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THE PROBABILISTIC REPRESENTATIVE VALUES

By Silvia Lorenzo-Freire, Balbina Casas-Méndez

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TILBURG UNIVERSITY

The probabilistic representative values*

Silvia Lorenzo-Freire[1,2] Balbina Casas-Méndez[3]

Abstract

In this paper we define a new family of solutions for the class of cooperative games with transferable utility, in which the set of players exhibits a structure of a priori unions. This family is deeply connected with the Shapley value for games with transferable utility but, moreover, we assume a solidarity strong connection among all the components of each union. As a consequence of this, they are disposed to delegate one coalition of members of the union to negotiate with the other unions, and, therefore, each union will have a representative coalition. Furthermore, three interesting solutions that belong to this family of values are studied, as well as the non cooperative selection of the best representative coalition for each union.

Key words: TU-games with unions, Shapley value, representative coalition.
JEL code: C71, C72.

1 Introduction

Classical model of cooperative games with transferable utility (TU-games, to abbreviate), involves a set of players in such a way that if a coalition of them decides to cooperate, they can guarantee a certain payoff. To share out the payoff of the total coalition, different solutions were studied in the literature. One of the most important of these solutions is the Shapley value (1953).

With the passing of time, due to the complexity of most of real situations, the traditional model of TU-games was enriched. Aumann and Dreze (1974) and Owen (1977) considered TU-games where there is a system of unions among the players, which is formed previously to the negotiation process, and that

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[1]Corresponding author. Postal address: Departamento de Estatística e IO, Facultade de Económicas, Universidade de Vigo, Campus Universitario Lagoas-Marcosende, 36310 Vigo, Spain. E-mail: silvia_lorenzo@uvigo.es.

[2]Departamento de Estatística e IO, Universidade de Vigo.

[3]Departamento de Estatística e IO, Universidade de Santiago de Compostela.

conditions it. Aumann and Dreze's value for TU-games with unions is defined with the condition of no transference of payoff among the unions, that is, each union only obtains the payoff that it can guarantee itself. The Owen value for TU-games with unions is constructed in two steps. In the first step, the unions decide the quantity each union receives, with the possibility of transference of payoff among them. In the second step, the division of the allocation of each union among its members is determined, taking into account the chances of each player to join other unions.

In this framework, we valuate the TU-games with unions from an alternative point of view. Our first assumption is that each union implies a strong compromise among their members in such a way that it is no possible to reach an agreement among a proper subset of a union and members of different unions. Consider, for example, a situation involving a set of users that have to pay the costs of a public good and these users are grouped by the company they belong to. In this situation, a contract among the members of each union can forbid all the users to join another company to share expenses. Our second assumption is that the strong contract which generates a union is also a solidarity contract that leads to the players within a union to divide equally benefits or losses. The last assumption consists on accepting that each union participates in the negotiation process by means of a representative coalition. For example, it is usual that a bank account involves a group of persons and that the associated contract allows one member to carry out all the transactions concerning the account. With these assumptions, we define a new family of solutions.

The paper is organized as follows. In section 2, we define the family of solutions by means of marginal contributions in all the possible arrival orders of players. Moreover, we identify the solutions as the Shapley values of TU-games. In section 3, we study three interesting solutions of this family and finally, in section 4, we give a non cooperative justification of the strength of some of these solutions.

2 Probabilistic representative values

A *cooperative game with transferable utility* (TU-game) is a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function, which assigns to each coalition $S \in 2^N$ the value $v(S)$, i.e., the benefits that S can generate independently of the players in $N \setminus S$. We assume that $v(\emptyset) = 0$. $G(N)$ denotes the set of TU-games with set of players N. *A solution for TU-games* is a function $f : G(N) \to \mathbb{R}^n$, where the payoff vector $f(N, v)$ is allocated to each TU-game (N, v).

A *TU-game with unions* is a triple (N, v, P), where (N, v) is a TU-game and $P = \{P_1, \dots, P_m\}$ is a partition of the set of players N. We denote $M = \{1, \dots, m\}$. For a TU-game with unions (N, v, P), the associated *quotient game* is the TU-game, (M, v^P), where the set of players can be identified with the set of unions, and for all $L \subset M$, $v^P(L) = v\left(\bigcup_{l \in L} P_l\right)$. We denote by $U(N)$ the set of TU-games with unions where the set of players is N. A *solution for TU-games with unions* is a function $f : U(N) \to \mathbb{R}^n$ that assigns to each TU-game with unions (N, v, P) the payoff vector $f(N, v, P)$.

Games with unions were first studied in Owen (1977). In that paper, Owen considers that the partition given by the unions modifies the negotiation possibilities of the players and defines a variation of the Shapley value which takes this fact into account: the Owen value.

The meaning of the unions in this paper is different from Owen's. We consider that a union is a set of solidary players, in the sense that:
a) the negotiation takes place among the unions,
b) the players of each union are anonymous, and
c) the benefits are equally allocated to its members.
Our aim is to modify the Shapley value for TU-games with unions under this new interpretation of the unions.

The main idea we take into account to provide such a modification is the natural feature that unions can only negotiate through *representatives.* Notice that players are solidary within unions, so we will not consider that a union is represented by a precise set of its members, but by a portion of its members. Since the representation system that each union will use is not fixed a priori, we will introduce a family of values instead of a particular value: the family of *probabilistic representative values.*

2.1 The definition of the family

The family of probabilistic representative values assumes that each P_l can be represented by any subset of k players of the union $(1 \leq k \leq |P_l|)$, where k is selected with probability q_k^l $\left(\sum_{k=1}^{|P_l|} q_k^l = 1, \text{for all } l \in M\right)$. Notice that this family includes values based on very natural representation systems. For instance, a union P_l can agree that all its members must sign in order that P_l is able to make a decision in the negotiation process (in which case $q_{|P_l|}^l = 1$, $q_k^l = 0$ for all $k \in \{1, \dots, |P_l| - 1\}$). When a particular representation system for each union is fixed, we have a particular probabilistic representative value. Next, we formally introduce the family of probabilistic representative values. In the next section, we study three specially relevant values in this family.

