{-x_0(\mu_*+\pi-\phi)}{\sigma_v^2}}N\left(\frac{-x_0-\phi T}{\sigma_V\sqrt{T}}\right)\right] \\ \text{where } \phi \equiv \sqrt{\left(\mu_*+\pi\right)^2 + 2\sigma_V^2 y^*_{RFV}} \text{ and } \mu_* \equiv r-\delta-\frac{\sigma_V^2}{2}\end{gathered} \tag{D9}$$

The RFV expected return premium follows

$$RP_{RFV} = y^*_{RFV} - y^*_{rf} \tag{D10}$$

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