Take a TU-game with unions (N, v, P) with $P = \{P_1, \ldots, P_m\}$ and suppose that a *representation system* for each union is provided by $q = \big(q^1, \ldots, q^m\big)$, where each q^l is interpreted as above and satisfies that $q^l_k \geq 0$, for all $k \in \{1, \ldots, |P_l|\}$, $\sum_{k=1}^{|P_l|} q^l_k = 1$. An order for N is defined as a bijection $\sigma : N \to N$, where $\sigma(k)$ denotes the player in N which occupies the k-th position, and $\sigma^{-1}(i)$ denotes the position of player i in the order given by σ. We denote by $\Pi(N)$ the set of all $n!$ orders for N. To define the probabilistic representative value for (N, v, P) with representative system q, we use the heuristic interpretation of the Shapley value. So, we consider that the contribution of a player $i \in P_l$ to a coalition S must be taken into account only if $(S \cap P_l) \cup \{i\}$ fully represents union P_l and $S \cap P_l$ does not; in this case, the contribution of i to S is the contribution in the quotient game of P_l to the unions fully represented by the players in S. To write this in a formal way, we need to introduce some notations.

Take $\sigma \in \Pi(N)$ and $\sigma(k) \in N$; suppose that $\sigma(k) \in P_l$.

- $n^\sigma_k(r) = \big|\big\{i \in P_r : \sigma^{-1}(i) < k\big\}\big|$ is the number of players in the union P_r satisfying the condition of preceding the position k.
- $R^\sigma_k = \{r \in M : n^\sigma_k(r) > 0\}$ is the set of unions with players placed before position k.
- $Q^\sigma_k = \{r \in M : n^\sigma_k(r) = |P_r|\}$ is the set of unions whose players precede the position k.
- $P^\sigma_k = R^\sigma_k \setminus Q^\sigma_k$.
- For every $L \subset M$, $p^{\sigma,q}_1(L) = \prod_{r \in M\setminus\{l\} : r \in L} \sum_{j=1}^{n^\sigma_k(r)} q^r_j$ indicates the probability for all the unions of $L \setminus \{l\}$ of being represented before position k.
- For every $L \subset M$, $p^{\sigma,q}_2(L) = \prod_{r \in M\setminus\{l\} : r \notin L} \sum_{j=n^\sigma_k(r)+1}^{|P_r|} q^r_j$ is the probability for all the unions that do not belong to $L \cup \{l\}$ of being represented after position k.

Definition 1 *The probabilistic representative value for the TU-game with unions (N, v, P) and the representation system q is defined by*

$$\beta^q(N, v, P) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma, \beta^q}(v)$$

where, for all $\sigma \in \Pi(N)$, for all $P_l \in P$, for all player $\sigma(k) \in P_l$, $m^{\sigma,\beta^q}_{\sigma(k)}(v)$ is the expected contribution of the k-th player to the unions represented before this player, and it can be expressed as [1]

$$q^l_{n^\sigma_{k+1}(l)} \sum_{L \subset P^\sigma_k} p^{\sigma,q}(L) \left[v\left(\bigcup_{r \,\in\, L \,\cup\, Q^\sigma_k} P_r \cup P_l \right) - v\left(\bigcup_{r \,\in\, L \,\cup\, Q^\sigma_k} P_r \right) \right] {}^{2}$$

with

$$p^{\sigma,q}(L) = \begin{cases} p^{\sigma,q}_1(L) & \text{if } L = M \setminus \{l\} \\ p^{\sigma,q}_2(L) & \text{if } L = \emptyset \\ p^{\sigma,q}_1(L) p^{\sigma,q}_2(L) & \text{otherwise.} \end{cases}$$

To clarify this definition, we introduce two examples. In the first one, we illustrate how to calculate these solutions and, in the other one, we show a real situation that verifies all the assumptions we imposed at the beginning.

Example 1 Consider $N = \{1, 2, 3\}$, $P_1 = \{1, 2\}$ and $P_2 = \{3\}$ with $q = ((\frac{1}{3}, \frac{2}{3}), 1)$. In addition, $v(P_1) = 2$, $v(P_2) = 1$, and $v(N) = 4$. In Table 1, we calculate the expected contributions of the probabilistic representative value.

Order σ	m^{σ,β^q}_1	m^{σ,β^q}_2	m^{σ,β^q}_3
123	$\frac{2}{3}$	$\frac{4}{3}$	2
132	$\frac{2}{3}$	2	$\frac{4}{3}$
213	$\frac{4}{3}$	$\frac{2}{3}$	2
231	2	$\frac{2}{3}$	$\frac{4}{3}$
312	1	2	1
321	2	1	1

Table 1. The expected contributions

[1] It is convenient to point out that in the probabilistic representative values we only need the value of the game for groups of unions.

[2] Note that $q^l_{n^\sigma_{k+1}(l)}$ is the probability assigned in the union P_l to the player $\sigma(k)$. This probability must be considered because the union P_l is excluded in the definition of $p^{\sigma,q}(L)$.

Then, the corresponding probabilistic representative value is

$$\beta^q(N,v,P) = \frac{1}{n!}\sum_{\sigma\in\Pi(N)} m^{\sigma,\beta^q}(v) = \frac{1}{6}\left(\frac{23}{3},\frac{23}{3},\frac{26}{3}\right) = \left(\frac{23}{18},\frac{23}{18},\frac{13}{9}\right).$$

Example 2 At the moment, the Security Council of the United Nations consists of 15 members. They are disposed so that 5 of them are permanent and the other ones are elected by the General Assembly for a period of two years. Each member has a vote and decisions on substantive matters require 9 votes. Moreover, all the 5 permanent members have the "veto"power, that is, all their votes are necessary to pass a resolution.

Therefore, we may consider that the members are grouped in two unions $P_1 = \{\text{Permanent members}\}$ and $P_2 = \{\text{Non permanent members}\}$, and $q = (q^1, q^2)$ with $q^1 = (0,0,0,0,1)$ and $q^2 = (0,0,0,1,0,0,0,0,0,0)$.

If we calculate the associated probabilistic representative value, we obtain that

$$\beta_i^q(N,v,P) = \begin{cases} 0.19627, & i \in P_1 \\ 0.001865, & i \in P_2. \end{cases}$$

As was to be expected, the permanent members have much more power than the non permanent members.

2.2 The connection with the Shapley value

The family of probabilistic representative values of a TU-game with unions can not only be expressed as we have defined before, but also as the Shapley values of a TU-game related to the TU-game with unions and the representation system. To see this relation, we consider the following notation:

Given a coalition $S \subset N$, a set of unions $P = \{P_1, \ldots, P_m\}$, and a representation system q,

- $n_S(r) = |P_r \cap S|$ is the number of players in the union P_r belonging to S.
- $R_S = \{r \in M : P_r \cap S \neq \emptyset\} = \{r \in M : n_S(r) > 0\}$ is the set of unions with players in S.
- $Q_S = \{r \in M : P_r \subset S\} = \{r \in M : n_S(r) = |P_r|\}$ is the set of unions with all the players in S.
- $P_S = R_S \setminus Q_S$.

- For every $L \subset M$, $p_1^{S,q}(L) = \prod_{r \in L} \sum_{k=1}^{n_S(r)} q_k^r$ indicates the probability for all the unions of L of being represented in the coalition S.
- For every $L \subset M$, $p_2^{S,q}(L) = \prod_{r \in M \setminus L} \sum_{k=n_S(r)+1}^{|P_r|} q_k^r$ is the probability for all the unions that do not belong to L of being represented in the coalition $N \setminus S$.

In the next theorem, we establish that the probabilistic representative value of a TU-game with unions is the Shapley value ϕ of a TU-game in which the value of each coalition is the expectation of the combination of unions represented by the coalition.

Theorem 1 *For all $(N, v, P) \in U(N)$ and for all associated representation system q,*

$$\beta^q(N, v, P) = \phi(N, v^q)$$

where the game (N, v^q) is defined by

$$v^q(S) = \sum_{L \subset P_S} p^{S,q}(L) v\left(\bigcup_{r \,\in\, L \,\cup\, Q_S} P_r\right) \text{ for all } S \subset N,$$

with

$$p^{S,q}(L) = \begin{cases} p_1^{S,q}(L) & \text{if } L = M \setminus \{l\} \\ p_2^{S,q}(L) & \text{if } L = \emptyset \\ p_1^{S,q}(L) p_2^{S,q}(L) & \text{otherwise.} \end{cases}$$

Note that if $(N, v, P) \in U(N)$ is such that P is the trivial structure of unions $\{\{1\}, \ldots, \{n\}\}$, then $\beta^q(N, v, P) = \phi(N, v)$.

Example 3 If we consider Example 1 and calculate the associated game, we obtain that
$v^q(\{1\}) = v^q(\{2\}) = \frac{1}{3}v(P_1) = \frac{2}{3}$, $v^q(\{3\}) = v(P_2) = 1$,
$v^q(\{1, 2\}) = v(P_1) = 2$, $v^q(\{1, 3\}) = v^q(\{2, 3\}) = \frac{2}{3}v(P_2) + \frac{1}{3}v(N) = 2$, and
$v^q(N) = v(N) = 4$.

3 On three probabilistic representative values

In this section we will study three particular cases in the family of probabilistic representative values: the β, γ and δ values.

3.1 The β value

The β *value* is the probabilistic representative value where, fixed an order in the players, we assign the uniform discrete probability to the union. Then, all the players in a union have the same probability of being the representative player. In this way, the expression for each TU-game with unions (N, v, P), with $q_k^l = \frac{1}{|P_l|}$ for all $k \in \{1, \ldots, |P_l|\}$ and $l \in M$, is

$$\beta^q(N, v, P) = \beta(N, v, P) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma,\beta}(v)$$

where for all $\sigma \in \Pi(N)$, for all $P_l \in P$, for all player $\sigma(k) \in P_l$,

$$m_{\sigma(k)}^{\sigma,\beta}(v) = \frac{1}{|P_l|} \sum_{L \subset P_k^\sigma} p^{\sigma,\beta}(L) \left[v \left(\bigcup_{r \in L \cup Q_k^\sigma} P_r \cup P_l \right) - v \left(\bigcup_{r \in L \cup Q_k^\sigma} P_r \right) \right]$$

with $p^{\sigma,\beta}(L)$ as

$$p^{\sigma,\beta}(L) = \begin{cases} p_1^{\sigma,\beta}(L) & \text{if } L = M \setminus \{l\} \\ p_2^{\sigma,\beta}(L) & \text{if } L = \emptyset \\ p_1^{\sigma,\beta}(L) p_2^{\sigma,\beta}(L) & \text{otherwise,} \end{cases}$$

taking

$$p_1^{\sigma,\beta}(L) = \prod_{r \in M \setminus \{l\} \,:\, r \in L} \frac{|P_r \cap \{\sigma(1), \ldots, \sigma(k-1)\}|}{|P_r|} = \prod_{r \in M \setminus \{l\} \,:\, r \in L} \frac{n_k^\sigma(r)}{|P_r|}$$

and

$$p_2^{\sigma,\beta}(L) = \prod_{r \in M \setminus \{l\} \,:\, r \notin L} \frac{|P_r \cap \{\sigma(k+1), \ldots, \sigma(n)\}|}{|P_r|} = \prod_{r \in M \setminus \{l\} \,:\, r \notin L} \frac{|P_r| - n_k^\sigma(r)}{|P_r|}.$$

The next proposition shows that the β value can be calculated by computing the Shapley value in the quotient game, which is the game related to the unions, and dividing the value of each union equally among their members.

Proposition 1 *For all $(N, v, P) \in U(N)$, for all $i \in P_l \in P$, and for all $P_l \in P$, $\beta_i(N, v, P) = \frac{1}{|P_l|} \phi_l(M, v^P)$.*

3.2 The γ and δ values

The γ *value* is the probabilistic representative value where, given an order of the players, the last player is chosen as the representative player. So, for each TU-game with unions (N, v, P), taking $q^l_{|P_l|} = 1$ for all $l \in M$, we obtain that

$$\beta^q(N, v, P) = \gamma(N, v, P) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma,\gamma}(v)$$

where for all $\sigma \in \Pi(N)$, for all $P_l \in P$, for all player $\sigma(k) \in P_l$,

$$m^{\sigma,\gamma}_{\sigma(k)}(v) = \begin{cases} v\left(\left(\bigcup_{r \in Q^\sigma_k} P_r\right) \cup P_l\right) - v\left(\bigcup_{r \in Q^\sigma_k} P_r\right) & \text{if } P_l \subset \{\sigma(1), \ldots, \sigma(k)\} \\ 0 & \text{otherwise.} \end{cases}$$

The δ *value* is the probabilistic representative value where, for each order, the first player who arrives is the representative player. For each TU-game with unions (N, v, P), considering $q^l_1 = 1$ for all $l \in M$, we have that

$$\beta^q(N, v, P) = \delta(N, v, P) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma,\delta}(v)$$

where for all $\sigma \in \Pi(N)$, for all $P_l \in P$, for all player $\sigma(k) \in P_l$,

$$m^{\sigma,\delta}_{\sigma(k)}(v) = \begin{cases} v\left(\left(\bigcup_{r \in R^\sigma_k} P_r\right) \cup P_l\right) - \\ -v\left(\bigcup_{r \in R^\sigma_k} P_r\right) & \text{if } P_l \cap \{\sigma(1), \ldots, \sigma(k-1)\} = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Both values can be computed as weighted Shapley values in the quotient game and dividing the quantity obtained for a union among the players of this union (see Kalai and Samet, 1987). The same happens in the case of the β value (see Proposition 1). But, however, not all the probabilistic representative values coincide with weighted Shapley values in the quotient game. In the next example we show it.

Example 4 Consider $N = \{1, 2, 3, 4, 5, 6\}$ and $P = \{P_1, P_2, P_3\}$, with $P_1 = \{1\}$, $P_2 = \{2, 3\}$, and $P_3 = \{4, 5, 6\}$. The representation system is given by $q = (1, (1, 0), (0, 1, 0))$.

If we apply Definition 1 to the unanimity game[3] (N, u_N), $\beta^q(N, u_N, P) = \left(\frac{13}{30}, \frac{1}{12}, \frac{1}{12}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}\right)$.

Suppose that for all $(N, v, P) \in U(N)$, for all $i \in P_l \in P$ and for all $P_l \in P$, $\beta_i^q(N, v, P) = \frac{1}{|P_l|}\phi_l^w(M, v^P)$ for some $w \in \mathbb{R}^m_{++}$. Then, for all $i \in P_l \in P$ and for all $P_l \in P$, $\beta_i^q(N, u_N, P) = \frac{1}{|P_l|}\phi_l^w(M, u_N^P) = \frac{1}{|P_l|}\phi_l^w(M, u_M) = \frac{1}{|P_l|}\frac{w_l}{\sum_{r\in M} w_r}$, where the last equality is a consequence of the definition of the weighted Shapley values. Hence, for this set of a priori unions and this representation system, the vector of weights should be $w = \left(\frac{13}{30}, \frac{1}{6}, \frac{2}{5}\right)$.

Moreover, if we take (N, u_S) with $S = \{1, 2, 3\}$, we obtain that for all $i \in P_l \in P$ and for all $P_l \in P$, $\beta_i^q(N, u_S, P) = \frac{1}{|P_l|}\phi_l^w(M, u_S^P) = \frac{1}{|P_l|}\phi_l^w(M, u_L)$, with $L = \{1, 2\}$. By the definition of the weighted Shapley values, we obtain that $\phi^w(M, u_L) = \left(\frac{13}{18}, \frac{5}{18}, 0\right)$ and then $\beta^q(N, u_S, P) = \left(\frac{13}{18}, \frac{5}{36}, \frac{5}{36}, 0, 0, 0\right)$. But, if we use Definition 1, we obtain $\beta^q(N, u_S, P) = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}, 0, 0, 0\right)$. This contradiction proves that the supposition is not true.

To conclude the section, we give an example that illustrates the behavior of the three values defined above when we consider unanimity games.

Example 5 Consider the unanimity game $\left(N, u_{\{123\}}\right)$ with $N = \{1, 2, 3, 4\}$ and the sets of unions given in Table 2. This table shows the proposal of the β value, the γ value, the δ value, and the Owen value for the three partitions of the set of players.

P	β	γ	δ	Owen
$\{\{1\}, \{2, 3, 4\}\}$	$\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{3}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)$	$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0\right)$
$\{\{1\}, \{2\}, \{3, 4\}\}$	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{2}{5}, \frac{2}{5}, \frac{1}{10}, \frac{1}{10}\right)$	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$
$\{\{1, 2\}, \{3, 4\}\}$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0\right)$

Table 2. On three probabilistic representative values

Notice that, in the three cases, $\sum_{i\in P_r}\beta_i\left(N, u_{\{1,2,3\}}, P\right) = \sum_{i\in P_l}\beta_i\left(N, u_{\{1,2,3\}}, P\right)$,

[3]Given $S \subset N$, the unanimity game (N, u_S) is defined by $u_S(T) = \begin{cases} 1 & \text{if } S \subset T \\ 0 & \text{otherwise.} \end{cases}$ Moreover, S is the carrier of this unanimity game.

$$\frac{\sum_{i\in P_r}\gamma_i\left(N,u_{\{1,2,3\}},P\right)}{\sum_{i\in P_l}\gamma_i\left(N,u_{\{1,2,3\}},P\right)} = \frac{|P_r|}{|P_l|}, \text{ and } \frac{\sum_{i\in P_r}\delta_i\left(N,u_{\{1,2,3\}},P\right)}{\sum_{i\in P_l}\delta_i\left(N,u_{\{1,2,3\}},P\right)} = \frac{|P_l|}{|P_r|} \text{ for all } P_r,$$

$P_l \in P$. This always happens with the unions that have at least one player in the carrier of the unanimity game. Then, in the case of the γ value, all the players receive the same. It does not always happen with the other values, although in the three cases the players in each union obtain the same because the three values belong to the family of probabilistic representative values.

In spite of the fact that the payoffs in the Owen value for the unions are equal to the payoffs in the case of the β value, the Owen value does not give the same payoff to all the players in each union (the Owen value is not a probabilistic representative value) because the players who do not contribute to the other coalitions are always paid zero.

4 A strategic non cooperative selection of the representation system

This section is devoted to study what happens when the representation system is not fixed and the unions are able to choose the representation system more convenient to maximize their payoffs. For this purpose, we consider a negotiation process among the unions where each union decides its representation system independently of the other unions and such that no binding agreements can be reached among unions. So, we will define a strategic game to describe this procedure.

Definition 2 *Let (N, v, P) be a TU-game with unions such that the set of unions is $P = \{P_1, \ldots, P_m\}$.*

The representation strategic game $\Gamma(N, v, P) = (\{X_l\}_{l\in M}, \{U_l\}_{l\in M})$ is an m-person strategic game with players set $M = \{1, \ldots, m\}$, where

- $X_l = \left\{ x^l \in \mathbb{R}_+^{P_l} : \exists k \in \{1, \ldots, |P_l|\} \text{ with } x_k^l = 1 \text{ and } \sum_{j=1}^{|P_l|} x_j^l = 1 \right\}$ *is the non-empty strategy set of union $P_l \in P$. We will denote by $x^{(l,k)}$ the strategy of union P_l where $x_k^l = 1$.*
- $U_l : X = \prod_{l=1}^{m} X_l \longrightarrow \mathbb{R}$ *is the payoff function of the union P_l, which assigns to each combination of strategies $x = (x^1, \ldots, x^m) \in X$ the payoff*

$$U_l(x^1, \ldots, x^m) = \sum_{i \in P_l} \beta_i^x(N, v, P).$$

Proposition 2 *Let $G(N,v,P) = (\{\Delta(X_l)\}_{l \in M}, \{U_l\}_{l \in M})$ be the mixed extension of the strategic game associated to (N,v,P). For all $l \in M$*

a) $\Delta(X_l) = \left\{ q^l \in \mathbb{R}_+^{P_l} : \sum_{j=1}^{|P_l|} q_j^l = 1 \right\}.$

b) For all $q = (q^1, \ldots, q^m) \in \prod_{k=1}^{m} \Delta(X_k), U_l(q) = \sum_{i \in P_l} \beta_i^q(N, v, P).$

Theorem 2 *Let (N,v,P) be a TU-game with unions such that (N,v) is convex*[4] *and let $G(N,v,P) = (\{\Delta(X_l)\}_{l \in M}, \{U_l\}_{l \in M})$ be the mixed extension of the corresponding strategic game. The combination of strategies $q = (q^1, \ldots, q^m)$, such that $q^l = x^{(l,|P_l|)}$ for all $l \in M$, is a dominant combination of $G(N,v,P)$.*

According to the theorem, independently of the election of the other unions, the best choice for a union consists on being a strong union, that is, being only represented by all the players of this union. This means that the best element of the family of the probabilistic representative values when the game is convex for the unions is the γ value. On the contrary, the best for a union in a concave game consists on being represented by just a player of this union, as the next result establishes.

Corollary 1 *Let (N,v,P) be a TU-game with unions such that (N,v) is concave*[5] *and let $G(N,v,P) = (\{\Delta(X_l)\}_{l \in M}, \{U_l\}_{l \in M})$ be the mixed extension of the corresponding strategic game. The combination of strategies $q = (q^1, \ldots, q^m)$, such that $q^l = x^{(l,1)}$ for all $l \in M$, is a dominant combination of $G(N,v,P)$.*

[4] (N,v) is a convex game when $v(S \cup T) - v(S) \leq v(R \cup T) - v(R)$ for all $S \subset R \subset N \backslash T$.

[5] (N,v) is a concave game when $v(S \cup T) - v(S) \geq v(R \cup T) - v(R)$ for all $S \subset R \subset N \backslash T$.

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5 Appendix

5.1 Proof of Theorem 1

We know that the Shapley value can be interpreted as an average of the marginal contributions of the players in each order. Then

$$\phi(N, v^q) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(v^q)$$

with

$$m^{\sigma}_{\sigma(k)}(v^q) = v^q(\sigma(1), \ldots, \sigma(k)) - v^q(\sigma(1), \ldots, \sigma(k-1)) \text{ for all } 1 \leq k \leq n.$$

Therefore, if the marginal contribution for each order of the Shapley value of the game v^q coincides with the marginal contribution of the same order related to the solution β^q, then this Shapley value coincides with the solution. So, we will prove that

$$m^{\sigma}_{\sigma(k)}(v^q) = m^{\sigma,\beta^q}_{\sigma(k)}(v) \text{ for all } \sigma \in \Pi(N) \text{ and for all } 1 \leq k \leq n.$$

By the definition of v^q, $m^{\sigma}_{\sigma(k)}(v^q) =$

$$\sum_{L \subset P_{\{\sigma(1),\ldots,\sigma(k)\}}} p^{\{\sigma(1),\ldots,\sigma(k)\},q}(L) v\left(\bigcup_{r \in L \cup Q_{\{\sigma(1),\ldots,\sigma(k)\}}} P_r \right) -$$

$$\sum_{L \subset P_{\{\sigma(1),\ldots,\sigma(k-1)\}}} p^{\{\sigma(1),\ldots,\sigma(k-1)\},q}(L) v\left(\bigcup_{r \in L \cup Q_{\{\sigma(1),\ldots,\sigma(k-1)\}}} P_r \right) =$$

$$\sum_{L \subset P^{\sigma}_{k+1}} p^{\{\sigma(1),\ldots,\sigma(k)\},q}(L) v\left(\bigcup_{r \in L \cup Q^{\sigma}_{k+1}} P_r \right) -$$

$$\sum_{L \subset P^{\sigma}_{k}} p^{\{\sigma(1),\ldots,\sigma(k-1)\},q}(L) v\left(\bigcup_{r \in L \cup Q^{\sigma}_{k}} P_r \right).$$

On the assumption that $\sigma(k) \in P_l$, we obtain that

$$\sum_{L \subset P^{\sigma}_{k+1}} p^{\{\sigma(1),\ldots,\sigma(k)\},q}(L) v\left(\bigcup_{r \in L \cup Q^{\sigma}_{k+1}} P_r \right) =$$

$$\sum_{L \subset P^{\sigma}_{k} \setminus \{l\}} \sum_{j=1}^{n^{\sigma}_k(l)+1} q^l_j \left(\prod_{r \in M : r \in L} \sum_{j=1}^{n^{\sigma}_k(r)} q^r_j \right) \left(\prod_{r \in M : r \notin L \cup \{l\}} \sum_{j=n^{\sigma}_k(r)+1}^{|P_r|} q^r_j \right)$$

$$v\left(\bigcup_{r \in L \cup Q^{\sigma}_{k}} P_r \cup P_l \right) +$$

$$\sum_{L\subset P_k^\sigma\setminus\{l\}} \sum_{j=n_k^\sigma(l)+2}^{|P_l|} q_j^l \left(\prod_{r\in M: r\in L} \sum_{j=1}^{n_k^\sigma(r)} q_j^r\right) \left(\prod_{r\in M: r\notin L\cup\{l\}} \sum_{j=n_k^\sigma(r)+1}^{|P_r|} q_j^r\right)$$

$$v\left(\bigcup_{r\,\in\,L\,\cup\,Q_k^\sigma} P_r\right)$$

and

$$\sum_{L\subset P_k^\sigma} p^{\{\sigma(1),\ldots,\sigma(k-1)\},q}(L) v\left(\bigcup_{r\,\in\,L\,\cup\,Q_k^\sigma} P_r\right) =$$

$$\sum_{L\subset P_k^\sigma\setminus\{l\}} \sum_{j=1}^{n_k^\sigma(l)} q_j^l \left(\prod_{r\in M: r\in L} \sum_{j=1}^{n_k^\sigma(r)} q_j^r\right) \left(\prod_{r\in M: r\notin L\cup\{l\}} \sum_{j=n_k^\sigma(r)+1}^{|P_r|} q_j^r\right)$$

$$v\left(\bigcup_{r\,\in\,L\,\cup\,Q_k^\sigma} P_r \cup P_l\right) +$$

$$\sum_{L\subset P_k^\sigma\setminus\{l\}} \sum_{j=n_k^\sigma(l)+1}^{|P_l|} q_j^l \left(\prod_{r\in M: r\in L} \sum_{j=1}^{n_k^\sigma(r)} q_j^r\right) \left(\prod_{r\in M: r\notin L\cup\{l\}} \sum_{j=n_k^\sigma(r)+1}^{|P_r|} q_j^r\right)$$

$$v\left(\bigcup_{r\,\in\,L\,\cup\,Q_k^\sigma} P_r\right).$$

Thus

$$m_{\sigma(k)}^\sigma(v^q) = \sum_{L\subset P_k^\sigma\setminus\{l\}} q_{n_k^\sigma(l)+1}^l \left(\prod_{r\in M: r\in L} \sum_{j=1}^{n_k^\sigma(r)} q_j^r\right) \left(\prod_{r\in M: r\notin L\cup\{l\}} \sum_{j=n_k^\sigma(r)+1}^{|P_r|} q_j^r\right)$$

$$\left[v\left(\bigcup_{r\,\in\,L\,\cup\,Q_k^\sigma} P_r \cup P_l\right) - v\left(\bigcup_{r\,\in\,L\,\cup\,Q_k^\sigma} P_r\right)\right] = m_{\sigma(k)}^{\sigma,\beta^q}(v).$$

5.2 Proof of Proposition 1

Let $(N, v, P) \in U(N)$. If we define the set of the permutations where the player $i \in P_l$ is in the position k, with $1 \le k \le n$, as $\Pi_i^k(N) = \{\sigma \in \Pi(N) : \sigma(k) = i\}$, we know that

$$\beta_i(N, v, P) = \frac{1}{n!} \sum_{\sigma\in\Pi(N)} m_i^{\sigma,\beta}(v) = \frac{1}{n!} \sum_{k=1}^{n} \sum_{\sigma\in\Pi_i^k(N)} m_{\sigma(k)}^{\sigma,\beta}(v) =$$

$$\frac{1}{n!} \frac{1}{|P_l|} \sum_{k=1}^{n} \sum_{\sigma\in\Pi_i^k(N)} \sum_{L\subset P_k^\sigma} p^{\sigma,\beta}(L) \left[v\left(\bigcup_{r\,\in\,L\,\cup\,Q_k^\sigma} P_r \cup P_l\right) - v\left(\bigcup_{r\,\in\,L\,\cup\,Q_k^\sigma} P_r\right)\right].$$

We are only interested in the cases where $\bigcup_{r\,\in\,L\,\cup\,Q_k^\sigma} P_r = \bigcup_{l'\,\in\,M'} P_{l'}$, with $M' \subset M \setminus \{l\}$. Then

$$\beta_i(N, v, P) =$$

$$\frac{1}{n!}\ \frac{1}{|P_l|}\sum_{M'\subset M\setminus\{l\}}\left[v\left(\bigcup_{l'\ \in\ M'}P_{l'}\cup P_l\right)-v\left(\bigcup_{l'\ \in\ M'}P_{l'}\right)\right]$$

$$\sum_{k=m'+1}^{n-m+m'+1}\ \sum_{\sigma\in\Pi_i^k(N):[\exists L\subset P_k^\sigma:L\cup Q_k^\sigma\cup\{l\}=M'\cup\{l\}]}\ \sum_{L\subset P_k^\sigma:L\cup Q_k^\sigma\cup\{l\}=M'\cup\{l\}} p^{\sigma,\beta}(L).$$

Moreover, given $M'\subset M\setminus\{l\}$,

$$\sum_{k=m'+1}^{n-m+m'+1}\ \sum_{\sigma\in\Pi_i^k(N):[\exists L\subset P_k^\sigma:L\cup Q_k^\sigma\cup\{l\}=M'\cup\{l\}]}\ \sum_{L\subset P_k^\sigma:L\cup Q_k^\sigma\cup\{l\}=M'\cup\{l\}} p^{\sigma,\beta}(L)=$$

$$m'!\sum_{k=0}^{n-m}\left[\binom{k+m'}{k}(n-1-k-m')!\prod_{t=0}^{k-1}(n-m-t)\right]=$$

$$m'!(n-m)!(m-m'-1)!\sum_{k=0}^{n-m}\binom{k+m'}{k}\binom{n-1-k-m'}{m-m'-1}=$$

$$m'!(n-m)!(m-m'-1)!\sum_{k=1}^{n-m+1}\binom{k+m'-1}{k-1}\binom{n-k-m'}{m-m'-1}.$$

Now, we will show that

$$\sum_{k=1}^{n-m+1}\binom{k+m'-1}{k-1}\binom{n-k-m'}{m-m'-1}=\binom{n}{m}. \tag{1}$$

To obtain (1), we have to prove that

$$\sum_{k=1}^{n-m+1}\binom{k+m'-1}{k-1}\binom{n-k-m'}{m-m'-1}=\sum_{k=1}^{n-m+1}\binom{k+m'-2}{k-1}\binom{n-m'-k+1}{m-m'}. \tag{2}$$

To prove it, we will apply Stifel's formula.[6] So, using this formula several times, we know that for $1\le k\le n-m+1$,

$$\binom{k+m'-1}{k-1}=\sum_{k'=1}^{k}\binom{k+m'-1-k'}{k-k'}$$

and, therefore,

$$\sum_{k=1}^{n-m+1}\binom{k+m'-1}{k-1}\binom{n-k-m'}{m-m'-1}=$$

[6] Given $n_1,\ n_2\in\mathbb{N}$, Stifel's formula says that $\binom{n_1}{n_2}=\binom{n_1-1}{n_2}+\binom{n_1-1}{n_2-1}$.

$$\sum_{k=1}^{n-m+1} \binom{n-k-m'}{m-m'-1} \sum_{k'=1}^{k} \binom{k+m'-1-k'}{k-k'} =$$

$$\sum_{k=1}^{n-m+1} \binom{k+m'-2}{k-1} \sum_{k'=k}^{n-m+1} \binom{n-m'-k'}{m-m'-1}.$$

On the other hand, by Stifel's formula

$$\sum_{k'=k}^{n-m+1} \binom{n-m'-k'}{m-m'-1} = \binom{n-m'-k+1}{m-m'}$$

and, hence, we have (2).

To prove (1), we repeat $m'-1$ times the process used to obtain (2). Thus

$$\sum_{k=1}^{n-m+1} \binom{k+m'-1}{k-1} \binom{n-k-m'}{m-m'-1} = \sum_{k=1}^{n-m+1} \binom{k-1}{k-1} \binom{n-k}{m-1} = \binom{n}{m}.$$

Finally, with all the results we have obtained, we conclude that

$$\sum_{k=m'+1}^{n-m+m'+1} \sum_{\sigma\in\Pi_i^k(N):[\exists L\subset P_k^\sigma : L\cup Q_k^\sigma\cup\{l\}=M'\cup\{l\}]} \sum_{L\subset P_k^\sigma : L\cup Q_k^\sigma \cup\{l\}=M'\cup\{l\}} p^{\sigma,\beta}(L) =$$

$$\frac{m'!n!(m-m'-1)!}{m!}.$$

As a result of this expression, we can assert that

$$\beta_i(N,v,P) =$$

$$\frac{1}{n!|P_l|} \sum_{M'\subset M\setminus\{l\}} \frac{m'!n!(m-m'-1)!}{m!} \left[v\left(\bigcup_{l'\in M'} P_{l'} \cup P_l \right) - v\left(\bigcup_{l'\in M'} P_{l'} \right) \right] =$$

$$\frac{1}{n!|P_l|} \sum_{M'\subset M\setminus\{l\}} \frac{n!}{m!} \sum_{\tau\in\Pi_l^{m'+1}(M):\tau(j)\in M'\ \forall j<m'+1} \left[v^P\left(M'\cup l\right) - v^P\left(M'\right)\right] =$$

$$\frac{1}{|P_l|}\phi_l(M,v^P).$$

5.3 Proof of Proposition 2

a) comes from the definition of $\Delta(X_l)$. From now on, we will denote $\Delta(X_l)=Q_l$. With regard to b), we will prove it defining the function $f: \prod_{r=1}^{m} Q_r \longrightarrow \mathbb{R}$ such that for all $q \in \prod_{r=1}^{m} Q_r$, $f(q) = \sum_{i \in P_l} \beta_i^q(N, v, P)$. This function f verifies that for all $r \in M$ and $\lambda \in [0,1]$,

$$f(q^1, \ldots, q^{r-1}, \lambda \hat{q^r} + (1-\lambda)\tilde{q}^r, q^{r+1}, \ldots, q^m) =$$

$$\lambda f(q^1, \ldots, q^{r-1}, \hat{q^r}, q^{r+1}, \ldots, q^m) + (1-\lambda) f(q^1, \ldots, q^{r-1}, \tilde{q}^r, q^{r+1}, \ldots, q^m).$$

To simplify, we denote $\bar{q} = (q^1, \ldots, q^{r-1}, \lambda \hat{q^r} + (1-\lambda)\tilde{q}^r, q^{r+1}, \ldots, q^m)$, $\hat{q} = (q^1, \ldots, q^{r-1}, \hat{q^r}, q^{r+1}, \ldots, q^m)$, and $\tilde{q} = (q^1, \ldots, q^{r-1}, \tilde{q}^r, q^{r+1}, \ldots, q^m)$. By Theorem 1, we know that $f(\bar{q}) = \sum_{i \in P_l} \phi_i(N, v^{\bar{q}})$. Moreover, it can be easily seen that for all $S \subset N$, $v^{\bar{q}}(S) = \lambda v^{\hat{q}}(S) + (1-\lambda) v^{\tilde{q}}(S)$. Thus, as the Shapley value satisfies the property of linearity, we know that $f(\bar{q}) = \sum_{i \in P_l} \phi_i(N, \lambda v^{\hat{q}} + (1-\lambda) v^{\tilde{q}}) = \lambda \sum_{i \in P_l} \phi_i(N, v^{\hat{q}}) + (1-\lambda) \sum_{i \in P_l} \phi_i(N, v^{\tilde{q}}) = \lambda f(\hat{q}) + (1-\lambda) f(\tilde{q})$.

Applying this result, we obtain that for all $q = (q^1, \ldots, q^m) \in \prod_{r=1}^{m} Q_r$,

$$U_l(q^1, \ldots, q^m) = \sum_{j_1=1}^{|P_1|} \cdots \sum_{j_m=1}^{|P_m|} q^1_{j_1} : . : q^m_{j_m} U_l(x^{(1,j_1)}, \ldots, x^{(m,j_m)}) =$$

$$\sum_{j_1=1}^{|P_1|} \cdots \sum_{j_m=1}^{|P_m|} q^1_{j_1} : . : q^m_{j_m} \sum_{i \in P_l} \beta_i^{(x^{(1,j_1)}, \ldots, x^{(m,j_m)})}(N, v, P) = \sum_{i \in P_l} \beta_i^q(N, v, P).$$

5.4 Proof of Theorem 2

Let (N, v, P) be a TU-game with unions and take $l \in M$. Then, if we denote $(x \setminus x^{(l,k)}) = (x^1, \ldots, x^{k-1}, x^{(l,k)}, x^{k+1}, \ldots, x^m)$ for all $k \in \{1, \ldots, |P_l|\}$, we have to prove that

$$U_l(x \setminus x^{(l,|P_l|)}) \geq U_l(x \setminus x^{(l,k)}) \text{ for all } x \in X \text{ and all } k \in \{1, \ldots, |P_l| - 1\}.$$

For it, it is enough to check that for $i \in P_l$,

$$\beta_i^{(x \setminus x^{(l,|P_l|)})}(N, v, P) \geq \beta_i^{(x \setminus x^{(l,k)})}(N, v, P). \tag{3}$$

So, we define for each $x \in X$ the subset of orders of players $O_i^x = \{\sigma \in \Pi(N) \text{ such that } i \text{ represents } P_l \text{ with the representation system } x\}$. Now, we consider the function $g : O_i^{(x \setminus x^{(l,|P_l|)})} \longrightarrow O_i^{(x \setminus x^{(l,k)})}$ which assigns to an order where the player i is the last in the union who arrives, $\sigma \in O_i^{(x \setminus x^{(l,|P_l|)})}$,

the order $g(\sigma) = \tilde{\sigma}$, where only the positions given by σ of player i and the k-th player in the union that arrives are exchanged.
Due to the fact that (N, v) is convex[7], if we take $\sigma \in O_i^{(x \setminus x^{(l,|P_l|)})}$ and $\tilde{\sigma} = g(\sigma)$, we have that $m_i^{\sigma, \beta^{(x \setminus x^{(l,|P_l|)})}}(v) \geq m_i^{\tilde{\sigma}, \beta^{(x \setminus x^{(l,k)})}}(v)$.
On the other hand, $\beta_i^x(N, v, P) = \frac{1}{n!} \sum_{\sigma \in O_i^x} m_i^{\sigma, \beta^x}(v)$. Then, as g is a bijection, we can conclude that (3) is true.

[7]In fact, we only need the convexity of the quotient game (N, v^P).